THE n-MOMENT INSURANCE CAPM

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Abstract

Until recently, the importance of skewness in the rate of return distribution has been largely unrecognized in financial journals. The re-emergence of skewness in financial literature is particularly relevant to catastrophe insurance products where some of the most extremely skewed distributions occur. This paper presents an argument for including a provision in the equilibrium premium to cover the cost of skewness. It also generalizes the insurance CAPM to n *moments. This extension permits explicitly determining the impact that skewness and other higher moments have on the needed premium.*

1. ASYMMETRY AND ITS IMPLICATIONS

In much of modern finance theory, it is presumed that the standard deviation of the rate of return is the appropriate measure of risk to the investor. The Capital Asset Pricing Model (CAPM), for example, assumes this to be true. It is then a mathematical consequence of this and a few other assumptions that only the systematic component (beta) of this risk is rewarded in financial markets. This seems quite reasonable for returns that are symmetrically distributed. It does not seem so reasonable, however, for returns that are asymmetrically distributed. Consider that, although investors dislike unexpected large losses, they like unexpected large gains. It seems reasonable then that investors place different values on two different securities that promise the same expected return and the same standard deviation of return but differ in that the return on one is symmetrically distributed while the return on the other is positively skewed.[1] In fact, there are reasons to believe, and evidence which corroborates, that the latter security is preferred to the former.

For example, Arditti (1967, p. 21) argues that it is reasonable to expect risk aversion to decrease with wealth. He gives an example of a bet with two equally likely outcomes: either a loss of \$10,000 or a gain of \$20,000. Since both outcomes are equally likely the expected value is \$5,000. He then asks if a wealthy man or a poor man would more likely pay a higher price for this bet. Arditti concludes that it is reasonable to expect a wealthy man to pay more for this bet since in his words "a loss of \$10,000 to him would be trivial while a similar loss to the poor man would render him assetless." Arditti goes on to show that whenever risk aversion decreases with wealth, it necessarily follows that positive skewness is preferred. That is, investors are willing to pay a premium, or give up expected return, in exchange for positive skewness.

One does not have to go any farther than to consider all of the various state-run lotteries as corroborating examples. Lottery players face an almost certain loss of a trivial amount in exchange for a trivial probability of a very large gain. The expected return on lottery tickets is, of course, negative since government extracts a significant portion of the revenues. Lottery players, thus, pay a premium in exchange for positive skewness.

Others have reached the same conclusions for opportunities similar to the lottery. In a discussion trying to explain Internet stock price increases, Alan Greenspan (1999, p. C1) described this "lottery premium" in the *Wall Street Journal*:

> What lottery managers have known for centuries is that you could get somebody to pay for a one-in-a-

[1]For purposes of this paper, we are using William Sharpe's (1985) definition of security, i.e., a security is "a legal representation of the right to receive prospective future benefits under stated conditions."

> million shot more than the [pure economic] value of that chance.

Consider, for a moment, the lottery as a relevant analogy to understand the skewness associated with catastrophes. Catastrophe insurance can be thought of from the policyholder's perspective as a conditional lottery. This provides a concrete example of the cost of skewness. With this lottery, if the catastrophe occurs then there is a huge payoff. Of course, there is also a large loss that offsets the payoff. But the loss is there regardless of insurance. Thus, if the loss is going to happen, it is preferable to have insurance.

Imagine a security that trades in financial markets and promises a large payoff in the event of a catastrophe somewhere else in the world. The details don't really matter for this example, as long as the payoff is triggered by a rare, random event. Since the cash flows are similar to those of a lottery, we can expect that the purchasers, as is true with a lottery, would pay a skewness premium. One implication of the Capital Asset Pricing Model is that all investors hold the same portfolio of risky assets, the market portfolio, even if it might include lottery tickets. Since investors are holding the market portfolio, the skewness premium would reflect only systematic components of skewness, i.e., that portion of skewness that cannot be diversified away. But the cash flows on this security are also similar to those of catastrophe insurance. Hence, the free market price of this security, which includes the cost of skewness, must also equal the equilibrium price for a perfectly corresponding catastrophe insurance contract, i.e., a contract with the same expected cash flows, the same systematic risk of receiving those cash flows, the same systematic skewness, etc.

One might argue that this analogy is inappropriate, since there is a fundamental difference between the demand for lottery tickets and catastrophe insurance. The cost of skewness, however, is unaffected. Consider that a person might be willing to buy a lottery ticket for a dollar, but unwilling to buy

1,000,000 of them. Clearly a person's willingness to buy tickets depends on his overall wealth as well as his preference for skewness and other factors. Certainly he would be more willing to buy one lottery ticket rather than say 200 (the price of the catastrophe insurance). With a single lottery ticket there is only one dollar at risk. With 200 tickets, there are 200 dollars at risk. What motivates people to buy the catastrophe insurance, though, is that the lottery is contingent on an otherwise bad event. It is offsetting the risk of that bad event that motivates them to buy catastrophe insurance. Accordingly, we can expect that people are more willing to buy 200 dollars worth of catastrophe insurance than 200 dollars worth of lottery tickets. But the cash flows in the catastrophe insurance are identical to the cash flows in the lottery, so the cost of skewness must be the same for both. Preference for skewness varies from individual to individual in a complex and unknown way. It is certainly multi-variate, with wealth being one of the variables. But in the aggregate, the market determines the price for skewness in such a way that the markets clear. Demand is also a variable that depends upon price, and so supply and demand are in balance at the equilibrium price.

Hence, the equilibrium returns implied by the CAPM may be inadequate for securities with heavily skewed returns. Accordingly, to adequately charge for an insurance policy covering hurricane and other catastrophic risks, a provision covering the cost of skewness must be added to the otherwise needed premium to compensate investors for the extremely skewed loss distributions of catastrophes.

Others have also recognized this shortcoming of the CAPM. For example, Yehuda Kahane (1979) notes the need for analyzing higher moments of profit distributions for certain utility assumptions in his paper deriving the insurance CAPM. He states on page 237:

> All distributions were assumed to be characterized by the first two moments. This makes the model ac-

> ceptable only for certain utility assumptions. … Thus, measures of asymmetry, like the skewness and semi-variance, may be needed in a loading formula (especially for risks with catastrophic nature—which are represented by extremely skewed distributions).

Alan Kraus and Robert Litzenberger (1976) go even further by stating on page 1086 that:

> The evidence suggests that prior empirical findings that are interpreted as inconsistent with the traditional theory can be attributed to misspecification of the capital asset pricing model by omission of systematic (non-diversifiable) skewness.

Campbell Harvey and Akhtar Siddique (2000) define systematic skewness, or coskewness on page 1265:

> [Coskewness is] the component of an asset's skewness related to the market portfolio's skewness.

In order to capture the contribution of the cost of skewness to the equilibrium return, it is necessary to generalize the CAPM. Section 2 presents the three-moment CAPM derived by Rubinstein (1973) and Kraus and Litzenberger (1976). Section 3 derives the three-moment insurance CAPM. Section 4 derives the n-moment insurance CAPM. This derivation depends on the n-moment CAPM that is derived in the Appendix. Section 5 presents conclusions and implications.

2. THE THREE-MOMENT CAPM

2.1. The Model

Kraus–Litzenberger (1976) follow Rubinstein's lead (1973) in their development of a three-moment capital asset pricing model that incorporates the coskewness of an asset. (See the Appendix for a formal derivation of the model.) Their model of equilibrium returns, assuming the rate of return on the market portfolio is nonsymmetrically distributed, is given below:

$$\mathrm{E}(R_i) - R_f = b_1\beta_i + b_2\gamma_i \tag{2.1}$$

where

$R_f = 1 + r_f$ = one plus the risk-free rate of return,

$R_i = 1 + r_i$ = one plus the rate of return on ith asset,

$R_M = 1 + r_M$ = one plus the rate of return on market portfolio,

$$\beta_i = \frac{\sigma_{R_i R_M}}{\sigma^2_{R_M}} = \frac{\mathrm{E}([R_i - \mathrm{E}(R_i)][R_M - \mathrm{E}(R_M)])}{\mathrm{E}([R_M - \mathrm{E}(R_M)]^2)},$$

$$\gamma_i = \frac{\tau_{R_i R_M R_M}}{\tau^3_{R_M}} = \frac{\mathrm{E}([R_i - \mathrm{E}(R_i)][R_M - \mathrm{E}(R_M)]^2)}{\mathrm{E}([R_M - \mathrm{E}(R_M)]^3)},$$

$\tau_{R_M} = (\mathrm{E}[(R_M - \mathrm{E}(R_M))^3])^{1/3}$,

b_1 = market risk premium, and

b_2 = market skewness premium.

Simplifying (2.1) leads to:

$$\mathrm{E}(r_i) - r_f = b_1\beta_i + b_2\gamma_i. \tag{2.2}$$

One final simplification leads to the intercept form of the equation:

$$\mathrm{E}(r_i) = r_f + b_1\beta_i + b_2\gamma_i. \tag{2.3}$$

Kraus and Litzenberger's derivation assumes that all investors have the same probability beliefs, and further, that each investor's risk tolerance is a linear function of wealth, $(a_i + bW_i)$, with the same cautiousness, b, for all investors. These assumptions are required to ensure that each investor's optimal risk asset portfolio is the same, that is, the market portfolio. These assumptions are very strong and arguably unreasonable. However, if one's purpose is to estimate equilibrium returns, then it is not essential that all investors have the same optimal risk asset portfolio. In the case of disagreement, b_1 and b_2 may still be interpreted as the market price of risk and the market price of skewness, respectively, as will be shown in a later section of this paper.

Kraus and Litzenberger empirically tested the three-moment model using monthly, deflated excess rates of return. That is, their measure of the rate of return for the ith security is $(R_i - R_f)/R_f$, where the returns are measured over a monthly holding period. They state on page 1098:

> Empirical evidence is presented that is consistent with a three moment valuation model. Investors are found to have an aversion to variance and a preference for positive skewness.

Specifically, they found the values of b_1 (the market risk premium) and b_2 (the market skewness premium) to be 1.119 and -0.212, respectively. Moreover, both were significant. As Arditti shows, whenever risk aversion decreases with wealth, it follows that positive skewness is preferred. This further implies that b_2 and τ_{R_M} are of opposite sign. For example, if the market is positively skewed, or τ_{R_M} is positive, then investors will give up return, which implies a negative b_2, in exchange for this positive skewness. Kraus and Litzenberger's results confirm this expectation. Since β and γ for the market portfolio are both equal to one, a negative value for b_2 and a positive value for τ_{R_M} necessarily increases the market risk premium, and thus, the significance of risk.

The following hypothetical example demonstrates the impact of coskewness on the traditional CAPM estimate. In the traditional two-moment CAPM, the excess of the expected return on the market portfolio over the risk-free rate is the market risk premium, but in the three-moment model this excess amount is the sum of the market risk premium and the market skewness premium. By definition, the beta and gamma of the market portfolio are one. Hence, from Equation (2.2) for the market portfolio we have:

$$E(r_m) - r_f = b_1 + b_2.$$

As mentioned earlier, Kraus and Litzenberger estimated b_1 and b_2 to be 1.119% per month and –0.212% per month, respectively. Using the sum of these values of the risk premium and the skewness premium, respectively, to estimate the excess of the expected return on the market portfolio over the risk-free rate, we get:

$$E(r_m) - r_f = 1.119\% - 0.212\% = 0.91\% \text{ per month.}$$

The excess of the expected return on the market portfolio over the risk-free rate must be the same for both the traditional two-moment CAPM and the three-moment CAPM. In the two-moment model, however, this quantity is simply the market risk premium:

$$E(r_m) - r_f = b_1' = 0.91\% \text{ per month.}$$

Hence, the failure to include skewness in the two-moment CAPM results in understating the market risk premium by 19% (i.e., $1.0 - .91/1.119$).

There are two implications of this theoretical example for a negatively skewed market such as the market for catastrophe insurance. First, the market risk premium is understated in the traditional two-moment CAPM. Second, additional return is required to compensate insurers and their investors for the negative skewness of catastrophe insurance products. Therefore, the three-moment CAPM is of particular significance to the insurance industry.

In an exercise on pages 1276–1278, Harvey and Siddique (2000) estimate the risk premium for coskewness. They rank stocks based on their past coskewness and create three value-weighted portfolios using 60 months of returns: 30 percent with the most negative skewness, 40 percent with medium values of skewness, and 30 percent with the highest skewness. Harvey and Siddique conclude on page 1263 that "Systematic skewness is economically significant and commands a risk premium, on average, of 3.60 percent per year." They estimate a skewness premium for coskewness of 3.60 percent by taking the difference in annual excess returns between the portfolio with the most negative coskewness and the portfolio with the highest coskewness.

Moreover, Harvey and Siddique (2000) conclude (pp. 1287–1288) that systematic skewness is not only statistically significant but also economically significant. They reached this conclusion by analyzing pricing errors with the model containing coskewness as a variable relative to the traditional CAPM and by measuring the expected return implied by a change in coskewness.

Friend and Westerfield (1980) also found evidence that investors prefer skewness; however, they did not find that evidence to be compelling. They state on page 913:

> Our analysis provides some but not conclusive evidence … suggesting that investors may be willing to pay a premium for positive skewness in their portfolios.

Kian-Guan Lim (1989), though, found strong evidence that confirms Kraus and Litzenberger's earlier conclusions. Lim divided the fifty-year period from January 1933 through December 1982 into ten consecutive five-year periods. The model was then tested using data from each of the sub-periods as well as for the entire period. Lim concluded that investors prefer coskewness when market returns are positively skewed, and dislike coskewness when market returns are negatively skewed. Moreover, in all of the subperiods in which the model was not rejected at the one percent level of significance, the skewness premium and the skewness of the market return were of opposite sign. Further, Lim found the evidence to be particularly strong when data from the entire period was used.

2.2. Properties of Covariance and Coskewness

As is the case with the traditional two-moment CAPM, beta in the three-moment CAPM is the measure of systematic risk. As a measure of risk, beta is linear in the sense that the beta of a linear combination of securities is the linear combination of the betas of the securities themselves. Specifically, the beta of a portfolio is equal to the weighted average of the betas of the securities in the portfolio.

Let

Z = a portfolio of n securities,

S_i = the dollars invested in the ith security,

r_i = the rate of return on the ith security,

r_Z = the rate of return on the portfolio,

r_M = the return on the market portfolio, and

$S = \sum_i S_i$;

then

$$\beta_Z = \frac{\sigma_{R_Z R_M}}{\sigma^2_{R_M}} = \frac{\operatorname{Cov}(r_Z, r_M)}{\operatorname{Var}(r_M)} = \frac{\operatorname{Cov}\left(\left(\frac{\sum S_i r_i}{S}\right), r_M\right)}{\operatorname{Var}(r_M)}$$

$$= \frac{(\sum S_i \operatorname{Cov}(r_i, r_M))}{S \operatorname{Var}(r_M)}$$

$$= \frac{\sum S_i \beta_i}{S}.$$

For Z equal to the market portfolio, the covariance of the rate of return on the market portfolio with itself is equal to the variance of the rate of return on the market portfolio. Therefore, the weighted sum of covariances of the rates of return on all of the securities in the market portfolio is equal to the variance of the rate of return on the market portfolio.

Similarly, the gamma of a portfolio is the weighted average of the gammas of the individual securities.

$$
\begin{aligned}
\gamma_Z &= \frac{\tau_{r_Z r_M r_M}}{\tau^3_{r_M}} = \frac{\mathrm{E}((r_Z - \mathrm{E}(r_Z))(r_M - \mathrm{E}(r_M))^2)}{\mathrm{E}((r_M - \mathrm{E}(r_M))^3)} \\
&= \frac{\mathrm{E}\left(\left[\left(\sum \frac{S_i r_i}{S}\right) - \mathrm{E}\left(\sum \frac{S_i r_i}{S}\right)\right][r_M - \mathrm{E}(r_M)]^2\right)}{\mathrm{E}((r_M - \mathrm{E}(r_M))^3)} \\
&= \frac{\sum \left(\frac{S_i}{S}\right) \mathrm{E}([r_i - \mathrm{E}(r_i)][r_M - \mathrm{E}(r_M)]^2)}{\mathrm{E}([r_M - \mathrm{E}(r_M)]^3)} \\
&= \sum \frac{S_i \gamma_i}{S}.
\end{aligned}
$$

The coskewness of the return on the market portfolio with itself is equal to the skewness of the return on the market portfolio. Hence, the weighted sum of the coskewnesses of the returns on all of the securities in the market portfolio is equal to the skewness of the return on the market portfolio.

2.3. Disagreement

As noted earlier, under the assumptions of complete agreement on the part of investors about expected returns and identical risk tolerance functions, the optimal combination of risky assets is the same for each investor. It necessarily follows that the optimal portfolio is the market portfolio. These are very strong assumptions. But they are not intrinsic to the three-moment CAPM. Rather, they also apply to the traditional two-moment CAPM. Sharpe relaxes these assumptions in Appendix D of his book.

He concludes on page 291:

> [T]he equilibrium relationships derived for a world of complete agreement can be said to apply to a world in which there is disagreement, if certain values are considered to be averages.

In this section, we will relax these assumptions and investigate the implications.

In the case of disagreement, each investor has his own optimal risk asset portfolio, which depends entirely on his expectations. Different investors do not necessarily have the same optimal risk asset portfolios. For simplicity, assume that there are only two investors. The arguments presented here can be extended to any finite number of investors.

Suppose that M_1 and M_2 are the optimal risk asset portfolios of the two investors. Let M be the market portfolio.

Then

$$M = M_1 + M_2.$$

Let

r_{ij} = the rate of return for security i that is expected by the jth investor,

S_{ij} = the dollars invested in security i by the jth investor,

$$S_1 = \sum_i S_{i1},$$

$$S_2 = \sum_i S_{i2},$$

$$r_{M_1} = \frac{\sum_i S_{i1} r_{i1}}{S_1}, \text{ and}$$

$$r_{M_2} = \frac{\sum_i S_{i2} r_{i2}}{S_2}.$$

Then, the average expected returns are given by:

$$r_i = \frac{(S_{i1}r_{i1} + S_{i2}r_{i2})}{(S_{i1} + S_{i2})},$$

and

$$r_M = \frac{(S_1 r_{M_1} + S_2 r_{M_2})}{(S_1 + S_2)}.$$

Thus,

$$\begin{aligned}\mathrm{Cov}(r_i, r_M) &= \mathrm{Cov}\left(\frac{(S_{i1}r_{i1} + S_{i2}r_{i2})}{(S_{i1} + S_{i2})}, r_M\right)\\ &= \left(\frac{S_{i1}}{S_{i1} + S_{i2}}\right)\mathrm{Cov}(r_{i1}, r_M) + \left(\frac{S_{i2}}{S_{i1} + S_{i2}}\right)\mathrm{Cov}(r_{i2}, r_M).\end{aligned}$$

Hence, recalling that

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)} \qquad \text{implies that:}$$

$$\beta_i = \left(\frac{S_{i1}}{S_{i1} + S_{i2}}\right)\beta_{i1} + \left(\frac{S_{i2}}{S_{i1} + S_{i2}}\right)\beta_{i2}.$$

Note that β_{i1} and β_{i2} are computed with respect to the total market portfolio, rather than with respect to each investor's optimal portfolio. Thus, in a world of agreement everybody has the same estimate of β, and in a world of disagreement, β turns out to be a weighted average over all investors.

The same relationship holds true for coskewness and gamma. Let the coskewness be denoted by:

$$\tau_{abb} = \mathrm{Cosk}(a, b, b) = \mathrm{E}([a - \mathrm{E}(a)][b - \mathrm{E}(b)]^2).$$

Assume again that there are only two investors who disagree. Then for any security:

$$\mathrm{Cosk}(r_i, r_M, r_M) = \mathrm{Cosk}\left(\frac{(S_{i1}r_{i1} + S_{i2}r_{i2})}{(S_{i1} + S_{i2})}, r_M, r_M\right).$$

It can be shown using the results from Section 2.2 and the linearity of the expected value operator that for any three random variables, x, y, and z, and any two constants, a and b, that:

$$\mathrm{Cosk}(ax + by, z, z) = a\,\mathrm{Cosk}(x, z, z) + b\,\mathrm{Cosk}(y, z, z).$$

Hence,

$$\mathrm{Cosk}(r_i, r_M, r_M) = \left(\frac{S_{i1}}{S_{i1} + S_{i2}}\right)\mathrm{Cosk}(r_{i1}, r_M, r_M) + \left(\frac{S_{i2}}{S_{i1} + S_{i2}}\right)\mathrm{Cosk}(r_{i2}, r_M, r_M).$$

And since,

$$\gamma_i = \frac{\mathrm{Cosk}(r_i, r_M, r_M)}{\tau^3_{R_M}},$$

it follows that:

$$\gamma_i = \left(\frac{S_{i1}}{S_{i1} + S_{i2}}\right)\gamma_{i1} + \frac{S_{i2}}{(S_{i1} + S_{i2})}\gamma_{i2},$$

where γ_{i1} and γ_{i2} are computed with respect to the total market rather than with respect to each investor's optimal portfolio. Hence, in a world of agreement everybody has the same estimate of γ, and in a world of disagreement, γ turns out to be a weighted average over all investors.

3. THE THREE-MOMENT INSURANCE CAPM

Following D'Arcy and Doherty's (1988) derivation of the insurance CAPM, the rate of return to the insurer, r_e, is composed of a linear combination of both an underwriting rate of return, r_u, and an investment rate of return, r_i.

$$r_e = \frac{r_u P(1 - t_u)}{S} + \frac{r_i(S + kP)(1 - t_i)}{S}, \tag{3.1}$$

where

r_e = rate of return on equity,

P = premiums in a given year,

S = shareholders' equity,

r_u = underwriting return per dollar of premium,

t_u = tax rate on underwriting income,

k = funds generating coefficient,[2]

r_i = investment return per dollar invested, and

t_i = tax rate on investment income.

At equilibrium based on Equation (2.3) and assuming that shareholders' equity, S, is valued at its expected market value, rather than at its statutory accounting or GAAP accounting value:

$$\mathrm{E}(r_e) = r_f + b_1\beta_e + b_2\gamma_e. \tag{3.2}$$

Further,

$$\mathrm{E}(r_i) = r_f + b_1\beta_i + b_2\gamma_i. \tag{3.3}$$

Moreover, the equity beta (gamma) can be expressed as a linear combination of an underwriting beta (gamma) and an investment beta (gamma) as follows:

$$\beta_e = \frac{P\beta_u(1-t_u)}{S} + \frac{(S+kP)\beta_i(1-t_i)}{S}, \text{ and} \tag{3.4}$$

$$\gamma_e = \frac{P\gamma_u(1-t_u)}{S} + \frac{(S+kP)\gamma_i(1-t_i)}{S}. \tag{3.5}$$

Setting Equation (3.1) equal to Equation (3.2) results, at equilibrium, in:

$$\frac{\mathrm{E}(r_u)P(1-t_u)}{S} + \frac{\mathrm{E}(r_i)(S+kP)(1-t_i)}{S} = r_f + b_1\beta_e + b_2\gamma_e.$$

[2]This is sometimes estimated by the ratio of the invested portion of reserves to premiums.

Substituting with the above three expressions for $\mathrm{E}(r_i)$, β_e and γ_e from Equations (3.3), (3.4) and (3.5) gives:

$$\frac{\mathrm{E}(r_u)P(1-t_u)}{S} + \frac{(S+kP)(r_f + b_1\beta_i + b_2\gamma_i)(1-t_i)}{S}$$
$$= r_f + \frac{P(1-t_u)(b_1\beta_u + b_2\gamma_u)}{S}$$
$$+ \frac{(S+kP)(1-t_i)(b_1\beta_i + b_2\gamma_i)}{S}.$$

Simplifying and solving for the after-tax equilibrium underwriting return yields:

$$\mathrm{E}(r_u)(1-t_u) = -kr_f(1-t_i) + \frac{t_i r_f S}{P} + (1-t_u)b_1\beta_u + (1-t_u)b_2\gamma_u. \tag{3.6}$$

Thus the equilibrium after-tax underwriting return consists of four components: the first effectively represents interest paid to policyholders for the use of their funds; the second is to recapture the tax penalty of being an insurer;[3] the third component is a provision to compensate for risk; and the fourth component is a provision to compensate for skewness.

4. THE n-MOMENT INSURANCE CAPM

There is strong evidence as reported in this paper that including the third moment significantly improves the CAPM and the insurance CAPM. Any benefits of including moments beyond the third are unclear now and await further research. Nevertheless, generalizing the model to n moments is simple and straightforward and is presented here.

[3]The tax penalty is the double taxation of investment income—once at the corporate level and once at the personal level—on underlying equity. Mutual funds, in contrast, are not subject to corporate income taxes. Accordingly, investors will not invest in an insurance company unless the underwriting operation is expected to at least recover the tax penalty.

At equilibrium based on Equation (A.6) and assuming that shareholder's equity, S, is valued at its expected market value, rather than at its statutory accounting or GAAP accounting value:

$$\mathrm{E}(r_e) = r_f + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_e}. \tag{4.1}$$

Further,

$$\mathrm{E}(r_i) = r_f + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_i}. \tag{4.2}$$

Moreover, for $n = 2, \ldots, \infty$,

$$\nu_{n_e} = \frac{P\nu_{n_u}(1-t_u)}{S} + \frac{(S+kP)\nu_{n_i}(1-t_i)}{S}. \tag{4.3}$$

Setting Equation (3.1) equal to Equation (4.1) results, at equilibrium, in:

$$\frac{\mathrm{E}(r_u)P(1-t_u)}{S} + \frac{\mathrm{E}(r_i)(S+kP)(1-t_i)}{S} = r_f + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_e}.$$

Substituting with the above expressions for $\mathrm{E}(r_i)$ and ν_{n_e}, for $n = 2, \ldots, \infty$ from Equations (4.2) and (4.3) gives:

$$\frac{\mathrm{E}(r_u)P(1-t_u)}{S} + \frac{(S+kP)(r_f + \sum_{n=2}^{\infty} b_{(n-1)}\nu_{n_i})(1-t_i)}{S}$$
$$= r_f + \sum_{n=2}^{\infty} b_{(n-1)} \left[\frac{P\nu_{n_u}(1-t_u)}{S} + \frac{(S+kP)\nu_{n_i}(1-t_i)}{S}\right].$$

Simplifying and solving for the after-tax equilibrium underwriting return yields:

$$\mathrm{E}(r_u)(1-t_u) = -kr_f(1-t_i) + \frac{t_i r_f S}{P} + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_u}(1-t_u).$$

5. CONCLUSIONS

Until recently the importance of skewness in the rate of return distribution has largely been unrecognized in financial journals. But it is in the actuarial realm that some of the most extremely skewed return distributions occur, particularly those for catastrophe insurance products. Because some of those distributions are so overwhelmingly skewed, it is essential to assess systematic skewness when determining equilibrium returns and needed premiums.

This paper presents an argument for including a provision in the equilibrium premium to cover the cost of skewness. It also generalizes the insurance CAPM to include the cost of skewness. This permits an explicit determination of the impact that skewness has on the equilibrium premium, at least theoretically. Practical application awaits further empirical studies that measure the amount of systematic skewness in the insurance industry as well as further investigation into the magnitude of the market skewness premium and the market risk premium in the context of a three-moment model.

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APPENDIX

DERIVATION OF THE n-MOMENT CAPM

This appendix presents Rubinstein's derivation of the n-moment CAPM and extends it to derive the market risk premium and the market skewness premium.

Let W_i be the initial wealth of the ith individual. Assume that every dollar of that wealth is invested in one of j securities. Let S_{ij} be the amount that the ith individual has invested in the jth security. Then

$$W_i = \sum_j S_{ij},$$

and the wealth at the end of the year is:

$$\tilde{W}_i = \sum_j S_{ij} R_j,$$

where $R_j = (1 + r_j) =$ one plus the rate of return on the jth security.

Let U_i be the continuously differentiable utility of wealth function for the ith individual. Assume that every individual maximizes $\mathrm{E}_i(U_i(\tilde{W}_i))$ subject to the constraint $W_i = \sum_j S_{ij}$.

Taking the expected value of the Taylor series expansion of $U_i(\tilde{W}_i)$ around $\mathrm{E}_i(\tilde{W}_i)$ gives:

$$\mathrm{E}_i(U_i(\tilde{W}_i)) = \sum_{n=0}^{\infty} \frac{U_i^{(n)} \mu_{in}}{n!},$$

where $U_i^{(n)}$ is the nth derivative of U_i evaluated at $\mathrm{E}_i(\tilde{W}_i)$, and $\mu_{in} = \mathrm{E}_i(\tilde{W}_i - \mathrm{E}_i(\tilde{W}_i))^n$ is the nth central moment of $\tilde{W}_i$. Forming the Lagrangian, the individual's problem is to maximize Z, where

$$Z = \sum_{n=0}^{\infty} \frac{U_i^{(n)} \mu_{in}}{n!} + L_i \left(W_i - \sum_j S_{ij} \right).$$

Dropping the subscript i for simplicity and differentiating gives:

$$\frac{\partial Z}{\partial S_j} = \sum_n \left\{ \frac{\partial}{\partial S_j} \left(\frac{U^{(n)}}{n!} \right) (\mu_n) + \left(\frac{U^{(n)}}{n!} \right) \left(\frac{\partial \mu_n}{\partial S_j} \right) \right\} - L = 0,$$

and

$$W = \sum_j S_j.$$

Let

$$\bar{W} = \mathrm{E}(\tilde{W}) = \sum_j S_j \mathrm{E}(R_j) \Rightarrow \frac{\partial \bar{W}}{\partial S_j} = \mathrm{E}(R_j).$$

So

$$\frac{\partial}{\partial S_j} \left(\frac{U^{(n)}}{n!} \right) = \frac{U^{(n+1)}}{n!} \left(\frac{\partial \bar{W}}{\partial S_j} \right) = \frac{U^{(n+1)}}{n!} \mathrm{E}(R_j).$$

Thus,

$$\frac{\partial Z}{\partial S_j} = \mathrm{E}(R_j) \left(\sum_n \frac{U^{(n+1)} \mu_n}{n!} \right) + \sum_n \frac{U^{(n)}}{n!} \left(\frac{\partial \mu_n}{\partial S_j} \right) - L = 0.$$

But the term $\sum_n U^{(n+1)} \mu_n / n!$ is the Taylor series expansion of $U^{(1)}$ around $\bar{W}$. And,

$$\begin{aligned} \frac{\partial}{\partial S_j}(\mu_n) &= \frac{\partial}{\partial S_j} \left(\mathrm{E} \left(\sum_j S_j R_j - \sum_j S_j \mathrm{E}(R_j) \right)^n \right) \\ &= n\mathrm{E} \left\{ (\tilde{W} - \bar{W})^{n-1} \left(\frac{\partial}{\partial S_j} \right) \left(\sum_j S_j R_j - \sum_j S_j \mathrm{E}(R_j) \right) \right\} \\ &= n\mathrm{E}\{(\tilde{W} - \bar{W})^{n-1} (R_j - \mathrm{E}(R_j))\}. \end{aligned}$$

Hence,

$$\mathrm{E}(R_j) U^{(1)} + \sum_{n=2}^{\infty} \frac{U^{(n)} \mathrm{E}[(R_j - \mathrm{E}(R_j))(\tilde{W} - \bar{W})^{n-1}]}{(n-1)!} = L. \tag{A.1}$$

Since $\mu_0 = 1$ and $\mu_1 = 0 \Rightarrow \partial/\partial S_j(\mu_0) = \partial/\partial S_j(\mu_1) = 0$.

The expression in (A.1) is true for all j. Subtracting the expression for the kth security from the expression for the jth security gives:

$$\mathrm{E}(R_j - R_k)U^{(1)} + \sum_{n=2}^{\infty} \frac{U^{(n)}\mathrm{E}[(R_j - R_k - (\mathrm{E}(R_j) - \mathrm{E}(R_k)))(\tilde{W} - \bar{W})^{n-1}]}{(n-1)!} = 0.$$

Hence,

$$\mathrm{E}(R_j) = \mathrm{E}(R_k) - \sum_{n=2}^{\infty} \frac{U^{(n)}\mathrm{E}[(R_j - R_k - (\mathrm{E}(R_j) - \mathrm{E}(R_k)))(\tilde{W} - \bar{W})^{n-1}]}{U^{(1)}(n-1)!}.$$

Let $\theta_n = -U^{(n)}/U^{(1)}(n-1)!$. Then,

$$\mathrm{E}(R_j) = \mathrm{E}(R_k) + \sum_{n=2}^{\infty} \theta_n \mathrm{E}[(R_j - R_k - (\mathrm{E}(R_j) - \mathrm{E}(R_k)))(\tilde{W} - \bar{W})^{n-1}]. \tag{A.2}$$

Assume that a risk-free security exists. Let R_f be one plus the rate of return on the risk-free security.

Equation (A.2) applies to all securities, so substituting R_f for R_k gives:

$$\mathrm{E}(R_j) = R_f + \sum_{n=2}^{\infty} \theta_n \mathrm{E}[(R_j - \mathrm{E}(R_j))(\tilde{W} - \bar{W})^{n-1}]. \tag{A.3}$$

Let S_f denote the amount that the individual has invested in the risk-free security,

$P = W - S_f$ denote the amount that the individual has invested in his portfolio of risky securities, and

R_p = one plus the rate of return on the portfolio of risky securities.

Then, $\tilde{W} = PR_p + S_f R_f$, and

$$\mathrm{E}(\tilde{W}) = P\mathrm{E}(R_p) + S_f R_f.$$

Thus,

$$\mathrm{E}(R_j) = R_f + \sum_{n=2}^{\infty} \theta_n P^{n-1} \mathrm{E}[(R_j - \mathrm{E}(R_j))(R_p - \mathrm{E}(R_p))^{n-1}].$$

Under the assumptions of complete agreement among individuals and identical risk tolerance functions, it follows that every individual has the same optimal portfolio of risky assets. Moreover, that portfolio is the market portfolio. Hence,

$$\mathrm{E}(R_j) = R_f + \sum_{n=2}^{\infty} \theta_n P^{n-1} \mathrm{E}[(R_j - \mathrm{E}(R_j))(R_M - \mathrm{E}(R_M))^{n-1}], \tag{A.4}$$

where R_M = one plus the rate of return on the market portfolio. Let

$$\nu_{n_j} = \frac{\mathrm{E}[(R_j - \mathrm{E}(R_j))(R_M - \mathrm{E}(R_M))^{(n-1)}]}{\mathrm{E}[(R_M - \mathrm{E}(R_M))^n]} \qquad \text{for} \quad n = 2, \ldots, \infty,$$

and

$$b_{(n-1)} = \theta_n P^{(n-1)} \mathrm{E}(R_M - \mathrm{E}(R_M))^n.$$

Then, the n-moment CAPM is:

$$\mathrm{E}(R_j) = R_f + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_j}. \tag{A.5}$$

Equivalently,

$$\mathrm{E}(r_j) = r_f + \sum_{n=2}^{\infty} b_{(n-1)} \nu_{n_j}. \tag{A.6}$$

For the three-moment CAPM, the traditional notation is given by:

$$\beta_j = \nu_{2_j}, \text{ and}$$

$$\gamma_j = \nu_{3_j}.$$

Then the three-moment CAPM is:

$$\mathrm{E}(R_j) = R_f + b_1 \beta_j + b_2 \gamma_j. \tag{A.7}$$

Additional insight into the coefficients b_1 and b_2 can be gained as follows.

Let R_W denote one plus the rate of return on the individual's entire portfolio, and let σ_{R_W} and τ_{R_W} denote the standard deviation and the skewness, respectively, of the rate of return on the individual's entire portfolio.

Then, in conjunction with the results from Section 2.2,

$$\sigma_{R_W} = \sum_j \frac{S_j \beta_j \sigma_{R_M}}{W}, \text{ and}$$

$$\tau_{R_W} = \sum_j \frac{S_j \gamma_j \tau_{R_M}}{W}.$$

Let β_W and γ_W denote the beta and the gamma of the individual's entire portfolio. It follows that

$$\beta_W = \frac{\sigma_{R_W}}{\sigma_{R_M}} \qquad \text{and} \qquad \gamma_W = \frac{\tau_{R_W}}{\tau_{R_M}}.$$

Moreover,

$$\sigma_W = W \sigma_{R_W} \qquad \text{and} \qquad \tau_W = W \tau_{R_W}.$$

Consider that

$$\bar{W} = W\mathrm{E}(R_W) = WR_f + Wb_1\beta_W + Wb_2\gamma_W$$
$$= W(R_f) + \frac{b_1\sigma_W}{\sigma_{R_M}} + \frac{b_2\tau_W}{\tau_{R_M}}.$$

Since the market portfolio is unchanging, σ_{R_M} and τ_{R_M} are constants. It follows that

$$b_1 = \frac{\partial\bar{W}}{\partial\sigma_W}(\sigma_{R_M}), \text{ and}$$
$$b_2 = \frac{\partial\bar{W}}{\partial\tau_W}(\tau_{R_M}).$$

Thus, the coefficients are the additional required returns per unit of risk and skewness, respectively, times the units of risk and skewness, respectively.