Insights into Present Value and Duration

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ABSTRACT

The current syllabus of the Casualty Actuarial Society, especially parts four and ten thereof, exposes actuaries to mathematical finance, particularly to the valuation and management of cash flows. The Society believes that financial matters will be even more important to actuaries of the twenty-first century. However, the syllabus readings do not take advantage of the mathematical proficiency of actuaries. As a result, their understanding is not as clear as it could be. This paper will apply first-year calculus to the concepts of present value and duration. Also, the calculus will permit the definition of a powerful concept that may yet be unfamiliar to many actuaries, the forward rate. Perhaps actuaries will gain fresh insights into these concepts, be more confident and competent to use them, and be better equipped to study more advanced financial theory.

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1. The Formulation of Present Value

The common expression of a cash flow is discrete: an amount of cash c_i is received a time t_i. Normally the interest rate is denoted by i; however, since i will be a subscripting index, we will employ r for the interest rate. This is apt, not only because r suggests 'rate', but also because it suggests 'risk-free rate'.[1] Then we have a familiar formula for the present value of the cash flow at time 0: $PV = \sum_i \frac{c_i}{(1+r)^{t_i}}$. For now we will assume the yield curve to be flat; in Section 5 we will relax this assumption.

The formula can be improved in two ways. Most obviously, it can be improved to handle continuous cash flows. Let $C(t)$ represent the cumulative amount of cash received up to and including time t. If a discrete amount is received at time t, then $C(t)$ will be discontinuous from the left. But we may safely assume that $C(t)$ is continuous from the right.[2] $C(t)$ can decrease as well as increase, a decrease representing a negative cash receipt, or a positive cash disbursement.

The other improvement is to express present value in terms of force of interest (ρ), rather than in terms of interest (r). The equivalency is that $e^{-\rho t} = (1+r)^{-t}$, which implies that $e^{\rho} = 1 + r$ and that $\rho = \ln(1+r)$. That this expression is indeed an improvement will

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[1] The cash flows of this paper are certain, or non-stochastic. Therefore, it is proper to present-value them at the risk-free interest rate. Appendix A touches on the valuation of stochastic cash flows.

[2] For the adviseability of continuity from the right see Appendix B.

be evident in Section 4, where we will differentiate PV with respect to ρ. Moreover, treatments of mathematical finance, e.g., [9], use the force-of-interest notation.

Thus we arrive at the formula: $PV = \int_0^\infty e^{-\rho u} dC(u)$. The summation has been replaced with an integral, in particular, with a Stieltjes integral. If C is differentiable over $[0, \infty)$, this can be expressed as a Riemann integral: $PV = \int_0^\infty e^{-\rho u} C'(u) du$. But the Stieltjes form is more versatile in that it can accommodate discrete cash flows as easily as it can continuous, as is discussed in [1: 12f.] and [2: 21]. One should notice that present value is unaffected by the level of C, since for any constant k, $d(k + C(u)) = dC(u)$.

The first insight afforded by this formulation is the linearity of present value. If we have a grand cash flow composed of other cash flows, i.e., $C(u) = \sum \alpha_i C_i(u)$, then:

$$\begin{aligned} PV &= \int_0^\infty e^{-\rho u} dC(u) \\ &= \int_0^\infty e^{-\rho u} d\left(\sum_i \alpha_i C_i(u)\right) \\ &= \sum_i \alpha_i \left(\int_0^\infty e^{-\rho u} dC_i(u)\right) \\ &= \sum_i \alpha_i PV_i \end{aligned}$$

We know that present value must be a linear operator in order for pricing to be free of arbitrage; but now we know also that the linearity of present value follows from the linearity of integration.

Since we will be interested in present value not only when the present is time 0, we will elaborate our formulation so as to mean the present value at time t of the future cash flow: $PV(C;t,\rho)=\int_t^\infty e^{-\rho(u-t)}dC(u)$. The form on the left side of the equation makes explicit that PV is an operator on C with arguments t and ρ.

2. An Insight into Total Return

In order to see how PV changes with time, we will take the derivative of PV with respect to t. But t appears in the lower limit of the integral, as well as in the integrand. Rather than just claim a result from calculus, we will do the derivative from its definition:

$$
\begin{aligned}
PV(t+\Delta t) &= \int_{t+\Delta t}^{\infty} e^{-\rho(u-(t+\Delta t))}dC(u) \\
&= \int_t^\infty e^{-\rho(u-(t+\Delta t))}dC(u) - \int_t^{t+\Delta t} e^{-\rho(u-(t+\Delta t))}dC(u)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\Delta PV(t) &= PV(t+\Delta t) - PV(t) \\
&= \int_t^\infty e^{-\rho(u-(t+\Delta t))}dC(u) - \int_t^{t+\Delta t} e^{-\rho(u-(t+\Delta t))}dC(u) - \int_t^\infty e^{-\rho(u-t)}dC(u) \\
&= \int_t^\infty \left(e^{-\rho(u-(t+\Delta t))} - e^{-\rho(u-t)}\right)dC(u) - \int_t^{t+\Delta t} e^{-\rho(u-(t+\Delta t))}dC(u) \\
&= \int_t^\infty \Delta\left(e^{-\rho(u-t)}\right)dC(u) - \int_t^{t+\Delta t} e^{-\rho(u-(t+\Delta t))}C'(u)du
\end{aligned}
$$

In the last equation we have assumed that C is differentiable in $[t, t+\Delta t]$. If this is true, then by the mean value theorem there is a $\xi\in[t, t+\Delta t]$, such that:

$$\Delta PV(t) = \int_t^{\infty} \Delta\left(e^{-\rho(u-t)}\right) dC(u) - \int_t^{t+\Delta t} e^{-\rho(u-(t+\Delta t))} C'(u)du$$

$$= \int_t^{\infty} \Delta\left(e^{-\rho(u-t)}\right) dC(u) - e^{-\rho(\xi-(t+\Delta t))} C'(\xi)\Delta t$$

$$\frac{\Delta PV(t)}{\Delta t} = \int_t^{\infty} \frac{\Delta\left(e^{-\rho(u-t)}\right)}{\Delta t} dC(u) - e^{-\rho(\xi-(t+\Delta t))} C'(\xi)$$

In the limit as $\Delta t \to 0$, $\xi \to t$ and $\frac{\Delta\left(e^{-\rho(u-t)}\right)}{\Delta t} \to \frac{\partial}{\partial t}\left(e^{-\rho(u-t)}\right) = \rho e^{-\rho(u-t)}$. Therefore, provided that C be differentiable at t:

$$\frac{\partial}{\partial t} PV(t) = \int_t^{\infty} \rho e^{-\rho(u-t)} dC(u) - e^{-\rho(t-t)} C'(t)$$

$$= \rho \int_t^{\infty} e^{-\rho(u-t)} dC(u) - C'(t)$$

$$= \rho PV(t) - C'(t)$$

Rearranging terms and dividing by $PV(t)$, we have:

$$\frac{1}{PV(t)} \frac{\partial}{\partial t} PV(t) + \frac{C'(t)}{PV(t)} = \rho$$

The first term of the left side represents the instantaneous capital appreciation, and the second represents the instantaneous dividend yield. The sum of these two must equal the force of interest at which the cash flow is present-valued.

So we have attained to the insight that a differentiable cash flow is always earning its cost of capital (ρ), or working at "ρ-power." Less rigorously but perhaps more vividly, we can understand this from the standpoint of arbitrage. If there were times during which a cash flow were earning more than its cost of capital, the flows of these times would be working harder than they were paid to work. And were a cash flow earning less than its cost of capital, it would not be working as hard as it was paid to work. These times could be packaged and traded at unfair prices. Such situations are intolerable in economics, where one should neither get nor let something for nothing.

3. An Insight into Discounting Liabilities

The fundamental equation of a balance sheet is that assets equal liabilities plus surplus, or $A = L + S$. The liabilities, L, are normally booked at nominal, or undiscounted, values. If L were discounted, it would be smaller and S would be larger. In particular, property/casualty insurance companies book nominal values for most of their unpaid losses. However, it may be years, even decades, before these losses are paid. Therefore, discounting losses would greatly increase the surplus of some insurance companies. Not assessing the pros and cons of discounting insurance losses, the author simply comments that three reasons argue for the eventual acceptance of discounting into insurance accounting: (1) the consideration of investment income in pricing, (2) the demand of the 1986 Tax Reform Act that taxable income be based on discounted liabilities, and (3) that discounting underlies the NAIC's risk-based capital calculation. This paper will show that accounting can handle the discounting of liabilities in two ways.

Consider a balance sheet at time 0, with liabilities $L(0)$. As these liabilities run off, $L(t)$ will approach zero; in fact, $\lim_{t\to\infty} L(t) = L(\infty) = 0$. The running off of liabilities represents a payout of cash; so there exists a cumulative cash payout function $C(t)$ such that for all time $t \geq 0$ $C(t) + L(t)$ is constant. As mentioned in Section 1, the level of C is irrelevant;

nonetheless, the simplest qualifier for $C(t)$ is $L(0) - L(t)$. But whatever $C(t)$ is chosen, $dC(t) = -dL(t)$. The present value at time 0 of the liabilities is $PV = \int_0^\infty e^{-\rho u} dC(u)$. However, by integration by parts:

$$\begin{aligned}
PV &= \int_0^\infty e^{-\rho u} dC(u) \\
&= -\int_0^\infty e^{-\rho u} dL(u) \\
&= -e^{-\rho u} L(u)\Big|_0^\infty + \int_0^\infty L(u) de^{-\rho u} \\
&= -e^{-\rho\infty} L(\infty) + e^{-\rho 0} L(0) + \int_0^\infty e^{-\rho u}(-\rho) L(u) du \\
&= L(0) - \int_0^\infty e^{-\rho u} \rho L(u) du \\
&= L(0) - D
\end{aligned}$$

PV represents the present value of the liabilities, or the discounted liabilities; but the D of the last equation represents the discount from the nominal value. By definition, present value equals nominal value minus the discount.

But consider the meaning of the discount $D = \int_0^\infty e^{-\rho u} \rho L(u) du$: $\rho L(u)du$ represents the income earned during $[u, u + du]$ on the liability remaining at time u, and $e^{-\rho u}$ values this income to the present (time 0). Therefore, the present value of a liability equals the nominal value of the liability minus the present value of the investment income that can be earned on the liability while it runs off.

Booking a liability now at its nominal value is not the same as paying the liability now. The liable entity would be averse to paying now, since it would forfeit investment income. Only if it received credit for the present value of the investment income would it be willing to pay the liability now. If accounting were to recognize this economic reality, it could do so in two ways. First, it could prescribe that the liabilities be booked at their present (discounted) value, i.e., at nominal value *net of the present value of the investment income*. Or second, it could prescribe that the liabilities be booked at nominal value, but that the assets be booked *gross of the present value of the investment income*. Letting D be the present value of the investment income, i.e., the discount, we see that in either case the resulting surplus will be the same: $(A+D)-L=A-(L-D)=S$. Accounting policy regarding whether to gross up assets or to net down liabilities would be similar in discounting as it is in reinsurance.

4. Insights into Duration

Many treatments of duration, particularly those on the actuarial syllabus, are guilty of a bait-and-switch tactic. They introduce the notion of duration by discussing the effect of a small discontinuous jump in the interest rate on the future value of a cash flow. If the rate decreases, the cash flow appreciates. But reinvestment is at a lower rate, so eventually the future value will fall behind what it otherwise would have been. And if the rate increases, the cash flow depreciates. But reinvestment is at a higher rate, so eventually the future value will exceed what it otherwise would have been. By the intermediate value theorem, there must be some time at which the future value of the cash flow is immune to interest rate changes. However, duration is commonly defined to be a weighted-average time, viz.:

$$\frac{\int_0^\infty ue^{-\rho u}dC(u)}{\int_0^\infty e^{-\rho u}dC(u)}$$

The time of immunity (the bait) is conceptually different from the weighted-average time (the switch); and most treatments do not prove that the two are equal, perhaps because the proof requires calculus.

To prove the equality, let *FV*(*t*, ρ) denote the future value of the cumulative cash flow *C* at time *t* (the present is time 0). Thus $FV(t,\rho) = e^{\rho t}PV = e^{\rho t}\int_0^\infty e^{-\rho u}dC(u) = \int_0^\infty e^{\rho(t-u)}dC(u)$.

Differentiating with respect to ρ, we have:

$$\begin{aligned}\frac{\partial FV}{\partial \rho} &= \frac{\partial}{\partial \rho}\int_0^\infty e^{\rho(t-u)}dC(u)\\ &= \int_0^\infty (t-u)e^{\rho(t-u)}dC(u)\\ &= e^{\rho t}\int_0^\infty (t-u)e^{-\rho u}dC(u)\\ &= e^{\rho t}\left(t\int_0^\infty e^{-\rho u}dC(u) - \int_0^\infty ue^{-\rho u}dC(u)\right)\end{aligned}$$

At the time of immunity (call it *d*(ρ), a function of ρ) this derivative will be zero. But the derivative is zero if and only if $d(\rho) = \frac{\int_0^\infty ue^{-\rho u}dC(u)}{\int_0^\infty e^{-\rho u}dC(u)}$, which proves the equality.

If we take the second derivative with respect to ρ, we have $\frac{\partial^2 FV}{\partial \rho^2} = \int\limits_0^\infty (t-u)^2 e^{\rho(t-u)} dC(u)$.

If C is non-decreasing, but at least sometimes increasing, as is normal for an asset, this second derivative will be positive. This means that $FV(d(\rho), \rho + \Delta\rho)$, considered as a function of Δρ, has a local minimum at Δρ = 0. A small instantaneous change in the interest rate will increase the future value of the cash flow at the duration time.

In fairness, the bait-and-switch tactic of many treatments of duration is inadvertent, since the context of these treatments is asset/liability matching. But even in this context, a disregard for calculus hinders insight into the meaning of duration and asset/liability matching.

Consider the present value (the present being time 0) of a cash flow as a function of the force of interest: $PV(\rho) = \int_0^\infty e^{-\rho u} dC(u)$. If we differentiate this function with respect to the force of interest, we will have $\frac{\partial PV}{\partial \rho} = PV'(\rho) = -\int_0^\infty u e^{-\rho u} dC(u)$. The higher-order derivatives are easily obtained: $\frac{\partial^n PV}{\partial \rho^n} = PV^{[n]}(\rho) = (-1)^n \int_0^\infty u^n e^{-\rho u} dC(u)$. From this we will define the n^{th} duration moment d_n as:

$$d_n = \frac{\int_0^\infty u^n e^{-\rho u} dC(u)}{\int_0^\infty e^{-\rho u} dC(u)} = (-1)^n \frac{PV^{[n]}(\rho)}{PV(\rho)}$$

So d_n is the weighted-average n^{th} power of time, and it is proportional to the n^{th} derivative of PV. d_0 is unity, d_1 is called the Macaulay duration, and d_2 is the called the convexity.

These expressions are unnecessarily complicated, even cluttered, when interest rates are used. If we express present value in terms of rate r compounded s times per time period, the present value of a cash flow will be $PV(r)=\int_0^\infty \left(1+\frac{r}{s}\right)^{-su} dC(u)$. Differentiate:

$$\begin{aligned}
PV'(r) &= \frac{\partial}{\partial r}\int_0^\infty \left(1+\frac{r}{s}\right)^{-su} dC(u) \\
&= \int_0^\infty (-su)\left(1+\frac{r}{s}\right)^{-su-1} \frac{1}{s} dC(u) \\
&= \frac{-1}{\left(1+\frac{r}{s}\right)}\int_0^\infty u\left(1+\frac{r}{s}\right)^{-su} dC(u)
\end{aligned}$$

Define the n^{th} modified-duration moment md_n as $(-1)^n \frac{PV^{[n]}(r)}{PV(r)}$. Then:

$$md_1 = -\frac{PV'(r)}{PV(r)} = \frac{1}{\left(1+\frac{r}{s}\right)} \frac{\int_0^\infty u\left(1+\frac{r}{s}\right)^{-su} dC(u)}{\int_0^\infty \left(1+\frac{r}{s}\right)^{-su} dC(u)} = \frac{1}{\left(1+\frac{r}{s}\right)} d_1$$

Thus, modified duration is the Macaulay duration with an adjustment factor [4: 6-8], an adjustment that approaches unity as $s \to \infty$, the limit yielding the force-of-interest formulation. And the higher-order derivatives with respect to r are even more unwieldy, since the adjustment factor itself is a function of r. Therefore, it is better to employ force-of-interest formulations in cash-flow mathematics.

Now, to borrow from Keith Holler [6], duration "hides a Taylor series:"

$$
\begin{aligned}
PV(\rho+\Delta\rho) &= \sum_{j=0}^{\infty} \frac{PV^{[j]}(\rho)}{j!}\Delta\rho^{j} \\
\Delta PV(\rho) &= PV(\rho+\Delta\rho) - PV(\rho) \\
&= \sum_{j=1}^{\infty} \frac{PV^{[j]}(\rho)}{j!}\Delta\rho^{j} \\
\frac{\Delta PV(\rho)}{PV(\rho)} &= \sum_{i=1}^{\infty} \frac{1}{j!}\frac{PV^{[j]}(\rho)}{PV(\rho)}\Delta\rho^{j} \\
&= \sum_{j=1}^{\infty} (-1)^{j} \frac{d_j(\rho)}{j!}\Delta\rho^{j}
\end{aligned}
$$

So the relative change in the present value of a cash flow is a polynomial in $\Delta\rho$, whose coefficients involve the duration moments.

We will now combine cash flows. Let $C_i(u)$ be the i^{th} cash flow, whose present value is $PV_i(\rho)$ and whose j^{th} duration moment is $d_{ij}(\rho)$. Hence, $\dfrac{\Delta PV_i(\rho)}{PV_i(\rho)} = \sum_{j=1}^{\infty} (-1)^{j} \dfrac{d_{ij}(\rho)}{j!}\Delta\rho^{j}$.

Form cash flow $C(u)$ by combining α_i units (e.g., dollars) worth of each $C_i(u)$. Therefore, $C(u) = \sum_i \dfrac{\alpha_i}{PV_i(\rho)} C_i(u)$ and, as a check, $PV(\rho) = \sum_i \dfrac{\alpha_i}{PV_i(\rho)} PV_i(\rho) = \sum_i \alpha_i$. If the moment after the combination the force of interest changes by $\Delta\rho$:

$$
\begin{aligned}
\Delta PV(\rho) &= \sum_i \frac{\alpha_i}{PV_i(\rho)} \Delta PV_i(\rho) \\
&= \sum_i \alpha_i \frac{\Delta PV_i(\rho)}{PV_i(\rho)} \\
&= \sum_i \alpha_i \sum_{j=1}^{\infty} (-1)^{j} \frac{d_{ij}(\rho)}{j!}\Delta\rho^{j} \\
&= \sum_{j=1}^{\infty} (-1)^{j} \frac{\sum_i \alpha_i d_{ij}(\rho)}{j!}\Delta\rho^{j}
\end{aligned}
$$

An even more aesthetic form follows, in which $d_j(\rho) = \frac{\sum_i \alpha_i d_{ij}(\rho)}{\sum_i \alpha_i}$:

$$
\begin{aligned}
\frac{\Delta PV(\rho)}{PV(\rho)} &= \frac{\Delta PV(\rho)}{\sum_i \alpha_i} \\
&= \frac{\sum_{j=1}^{\infty} (-1)^j \frac{\sum_i \alpha_i d_{ij}(\rho)}{j!} \Delta\rho^j}{\sum_i \alpha_i} \\
&= \sum_{j=1}^{\infty} (-1)^j \frac{\left(\frac{\sum_i \alpha_i d_{ij}(\rho)}{\sum_i \alpha_i} \right)}{j!} \Delta\rho^j \\
&= \sum_{j=1}^{\infty} (-1)^j \frac{d_j(\rho)}{j!} \Delta\rho^j
\end{aligned}
$$

This demonstrates that the duration moments of a combination of cash flows are equal to the present-value-weighted averages of the duration moments of the cash flows.

Asset/liability matching is a form of combining cash flows. Both the assets and the liabilities are packages of cash flows, whose prices are the present values of those cash flows. But the purpose of this matching is to create a cash flow whose $\frac{\Delta PV(\rho)}{PV(\rho)}$ is as close as possible to zero. Since $\frac{\Delta PV(\rho)}{PV(\rho)} = \sum_{j=1}^{\infty} (-1)^j \frac{d_j(\rho)}{j!} \Delta\rho^j$ is a polynomial in $\Delta\rho$, "as close as possible to zero" means to set as many $d_j(\rho)$ as possible to zero in ascending order of j. Since $d_j(\rho) = \frac{\sum_i \alpha_i d_{ij}(\rho)}{\sum_i \alpha_i} = \sum_i w_i d_{ij}(\rho)$, where the d_{ij}s are given, the task reduces to finding the weights of the assets and liabilities so that as many $d_j(\rho)$ as possible are zero. In the usual treatment of asset/liability matching the assets as a whole are treated as one cash flow, as are the liabilities as a whole. In this case $i = 2$, so at most only the first duration moment can be zeroed. But in general, i cash flows can be weighted so as to zero as many as $i - 1$ duration moments.

Note that the expression $PV(\rho) = \int_0^{\infty} e^{-\rho u} dC(u)$ is like a moment generating function. Hence, claiming that $\frac{\Delta PV(\rho)}{PV(\rho)}$ is zero because it is zero to an i^{th} order approximation is like claiming that a random variable X is zero because its first i moments are zero. But precisely, $PV(\rho) = \int_0^{\infty} e^{-\rho u} dC(u)$ is the Laplace transform of C'. Appendix B will demonstrate that if for all ρ $PV(C; \rho) = 0$, then $C(u)$ must be constant. Finally, if $C(u) = u^n$, $PV(\rho) = n\int_0^{\infty} e^{-\rho u} u^{n-1} du = n\frac{\Gamma(n)}{\rho^n}\int_0^{\infty} \frac{\rho^n}{\Gamma(n)} e^{-\rho u} u^{n-1} du = n\frac{\Gamma(n)}{\rho^n} = \frac{\Gamma(n+1)}{\rho^n}$. So the gamma function can be useful in certain present value problems.

5. The Complication of a Non-Flat Yield Curve

Let $v(t)$ represent the present value of the guaranteed receipt of one unit of value at time t. Therefore, one unit now (at time 0) is equal to the guaranteed receipt of $a(t) = \frac{1}{v(t)}$ units at time t. One unit now will accumulate to $a(t)$ at time t, and to $a(t+\Delta t)$ at time $t+\Delta t$.

Therefore, the instantaneous forward rate of return at time t is [9: 5-7]:

$$\phi(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{a(t+\Delta t) - a(t)}{a(t)} = \frac{a'(t)}{a(t)} = \frac{\partial \ln(a(t))}{\partial t} = \frac{\partial \ln\left(v(t)^{-1}\right)}{\partial t} = -\frac{\partial \ln(v(t))}{\partial t} = -\frac{v'(t)}{v(t)}.$$

Until now, it has been assumed that $v(t) = e^{-\rho t}$, implying that, for all t, $\phi(t) = \rho$. The yield-to-maturity-t is the average forward rate during [0, t], so $YTM(t) = \frac{1}{t}\int_0^t \phi(u)du$. A flat yield curve is thus equivalent to a constant forward rate.

Every insight of the previous sections has relied on the assumption of a flat yield curve; but real yield curves are not flat.[3] Now we will relax this assumption and see how our insights are affected by a general $v(t)$. The forward rate will come in handy, so it will be useful to remember that $v(t)\phi(t) = -v'(t)$.

As to total return (Section 2), the present value at time t of the future cash flow is $PV(t) = \int_t^\infty v(u-t)dC(u)$. Passing over the details, we differentiate:

$$\begin{aligned}
\frac{\partial}{\partial t} PV(t) &= \frac{\partial}{\partial t}\int_t^\infty v(u-t)dC(u) \\
&= \int_t^\infty \left(\frac{\partial}{\partial t} v(u-t)\right)dC(u) - v(t-t)C'(t) \\
&= \int_t^\infty (-v'(u-t))dC(u) - v(0)C'(t) \\
&= \int_t^\infty \phi(u-t)v(u-t)dC(u) - C'(t)
\end{aligned}$$

[3] Malkiel [7] discusses the theories for the non-flatness of yield curves.

Again, as in Section 2, rearranging terms and dividing by $PV(t)$, we have:

$$\frac{1}{PV(t)}\frac{\partial}{\partial t}PV(t)+\frac{C'(t)}{PV(t)}=\frac{\int_t^{\infty}\phi(u-t)v(u-t)dC(u)}{\int_t^{\infty}v(u-t)dC(u)}=\overline{\phi}(t)$$

So, by means of capital appreciation and dividends, a differentiable cash flow is always earning its weighted-average forward rate of return, the weight being the present value of the differential cash flow.

As to discounting liabilities (Section 3),

$$\begin{aligned}
PV &= \int_0^{\infty} v(u)dC(u) \\
&= -\int_0^{\infty} v(u)dL(u) \\
&= -v(u)L(u)\Big|_0^{\infty} + \int_0^{\infty} L(u)dv(u) \\
&= -v(u)L(u)\Big|_0^{\infty} + \int_0^{\infty} L(u)v'(u)du \\
&= -v(\infty)L(\infty) + v(0)L(0) + \int_0^{\infty} L(u)\big(-v(u)\phi(u)\big)du \\
&= L(0) - \int_0^{\infty} v(u)\phi(u)L(u)du \\
&= L(0) - D
\end{aligned}$$

$L(u)$ is the liability remaining at time u, which earns investment income at the forward rate $\phi(u)$. So $\phi(u)L(u)du$ is the investment income earned in $[u, u + du]$, which is present-valued by $v(u)$. Therefore, the discount D is the present value of the investment income, whatever the yield curve may be.

Duration (Section 4) involves an instantaneous change in the whole yield curve, from $v(t)$ at time 0 to $v^*(t)$ an instant later. Let V be the set of all functions v: $[0, \infty) \rightarrow (0, 1]$ that are suitable for present-valuing. To be suitable, $v(t)$ must be continuous and strictly decreasing, $v(0) = 1$, and $\lim_{t\to\infty} v(t) = 0$. Furthermore, $v'(t)$ should exist, which since $v(t)$ is strictly decreasing implies that $v'(t) < 0$. Thus the forward rates $\phi(t) = -\frac{v'(t)}{v(t)}$ exist and are positive. Probably, a realistic ϕ is continuous from the right, and has at most a countably infinite number of discontinuities from the left. It seems that even with all these constraints on $v(t)$ the cardinality of V is greater than that of the real numbers $\Re$, and hence greater than that of any $\Re^n$. If so, then V cannot be parameterized, which means that the change in yield curve cannot be represented as a change in coordinates. And even if V could be parameterized, the parameterization would have to be continuous, so that a small change in yield curve would correspond with a small change in coordinates.

But one type of change in the yield curve that we can parameterize is the parallel shift. In this change $YTM(t)$ shifts to $YTM(t) + \rho$, or equivalently, $\phi(t)$ shifts to $\phi(t) + \rho$. Now:

$$v(t) = e^{\ln(v(t))-\ln(v(0))} = e^{\int_0^t d\ln(v(u))} = e^{-\int_0^t \phi(u)du}$$

Therefore, the present-value function after a parallel shift of ρ (a ρ-shift) is:

$$v(t;\rho) = e^{-\int_0^t (\phi(u)+\rho)du} = e^{-\int_0^t \phi(u)du-\rho t} = e^{-\rho t}v(t)$$

So, first let us examine the time of immunity. The future value of the cash flow C at time t (the present is time 0) after a ρ-shift will be:

$$FV(t,\rho)=\frac{\int_0^\infty v(u;\rho)dC(u)}{v(t;\rho)}$$

$$=\frac{\int_0^\infty e^{-\rho u}v(u)dC(u)}{e^{-\rho t}v(t)}$$

$$=\frac{1}{v(t)}\int_0^\infty e^{\rho(t-u)}v(u)dC(u)$$

Differentiating with respect to ρ, we have:

$$\frac{\partial FV}{\partial \rho}=\frac{\partial}{\partial \rho}\frac{1}{v(t)}\int_0^\infty e^{\rho(t-u)}v(u)dC(u)$$

$$=\frac{1}{v(t)}\int_0^\infty (t-u)e^{\rho(t-u)}v(u)dC(u)$$

$$=\frac{e^{\rho t}}{v(t)}\int_0^\infty (t-u)e^{-\rho u}v(u)dC(u)$$

$$=\frac{e^{\rho t}}{v(t)}\left(t\int_0^\infty e^{-\rho u}v(u)dC(u)-\int_0^\infty ue^{-\rho u}v(u)dC(u)\right)$$

The situation before the shift is represented by $\rho = 0$, and:

$$\left.\frac{\partial FV}{\partial \rho}\right|_{\rho=0}=\frac{e^{0t}}{v(t)}\left(t\int_0^\infty e^{-0u}v(u)dC(u)-\int_0^\infty ue^{-0u}v(u)dC(u)\right)$$

$$=\frac{1}{v(t)}\left(t\int_0^\infty v(u)dC(u)-\int_0^\infty uv(u)dC(u)\right)$$

So the time of immunity, at which this derivative is zero, is $\frac{\int_0^\infty uv(u)dC(u)}{\int_0^\infty v(u)dC(u)}$, which is the Macaulay duration.

The present value of the cash flow after the ρ-shift is $PV(\rho) = \int_0^\infty e^{-\rho u} v(u) dC(u)$. The n^{th} derivative is $\frac{\partial^n PV}{\partial \rho^n} = PV^{[n]}(\rho) = (-1)^n \int_0^\infty u^n e^{-\rho u} v(u) dC(u)$. Before the shift ρ = 0, at which $PV^{[n]}(0) = (-1)^n \int_0^\infty u^n v(u) dC(u)$. So, if we define the general n^{th} duration moment as $d_n = \frac{\int_0^\infty u^n v(u) dC(u)}{\int_0^\infty v(u) dC(u)}$, it will remain true in the case of the ρ-shift that $d_n = (-1)^n \frac{PV^{[n]}(0)}{PV(0)}$. Therefore, everything stated in Section 4 about Taylor series and asset/liability matching will apply to the case of a parallel shift of a non-flat yield curve.

Let $v(t;\ \rho)$ be the yield curve after some unspecified change. The current state corresponds to ρ = 0, so $v(t;\ 0) = v(t)$. Then the present value of a cash flow after a ρ-change is $PV(\rho) = \int_0^\infty v(u;\rho) dC(u)$. Therefore, $PV^{[n]}(\rho) = \int_0^\infty v_\rho^{[n]}(u;\rho) dC(u)$, and $PV^{[n]}(0) = \int_0^\infty v_\rho^{[n]}(u;0) dC(u)$. Asset/liability matching to the p^{th} order is theoretically justified only if $\int_0^\infty (-u)^n v(u;0) dC(u) = \int_0^\infty v_\rho^{[n]}(u;0) dC(u)$ for $n = 1, 2, \ldots, p$. Only then will $d_n = (-1)^n \frac{PV^{[n]}(0)}{PV(0)}$. For this to be independent of the cash flow, $v_\rho^{[n]}(u;0) = (-u)^n v(u;0)$. It would seem that the only solution to this partial differential equation is $v(t;\rho) = e^{-\rho t} v(t;0) = e^{-\rho t} v(t)$, implying the ρ-change to be a ρ-shift.

6. Conclusion

The lack of a calculus-level treatment of finance on the actuarial syllabus impedes actuaries from important insights. This paper has presented insights into total return, discounting, and duration. The two following appendices will elaborate even more on discounting and duration. The more involved actuaries become in financial matters, the more beneficial these insights will be.

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APPENDIX A

A Comparison of Valuation Models

Robert Sturgis introduced casualty actuaries to the subject of valuing an insurance company. The value of a company is its economic value, which is "the book value plus the present worth (i.e., the capitalized value) of expected future earnings." [11: 148] He continued:

> This general valuation concept [economic value] ... suggests two alternative formulas:
> 1. The discounted value of maximum stockholder dividends; and,
> 2. Current net worth plus the discounted value of future earnings less cost of capital. [11: 148f.]

Miccolis calls the first of these alternatives the cash-flow model (*CF*), and the second the income (*IN*) model, and states:

> A cash flow model has also been suggested by Rothman and Deutsch which Sturgis shows to be a special case of the income model (when the discount rate equals the investment yield). [8: 301]

Rothman and Deutch themselves agree:

> If the only cash flow available to the investor is the dividend stream, and dividends are limited to statutory earnings, then the two definitions of economic value result in essentially the same valuation. [10: 127]

The author does not see how Sturgis showed the cash-flow model to be a special case of the income model; so he cannot agree with that portion of Miccolis' statement. Nevertheless, all three agree that the two models are equivalent under certain conditions. But no one gives a demonstration of the equivalence, and many readers probably feel themselves to be missing the point. So this appendix will apply the concepts of this paper in order, hopefully, to illuminate the issue.

We will need some notation:

$A(t)$	Assets at time t ($A = L + S$)
$L(t)$	Liabilities
$S(t)$	Surplus ($S = S_0 + S_1$)
$S_0(t)$	Surplus whose opportunity for investment is not restricted
$S_1(t)$	Surplus whose opportunity for investment is restricted
$Prem(t)$	Cumulative Premium (for simplicity, consider premium to be collected and earned when written)
$Incd(t)$	Cumulative Incurred Loss (and Expense)
$Paid(t)$	Cumulative Paid Loss
$Unpaid(t)$	Cumulative Unpaid Loss ($Incd = Paid + Unpaid$)
$Und(t)$	Cumulative Underwriting Profit ($Und = Prem - Incd$)
δ	Instantaneous rate of return on assets
$Inv(t)$	Cumulative Investment Income ($dInv(t) = \delta A(t)dt$)
$Earn(t)$	Cumulative Earnings ($Earn = Und + Inv$)
ρ	Selected present-valuing rate
κ	Opportunity cost of capital

Again, for simplicity, assume that the liabilities of the company arise solely from the insurance business, so $L(t) = Unpaid(t)$.

Perhaps a comment is in order about the three rates of return (δ, ρ, κ), and the so-called "cost of capital" in general. "Capital" just a fancy word for money. The question "How much does a dollar cost?" would seem to have an obvious answer: "One dollar." But the real question is, "How much will it cost me to use one of your dollars? When do I have to repay you, and how do I compensate you in the meantime?" The answer is expressed as an equation: $B(t+\Delta t) = B(t) + \gamma B(t)\Delta t - C(t)\Delta t$. At time t one has borrowed, or has a balance, of $B(t)$. The cost of borrowing this money, or of maintaining this balance, during $[t, t + \Delta t]$ is $\gamma B(t)\Delta t$, where γ is the cost of capital. $C(t)$ is the rate at which compensation is made to the lender. In the limit as $\Delta t \to 0$ we have the differential equation: $B'(t) = \gamma B(t) - C(t)$. If compensation is made at the cost of capital, or $C(t) = \gamma B(t)$, the balance remains constant. If compensation is deferred, or $C(t) = 0$, then $B'(t) = \gamma B(t)$, which means that the balance will grow exponentially at the cost of capital.

"Cost of capital" is a slippery concept, at times referring to either δ, ρ, or κ. The cost of the company's capital to the assets is δ, and the cost of the owners' capital to the company is ρ. In a risk-free world the two costs would have to be equal.[4] But Sturgis argues for an additional cost κ to the company for that portion of the owners' capital that must be tied up in sub-par investments. However, if the owners were getting their desired cost of capital ρ from the company, why would it matter if certain funds had to be tied up? If certain funds were tied up, so as to drag the investment income, say, from $\delta A(t)dt$ to $(\delta A(t) - \kappa S_1(t))dt$, why could not the $\kappa S_1(t)dt$ be treated as an expense?

Realistically, insurance regulations place no restriction on a company's investing in treasury securities. So is there even such as thing as tied-up, or sub-par, funds? The owners might argue that if it weren't for regulations they could make more money by concentrating the assets of the company in high-return stocks, rather than in low-return bonds. So the regulations supposedly deprive them of an opportunity. But this ignores that stocks are riskier than bonds. In the words of Miccolis, "unless there is a yield differential at the same level of risk, the 'opportunity' cost of capital would not be risk-neutral." [8: 299] So the owners' argument would be pre-CAPM, and thus naïve. But despite these problems, we will keep the three "costs of capital" in this comparison.

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[4] This is not to concede that in a risky world cash flows should be discounted at "risk-adjusted" rates of return. See the final paragraph of this appendix.

Now according to the cash-flow model the value of an insurance company equals the value of its assets plus the present value of its insurance cash flows:

$$CF(t) = A(t) + \int_t^\infty e^{-\rho(u-t)} d\left(Prem(u) - Paid(u)\right)$$

But according to the income model its value equals the value of its surplus plus the present value of its earnings minus the costs associated with its surplus:

$$IN(t) = S(t) + \int_t^\infty e^{-\rho(u-t)} dEarn(u) - \int_t^\infty e^{-\rho(u-t)} \delta S(u) du - \int_t^\infty e^{-\rho(u-t)} \kappa S_1(u) du$$

One might think that the last two terms are erroneous, that the value should consist of just the surplus and the subsequent earnings. However, the earnings include *all* the investment income, even the income from the surplus. The subtraction of $\int_t^\infty e^{-\rho(u-t)} \delta S(u) du$ avoids double counting. The last term is Sturgis' (opportunity) cost of capital [11: 149], whether justified or not.

Both models ignore whether the cash flows and the earnings are distributable, i.e., they assume that later cash flows and earnings do not depend on the reinvestment (plowing back) of earlier cash flows and earnings. If a company made $100 in one year and $100 in the second and final year, but had to use $50 of the first year's earnings in order to gear up for the second year's earnings, the company should be valued on an earnings' stream of $50 and $100.[5]

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[5] A third model, the "terminal value" model of Miccolis' [8: 301], may make the appraiser more aware of the plow-back assumptions. But the drawback of this model is that the company's present value depends on the calculation of its value at some future terminus.

So, performing a sequence of algebraic manipulations, which includes an integration by parts,[6] we transform the *CF* model to look as much as possible like the *IN* model:

$$
\begin{aligned}
CF(t) &= A(t) + \int_t^\infty e^{-\rho(u-t)} d\big(Prem(u) - Paid(u)\big) \\
&= A(t) + \int_t^\infty e^{-\rho(u-t)} d\big(Prem(u) - Paid(u) - Unpaid(u)\big) + \int_t^\infty e^{-\rho(u-t)} dUnpaid(u) \\
&= A(t) + \int_t^\infty e^{-\rho(u-t)} dUnd(u) + \int_t^\infty e^{-\rho(u-t)} dL(u) \\
&= A(t) + \int_t^\infty e^{-\rho(u-t)} dUnd(u) + \left(e^{-\rho(u-t)} L(u)\right)\Big|_t^\infty - \int_t^\infty (-\rho) e^{-\rho(u-t)} L(u)du \\
&= A(t) + \int_t^\infty e^{-\rho(u-t)} dUnd(u) + e^{-\rho(\infty-t)} L(\infty) - e^{-\rho(t-t)} L(t) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du \\
&= A(t) + \int_t^\infty e^{-\rho(u-t)} dUnd(u) - L(t) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du \\
&= S(t) + \int_t^\infty e^{-\rho(u-t)} dUnd(u) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du \\
&= S(t) + \int_t^\infty e^{-\rho(u-t)} d\big(Und(u) + Inv(u)\big) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du - \int_t^\infty e^{-\rho(u-t)} dInv(u) \\
&= S(t) + \int_t^\infty e^{-\rho(u-t)} dEarn(u) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du - \int_t^\infty e^{-\rho(u-t)} \delta A(u)du \\
&= S(t) + \int_t^\infty e^{-\rho(u-t)} dEarn(u) + \int_t^\infty e^{-\rho(u-t)} \rho L(u)du - \int_t^\infty e^{-\rho(u-t)} \delta\big(L(u) + S(u)\big)du \\
&= S(t) + \int_t^\infty e^{-\rho(u-t)} dEarn(u) + \int_t^\infty e^{-\rho(u-t)} (\rho - \delta) L(u)du - \int_t^\infty e^{-\rho(u-t)} \delta S(u)du \\
&= IN(t) + \int_t^\infty e^{-\rho(u-t)} \kappa S_1(u)du + \int_t^\infty e^{-\rho(u-t)} (\rho - \delta) L(u)du
\end{aligned}
$$

Thus, if $\kappa = 0$ and $\rho = \delta$, the cash-flow model is equivalent to the income model. But in practice, appraisers add an opportunity cost ($\kappa > 0$) and/or demand a rate of return greater

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[6] It is also reasonably assumed that $\lim_{u \to \infty} e^{-\rho(u-t)} L(u) = 0$.

than that of the assets ($\rho > \delta$), both of which lead to the cash-flow model's giving a greater value than that of the income model.

The issue of valuing a stochastic cash flow has arisen several times in this paper. The common practice is to discount the expected cash flow at some "risk-adjusted" rate of return, the rate being higher as the cash flow is more variable. This is the justification for ρ's being greater than the risk-free rate of return. It has not been the purpose of the paper to dispute this practice. But Appendix F of [5] offers a promising alternative.

APPENDIX B

The Laplace Transform as the Signature of a Cash Flow

In Section 4 we mentioned that $PV(\rho) = \int_0^\infty e^{-\rho u} dC(u)$ is the Laplace transform of C'. In this appendix we will prove that if $PV(\rho) = 0$ for all ρ, then $C(u)$ must be constant. This has an important corollary, viz., that if $PV(C_1; \rho) = PV(C_2; \rho)$ for all ρ, then $C_1(u) - C_2(u)$ must be constant. This owes to the fact that the Laplace transform is linear, so that $PV(C_1 - C_2; \rho) = PV(C_1; \rho) - PV(C_2; \rho)$. Inferring that two cash flows are equivalent from the equality of their Laplace transforms is like inferring that two random variables have the same cumulative distribution function from the equality of their characteristic functions. However, there are some subtleties of real analysis that complicate the inference. For example, define $C^*(u; u_0)$ as $C(u)$ for all $u \neq u_0$, and let $C^*(u_0)$ be anything other than $C(u_0)$. So C^* is different from C (at one point), but this difference will not show in their Laplace transforms. To eliminate such trivial differences Section 1 recommended that $C(u)$ be continuous from the right. This appendix will avoid the subtleties by working with finite (and thus discrete) cash flows. With finite cash flows we can approximate to within any tolerance all but the pathological cash flows.

The Laplace transform of a finite cash flow is $PV(\rho) = \sum_{i=1}^{n} e^{-\rho t_i} c_i$. Without loss of generality we can stipulate that all the t_is be unequal. Define a_i as e^{-t_i}. Since there is a one-to-one correspondence between t and e^{-t}, the a_is also must be unequal. Then the transform can be expressed as: $PV(\rho) = \sum_{i=1}^{n} a_i^{\rho} c_i = \begin{bmatrix} a_1^{\rho} & \dots & a_n^{\rho} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. If we choose n values of ρ, we can write the n simultaneous equations as:

$$\begin{bmatrix} PV(\rho_1) \\ \vdots \\ PV(\rho_n) \end{bmatrix} = \begin{bmatrix} a_1^{\rho_1} & \cdots & a_n^{\rho_1} \\ \vdots & \ddots & \vdots \\ a_1^{\rho_n} & \cdots & a_n^{\rho_n} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Since PV is always zero, we can choose ρ_i to be $i - 1$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

$$\mathbf{0}_{n\times 1} = \mathbf{A}_{n\times n}\mathbf{c}_{n\times 1}$$

Now **A** is known as a Vandermonde matrix. Eves [3: 126f.] shows that the determinant of such a matrix is $\prod_{i>j}(a_i - a_j)$. This determinant is zero if and only if some $a_i - a_j$ factor is zero. But all the a_is are unequal, so the determinant is not zero. Therefore, the matrix **A** is non-singular, $\mathbf{A}^{-1}$ exists, and there is one and only one **c** that satisfies the equation, viz.:

$$\mathbf{c} = \mathbf{I}_n\mathbf{c} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{c} = \mathbf{A}^{-1}(\mathbf{A}\mathbf{c}) = \mathbf{A}^{-1}(\mathbf{0}_{n\times 1}) = \mathbf{0}_{n\times 1}$$

If for every discount rate the Laplace transform of a finite cash flow is zero, then the cash flow must be a zero, or null, cash flow. And if the Laplace transforms of two finite cash flows are always equal, then the cash flows must be identical. If the Laplace transforms of *any* two cash flows are always equal, then any differences between the cash flows must be trivial.

How does asset/liability management work? Assets and liabilities are combined so as to make $\frac{\Delta PV(\rho)}{PV(\rho)}$ zero to within an error of order $O(\rho^n)$. Then $PV(\rho)$ will be approximately constant, e.g., $PV(\rho) \approx c = \sum_{i=1}^{1} e^{-\rho 0} c$. The right side of this equation is the Laplace transform of the cash flow of the receipt at time 0 of amount c, which is immune to a change in ρ.