Random Effects Linear Statistical Models and Bühlmann-Straub Credibility

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Abstract

The definition and application of random effects linear models as a better alternative to empirical Bayesian credibility will be presented. A short review of Bühlmann-Straub credibility is contained in section 2. The author presents tractable formulas for quantifying the variability of credibility estimates. The variability of credibility estimates is produced without having to make distribution assumptions. However, if one assumes normality, hypothesis tests and confidence intervals can be constructed.

I would like to thank Rhonda Puda, Brandon Keller and Willam R. Gillam for their critiques of the paper. I would also like to thank John Wiley & Sons, Inc., NCCI, Inc. and Gary Venter for use of their data.

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1. INTRODUCTION

Credibility Theory allows casualty actuaries to answer the question: "How much of the difference in experience of a given policyholder is due to random variation in the underlying claims experience and how much is due to the fact that the policyholder is really a better or worse risk than the average for a given rating class?" [1:385]. Of course the difference among states, territories, and classes can also be the item of interest.

Random effects statistical models allow casualty actuaries to answer the above question and others. This model provides valuable information about the variability of the credibility estimate without having to assume a particular distribution. Moreover, the estimated parameters are Best Linear Unbiased Estimates of the true, but unknown parameters.

The definition of the linear model, applicable results, and several applications of the model will be presented. The estimation of K in the Whitney credibility formula $Z = E / (E+K)$, as proposed by Bühlmann-Straub, will also be reviewed.

2. REVIEW OF BÜLHMANN-STRAUB CREDIBILITY

Assume $Y_1, ..., Y_n$ are independent conditional on Θ, with common mean $\mu(\theta) = E(Y_i | \Theta = \theta)$, and with conditional variances $\upsilon(\theta)/E_i = Var(Y_i | \Theta = \theta)$. E_i is a known constant measuring exposure. The credibility formula, $Z_i = E_i / (E_i+K)$, is derived from those assumptions. When each risk has the same number of exposure units, the credibility formula is $Z = n / (n + K)$, where n is the number of observations per risk (Bühlmann credibility). The credibility estimate is equal to $\bar{Y}_i Z_i + (1-Z_i)\hat{\mu}$. $\bar{Y}_i$ is the weighted average for the ith risk, and $\hat{\mu}$ is the credibility weighted average of each $\bar{Y}_i$.

For an excellent history of credibility please see Venter's Credibility Chapter in *Foundations of Casualty Actuarial Science* [2:375-387]. Also see *Loss Models* [1:385-510] for a concise presentation of the principal components of credibility theory.

3. RANDOM EFFECTS LINEAR STATISTICAL MODELS

In using linear models to study the variability in data, we are interested in assigning that variability to the various categorizations of the data. The classifications that identify the source of observations are called factors. Usually there is more than one level of each factor. In classifying data in terms of factors and their levels, we are interested in the extent the different levels of each factor impacts the variable of interest. This is referred to as the *effect* of a level of a factor on that variable. The effects of a factor are classified as fixed effects or as random effects. *Fixed effects* are the effects from a finite set of levels of a factor that occur in the data and which are there because we are interested in them. *Random effects* are the effects from an infinite (usually) set of levels of a factor, of which only a random sample are deemed to occur in the data. For example, to test the tread-wear on sports cars compared to luxury sedans, four high perfomance tires were taken from each of seven batches. Whereas the effects due to type of car would be considered fixed effects (presumably we are interested in the particular cars), the effects due to batches would be considered a random sample of batches from some hypothetical, infinite population of batches. Since there is definite interest in the particular type of car used, the statistical concern is to estimate those car effects; they are fixed effects. Each individual tire is of no particular interest of itself to the trial; it is of interest solely as being one of twenty-eight tires randomly chosen from a larger population of tires. Inferences can and will be made about that population.

No assumption has been made that the type of cars are selected at random from a distribution of car types. In contrast, this kind of assumption has been made about the batch effects; interest in them lies in estimating the variance of those effects. Therefore the data are considered as having two sources of random variation: batch variance and, as usual, error variance. These two sources are known as *variance components*: their sum is the variance of the variable being observed. Models having only fixed effects are called fixed models. Models that contain both fixed and random effects are called mixed models. Finally, those having (apart from a single general mean common to all observations) only random effects are called random models.

Table 3.01 taken from [9:17] summarizes the mathematical characteristics of both classes of models.

Table 3.01

Characteristics of the fixed effects model and the random effects model for the 1-way classification

Characteristic	Fixed Effects Model	Random Effects Model
Model equation	$y_{ij} = \beta + \alpha_i + e_{ij}$	$y_{ij} = \beta + \alpha_i + e_{ij}$
Mean of y_{ij}	$E(y_{ij}) = \beta + \alpha_i$	$E(y_{ij}\mid\alpha_i) = \beta + \alpha_i$ $E(y_{ij}) = \beta$
α_i	Fixed, unknowable constant	$\alpha_i \sim$ i.i.d. $(0,\sigma_\alpha^2)$
e_{ij}	$e_{ij} = y_{ij} - E(y_{ij})$ $= y_{ij} - (\beta + \alpha_i)$ $e_{ij} \sim$i.i.d. $(0,\sigma_e^2)$	$e_{ij} = y_{ij} - E(y_{ij}\mid\alpha_i)$ $= y_{ij} - (\beta + \alpha_i)$ $e_{ij} \sim$i.i.d. $(0,\sigma_e^2)$
$E(e_{ij}\alpha_i)$	$E(e_{ij}\alpha_i) = \alpha_i E(e_{ij}) = 0$	$E(e_{ij}\alpha_i) = 0$, $\text{cov}(\alpha_i\alpha_k) = 0$
$\text{var}(y_{ij})$	$\text{var}(y_{ij}) = \sigma_e^2$	$\text{var}(y_{ij}) = \sigma_\alpha^2 + \sigma_e^2$
$\text{cov}(y_{ij}, y_{i'j'})$	$\text{cov}(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_e^2 & \text{for } i = i' \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$	$\text{cov}(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_\alpha^2 + \sigma_e^2 & \text{for } i = i' \text{ and } j = j' \\ \sigma_\alpha^2 & \text{for } i = i' \text{ and } j \neq j' \\ 0 & \text{otherwise} \end{cases}$

To illustrate via the question posed in the introduction; assume the policyholders are in the same class. The classification plan attempts to group risks with similar characteristics. If the class plan is effective, the overall class mean, β, can be considered common to all the risks. Some risks will have better experience than the average risk and others will have worse. The actual experience of the risks are samples from a random variable representing the experience of each risk. How the actual experience of each risk varies from the class' average/expected experience can be modeled as the random effects, α_i, in the linear statistical model. Thus we seek to estimate the conditional mean $E(\beta + \alpha_i \mid \mathbf{Y})$, where $\mathbf{Y}$ is the vector of observed experience for the risks.

A matrix presentation of the random effects model which includes exposure weights follows:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{W}\mathbf{A} + \mathbf{e},\ \mathbf{A} \sim (\mathbf{O}, \sigma_{A}^{2}\mathbf{I}_{r}),\ \mathbf{e} \sim (\mathbf{O}, \sigma_{e}^{2}\mathbf{I}_{N}) \tag{3.1}$$

Y a Nx1 matrix, contains the experience; losses or number of claims. N is the total number of observations, N = rn, where n is the number of observations per risk, and r is the number of risks. **X** a Nx1 matrix, contains exposures. **W** is a Nxr block diagonal matrix of exposures.

β is the overall class mean and **A** is a rx1 matrix of random effects parameters, α_i, i = 1,...,r. From Table 3.01, the random effects are independent of the error terms **e**, and also independent across risks.

$$\mathrm{Var}(\mathbf{Y}) = \mathbf{V} = \mathbf{W}\sigma_{A}^{2}\mathbf{W}' + \sigma_{e}^{2}(\mathrm{Diag}(\mathbf{X}))\ , \tag{3.2}$$

V is block diagonal across risks and is the sum of the familiar terms: Variance of the Hypothetical Means (VHM), $\mathbf{W}\sigma_{A}^{2}\mathbf{W}'$ and Expected Value of the Process Variance (EVPV), $\sigma_{e}^{2}(\mathrm{Diag}(\mathbf{X}))$.

If Var(**A**) and Var(**e**) are known, the estimators of β and **A** shown below are the best linear unbiased estimators (given the observations in **Y**). In most cases, Var(**A**) and Var(**e**) are also estimated. Hence, the following generalized least squares [5:597] formulas for $\hat{\beta}$ and $\hat{\mathbf{A}}$ produce empirical best linear unbiased estimators. Here, "best" means minimum mean squared error.

$$\hat{\beta} = (\mathbf{X}'\,\hat{\mathbf{V}}^{-1}\,\mathbf{X})^{-1}\,(\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{Y}) \tag{3.3}$$

$$\hat{\mathbf{A}} = \hat{\sigma}_{\mathbf{A}}^2 \mathbf{W}' \hat{\mathbf{V}}^{-1} (\mathbf{Y} - \mathbf{X}\hat{\beta}) \tag{3.4}$$

If $(\mathbf{X}' \hat{\mathbf{V}}^{-1} \mathbf{X})$ is singular, the Moore-Penrose (generalized) inverse,"-" instead of the regular inverse "-1", can be used in $\hat{\beta}$ [3:1-28].

Unbiased estimators of Var(**A**) and Var(**e**) are estimated by using a weighted analysis of variance (ANOVA) table and equating mean squares to their expected value [3:388-389, 452]. The derivation is presented below.

Table 3.02: Weighted ANOVA Table for a One Way Classification

Source of Varation	d.f.	Sum of Squares	Mean Square
Rows	r-1	$SSR = \sum_{i=1}^{r} E_i \, (\bar{Y}_i - \bar{Y})^2$	$\mathrm{MSR} = {}^{\mathrm{SSR}}/_{\mathrm{r-1}}$
Residual Error	N-r	$SSE = \sum_{i=1}^{r} \sum_{j=1}^{n} E_{ij} \, (Y_{ij} - \bar{Y}_i)^2$	$\mathrm{MSE} = {}^{\mathrm{SSE}}/_{\mathrm{N-r}}$

$$E(MSE) = \sigma_e^2 \tag{3.5}$$

$$E(MSR) = \frac{1}{r-1}\left(\sum_{i=1}^{r} E_i - \sum_{i=1}^{r} E_i^2 / \sum_{i=1}^{r} E_i\right)\sigma_A^2 + \sigma_e^2 \tag{3.6}$$

Substituting the estimate of σ_e^2 into equation (3.7) produces an unbiased estimator of $\sigma_{\mathbf{A}}^2$.

$$\hat{\sigma}_{\mathbf{A}}^2 = \left[\sum_{i=1}^{r} E_i - \sum_{i=1}^{r} E_i^2 / \sum_{i=1}^{r} E_i\right]^{-1} \left[\sum_{i=1}^{r} E_i(\bar{Y}_i - \bar{Y})^2 - \hat{\sigma}_e^2(r-1)\right] \tag{3.7}$$

It turns out equation (3.7) is the same formula for the estimated variance of the hypothetical means found in Herzog [4] and Klugman *et al* [1]. These estimates are used in $\hat{\mathbf{V}}$ and the estimates of β and **A** are then produced.

The variance-covariance matrix, $\hat{\mathbf{C}}$, of $\hat{\beta}$ and $\hat{\mathbf{A}}$ can be used to analyze the variability of linear combinations of $\hat{\beta}$ and $\hat{\mathbf{A}}$.

$$\mathrm{Cov}(\hat{\beta},\hat{\mathbf{A}}) = \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{X}'\Sigma_e^{-1}\mathbf{X} & \mathbf{X}'\Sigma_e^{-1}\mathbf{W} \\ \mathbf{W}'\Sigma_e^{-1}\mathbf{X} & \mathbf{W}'\Sigma_e^{-1}\mathbf{W}+\Sigma_{\mathbf{A}}^{-1} \end{bmatrix}^{-} \tag{3.8}$$

$\Sigma_e = \hat{\sigma}_e^2(\mathrm{Diag}(\mathbf{X}))$ and $\Sigma_{\mathbf{A}} = \hat{\sigma}_{\mathbf{A}}^2\mathbf{I}_r$.

If the reasonable assumption that $\mathbf{A} \sim \mathbf{N}(\mathbf{O},\sigma_{\mathbf{A}}^2\mathbf{I}_r)$ and $\mathbf{e} \sim \mathbf{N}(\mathbf{O},\sigma_e^2\mathbf{I}_N)$ is invoked, then hypothesis tests and confidence intervals of linear combinations of the parameters can be evaluated. For example, the following hypothesis test

$$\mathrm{H_o} : \mathbf{L}\begin{bmatrix} \hat{\beta} \\ \hat{\mathbf{A}} \end{bmatrix} = 0 \quad \text{compared to} \quad \mathrm{H_a} : \mathbf{L}\begin{bmatrix} \hat{\beta} \\ \hat{\mathbf{A}} \end{bmatrix} \neq 0$$

can be performed by calculating the following t-statistic, $t = \dfrac{\mathbf{L}\begin{bmatrix} \hat{\beta} \\ \hat{\mathbf{A}} \end{bmatrix}}{\sqrt{\mathbf{L}\hat{\mathbf{C}}\mathbf{L}'}}$. This t-statistic has degrees of freedom, N-rank(**X**). If a confidence interval is of interest, then use $\mathbf{L}\begin{bmatrix} \hat{\beta} \\ \hat{\mathbf{A}} \end{bmatrix} \pm t_{\alpha/2}\sqrt{\mathbf{L}\hat{\mathbf{C}}\mathbf{L}'}$. In addition, the coefficient of variation (CV) of the estimates can be used to assess variability without making the normality assumption.

4. APPLICATIONS

The first application, like Halliwell's [6] paper, uses an example from the *Foundations of Casualty Actuarial Science* [2:433]. **Y** contains six pure premiums for nine risks, all with the same number of exposure units. The objective is to calculate credibility weighted pure premiums for each state, i.e., the predicted pure premium for each risk given all the pure premiums. The overall mean will serve as the fixed effect across risks, and the individual experience is the random effects. The model and the results are presented in Exhibits 1- 2.

First, σ_ε^2 and σ_A^2 are estimated from equations (3.5) and (3.7). Next, these values are used in $\hat{\mathbf{V}}$ to produce $\hat{\beta}$ and $\hat{\mathbf{A}}$ in Exhibit 2. Each $\mathbf{L}_i$, $\hat{\beta}$ and $\hat{\mathbf{A}}$ is used to calculate each value in $\hat{\mathbf{Y}}$. Lastly, $\hat{\mathbf{C}}$ is used conduct significance tests of the linear combinations of the parameters. Each risk parameter combined with the fixed effect is significantly different from 0 at the 5.0% level. The key addition to this analysis is the variance of the linear combination of the mean and random effects. The variability of the credibility estimates is now quantified; via the confidence interval and the coefficient of variation. T-tests of each risk parameter, $\hat{\mathbf{A}}$, can also be calculated by redefining each $\mathbf{L}_i$, starting with 0 instead of 1. Each risk parameter and all risk parameters together are not significantly different from 0. If a particular risk parameter is significant, chances are that risk(s) should be reclassified; remember $\mathbf{A} \sim (\mathbf{O}, \sigma_A^2)$. The credibility estimates are also provided in Exhibit 1. There should be no surprise that the random effects model produced the same values as the credibility weighted estimates.

Another example using varying exposures is the case study presented in *Loss Models* [1:504]. In Exhibit 3, four years of claims and exposures are presented for professional liability coverage of life / health, pension, and property / liability actuaries. The objective is to calculate a credibility weighted frequency for each group of actuaries, i.e., the predicted frequency for each type of liability coverage given all the observed frequencies. The same steps as in example 1 are followed.

However, two credibility estimates are calculated; one using a weighted average for the complement and the other using a credibility weighted average of each Y_i as the complement. The credibility weighted average was introduced in *Loss Models* [1:468] so that the total experience is reproduced using the credibility estimates; 221 claims. Notice that $\hat{\mu}$ and $\hat{\beta}$ are the same and both differ from the weighted average of the individual frequencies. The variability of the frequency predictions again are a valuable addition to the analysis of this data.

Last, the method was applied in the initial stages of designing a frequency based experience rating system for smaller workers compensation risks. Again, the objective is to calculate a credibility weighted frequency for each risk. Data for State D is partitioned among risks in a particular class code where their 3 year average earned premium is between 3,000 and 5,000. The individual experience of each group is modeled as random effects. The data and results are in Exhibit 4. **Y** contains first, second and third reports of the number of claims for 22 risks. **X** contains the payroll (in hundreds). Again, the random effects linear model produced the same frequency as the credibility model.

Now for a few directions on the analysis that can be performed. Risk 16 has the highest credibility, but it also has the highest CV. As a result of the high CV and no claims, it fails the t-test. Risk 11 has the smallest credibility, and it also fails the t-test. The credibilities of both these risks are driven primarily by volume: Risk 16 the most, Risk 11 the least. Risk 12 has the smallest CV and above average credibility. Risk 12 has produced one claim for each year while its exposures have been relatively steady. All claim free risks fail the t-test, while all risks with at least one claim pass the t-test. These results make intuitive sense, because failing the t-test suggests that the predicted frequency is not significantly different from zero. All the claim free risks have a predicted frequency less than the average but not equal to zero. The CV and confidence intervals provide an objective quantification of the variability underlying the potential frequencies. For instance, the upper end point of the confidence interval for Risk 16 is 49% higher than the overall frequency. This type of analysis aids the use of judgment needed to place swing limits on the experience modification for the small risks.

5. CONCLUSION

Credibility models are only a subset of the applications of random effects linear models to actuarial science. This paper provides a complete method for quantifying the variability of credibility estimates. The random effects model is relevant wherever credibility is required. Hopefully, others will see the great benefit of this technique, and start the climb out of the Flatlands regarding our statistical modeling skills.

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EXHIBIT 1

Risk	Y	X	W								
1	0.430	1	1								
1	0.375	1	1								
1	2.341	1	1								
1	0.175	1	1								
1	1.016	1	1								
1	0.466	1	1								
2	0.247	1		1							
2	1.587	1		1							
2	1.939	1		1							
2	0.712	1		1							
2	0.054	1		1							
2	0.261	1		1							
3	0.661	1			1						
3	0.237	1			1						
3	0.063	1			1						
3	0.250	1			1						
3	0.602	1			1						
3	0.700	1			1						
4	0.182	1				1					
4	0.351	1				1					
4	0.011	1				1					
4	0.022	1				1					
4	0.019	1				1					
4	0.252	1				1					
5	0.311	1					1				
5	0.664	1					1				
5	1.002	1					1				
5	0.038	1					1				
5	0.370	1					1				
5	2.502	1					1				
6	0.301	1						1			
6	0.253	1						1			
6	0.044	1						1			
6	0.109	1						1			
6	2.105	1						1			
6	0.891	1						1			
7	0.219	1							1		
7	1.186	1							1		
7	0.431	1							1		
7	1.405	1							1		
7	0.241	1							1		
7	0.804	1							1		
8	0.002	1								1	
8	0.058	1								1	
8	0.235	1								1	
8	0.018	1								1	
8	0.713	1								1	
8	0.208	1								1	
9	0.796	1									1
9	0.260	1									1
9	0.932	1									1
9	0.857	1									1
9	0.129	1									1
9	0.349	1									1

Var(e)	Var(A)	K	Zi
0.35701	0.00669	53.33244	0.10113

Risk	$\bar{Y}i$	Credibility Weighted Pure Prem.
1	0.80050	0.58675
2	0.80000	0.58670
3	0.41883	0.54815
4	0.13950	0.51991
5	0.81450	0.58817
6	0.61717	0.56821
7	0.71433	0.57804
8	0.20567	0.52660
9	0.55383	0.56181
$\bar{Y}$	0.56270	

EXHIBIT 2

B
0.563

Risk	A	L1`	L2`	L3`	L4`	L5`	L6`	L7`	L8`	L9`
		1	1	1	1	1	1	1	1	1
1	0.024	1	0	0	0	0	0	0	0	0
2	0.024	0	1	0	0	0	0	0	0	0
3	-0.015	0	0	1	0	0	0	0	0	0
4	-0.043	0	0	0	1	0	0	0	0	0
5	0.025	0	0	0	0	1	0	0	0	0
6	0.006	0	0	0	0	0	1	0	0	0
7	0.015	0	0	0	0	0	0	1	0	0
8	-0.036	0	0	0	0	0	0	0	1	0
9	-0.001	0	0	0	0	0	0	0	0	1

E(A) = -8.0E-07

C

0.007355	-0.000744	-0.000744	-0.000744	-0.000744	-0.000744	-0.000744	-0.000744	-0.000744	-0.000744
-0.000744	0.006092	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075
-0.000744	0.000075	0.006092	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075
-0.000744	0.000075	0.000075	0.006092	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075
-0.000744	0.000075	0.000075	0.000075	0.006092	0.000075	0.000075	0.000075	0.000075	0.000075
-0.000744	0.000075	0.000075	0.000075	0.000075	0.006092	0.000075	0.000075	0.000075	0.000075
-0.000744	0.000075	0.000075	0.000075	0.000075	0.000075	0.006092	0.000075	0.000075	0.000075
-0.000744	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.006092	0.000075	0.000075
-0.000744	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.006092	0.000075
-0.000744	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.000075	0.006092

Risk	Predicted Pure Prem.	Var of Pure Prem.	Coeff of Variation	t-statistic	t 0.025	Degrees of Freedom	Confidence Interval Lower pt	Upper pt
1	0.58675	0.01196	0.18639	5.36524	2.00575	53	0.36740	0.80610
2	0.58670	0.01196	0.18640	5.36478	2.00575		0.36735	0.80605
3	0.54815	0.01196	0.19951	5.01232	2.00575		0.32880	0.76751
4	0.51991	0.01196	0.21035	4.75402	2.00575		0.30055	0.73926
5	0.58817	0.01196	0.18594	5.37818	2.00575		0.36881	0.80752
6	0.56821	0.01196	0.19247	5.19571	2.00575		0.34886	0.78756
7	0.57804	0.01196	0.18920	5.28556	2.00575		0.35869	0.79739
8	0.52660	0.01196	0.20768	4.81520	2.00575		0.30725	0.74595
9	0.56181	0.01196	0.19466	5.13715	2.00575		0.34245	0.78116

EXHIBIT 3

Group	Year	Y	X	W		
LH	1990	20	853	853		
LH	1991	14	1,105	1,105		
LH	1992	16	1,148	1,148		
LH	1993	21	1,270	1,270		
P	1990	27	1,446		1,446	
P	1991	35	1,780		1,780	
P	1992	36	1,717		1,717	
P	1993	24	2,065		2,065	
PL	1990	5	639			639
PL	1991	8	725			725
PL	1992	4	685			685
PL	1993	11	864			864
	Total	221	14,297	4,376	7,008	2,913

Var(e)	Var(A)	K	Group	Zi	Weighted Average Frequency	Credibility Weighted Frequency	û	Credibility Weighted Frequency, û
0.0209424	0.0000097	2151.668	L/H	0.67038	0.01622	0.01597	0.01478	0.01575
			P	0.76509	0.01741	0.01695		0.01679
			P/L	0.57516	0.00961	0.01210		0.01181
			Total Weighted Frequency		0.01546			
					Total Claims	224		221

B	Group	A	L1'	L2'	L3'
			1	1	1
0.014784	L/H	0.00097	1	0	0
	P	0.00201	0	1	0
	P/L	-0.00297	0	0	1

E(A) = 0.000674

C

4.8408E-06	-3.245E-06	-3.704E-06	-2.784E-06
-3.245E-06	5.3837E-06	2.4828E-06	1.8665E-06
-3.704E-06	2.4828E-06	5.12E-06	2.1302E-06
-2.784E-06	1.8665E-06	2.1302E-06	5.7364E-06

Group	Predicted Frequency	Var. of Frequency	Coeff. of Variation	t-statistic	t 0.025	Degrees of Freedom	Confidence Interval Lower pt	Confidence Interval Upper pt
L/H	0.01575	3.7342E-06	0.12269	8.15034	2.20099	11	0.01150	0.02000
P	0.01679	2.5535E-06	0.09516	10.50839	2.20099		0.01327	0.02031
P/L	0.01181	5.0087E-06	0.18951	5.27664	2.20099		0.00688	0.01674
Total Claims	221							

EXHIBIT 4

Risk	Report	Y	X
1	1	0	312.65
1	2	0	350.65
1	3	0	151.25
2	1	0	328.07
2	2	0	270.00
2	3	0	240.00
3	1	0	136.00
3	2	1	140.00
3	3	1	124.00
4	1	0	800.34
4	2	0	758.03
4	3	2	329.89
5	1	0	502.80
5	2	0	404.56
5	3	1	241.93
6	1	0	108.50
6	2	0	80.50
6	3	1	201.80
7	1	0	7.50
7	2	0	69.04
7	3	0	429.30
8	1	0	160.49
8	2	0	279.83
8	3	0	98.10
9	1	0	173.23
9	2	0	260.17
9	3	0	215.61
10	1	1	518.20
10	2	0	588.10
10	3	1	556.48
11	1	0	128.54
11	2	0	98.70
11	3	0	86.94
12	1	1	453.65
12	2	1	364.33
12	3	1	315.96
13	1	2	235.85
13	2	0	156.03
13	3	0	265.72
14	1	0	115.30
14	2	0	61.51
14	3	0	251.58
15	1	1	374.17
15	2	0	340.99
15	3	1	485.27
16	1	0	1353.45
16	2	0	1119.16
16	3	0	554.17
17	1	1	278.11
17	2	0	432.80
17	3	0	256.80
18	1	0	78.97
18	2	0	165.28
18	3	0	110.14
19	1	0	574.94
19	2	0	416.60
19	3	1	196.88
20	1	0	485.73
20	2	0	271.33
20	3	0	195.10
21	1	0	203.69
21	2	0	212.30
21	3	0	92.44
22	1	0	347.28
22	2	0	107.19
22	3	0	87.27

Var(e)	K
0.000942	5845.66
Var(A)	Degrees of Freedom
1.6116E-07	65

Risk	Zi	Weighted Average Frequency	μ	Credibility Weighted Frequency	Mod
1	0.122301	0.000000	0.000867	0.000761	0.88
2	0.125390	0.000000		0.000758	0.87
3	0.064045	0.005000		0.001132	1.31
4	0.244153	0.001059		0.000914	1.05
5	0.164303	0.000870		0.000868	1.00
6	0.062664	0.002559		0.000973	1.12
7	0.079641	0.000000		0.000798	0.92
8	0.084338	0.000000		0.000794	0.92
9	0.099930	0.000000		0.000780	0.90
10	0.221455	0.001203		0.000941	1.09
11	0.051005	0.000000		0.000823	0.95
12	0.162465	0.002646		0.001156	1.33
13	0.101119	0.003041		0.001087	1.25
14	0.068280	0.000000		0.000808	0.93
15	0.170368	0.001666		0.001003	1.16
16	0.341144	0.000000		0.000571	0.66
17	0.142031	0.001033		0.000891	1.03
18	0.057159	0.000000		0.000817	0.94
19	0.168952	0.000841		0.000863	1.00
20	0.140068	0.000000		0.000746	0.86
21	0.080016	0.000000		0.000798	0.92
22	0.084814	0.000000		0.000794	0.92
					1.00

β	Risk	A	Predicted Frequency	Coefficient of Variation	t-statistic	t 0.025	Confidence Interval Lower pt	Upper pt
0.000867	1	-0.000106	0.000761	0.565540	1.768220	1.997138	0.000000	0.001621
	2	-0.000109	0.000758	0.566302	1.765841	1.997138	0.000000	0.001616
	3	0.000265	0.001132	0.395768	2.526732	1.997138	0.000237	0.002026
	4	0.000047	0.000914	0.429760	2.326880	1.997138	0.000130	0.001698
	5	0.000001	0.000868	0.481326	2.077592	1.997138	0.000034	0.001702
	6	0.000106	0.000973	0.460730	2.170469	1.997138	0.000078	0.001868
	7	-0.000069	0.000798	0.555441	1.800370	1.997138	0.000000	0.001683
	8	-0.000073	0.000794	0.556516	1.796894	1.997138	0.000000	0.001676
	9	-0.000087	0.000780	0.560148	1.785244	1.997138	0.000000	0.001653
	10	0.000074	0.000941	0.424785	2.354132	1.997138	0.000143	0.001740
	11	-0.000044	0.000823	0.549078	1.821235	1.997138	0.000000	0.001725
	12	0.000289	0.001156	0.361709	2.764652	1.997138	0.000321	0.001991
	13	0.000220	0.001087	0.401858	2.488444	1.997138	0.000215	0.001959
	14	-0.000059	0.000808	0.552879	1.808715	1.997138	0.000000	0.001700
	15	0.000136	0.001003	0.414397	2.413147	1.997138	0.000173	0.001833
	16	-0.000296	0.000571	0.633215	1.579242	1.997138	0.000000	0.001294
	17	0.000024	0.000891	0.476477	2.098739	1.997138	0.000043	0.001738
	18	-0.000050	0.000818	0.550419	1.816797	1.997138	0.000000	0.001716
	19	-0.000004	0.000863	0.482365	2.073119	1.997138	0.000032	0.001694
	20	-0.000121	0.000746	0.569984	1.754434	1.997138	0.000000	0.001594
	21	-0.000069	0.000798	0.555527	1.800093	1.997138	0.000000	0.001683
	22	-0.000074	0.000794	0.556625	1.796541	1.997138	0.000000	0.001676

E(A) = -0.000022