APPLICATION OF THE OPTION MARKET PARADIGM TO THE SOLUTION OF INSURANCE PROBLEMS

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1. INTRODUCTION

Michael Wacek's paper is based on the well-known fact that the Black–Scholes call option price is the discounted expected excess value of a certain lognormal random variable.[1] Specifically, the Black–Scholes price can be written as

$$BS = e^{-r(T-t)}\mathrm{E}[(\tilde{S}(T)-k)^+],$$

where r is the risk-free rate of interest, T is the time when the option expires, t is the current time, $\tilde{S}(T)$ is a lognormal random variable related to the stock price $S(T)$ at time T, k is the exercise price, and $x^+ := \max(x,0)$. In insurance terms, $(L-k)^+$ represents the indemnity payment on a policy with a loss of L and a deductible k. The Black–Scholes price can also be regarded as the discounted insurance charge (see Gillam and Snader [18] or Lee [25]). It is easy to compute the insurance charge under

[1] The formula is explicit in virtually all financial economics derivations, for example, Merton [27, p. 283], Cox and Ross [4, p. 154, equation 19, which is essentially the author's Equation 1.3], Harrison and Kreps [19, Corollary to Theorem 3], Karatzas and Shreve [23, p. 378], Hull [20, p. 223 (for forward contracts on a stock)], as well as more overtly actuarial works, such as Gerber and Shiu [17, p. 104] and Kellison [24, Appendix X].

the lognormal assumption to arrive at—but not to derive—the explicit Black–Scholes formula.[2]

Even without reference to the Black–Scholes formula, there are obvious analogies between insurance and options because both are *derivatives*. An insurance payment is a function of—is *derived* from—the insured's actual loss; similarly, the terminal value of an option is a function of the value of some underlying security. To the extent that options and insurance use the same functions to derive value, there will be a dictionary between the two. As Wacek points out, this is the case. For example, the excess function $(L-k)^+$ is used to derive the terminal value of a call and an insurance payment with a deductible k; $\min(L,l)$ determines the value of an insurance contract with a limit l; and $(k-L)^+ = k - \min(L,k)$ gives the terminal value of a put option as well as the insurance savings function. There are several other examples given in the paper, including a cylinder. The author explains how an insurance cylinder can be used to provide cheaper reinsurance and greater earnings stability for the cedent. The idea of regarding an insurance payment as a function of the underlying loss has been discussed previously in the *Proceedings* by Lee [25] and [26], and Miccolis [28]. The connection between insurance and options, based on the fact that both are derivatives, was also noted in D'Arcy and Doherty [9, p. 57].

Here are two other interesting correspondences between option structures and insurance. The first is the translation from put-call parity in options pricing to the relationship "one plus savings equals entry ratio plus insurance charge" from retrospective rating. The put option is equivalent to the insurance savings function and the call option to the insurance charge function (see Lee [25] and [26], which has the options profit diagrams, or Gillam and Snader [18] for more details).

The second correspondence applies Asian options to a model of the rate of claims payment or reporting in order to price catas-

[2]Kellison [24, Appendix X] gives all the details.

trophe index futures and options. This example is too involved to describe in detail here. The interested reader should look in the original papers by Cummins and Geman [7] and [8].

Insurance can also be regarded as a swap transaction. Hull [20] defines swaps generically as "private agreements between two companies to exchange cash flows in the future according to a prearranged formula." In insurance language one cash flow is the known premium payment, generally consisting of one or more installments during the policy period, and the other varies according to losses and continues for a longer period of time. Many recent securitization transactions have been structured as swaps. Indeed, in that context a swap is essentially insurance from a non-insurance company counter-party. Arguably, swaps are a better model for insurance than options because they involve a series of cash flows into the future rather than a single payment. Options, which involve a single payment when the option is exercised, are not a good model for a per occurrence insurance product that could cover many individual claims.

Despite the title of the paper, Wacek is more concerned with options *notation*—puts, calls, profit diagrams and so forth—than with the options market *paradigm*. The dictionary definition of a paradigm is a "philosophical and theoretical framework of a scientific school or discipline within which theories, laws, and generalizations…are formulated." Wacek's paper does not discuss the assumptions underlying Black–Scholes nor the derivation of the formula in any detail. Each is an important part of the options pricing paradigm. Moreover, the comments he offers on options prices tend to confuse a pure premium (loss cost) with a price (loss cost including risk charge, in this context). He rightly draws a distinction between the two but does not clearly state whether the Black–Scholes formula gives the former or the latter.

This review will focus on the theoretical framework, or paradigm, of options pricing. Section 2 will compare the Option Pricing Paradigm with the corresponding actuarial notion, and discuss how the former relies on hedging to remove risk while the latter relies on the law of large numbers to assume and manage risk. The distinction between using hedging and diversification to manage risk highlights an essential difference between the capital and insurance markets. Section 3 will determine the actuarial price for a stock option under the lognormal distribution assumption, and will compare the result to the Black–Scholes formula. Section 4 then discusses why the Black–Scholes result is different from the actuarial answer. It will also explain why the Black–Scholes formula gives a price rather than a pure premium. Section 5 will propose an application of the Option Pricing Paradigm to catastrophe insurance and discuss options on non-traded instruments. Finally, Section 6 will compare market prices with the Black–Scholes prices.

This review will only discuss applications of options pricing to individual contracts in a very limited way. The reader should be aware that there are many other important applications, including the pioneering work of Cummins [6], revolving around valuing the insurance company's option to default. The groundbreaking paper by Phillips, Cummins and Allen [30] gives an application of these ideas to pricing insurance in a multi-line company. The reader should refer to the recent literature for more information on these ideas.

2. OPTION PRICING PARADIGM AND ACTUARIALLY FAIR PRICES

The actuarial, or fair, value of an uncertain cash flow is defined to be its expected value. Insurance premiums are generally determined by loading the discounted actuarial value of the insured losses for risk and expenses. In this discussion it will be assumed that a risk charge is loaded into the pure premium by discounting at a risk-adjusted interest rate. Clearly this is neither the only choice nor is it necessarily the best choice. It will also be assumed that there are no expenses, and the word "price" will be used to refer to a risk-loaded pure premium.

The Option Pricing Paradigm defines the price of an option to be the smallest cost of bearing the risk of writing the option, which is completely different from the actuarial viewpoint. In this context, being able to *bear* the risk of writing an option (equivalent to writing insurance) means being able to respond to the holder of the option whatever contingency might occur. In actuarial-insurance language this implies a zero probability of ruin, for if there is a non-zero probability of ruin then there is a contingency under which the option writer cannot respond to the holder, and hence the writer is not able to bear the risk (according to the definition).

The insurance company approach to bearing risk is to charge a pure premium plus risk load, to have a substantial surplus, and to pool a large number of independent risks. If stock prices follow an unbounded distribution, such as the lognormal, then it is not possible to write an option and achieve a zero probability of ruin using this insurance approach to bearing risk. Thus, unlike insurance, pricing and risk bearing in the Option Pricing Paradigm do not rely on the law of large numbers—a crucial difference.

One way of bearing the risk of writing a stock option is to set up a hedging portfolio with the following four properties:

1. The portfolio consists of the stock underlying the option and risk-free borrowing or lending.
2. The terminal value of the hedging portfolio equals the terminal value of the option for all contingencies.
3. The hedging portfolio is *self-financing*: once it has been set up it generates no cash flows, positive or negative, until the option expires.
4. The hedging portfolio uses a deterministic trading strategy which only relies on information available when each trade is made. Trading only takes place between when the option is written and when it expires.

It is not clear that hedging portfolios exist. However, if they do then the Option Pricing Paradigm price of an option can be no greater than the smallest amount for which it is possible to set up (i.e., purchase) a hedging portfolio. Indeed, by setting up a hedging portfolio the writer of the option is able to bear the attendant risks, because the portfolio generates enough cash to respond to the holder no matter what contingency occurs. By definition, the price of the option is the smallest amount of money for which this is possible. Therefore, the actual option price can be no greater than the cost of the cheapest hedging portfolio.

On the other hand, if there are no arbitrage opportunities, the Option Pricing Paradigm price must be at least as large as the cost of setting up the cheapest hedging portfolio. Since the writer can bear the risk of writing the option, it must have a portfolio, purchased with the proceeds of writing the option, with an ending cash position at least as large as the terminal value of the option (and hence a hedging portfolio) in every contingency. Such a portfolio is said to *dominate* the hedging portfolio. If portfolio A dominates portfolio B then, in the absence of arbitrage, A must cost more than B. Here, the option price is used to purchase a portfolio which in turn dominates a hedging portfolio, and therefore the option price must be at least as great as the cost of a hedging portfolio.[3] Combining this with the previous paragraph shows that in the absence of arbitrage, the Option Pricing Paradigm price equals the smallest amount for which it is possible to set up a hedging portfolio.

The above argument relies on the absence of arbitrage opportunities in the market. An arbitrage is the opportunity to earn a

[3]The price of the option could simply be defined as the smallest cost of setting up a hedging portfolio. For example, in Karatzas and Shreve [23] the fair price for a contingent claim is defined as the smallest amount x which allows the construction of a hedging portfolio with initial wealth x. However, it is generally not possible for an insurer to set up a hedging portfolio because it cannot trade in the security underlying the insurance contract option. Thus, a definition in terms of hedging portfolios would not have transferred to insurance. On the other hand, "the cost of bearing the risk," albeit with a possibly weaker notion of bearing risk, makes perfect sense in an insurance setting and is equivalent to the hedging portfolio definition for options.

riskless profit. In general, the existence of arbitrage opportunities is not compatible with an equilibrium model of security prices, since informed agents would engage in arbitrage and hence modify market prices (see Dybvig and Ross [16, pp. 57–71] for an explanation of the close connections between no-arbitrage and options pricing). In options pricing, no-arbitrage is used to justify defining the price of an option as the smallest cost of a hedging portfolio; if the option sold for more or less than the cost of a hedging portfolio then risk-free arbitrage profits would be possible. Put another way, the option and the hedging portfolio are *comparables*, and no-arbitrage implies that comparables must have the same value. Since "the value of an asset is equal to the combined values of its constituent items of cash flow" [3], if two assets have the same cash flows then they are equivalent to an investor and must command the same price. The fact that one is an option and the other a synthetic option created from a portfolio of bonds and stocks is irrelevant. Obviously this only applies in a world where the Black–Scholes assumptions hold—so in particular there are no transaction costs, no discontinuous jumps in stock prices, and continuous trading. Finally, no-arbitrage is a *consequence* of the model framework, not an *assumption*; potential arbitrages are ruled out through restrictions on admissible trading strategies. This is a more advanced point; the interested reader should see Harrison and Kreps [19], Dothan [14], and Delbaen and Schachermayer [10], [11] and [12] for more detailed information.

To conclude, this section has introduced the notions of no-arbitrage and hedging portfolios, and explained how the Option Pricing Paradigm defines price to be the smallest cost of setting up a hedging portfolio. These beginnings are enough to point out some significant differences compared to actuarial methods of pricing, one of which is that option pricing does not rely on the law of large numbers. The question of whether hedging portfolios actually exist will be discussed in Section 4. First, we will look at how an actuary would price an option.

3. AN ACTUARIAL APPROACH TO OPTION PRICING

It is instructive to compare the Black–Scholes price with the actuarial price—including risk-load—for a call option. Before defining terms we must fix the notation. Assume all interest rates and returns are continuously compounded. Let r be the risk-free rate of interest, μ the expected return on the stock, $S(t)$ or S_t the stock price at time t, and r' a risk-adjusted interest rate for discounting the option payouts. To keep the notation as simple as possible, assume that the current time is $t = 0$ and that the option expires at time $t = T$. Assume also that the stock price process is a geometric Brownian motion,[4] so that $\ln(S(t)/S(0))$ is normally distributed with mean $t(\mu - \sigma^2/2)$ and variance $t\sigma^2$ for some $\sigma > 0$. Finally, let k be the exercise price.

The actuarial price for the option is the present value of the expected payouts discounted at a risk-adjusted interest rate:

$$e^{-r'T}\mathrm{E}((S(T) - k)^+). \tag{3.1}$$

With $r' = r$, Equation 3.1 is Equation 1.3 from the paper.[5] An actuary could compute Equation 3.1 after estimating appropri-

[4]This means that over a very short time interval dt, the return on the stock dS_t/S_t satisfies the stochastic differential equation $dS_t/S_t = \mu dt + \sigma dW_t$, where W_t is a Brownian motion. By definition W_t is normally distributed with mean zero and variance $t\sigma^2$. It follows from Ito's Lemma that the solution to the stock price stochastic differential equation can be written as $S_t = S_0\exp((\mu - \sigma^2/2)t + \sigma W_t)$. Hence S_t has a lognormal distribution and $\ln(S_t/S_0)$ is normal with mean $(\mu - \sigma^2/2)t$ and variance $t\sigma^2$. Since $E(S_t) = S_0 e^{\mu t}$ it is reasonable to call μ the expected rate of return on the stock. See Hull [20], or Karatzas and Shreve [23], for more details. In particular, Hull [20, Chapter 10.3] discusses the difference between expected returns over a short period of time and the expected continuously compounded rate of return. If there is variability in the rate of return, so $\sigma > 0$, then the former, μ, is greater than the latter, $\mu - \sigma^2/2$.

[5]Wacek justifies assuming $r' = r$ using the notion of a hedging portfolio. However, in this section an actuarial viewpoint is taken instead. To the actuary—and the financial economics community as a whole prior to Black–Scholes—we should have $r' > \mu > r$, since the option is more leveraged than the stock and hence more risky. Brealey and Myers [2] point out that the option has a higher beta and a higher standard deviation of return than the underlying stock. Clearly $\mu > r$, since the stock is more risky than a risk-free bond. Note that $r' > r$ assumes the actuary is buying an option and discounting the payout as income; if the actuary were writing the option and pricing the payout as a loss, then $r' < r$ would be appropriate.

ate values for μ, σ and r'. For μ the actuary might try using a historical average return, the Capital Asset Pricing Model price, or some other suitable tool. It is easy to estimate σ from a time series of stock prices. For the discounting rate, the actuary would mystically select some number $r' > \mu$ if buying an option (as is assumed here) or $r' < r$ if writing one. Using Equation 3.1 the actuary would then arrive at the following expression for the call price:

$$e^{(\mu - r')T} S_0 \Phi\left(\frac{\ln(S_0/k) + (\mu + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - e^{-r'T} k \Phi\left(\frac{\ln(S_0/k) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right). \tag{3.2}$$

This formula is identical to a pre–Black–Scholes result derived by Samuelson [31] following the same logic used here. It is also mentioned in Ingersoll [21, pp. 199–212], which includes a survey of earlier attempts to determine a formula to price options. Comparing the actuarial Equation 3.2 to the Black–Scholes equation

$$S_0 \Phi\left(\frac{\ln(S_0/k) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - e^{-rT} k \Phi\left(\frac{\ln(S_0/k) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{3.3}$$

and relating the variables back to Equation 3.1 highlights two differences:

1. The Black–Scholes model appears to assume the stock earns the risk-free rate of return, that is $\mu = r$;
2. The Black–Scholes model discounts at the risk-free rate of interest, so $r' = r$.

Clearly, substituting $\mu = r$ and $r' = r$ into Equation 3.2 gives Equation 3.3. Section 6 compares option prices computed with these two equations.

The expected rate of return μ on the stock is a function of investor risk preferences. Individual investors could differ as to their opinions of μ and yet the assumption underlying the first difference above says they will agree on the option price. In order to underline how remarkable and counter-intuitive this assumption is, it is instructive to translate it into insurance language. Suppose $S(T)$ is the value of a portfolio of losses at time T. Further, suppose $S(T)$ is lognormal with parameters μ and σ. The insurance analog of the call option pricing problem is to price an aggregate stop loss on $S(T)$ with attachment k. Assume all actuaries agree on σ, that is, they agree on the coefficient of variation of aggregate losses.[6] Now, the assumption discussed above says different actuaries could disagree on μ—and hence the mean of $S(T)$—and yet agree on the price of the aggregate stop loss! Clearly there is something significant going on behind the Black–Scholes formula.

No actuary would assume $\mu = r$ and $r' = r$ in pricing since they are not reasonable in the real world. In his paper, Wacek points out that option pricing discounts at the risk-free rate, $r' = r$, but he does not mention the first point. The next section examines why the two assumptions above can be made in the Option Pricing Paradigm.

4. THE HEDGING PORTFOLIO IN DISCRETE TIME AND THE BLACK–SCHOLES FORMULA

The Black–Scholes formula is best understood by considering a discrete time example. While the example may appear simplistic, it contains all of the key ideas in the Black–Scholes derivation. Cox, Ross and Rubenstein [5] give an explanation of how to derive the full Black–Scholes result from a limit of the binomial models considered here. Their explanation is considered in more detail by Nawalkha and Chambers [29], who show that

[6]For a lognormal distribution, the coefficient of variation is $\sqrt{\exp(\sigma^2) - 1}$, a function of σ alone.

preference independence in [5] depends on a particular choice of binomial parameters.

Consider writing a call option on a stock currently priced at \$100. When the option expires, assume the stock price will be either \$120 or \$90. The one period risk-free rate of interest is 5%; also, the option expires in one period and has an exercise price of \$105. Finally, assume the stock does not pay any dividends.

At this point it is important to understand what has, and what has not, been assumed. Underlying our assumptions are two unknowns: the probability p that the stock price will end at \$120, and an expected return μ on the stock. The expected return μ is

$$\mu = \ln\left(\frac{120p + 90(1-p)}{100}\right)$$

(using continuous compounding) or, equivalently,

$$100 = e^{-\mu}(120p + 90(1-p)). \qquad (4.1)$$

Equation 4.1 expresses the current price as an expected present value, discounted at a risk-adjusted interest rate. It gives one relationship between the two unknowns p and μ. It is impossible for us to know whether the current stock price is \$100 because there is a very good chance of an upward price movement (high p) but investors are all very risk-averse (giving a high μ), or because there is only a moderate chance of an upward movement in a largely risk-neutral market. The fact that the Black–Scholes formula is independent of the choice of μ and p, subject to the constraint Equation 4.1, is one of its most remarkable features and it leads to the notion of risk-neutral valuation.

Return now to pricing the \$105 call option under the Option Pricing Paradigm. From Section 2, the price of the call is the smallest cost for which it is possible to set up a hedging portfolio. An explicit hedging portfolio for the option will now be constructed, demonstrating that they exist, at least in this simple case. Suppose the hedging portfolio consists of a stocks and b

dollars in bonds. The replicating property of a hedging portfolio requires that, at expiration,

$$120a + b = 15 \qquad \text{and}$$
$$90a + b = 0.$$

The top line corresponds to an upward movement in the stock price, when the option is worth $\max(120 - 105, 0) = 15$ at expiration. The bottom line corresponds to a downward movement in the stock price, when the option is worth $\max(90 - 105, 0) = 0$. Solving gives $a = 1/2$ and $b = -45$, meaning borrowing of \$45. The cost today of setting up a portfolio which consists of half of one stock plus \$45 debt one period from now equals $100/2 - e^{-0.05}45 = 7.19$. The first term is the cost of buying half a stock and the second term is the present value of a debt of \$45 one period from now.

It is easy to confirm that this portfolio hedges the option. If the stock price moves up, then selling the half-stock yields \$60, exactly enough to pay off the \$45 debt and pay the owner of the option the \$15 terminal value. If the stock price moves down, then selling the half-stock realizes \$45, which is exactly the amount required to pay off the debt. There is nothing else to pay since the option expires worthless.

Using the hedging portfolio it is also easy to see why no-arbitrage implies \$7.19 is the appropriate price for the option. If the option sold for more than \$7.19, say \$7.25, then arbitrageurs would write (sell) the over-priced options. With \$7.19 of the proceeds they could set up a hedging portfolio, effectively closing out their option position. They would make a risk-free profit of six cents per option written.

On the other hand, suppose the option sold for only \$7.15. Then arbitrageurs would want to buy the under-price options. They could short one stock to get \$100 and use the proceeds to buy two options for \$14.30, put \$85.61 into bonds earning the risk-free rate, and skim off the remaining nine cents as arbitrage profit. If the stock price rises to \$120, they exercise the two options to yield a \$30 profit, which combined with $85.61e^{0.05} =$ \$90 in bonds gives \$120—exactly enough to close out the short position in the stock. If the stock price falls to \$90 then the options expire worthless, but the arbitrageurs still have \$90 in the bonds to close out the stock position.

To summarize, these arguments lead to essentially the same two conclusions already noted in Section 3 from comparing Black–Scholes and the actuarial option price:

1. The option price is \$7.19 regardless of the risk preferences of individuals, expressed through the unknown quantities μ and p, provided Equation 4.1 is satisfied;
2. The hedging portfolio consists of stock and risk-free borrowing only. Once it is set up there is no risk to the option writer because movements in the stock price and the option price are perfectly correlated. A portfolio consisting of the hedge and the underlying option must therefore earn, and hence be discounted at, the risk-free rate of return.

Black and Scholes showed these two results are still true when the stock price is allowed to follow a more complex path in continuous time. Under the lognormal stock price assumption, they proved the option price function is given by Equation 3.3, and derived the required trading strategy to use in the hedging portfolio—the so-called "delta-hedging" strategy.

Cox and Ross [4] used the risk preference independence in the first conclusion above to argue that an option could be priced assuming investors have any convenient risk preference. The simplest selection for preferences is risk-neutrality. In a risk-neutral world, all stocks are expected to earn the risk-free return because investors do not require a premium for uncertainty. Thus $\mu = r$ is determined. Of course, stocks do not earn the risk-free return in the real, risk-averse, world. In our simple discrete model, selecting risk-neutral preferences for investors is equivalent to setting $\mu = r$, so now we can solve Equation 4.1 for p, to get $\tilde{p} = (100e^r - 90)/30 = 0.50424$. A similar result holds in continuous time: it is possible to explicitly adjust the stock price process to that which would prevail in a risk-neutral world. The adjusted process is denoted $\tilde{S}(t)$. $\tilde{S}(T)$ and $S(T)$, the distribution of the actual stock price at time T, will be different since the risk-neutral assumption does not hold in the real world.

The adjustment that takes S to $\tilde{S}$ is an adjustment of underlying probabilities. Risk preferences can be understood as a subjective assessment of probabilities for future events. Risk-averse individuals will assign greater than actual probability to bad outcomes. The adjustment we are looking for will be an assessment of these subjective probabilities. In order to reduce the return on a stock from μ to r, the probability of bad outcomes is increased and the probability of good outcomes is decreased. The adjusted probabilities are called an equivalent martingale measure, because the discounted stock price process becomes a martingale with respect to the new probabilities.

Wang's proportional hazard transform method for computing risk-loads also works by altering probabilities (see [34], [35]). Wang's method is therefore in line with modern financial economic thinking and deserves serious consideration by actuaries.

In a risk-neutral world, an option will be valued as the present value of its expected payouts, discounted at the risk-free rate. For a call option with exercise price k this would be

$$e^{-rT}\mathrm{E}((\tilde{S}(T) - k)^+). \tag{4.2}$$

Equation 4.2 gives a pricing formula very similar to the author's equation

$$e^{-rT}\mathrm{E}((S(T) - k)^+). \tag{4.3}$$

The difference between the two is the use of S, the real stock price process, versus $\tilde{S}$, the process that holds in a risk-neutral world. $\tilde{S}$ will have an expected return equal to $r < \mu$ and is certainly different from S, the "probability distribution of market prices at expiry" which Wacek uses in his version of Equation 4.3 on page 703. Section 6 gives a comparison of prices from these two formulae with actual market prices.

Finally, it is now clear that the Black–Scholes formula gives a *price* rather than a *pure premium*. Once the hedging portfolio has been set up there is no risk to the option writer; therefore there is no need for a risk load. In fact, adding a risk load to the Black–Scholes price would create an arbitrage opportunity. Wacek makes this point in his footnote 2, but then obscures the issue by characterizing the rate as a pure premium to allow for an extra risk load when the hedging argument is not available. However, he does not discuss what conditions are necessary in order to use a hedging argument. It turns out the condition is precisely that there exists an equivalent martingale measure, as discussed above (see Duffie [15] for more details on this point). Also, see Gerber and Shiu [17], Cox and Ross [4] and Delbaen, Schachermayer and Schwizier [13] for a discussion of pricing options based on stock price processes other than the lognormal used in Black–Scholes.

5. TWO ACTUARIAL APPLICATIONS OF THE OPTION PRICING PARADIGM

Catastrophe Insurance

I believe the Option Pricing Paradigm view is useful in a situation where Black–Scholes will likely never apply: catastrophe insurance and reinsurance pricing. It tells us to price by computing the cost of being able to bear the risk for the contract period and not by loading the expected loss for risk. For catastrophe reinsurance this means having access to a large, liquid pool of cash. The Option Pricing Paradigm also tells us to move away from a "bank" mentality where reinsurance is providing inter-temporal smoothing, and consider the premium *spent* during the exposure period on bearing the risk. Most other lines of insurance are based on such a point-in-time, between-insured risk sharing, rather than inter-temporal, per-insured risk sharing. As noted in Jaffee and Russell [22], many institutional problems arise for catastrophe insurance precisely because of the inter-temporal way it is currently handled.

Using the Black–Scholes model, the writer of an option uses the option premium to maintain the hedging portfolio (the hedging trading strategy for a call is buy high, sell low, so it is guaranteed to lose money). When the option expires, the initial premium has been exactly used up in stock trading losses whether the option ends up in or out of the money. Similarly, working within this framework, a catastrophe insurance premium should be spent during the policy term, perhaps on maintaining a line of credit, or paying a higher than market interest rate on a cat-bond.

Interestingly, it does not make sense to ask for a contingency reserve with this viewpoint for two reasons. First because at the end of the contract period there is no remaining premium to put into a reserve, and secondly because there is little or no taxable income produced by the product. The need for catastrophe reserves is largely a product of taxation of insurance companies. In this model, the catastrophe risk and premium would pass through the insurance company to an entity, such as a hedge fund, more economically suited to bearing the risk and providing the necessary funding after a large event.

Options on Non-Traded Instruments

The Black–Scholes approach appears to rely on the possibility of taking a position in the underlying stock. This is partially true. More important, however, is the fact that the stock represents the only source of uncertainty, or stochastic behavior, in the system. Writing an option and maintaining a hedging portfolio cancels out the pricing uncertainty, leaving a risk-free portfolio—as discussed above.

Consider an option on an untraded quantity, such as interest rates or an insurance loss index. While, by definition, it is not possible to take a position in the underlying, there is still only one source of uncertainty. Therefore, a Black–Scholes type argument can be used to construct a risk-free portfolio consisting of two options with different exercise prices or expiration dates. The portfolio will use the fact that the two options have instantaneously perfectly correlated prices to cancel all the risk (stochastic behavior). The result is a partial differential equation similar to the Black–Scholes equation involving the prices of the two options as unknowns. Unfortunately, one equation between two unknowns does not give a unique solution. When the underlying is traded, its price is known already, giving one equation in one unknown, which is soluble. However, the partial differential equation can be separated into an expression of the form

$$f(C_1) = f(C_2)$$

for some function f, where C_1 and C_2 are the unknown option prices. Since the lefthand side depends on the expiration and exercise price of C_1 but not C_2, both sides must be a function of the risk-free rate r and time t alone. This implies there are Black–Scholes-like partial differential equations for the prices of C_1 and C_2 each with one extra unknown, called the market price of risk for the underlying index. Since all the option prices depend on the same extra parameter, there are strong consistency conditions put on the prices of a set of options on one underlying instrument. This approach could be useful in an insurance context to help price derivatives off an insurance-based index. For a more detailed explanation of how the approach is applied to price interest rate derivatives, see Wilmot, Howison and Dewynne [36].

6. BLACK–SCHOLES IN ACTION

How well does the Black–Scholes formula perform in practice? It is often asserted that the model is widely used in the industry and also that traders are aware of its weaknesses; it is

TABLE 1

BLACK–SCHOLES PRICES

Exercise Price	Market Price	Traded Volume	Intrinsic Value	BS Price	Percent Error	Implied σ	Actuarial Price
750	186	714	169.77	181.06	−2.7%	32.83%	201.98
805	135	3	114.77	130.90	−3.0	27.92	150.55
890	67 1/2	10	29.77	66.89	−0.9	23.86	82.04
900	64	6	19.77	60.88	−4.9	25.27	75.32
910	59 1/4	102	9.77	55.21	−6.8	25.73	68.93
930	44 1/4	3,291	0.00	44.96	1.6	23.12	57.18
935	41 3/4	5	0.00	42.62	2.1	23.04	54.46
940	42 1/2	264	0.00	40.36	−5.0	24.64	51.83
950	36 1/4	14	0.00	36.11	−0.4	23.57	46.82
960	31 1/2	2	0.00	32.20	2.2	23.12	42.16
990	21	5	0.00	22.37	6.5	22.69	30.19
995	20	107	0.00	20.98	4.9	22.91	28.48
1,025	11	7	0.00	14.07	27.9	21.28	19.71

rarer to see comparisons with market prices. Such a comparison will be given in this section. This is not a scientific test of the model; rather it is supposed to indicate roughly how well it performs.

Table 1 gives the closing prices for all December S&P 500 European call options on September 15, 1997. The calls expired on December 19, 1997. The risk-free force of interest was about 5.12%, giving a discount factor of 0.9868. The S&P 500 closed September 15 at 919.77.

The market price shows the last trade price for each option. The intrinsic value is given by the current index price minus the exercise price, if positive. The Black–Scholes formula price (BS Price) is computed using $\sigma = 23.50\%$, an estimate derived from a contemporaneous sample of S&P daily returns. The implied volatility is calculated by setting the Black–Scholes price equal to the market price and solving for σ. The actuarial price is computed using Equation 3.2 with $r' = r$, the risk-free rate of return and $\mu = 13.98\%$ for a 15% annual return. Assuming

$r' = r$ makes Equation 3.2 exactly the same as Equation 1.3 on page 703 of the paper for a lognormal stock price distribution. Thus, the last column shows the impact on the option price of using the approximate expected rate of return of the underlying instrument rather than the risk-free rate of return; this is the difference between using $\tilde{S}$ and S as the stock price process. Clearly market prices are much closer to the Black–Scholes price. Using a higher discount rate in place of r, but leaving μ unchanged, would bring the actuarial price closer to the Black–Scholes value.

The results are really quite spectacular, especially when compared to the range of reasonable values determined by many actuarial analyses. Remember there is only one free parameter underlying all the model values, and even that is easy to estimate. As Hull [20] points out, the last option trade may have occurred well before the market closed, so the option price may correspond to a different S&P index value than the close. Hull is also a good reference for more information on the mechanics of options markets and for reasons why market prices diverge from Black–Scholes prices.

Finally, Table 1 only provides evidence that the market prices using Black–Scholes or a very similar formula. It does not necessarily follow that this is the "correct" price!

7. CONCLUSION

The thrust of Wacek's paper is that options pricing and insurance pricing are essentially the same and that it should be possible for each discipline to learn from the other. In many ways this is true, particularly on a practical level. Examples include the author's sections on rate guarantees and multi-year contracts. The philosophy of "look for the option" is an important part of modern finance, and is well illustrated by the many applications in Brealey and Myers [2]. Given its central role in finance, actuaries should understand the Option Pricing Paradigm and be able to apply it, where relevant, in their own work. Wacek's paper is valuable because it helps point actuaries in this direction.

However there are some very significant differences between the Option Pricing Paradigm and insurance which should not be glossed over. The Option Pricing Paradigm is based on arbitrage and pricing comparables, and it relies on hedging to remove risk. Insurance assumes and manages specific risk (see Turner [33]). There are typically no liquid markets or close comparables for the specific assets underlying insurance liabilities, and so option pricing techniques do not apply.[7] The specialized underwriting knowledge that insurance companies develop is a key part of their competitive advantage; what they do is bear the resulting underwriting risk, they do not hedge it away.[8] This point is discussed by Santomero and Babbel [32] in their review of financial risk management by insurers. Obviously, how an individual company chooses to manage risk does not alter the market price; the existence of a hedge-based pricing mechanism does, however, determine a market price in the absence of arbitrage.

At a detailed level, Wacek's transformation from the Black–Scholes formula to his supposedly more general Equation 1.3 (Equation 4.3 here) is inappropriate. In this discussion, I have shown how an actuarial approach to option pricing produces a result similar to the Black–Scholes formula but with two important differences: the assumed return on the stock (expected market return compared to risk-free return) and the discount rate (a rate greater than the expected market return compared to the risk-free return). The Black–Scholes argument shows that writing options is risk-free (in the conceptual model) because of the possibility of setting up a self-financing hedging strategy with the proceeds from writing an option. No-arbitrage then implies

[7] As Babbel says in [1]: "When it comes to the valuation of insurance liabilities, the driving intuition behind the two most common valuation approaches—arbitrage and comparables—fails us."

[8] Unless they can use their specialized knowledge to arbitrage the reinsurance markets!

the price of the option must be the smallest amount for which it is possible to set up a hedging portfolio. It follows that the option price is independent of an individual investor's risk preferences. Cox and Ross [4] then argued that the option can be priced assuming risk-neutrality. In a risk-neutral world stocks earn the risk-free return, thus explaining the first assumption. The hedging portfolio argument also shows the risk-free rate is appropriate for discounting, which explains the second assumption. Wacek makes the latter point but does not mention the former. As shown in Table 1, there is a significant difference between the option prices with and without the former assumption. Moreover, market prices are consistently closer to the Black–Scholes prices. The discussion of hedging and options pricing also makes it clear Black–Scholes gives a price, not a pure premium. Finally, Wacek's assertion that "the pricing mathematics is basically the same" for options and insurance is not really the case. Doubts as to this point can be dispelled by looking in any more advanced text on options pricing, such as [14].

In this review, a simple discrete time example has been given to illustrate the hedging portfolio argument. It shows how the option price is independent of risk preferences *given the current stock price*. While the example is often reproduced in finance texts, the discussion of exactly how risk preferences fit in (through Equation 4.1) is less common. A new application of the Option Pricing Paradigm to catastrophe insurance was proposed, and how the paradigm works in the case of an underlying which is not traded was discussed. Finally, a comparison of market prices, Black–Scholes prices and actuarial prices for some S&P options has been given.

The Black–Scholes option pricing formula is an important and beautiful piece of mathematics and financial economics. On the surface the formula is just the discounted expected excess value of a lognormal random variable—the tricky part is *which* lognormal variable! Understanding some of the paradigm lying behind the formula, and some of its subtleties, gets us to the core of the differences between how insurers and other financial institutions bear and manage risk. Given the current convergence between insurance and banking it is important for insurance actuaries to understand and to be able to exploit these differences—our future livelihoods could depend upon it.

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