Underwriting Risk

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Abstract

In a competitive insurance market, insurers have limited influence on the premium charged for an insurance contract. They must decide whether or not to compete at the market price. This paper deals with one factor in this decision – risk.

From policyholder's standpoint, the only risk that matters is insurer insolvency. For the insurer to stay in business, it has to have sufficient capital to keep this risk below an acceptable level. Also, investors demand an acceptable return on this capital. The problem is that the return comes from premiums that are charged to individual insureds, each with their own risk characteristics.

This paper proposes a way to set standards for accepting individual insureds based on the risk each contributes to the insurer's portfolio. These standards will assign the marginal capital to the insured. These standards will be expressed in terms of the acceptable return on allocated surplus. They will take into account: (1) the variability of the insurer's loss; (2) the time it takes until all claims are paid; and (3) the correlation of the insured's losses with the insurer's other losses.

We start by illustrating the basic concepts with simple examples, and finish with a comprehensive example that shows how we can put these standards into practice.

1. Introduction

Insurance companies are financial institutions with financial objectives. This paper is about linking the insurer's financial objectives with the myriad of individual underwriting and pricing decisions it makes as it goes about its business.

From the financial point of view, the insurer's mission is the following.

- Under normal circumstances, the insurer expects its premium income to generate enough money to pay for the insured losses. On some occasions, insured losses will exceed premium income and the insurer will have to pay for losses out of its own capital.
- The insurer assumes the risk of financial loss to its customers, i.e. the insureds. While the insurance contract covers losses arising from accidents that occur in a predetermined period, the losses themselves can be payable over a much longer period of time.
- The insurer's owners provide the capital. In return for assuming the insureds' risk of loss, the owners expect to receive a return on their investment that is competitive with other investments with similar risk. This return on the owner's capital investment is the insurer's financial measure of success.
- In return for assuming this risk, the insurer collects premium from the insureds. This premium is used to pay underwriting expenses and set up the necessary reserves to pay future losses.

The income that provides the return on the owner's capital is derived from two principal sources: the underwriting profit from its insurance operations; and the investment income from the assets underlying its reserves and its capital. Quite often, the underwriting profit is negative. This is acceptable if the investment income generated from writing the business is large enough to provide a competitive return on the owner's investment.

More generally, the insurer's income is the result of numerous underwriting underwriting and decisions made by employees of the insurer. Each decision involves a consideration of the expected underwriting profit, the length of time that the reserve must be held, and the additional capital needed to protect the insurer's solvency. From the owner's perspective, the results of the individual underwriting decisions do not matter as long as the total income is large enough to provide a competitive return on the investment. But ultimately, the insurer must make individual underwriting decisions that contribute, in an actuarial sense, to its overall financial objectives.

This paper describes some actuarial considerations that an insurer can make to link its underwriting decisions to its financial objectives.

2. The Cost of Committing Capital

We begin with a simple model that illustrates how the cost of capital influences the price of an insurance policy.[1]

The risk is that an insurer will have to pay an amount of $1 at time, T, in interval [0,t]. The distribution of T is uniform throughout [0,t]. The probability of making a payment at some time in this interval is q.

Assume that the insurer has to hold $1 of capital until either the claim is paid, or the liability expires at time t. The capital is invested in a risk free interest bearing account with force of nterest δ_i. Interest on this investment is paid continuously to the insurer. In return for subjecting its capital to the risk, the insurer requires a higher rate of return with force of interest δ_r.

[1] This example was motivated by a "thought experiment" about risk suggested to the author by Leigh Halliwell.

Case 1. The claim occurs

The insurer receives continuous interest payments at an annual rate of δ_i until time T, when the claim is paid. The insurer's expected rate of return is δ_r. The present value of the insurer's investment is:

$$\text{PV with claim} = \delta_i \frac{1 - e^{-\delta_r T}}{\delta_r}$$

Then: $$E[\text{PV with claim}] = \int_0^t \delta_i \frac{1 - e^{-\delta_r \tau}}{\delta_r} \cdot \frac{q}{t} d\tau = q \frac{\delta_i}{\delta_r} \left(1 - \frac{1 - e^{-\delta_r t}}{t\delta_r}\right)$$

Case 2. The claim does not occur

The insurer receives continuous interest payments at an annual rate of δ_i until time t. At that time the insurer's capital of $1 is returned. The present value of the insurer's investment is:

$$\text{PV without claim} = \delta_i \frac{1 - e^{-\delta_r t}}{\delta_r} + e^{-\delta_r t}$$

Then: $$E[\text{PV without claim}] = (1 - q) \cdot \left(\delta_i \frac{1 - e^{-\delta_r t}}{\delta_r} + e^{-\delta_r t}\right)$$

If the insurer is to expose its capital to the risk of loss, it must receive at least an amount, P, so that the expected rate of return on its investment of $1 is at least δ_r, That is:

$$1 = P + E[\text{PV with claim}] + E[\text{PV without claim}].$$

It is the insured who must provide P, otherwise the insurer would not accept the risk.

When pricing insurance policies, actuaries are accustomed to comparing P, the cost of the insurance needed before they will voluntarily write the insurance, to q, the expected loss payments. Define the risk load R, as this difference. In this example:

$$R = P - q$$

The following table illustrates how the cost of insurance can increase with the length of time that the supporting capital must be held.

Table 2.1

t	1	2	3	4	5	6
δ_i	6%	6%	6%	6%	6%	6%
δ_r	10%	10%	10%	10%	10%	10%
q	0.100	0.100	0.100	0.100	0.100	0.100
E[PV with claim]	0.003	0.006	0.008	0.011	0.013	0.015
E[PV without claim]	0.866	0.835	0.807	0.781	0.758	0.738
Cost of Insurance	0.131	0.160	0.185	0.208	0.229	0.248
Risk Load	0.031	0.060	0.085	0.108	0.129	0.148

We leave it to the reader to investigate how the other factors δ_i , δ_r and q affect the cost of insurance.

This example represents a very simplified view of the insurance business. A more comprehensive example could include the following considerations.

1. δ_i should increase with t. This is the normal behavior for fixed-rate investments.
2. δ_r depends upon the return of other investments with comparable risk and time commitment, which in turn depends upon the probability and the timing of the insurer's loss.
3. The losses that are covered by a typical collection of insurance policies are unlimited. The insurance buying public appears willing to accept the remote possibility that the insurer won't be able to cover its claims. There are a number of regulatory and rating agencies that take on the job of determining the amount of capital that is necessary to assure that the probability of insolvency is indeed remote.
4. An insurer usually underwrites several insureds whose losses are of different amounts, are paid at different times and are, more or less, independent. The cost of providing the total coverage depends upon the entire portfolio while the premium that provides for this cost

comes piecemeal from individual insureds. Since individual insureds may differ in their variability of loss and payment times, their effect on the insurer's financial position may differ.

Although an insurer's management cannot deal with these issues separately, it is also true that a single paper cannot adequately cover all these issues adequately. So in the remaining discussion we will restrict our considerations by: (1) assuming a single fixed risk-free rate of return; (2) assuming a single fixed rate of return to the investors for bearing the insurer's risk; and (3) using three conventional actuarial formulas for determining the insurer's required capital. This paper deals primarily with the issues raised in (4) above.

3. Probabilistic Capital Requirement Formulas

In order to protect the policyholders, the business of insurance is subject to solvency regulation. The regulators have the authority to revoke the insurer's license. In addition, there are a number of private agencies that rate insurers on their ability to pay claims. These ratings are taken very seriously by the insurers because the ratings have a strong influence on their ability to attract business. These institutions put a lower bound on an insurer's capital.

The insurer's management will often attempt to duplicate the regulator's and the rating agencies' capital requirements. In addition, they may develop their own probabilistic capital requirement formulas that they use for planning purposes. We now give a description of three such formulas.

Let X be a random variable representing the insurer's aggregate loss. Let:

$$F(x) = \Pr\{X \leq x\}$$
$$f(x) = F'(x)$$
$$\sigma = \text{Standard Deviation of } X$$
$$C = \text{Required Insurer Capital}$$

Then the required capital can be defined by one of the following equations

1. Probability of Ruin Formula

$$F(C+E[X]) = 1-\varepsilon$$

2. Expected Policyholder Deficit Formula

$$\frac{\int_{C+E[X]}^{\infty}(x-C-E[X])\cdot f(x)dx}{E[X]} = \eta$$

3. Standard Deviation Formula

$$C = T\cdot\sigma$$

Each of these formulas depends upon a judgmental solvency threshold denoted by either ε, η or T. More often than not, the people making these judgments also pay close attention to the capital requirements of the regulatory and private rating agencies.

We summarize the rationale underlying each of the formulas.

1. The probability of ruin formula is the classic actuarial solvency formula. It represents interests of the insurer's stockholders who have limited liability. That is, once the insurer is insolvent, nothing else matters.
2. The Expected policyholder deficit is a refinement of the probability of ruin formula in that it takes the size of insolvency into account. This appeals to the interests of the policyholders.
3. The standard deviation principle is equivalent to the probability of ruin formula when the insurer's distribution of losses is normal. While the normal assumption is not realistic, there is nothing to prevent one from using the standard deviation formula on other loss distributions. It is popular because it is easy to work with.

We provide an illustrative example that can easily be programmed on a spreadsheet[2] with formulas found in Klugman, Panjer and Willmot [1998]. Let X be a random variable with a gamma distribution. That is:

Probability Density Function

$$f(x) = \frac{(x/\theta)^{\alpha} e^{-x/\theta}}{x\Gamma(\alpha)} \tag{3.1}$$

Cumulative Distribution Function

$$\begin{aligned} F(x) &= \Gamma(\alpha; x/\theta) \\ &\equiv \text{GammaDist}(x, \alpha, \theta, \text{TRUE}) \end{aligned} \tag{3.2}$$

Expected Value

$$E[X] = \frac{\theta\Gamma(\alpha+1)}{\Gamma(\alpha)} \equiv \theta \cdot \exp(\text{GammaLn}(\alpha+1) - \text{GammLn}(\alpha)) \tag{3.3}$$

Limited Expected Value Function

$$\begin{aligned} E[X \wedge x] &= \frac{\theta\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1; x/\theta) + x \cdot (1 - \Gamma(\alpha+1; x/\theta)) \\ &\equiv \theta \cdot \exp(\text{GammLn}(\alpha+1) - \text{GammaLn}(\alpha)) \cdot \text{GammaDist}(x, \alpha+1, \theta, \text{TRUE}) \\ &\quad + x \cdot (1 - \text{GammaDist}(x, \alpha, \theta, \text{TRUE})) \end{aligned} \tag{3.4}$$

Variance

$$\begin{aligned} E[X^2] &= \frac{\theta^2\Gamma(\alpha+2)}{\Gamma(\alpha)} \equiv \theta^2 \cdot \exp(\text{GammaLn}(\alpha+2) - \text{GammLn}(\alpha)) \\ \sigma^2 &= \text{Var}[X] = E[X^2] - E[X]^2 \end{aligned} \tag{3.5}$$

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[2] The spreadsheet formulas given below are for MicroSoft Excel 97.

Using the relationship $\int_{C+E[X]}^{\infty}(x-C-E[X])\cdot f(x)dx = E\left[X\right]-E\left[X\wedge\left(C+E[X]\right)\right]$

and Equations 3.2-3.5, one can set up a spreadsheet to solve for the capital required for insurer loss distributions described by a gamma distribution. The following table shows the results for various solvency thresholds. In this table we set $\alpha = \theta = 100$. This yields a size of loss distribution with mean, E[X] = 10,000 and standard deviation, Std[X] = 1,000.

For reference, we have also included a premium to surplus ratio.[3] For comparison, the NAIC Early Warning Test penalizes any insurer who has a premium to surplus ratio that is higher than 3.0 to 1.

Table 3.1
Illustrative Capital Requirements

Probability of Ruin		
Threshold	Capital	P/S
1.0%	2,472	6.1 to 1
0.5%	2,763	5.4 to 1

Expected Policyholder Deficit		
Threshold	Capital	P/S
0.10%	2,091	7.2 to 1
0.05%	2,382	6.3 to 1

Standard Deviation		
Threshold	Capital	P/S
2.33	2,330	6.4 to 1
2.58	2,580	5.8 to 1

[3] In this paper we will use the term "surplus" and "capital" interchangeably, ignoring the formal distinctions there are between the two concepts. Also, we assume an expected loss ratio of 2/3.

We now add parameter uncertainty to this example. Let β be a random variable with $E[\beta] = 1$ and $Var[\beta] = b$. We then make Equations 3.1-3.5 conditional on β by replacing the parameter θ with $\theta \cdot \beta$. For example:

$$f(x|\beta) = \frac{\left(x/(\theta\beta)\right)^{\alpha} e^{-x/(\theta\beta)}}{x\Gamma(\alpha)} \tag{3.1′}$$

We then modify our three probabilistic capital requirement formulas to account for parameter uncertainty.

1. Probability of Ruin Formula

$$E_{\beta}\left[F(C + E[X]|\beta)\right] = 1 - \varepsilon$$

2. Expected Policyholder Deficit Formula

$$\frac{E_{\beta}\left[\int_{C+E[X]}^{\infty}(x - C - E[X]) \cdot f(x|\beta)dx\right]}{E[X]} = \eta$$

3. Standard Deviation Formula

$$C = T \cdot \sqrt{E_{\beta}\left[Var\left[X|\beta\right]\right] + Var_{\beta}\left[E\left[X|\beta\right]\right]}$$

We use a three-point distribution for β in this example. Let:

$$\beta_1 = 1 - \sqrt{3b},\ \beta_2 = 1,\ \beta_3 = 1 + \sqrt{3b},$$

$$Pr\{\beta = \beta_1\} = Pr\{\beta = \beta_3\} = 1/6 \text{ and } Pr\{\beta = \beta_2\} = 2/3. \tag{3.6}$$

We have that $E[\beta] = 1$ and $Var[\beta] = b$.

Let b = 0.02. Then: $\theta \cdot \beta_1 = 75.5051$, $\theta \cdot \beta_2 = 100.0000$, and $\theta \cdot \beta_3 = 124.4949$. Recall $\alpha = 100$.

If C = 4,443.25, then:

$$
\begin{aligned}
& E_\beta\left[F(C + E[X] \| \beta)\right] \\
& = \Gamma(\alpha; 14443.25/(\theta\beta_1))/6 + 2 \cdot \Gamma(\alpha; 14443.25/(\theta\beta_2))/3 + \Gamma(\alpha; 14443.25/(\theta\beta_3)) \\
& = 0.01
\end{aligned}
$$

This means that the required surplus to make the probability of ruin equal to 0.01 = 4,443.25. We similarly solved for the required surplus for the other formulas and parameters. The results are in the following table.

Table 3.2
Illustrative Capital Requirements with Parameter Uncertainty

Probability of Ruin		
Threshold	Capital	P/S
1.0%	4,443	3.4 to 1
0.5%	4,895	3.1 to 1

Expected Policyholder Deficit		
Threshold	Capital	P/S
0.10%	4,129	3.6 to 1
0.05%	4,557	3.3 to 1

Standard Deviation		
Threshold	Capital	P/S
2.33	4,049	3.7 to 1
2.58	4,484	3.3 to 1

4. The Marginal Cost of Capital

Consider the following situation. A single insured is up for renewal. An analysis of market conditions has determined the premium necessary to retain the insured. The expected loss and all other expenses are known. You must decide whether or not to renew the insured.

To make this decision, the insurer performs the following calculations.

C_T = the capital needed for its current business.

R_T = the total risk load (i.e. the total premium supplied by all insureds less the expected loss along with all underwriting and acquisition expenses) that is needed to attract the capital C_T.

$C_T - \Delta C_i$ = the total capital needed if the i^{th} insured is not renewed.

$R_T - \Delta R_i$ = the total risk load needed if the i^{th} insured is not renewed.

The insurer's decision to renew will depend other investment opportunities for ΔC_i. Under stable conditions, the insurer might decide to renew if $\frac{R_T - \Delta R_i}{C_T - \Delta C_i} < \frac{R_T}{C_T}$. However, if the insurer can find another prospect that requires the same marginal capital, ΔC_i, and will pay a premium that yields a higher profit, the insurer may decide not to renew.

Determining ΔC_i is complicated since, as the following examples will show, ΔC_i depends upon the characteristics of the insurer's total book of business.

In the following example, we assume that the insurer's distribution of losses has a gamma distribution with $\theta = 100$ and $\alpha = \alpha_T$. We also assume that the 1^{st} insured's distribution of losses has a gamma distribution with $\theta = 100$ and $\alpha = 1$. It is a property of the gamma distribution that the parameters of the insurer's distribution of losses are given by $\theta = 100$ and $\alpha = \alpha_T - 1$ when the 1st insured is removed. Since the insurer's expected loss is given by $\theta \cdot \alpha_T$, α_T can reasonably be viewed as an indicator of the size of the insurer.

Table 4.1
Illustrative Marginal Capital Calculations

Probability of Ruin @ 1.0%			
b	α_T	C_T	ΔC_1
	50	1,790.34	16.55
0.00	100	2,472.26	11.67
	200	3,436.22	8.24
	50	2,665.43	37.50
0.02	100	4,443.25	34.13
	200	7,693.44	31.35

Expected Policyholder Deficit @ 0.10%			
b	α_T	C_T	ΔC_1
	50	1,634.55	11.52
0.00	100	2,091.11	7.53
	200	2,684.89	4.84
	50	2,609.60	32.22
0.02	100	4,129.19	29.15
	200	6,915.78	27.00

Standard Deviation @ 2.33			
b	α_T	C_T	ΔC_1
	50	1,647.56	16.56
0.00	100	2,330.00	11.68
	200	3,295.12	8.25
	50	2,341.62	35.04
0.02	100	4,049.11	33.66
	200	7,382.83	33.16

We should note that parameter uncertainty generates correlation between the insured under consideration for renewal and the rest of the insurer's business. If the parameters of the loss distributions are mixed by the random variable β we have:

$$\mathrm{Var}[X] = E_\beta\left[\mathrm{Var}[X|\beta]\right] + \mathrm{Var}_\beta\left[E[X|\beta]\right]; \tag{4.1}$$

$$\mathrm{Cov}[X,Y] = E_\beta\left[\mathrm{Cov}[X,Y|\beta]\right] + \mathrm{Cov}_\beta\left[E[X|\beta], E[Y|\beta]\right]; \text{ and} \tag{4.2}$$

$$\rho = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\cdot\mathrm{Var}[Y]}} \tag{4.3}$$

In our example we have:

$$\begin{aligned}
&X|\beta_i \sim \mathrm{gamma}(1, \theta\cdot\beta_i)\\
&Y|\beta_i \sim \mathrm{gamma}(\alpha_T - 1, \theta\cdot\beta_i)\\
&E[X|\beta_i] = \theta\cdot\beta_i\\
&E[Y|\beta_i] = \theta\cdot\beta_i\cdot(\alpha_T - 1)\\
&\mathrm{Var}[X|\beta_i] = (\theta\cdot\beta_i)^2\cdot 2 - E[X|\beta_i]^2\\
&\mathrm{Var}[Y|\beta_i] = (\theta\cdot\beta_i)^2\cdot\alpha_T\cdot(\alpha_T - 1) - E[Y|\beta_i]^2
\end{aligned} \tag{4.4}$$

Using $E_\beta[g(X|\beta)] = \sum_{i=1}^{3} g(X|\beta_i)\cdot\Pr\{\beta_i\}$ applied to Equations 4.1-4.3 for the g's in Equations 4.4 and the $\Pr\{\beta_i\}$'s in Equation 3.6, we obtain the following coefficients of correlation for our example with b = 0.02.

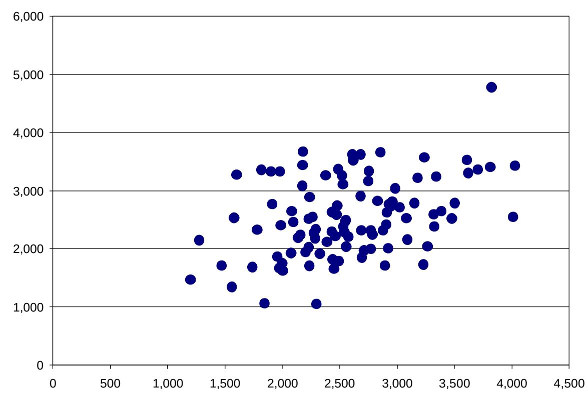
Table 4.2
Illustrative Coefficients of Correlation

α_T	ρ
50	0.137
100	0.195
200	0.277

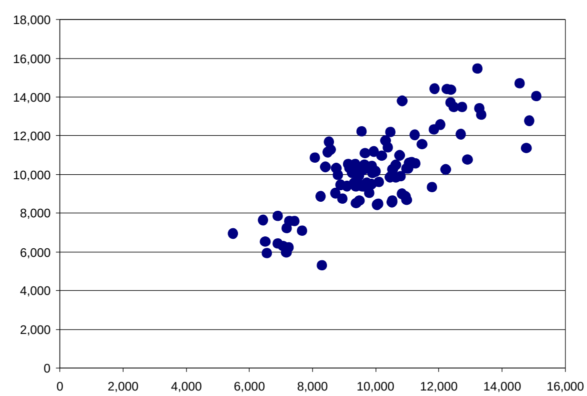
Note that the coefficient of correlation increases with α_T. This happens because the mean of the first insured's losses varies linearly with the mean of the insurer's remaining losses. But as the size of the insurer's remaining business increases, the insurer's random deviations decrease as a proportion of the mean. This leads to a higher coefficient of correlation.

This phenomenon can be seen clearly in the graphs below. The graphs below were generated by a simulation where θ was first selected at random. Then two random numbers were selected from a gamma distribution with the same α. Each graph shows 100 simulations.

Graph 4.1
a = 25, E[q] = 100



Graph 4.2
a = 100, E[q] = 100



These examples illustrate that the marginal capital an insurer must have to renew an insured depends upon properties of the insurer's entire book of business. In particular, we observed differences due to the insurer's size[4] and the correlation between the insured being considered for renewal, with the insurer's existing book of business.

This means that an insured, which is acceptable to one insurer may not be acceptable to another insurer with the same underwriting standards and financial goals. We see this happening in the current market for property insurance where there is an exposure to catastrophes. In property insurance, geographic proximity drives correlation in much the same way that parameter uncertainty does above. One insurer who is concentrated in an area will reject new business, while another insurer who is not concentrated in the same area will readily accept new business.

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[4] One might expect that an increase in the coefficient of correlation with the insurer's book of business might lead to an increase in the marginal cost of capital. As Tables 4.1 and 4.2 show, this is not necessarily the case.

5. Allocating Capital

In the last section we indicated that an insured might decide to renew if:

$$\frac{R_T - \Delta R_i}{C_T - \Delta C_i} \leq \frac{R_T}{C_T}. \tag{5.1}$$

After all, renewing in this case will maintain the return on the insurer's capital. With a little algebra, one can show that the above equation is true if and only if:

$$\Delta R_i \geq \frac{R_T}{C_T} \cdot \Delta C_i. \tag{5.2}$$

If the insurer plans to continue in its business, strict equality in Equation 5.2 for all insureds presents a problem. If (as is usually the case for insurers) $\sum_i \Delta C_i < C_T$, then:

$$\sum_i \Delta R_i = R_T \cdot \frac{\sum_i \Delta C_i}{C_T} < R_T.$$

Accepting all insureds at equality will not meet the insurers financial objectives. Therefore there must be a strict inequality for at least some insureds.

We assume that a strict inequality for some at the expense of others cannot exist in the long run. To solve this problem we propose a formula of the form:

$$\frac{\Delta R_i}{\Delta C_i} \geq K > \frac{R_T}{C_T} \text{ for all i.} \tag{5.3}$$

The insurer must have an expected return of R_T to keep its investors' capital. This means that:

$$\sum_i \Delta R_i \geq R_T. \tag{5.4}$$

Combining Equation 5.3 and 5.4 we find that K must be no smaller than:

$$K = \frac{R_T}{\sum_i \Delta C_i}. \tag{5.5}$$

Equations 5.3 and 5.5 do not provide the only solution to the problem posed by the strict equality of Equation 5.2. This problem is similar to what Mango [1998] refers to as the "ordering problem." Mango's other solutions to the ordering problem could also be used here.

The insurer's management could instruct its underwriters to give due consideration to Equations 5.3 and 5.5 when accepting insureds. But they often have another objective — to focus the underwriters' attention on maintaining an adequate return on capital. A common way to do this is to assign allocated capital, A_i, to individual insureds according to the following formula.[5] [6]

$$\frac{\Delta R_i}{A_i} \equiv \frac{R_T}{C_T}. \tag{5.6}$$

We combine Equations 5.3, 5.5 and 5.6 to arrive at a formula for allocating capital.

$$A_i = K \cdot \Delta C_i \cdot \frac{C_T}{R_T} = \Delta C_i \cdot \frac{C_T}{\sum_j \Delta C_j}. \tag{5.7}$$

That is, we allocate capital to individual insureds in proportion to their marginal capital.

We now continue with our illustrative example. We use our three capital requirements formulas on an insurer with $\alpha_T = 100$. We populate the insurer with insureds with $\alpha = 1, 2, 3$ and 4. We allow 10 insureds for each α. We chose $b = 0.02$. The results are in the following table.

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[5] Allocating capital has been controversial. Opponents to the idea say that the insurers' potential liability to the insured is limited by its entire capital, not the capital "allocated" to the insured. We agree. Conventional use of the term should be limited to communicating management financial goals to areas of underwriting responsibilty.

[6] This could be summarized to higher levels if desired, but for now we will allocate capital to individual insureds.

Table 5.1
Illustration of Capital Allocated to Individual Insureds

Probability of Ruin @ 1.0%			
α	Number of Insureds	ΔC per Insured	% Allocated to Insured
1	10	34.13	0.99865%
2	10	68.31	1.99865%
3	10	102.53	2.99999%
4	10	136.80	4.00270%
Total Marginal Capital		3417.68	

Expected Policyholder Deficit @ 0.10%			
α	Number of Insureds	ΔC per Insured	% Allocated to Insured
1	10	29.15	0.99871%
2	10	58.33	1.99868%
3	10	87.55	2.99992%
4	10	116.81	4.00269%
Total Marginal Capital		2918.35	

Standard Deviation @ 2.33			
α	Number of Insureds	ΔC per Insured	% Allocated to Insured
1	10	33.66	0.99964%
2	10	67.33	1.99964%
3	10	101.01	3.00000%
4	10	134.71	4.00072%
Total Marginal Capital		3367.10	

It is interesting to note that approximately the same proportion of capital is allocated to the insurer for each of the three capital requirement formulas. This is no accident, as we now demonstrate.

Express the capital as a function of the mean, μ, and the variance, σ^2, of the insurer's aggregate loss distribution. The capital requirement is $\mu + T \cdot \sigma^2$ for the standard deviation formula. In our example above, the probability of ruin and the expected policyholder deficit was a function of the parameters of the gamma distribution, which one can calculate from the mean and variance of the gamma distribution.

Let: $C = C(\mu, \sigma^2)$

$$\Delta C_i \approx \left.\frac{\partial C}{\partial \sigma^2}\right|_{\mu_T, \sigma_T^2} \cdot \Delta\sigma_i^2 \quad \text{(For small } \Delta\sigma_i^2\text{)} \tag{5.8}$$

$$\frac{A_i}{C_T} \approx \frac{\Delta C_i}{\sum_j \Delta C_j} = \frac{\left.\frac{\partial C}{\partial \sigma^2}\right|_{\mu_T, \sigma_T^2} \cdot \Delta\sigma_i^2}{\left.\frac{\partial C}{\partial \sigma^2}\right|_{\mu_T, \sigma_T^2} \cdot \sum_j \Delta\sigma_j^2} = \frac{\Delta\sigma_i^2}{\sum_j \Delta\sigma_j^2}$$

Equation 5.8 says that we can allocate capital in proportion to the marginal variance if:

1. We calculate the capital requirement with *any* differentiable function of the mean and variance of the insurer's aggregate loss distribution; and

2. The insured's variance of loss is small compared to the insurer's variance of loss.

If these conditions are met, allocating surplus becomes a simple task once one has the covariance matrix for all insureds. One calculates the marginal variance of the insured by summing all the covariances in the appropriate row and column of the covariance matrix.

But, as we shall see below, these conditions are not always met.

6. A Comprehensive Example

So far, this paper has developed the notion that there is a cost of insuring risk that depends on the insurer's cost of capital. Section 2 demonstrated that the cost of insuring depends upon the length of time that the capital must be held with a simple stochastic model. Section 3 introduced a more complex stochastic model but ignored the length of time that the capital must be held. Section 4 introduced the notion of marginal capital and Section 5 showed how to use marginal capital to allocate the cost of capital to a single insured.

This section combines both the time and stochastic elements of risk into a single comprehensive example.

The XYZ Insurance Company writes three lines of insurance: Property; General Liability; and Auto. To limit extraneous details, we shall assume that:

- All polices go into effect on January 1 and expire on December 31.
- The property losses are all paid by the end of the year.
- All Auto and General Liability losses are paid within three years.
- The lines of business have been stable for the last three years and are expected to remain so for the foreseeable future.
- XYZ has a conservative investment policy, so asset risk is not an issue.
- Invested assets earn interest at an annual rate of 6%.
- XYZ does not purchase reinsurance.
- The expected loss ratio is 2/3 for all lines.

The investors in XYZ demand a before-tax return on capital of 15%. XYZ's executives do not monitor the prices on individual insureds but they do hold their line managers/underwriters responsible for meeting profitability targets. XYZ's actuary, Jane, has the job of allocating surplus by line of insurance for use in evaluating the underwriting results of the year 2000.

Prior to doing this job, Jane projected XYZ's aggregate loss distribution for the year 2000. Noteworthy features of the aggregate loss distribution include:

- Losses for unpaid claims from accident years 1998 and 1999 are included as well as losses for the accident year 2000.
- The property claim severity distribution and claim count distributions are both very skewed.
- Auto and General Liability losses are correlated, but Property losses are independent of the liability losses. The correlation is generated by simultaneously varying the means of the claim count distributions in a manner analogous to that explained in Section 4 above.

The following table provides summary statistics for XYZ's aggregate loss distribution. A more detailed description is given in Appendix A. This description includes various percentiles of the aggregate loss distribution as well as the covariance matrix. We calculated the aggregate loss distributions with the Heckman/Meyers [1983] algorithm.

Table 6.2
Summary Statistics for XYZ's Aggregate Loss Distribution

Aggregate Mean	348,737,619
Aggregate Std. Dev	51,143,663

Line Statistics

Distribution Name	E[Count]	Std[Count]	E[Severity]	Std[Severity]	E[Total Loss]
Property AY 2000 Lag 0	2,400.00	760.53	10,999.77	224,488.75	26,399,448
G.L. AY 2000 Lag 0	1,200.00	243.67	40,348.87	160,218.51	48,418,644
G.L. AY 2000 Lag 1	600.00	123.06	59,798.30	194,452.18	35,878,980
G.L. AY 2000 Lag 2	300.00	62.74	79,247.73	221,803.85	23,774,319
G.L. AY 1999 Lag 1	600.00	123.06	59,798.30	194,452.18	35,878,980
G.L. AY 1999 Lag 2	300.00	62.74	79,247.73	221,803.85	23,774,319
G.L. AY 1998 Lag 2	300.00	62.74	79,247.73	221,803.85	23,774,319
A.L. AY 2000 Lag 0	1,800.00	315.67	27,620.60	83,875.82	49,717,080
A.L. AY 2000 Lag 1	600.00	107.11	40,705.79	102,382.07	24,423,474
A.L. AY 2000 Lag 2	200.00	37.52	53,790.97	116,562.06	10,758,194
A.L. AY 1999 Lag 1	600.00	107.11	40,705.79	102,382.07	24,423,474
A.L. AY 1999 Lag 2	200.00	37.52	53,790.97	116,562.06	10,758,194
A.L. AY 1998 Lag 2	200.00	37.52	53,790.97	116,562.06	10,758,194

Using this aggregate loss distribution, Jane calculated the needed capital under three different criteria with the following results.

Table 6.2

Capital Requirements for
XYZ Insurance Company

Standard Deviation @ 2.33	119,164,734
Probability of Ruin @ 1.0%	120,538,640
Expected Policyholder Deficit @ 0.05%	116,871,140

After consultation with the appropriate rating agencies, XYZ's management concluded that a capital of 120,000,000 would lead to an acceptable rating. Assuming an expected loss ratio of 2/3, this leads to a premium to surplus ratio of 2.7 to 1.[7]

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[7] The expected loss calculation did not include the reserves from prior years.

Now Jane went about her task of setting profitability targets by line of insurance. She proceeded as follows.

1. Since the agreed upon capital was close to her prior projections, she worked with the same capital requirements criteria as before.
2. For each capital requirement criteria, she calculated the marginal contribution to capital by, in turn, removing each of the lines and settlement lags from XYZ's portfolio.
3. She allocated XYZ's capital in proportion to the marginal capital for each line and settlement lag. The allocation proportions are shown in Tables 6.3a-c below.
4. Jane then calculated the capital, C_0 that was initially needed to support each line of insurance written in 2000 by multiplying the sum of the year 2000 allocation factors for each line by 120,000,000. Now as losses are paid, the capital needed to support the insurance written in 2000 can be released. Using the allocation factors, Jane similarly calculated the amount of capital, C_1 that was still needed at the beginning of 2001 and the amount of capital, C_2 that was needed at the beginning of 2002. These amounts are shown Tables 6.4a-c below.
5. The capital will be invested at a rate of i = 6%. Taking the investment earning into account, Jane then calculated the amount of capital that XYZ expects to release at the end of the first, second and third years with the formula:

$$\mathrm{Rel}_t = C_{t-1} \cdot (1+i) - C_t \tag{6.1}$$

6. Jane then calculated the risk load, R, that must be collected from the insureds at time t = 0, to give the investors a return of r = 15% on their investment of C_0. She used the formula:

$$C_0 = R + \sum_{t=1}^{3} \frac{\mathrm{Rel}_t}{(1+r)^t}$$

The resulting profitability targets, R, are given in Tables 6.4a-c.

Table 6.3a
Marginal Surplus
Standard Deviation @ 2.33

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	4,431,638	0	0

General Liability

Year\Lag	0	1	2
1998			7,608,686
1999		11,687,172	7,608,686
2000	16,070,791	11,687,172	7,608,686

Auto Liability

Year\Lag	0	1	2
1998			3,252,286
1999		7,436,987	3,252,286
2000	15,358,640	7,436,987	3,252,286

Total Marginal Capital	106,692,300
Capital	119,164,734

Allocated Capital

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	0.0415	0	0

General Liability

Year\Lag	0	1	2
1998			0.0713
1999		0.1095	0.0713
2000	0.1506	0.1095	0.0713

Auto Liability

Year\Lag	0	1	2
1998			0.0305
1999		0.0697	0.0305
2000	0.1440	0.0697	0.0305

Table 6.3b
Marginal Surplus
Probability of Ruin @ 1.0%

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	6,954,945	0	0

General Liability

Year\Lag	0	1	2
1998			7,429,797
1999		10,877,505	7,429,797
2000	15,292,833	10,877,505	7,429,797

Auto Liability

Year\Lag	0	1	2
1998			2,794,383
1999		6,486,603	2,794,383
2000	13,753,638	6,486,603	2,794,383

Total Marginal Capital	101,402,172
Capital	120,538,640

Allocated Capital

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	0.0686	0	0

General Liability

Year\Lag	0	1	2
1998			0.0733
1999		0.1073	0.0733
2000	0.1508	0.1073	0.0733

Auto Liability

Year\Lag	0	1	2
1998			0.0276
1999		0.0640	0.0276
2000	0.1356	0.0640	0.0276

Table 6.3c
Marginal Surplus
Expected Policyholder Deficit @ 0.05%

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	8,170,468	0	0

General Liability

Year\Lag	0	1	2
1998			6,153,199
1999		9,772,421	6,153,199
2000	13,899,936	9,772,421	6,153,199

Auto Liability

Year\Lag	0	1	2
1998			2,389,980
1999		5,582,538	2,389,980
2000	11,964,669	5,582,538	2,389,980

Total Marginal Capital	90,374,524
Capital	116,892,764

Allocated Capital

Property

Year\Lag	0	1	2
1998			0
1999		0	0
2000	0.0904	0	0

General Liability

Year\Lag	0	1	2
1998			0.0681
1999		0.1081	0.0681
2000	0.1538	0.1081	0.0681

Auto Liability

Year\Lag	0	1	2
1998			0.0264
1999		0.0618	0.0264
2000	0.1324	0.0618	0.0264

Table 6.4a
Profitability Target Calculation
Standard Deviation @ 2.33

t	0	1	2	3
		Property		
C_t	4,984,395			
Rel_t		5,283,458		
R	390,083			
		General Liability		
C_t	39,777,920	21,702,624	8,557,715	
Rel_t		20,461,971	14,447,067	9,071,178
R	5,096,397			
		Auto Liability		
C_t	29,296,861	12,022,543	3,657,942	
Rel_t		19,032,131	9,085,953	3,877,419
R	3,327,431			
Total R	8,813,911			

Table 6.4b
Profitability Target Calculation
Probability of Ruin @ 1.0%

t	0	1	2	3
		Property		
C_t	8,230,527			
Rel_t		8,724,359		
R	644,128			
		General Liability		
C_t	39,762,622	21,664,982	8,792,471	
Rel_t		20,483,397	14,172,410	9,320,019
R	5,106,530			
		Auto Liability		
C_t	27,259,326	10,983,180	3,306,892	
Rel_t		17,911,705	8,335,279	3,505,305
R	3,076,466			
Total R	8,827,125			

Table 6.4c

Profitability Target Calculation
Expected Policyholder Deficit @ 0.05%

t	0	1	2	3
		Property		
C_t	10,848,811			
Rel_t		11,499,740		
R	849,037			
		General Liability		
C_t	39,602,606	21,146,162	8,170,265	
Rel_t		20,832,600	14,244,666	8,660,481
R	5,021,880			
		Auto Liability		
C_t	26,472,751	10,585,971	3,173,434	
Rel_t		17,475,145	8,047,696	3,363,840
R	2,979,979			
Total R	8,850,897			

Jane could have calculated profitability targets for individual insureds using the same methodology, but that was not her task. Nevertheless, she has a standing offer to calculate these targets, should she be asked.

Note that the three methods allocate surplus to the lines in different proportions in contradiction to Equation 5.8. This is because the capital requirements criteria are not all simple functions of the mean and variance of the aggregate loss distribution. When one does not derive the aggregate loss distribution from the first two moments, we should expect this to happen.

At XYZ, the underwriters' bonuses depend upon how well their lines of insurance perform relative to the targeted returns. The fact that the three capital requirements criteria produce different results has sparked a debate among XYZ's management. They have yet to inform us of their decision of which criterion to accept.

7. Data and Technology Requirements

You will need the following items to perform a capital allocation analysis like the one above.

1. Claim severity distributions by line and settlement lag
2. Claim count distributions by line and settlement lag
3. A correlation model between lines of insurance
4. An aggregate loss model

A large insurer might be able to analyze its own data to derive a claim severity distribution. Those who have neither the necessary volume of data nor the inclination to analyze their own data can obtain this information from an insurance advisory organization.

Claim count distributions and correlations between are harder to come by. The claim count depends upon exposure, which varies by observation. When one gets one observation per year, it is difficult to get a sufficient number of observations to get a reliable estimate of the claim count distribution parameters. A similar problem occurs when you are modeling the correlation structure. However, if one accepts the idea that similar claim count distribution and correlation structures apply to different insurers, more reliable estimates can be made. We are in the process of making such estimates and Meyers [1999b] outlines our methodology.

Wang [1998] and Meyers [1999a] have written papers about aggregate loss models that allow one to account for correlation. Between the two papers, there are a variety of correlation structures and calculation methods. Wang shows how to use the Fast Fourier Transform, and Meyers shows how to use the method of Heckman and Meyers [1983] to get the aggregate loss distribution with correlated lines of insurance.

To summarize, the data and the technology are available to do these analyses.

References

1. Heckman, Philip E. and Meyers, Glenn G., 1983, "The Calculation of Aggregate Loss Distributions from Claim Severity Distributions and Claim Count Distributions" *PCAS LXX*.
2. Klugman, Stuart A.; Panjer, Harry H.; and Willmot, Gordon E., 1998, Loss Models: From Data to Decisions. John Wiley & Sons.
3. Mango, Donald F., 1998, "An Application of Game Theory: Property Catastrophe Risk Load" to appear in the *Proceedings of the Casualty Actuarial Society*.
4. Meyers, Glenn G., 1999a, " A Discussion of Aggregation of Correlated Risk Portfolios: Models & Algorithms, by Shaun S. Wang," to appear in the *Proceedings of the Casualty Actuarial Society*.
5. Meyers, Glenn G., 1999b, "Estimating Between Line Correlations Generated by Parameter Uncertainty," submitted for publication.
6. Shaun S. Wang, 1998, "Aggregation of Correlated Risk Portfolios: Models & Algorithms," to appear in the 1998 *Proceedings of the Casualty Actuarial Society*.

Appendix A

XYZ Insurance Company Aggregate Loss Distribution for EPD

Aggregate Mean	348,737,619
Aggregate Std. Dev	51,143,663

Entry Ratio	Aggregate Loss	Excess Probability	Excess Pure Premium	Excess Pure Premium Ratio	Excess Std. Deviation
0.0500	17,436,881	1.00000	331,300,738	0.95000	51,143,663
0.1000	34,873,762	1.00000	313,863,857	0.90000	51,143,663
0.1500	52,310,643	1.00000	296,426,976	0.85000	51,143,663
0.2000	69,747,524	1.00000	278,990,095	0.80000	51,143,663
0.2500	87,184,405	1.00000	261,553,214	0.75000	51,143,663
0.3000	104,621,286	1.00000	244,116,333	0.70000	51,143,663
0.3500	122,058,167	1.00000	226,679,452	0.65000	51,143,663
0.4000	139,495,048	1.00000	209,242,571	0.60000	51,143,663
0.4500	156,931,929	1.00000	191,805,690	0.55000	51,143,663
0.5000	174,368,810	1.00000	174,368,810	0.50000	51,143,663
0.5500	191,805,691	1.00000	156,931,929	0.45000	51,143,661
0.6000	209,242,687	0.99997	139,495,048	0.40000	51,143,339
0.6500	226,686,696	0.99858	122,058,167	0.35000	51,125,716
0.7003	244,238,019	0.98430	104,621,286	0.30000	50,878,359
0.7522	262,337,792	0.93716	87,184,405	0.25000	49,639,303
0.8075	281,590,030	0.87687	69,747,524	0.20000	46,721,989
0.8659	301,983,999	0.83515	52,310,643	0.15000	42,717,920
0.9285	323,791,593	0.74502	34,873,762	0.10000	38,088,424
0.9646	336,406,976	0.62930	26,155,321	0.07500	34,870,906
1.0113	352,679,617	0.44134	17,436,881	0.05000	29,917,292
1.0233	356,846,506	0.39614	15,693,193	0.04500	28,546,354
1.0367	361,526,594	0.34991	13,949,505	0.04000	26,981,847
1.0520	366,869,090	0.30416	12,205,817	0.03500	25,181,542
1.0697	373,059,435	0.26092	10,462,129	0.03000	23,099,558
1.0905	380,304,532	0.22239	8,718,440	0.02500	20,697,967
1.1149	388,806,592	0.18956	6,974,752	0.02000	17,964,959
1.1436	398,804,143	0.16025	5,231,064	0.01500	14,915,778
1.1783	410,900,775	0.12800	3,487,376	0.01000	11,532,711
1.2258	427,473,858	0.08238	1,743,688	0.00500	7,584,599
1.3067	455,704,573	0.02324	348,738	0.00100	3,048,332
1.3351	465,608,759	0.01274	174,369	0.00050	2,151,270
1.3938	486,066,735	0.00293	34,874	0.00010	885,782

XYZ Insurance Company Aggregate Loss Distribution for Probability of Ruin

Aggregate Mean	348,737,619
Aggregate Std. Dev	51,143,663

Entry Ratio	Aggregate Loss	Cumulative Probability	Limited Pure Premium	Limited Pure Premium Ratio	Limited Std. Deviation
0.9003	313,985,816	0.20000	306,264,252	0.87821	18,419,083
0.9264	323,087,734	0.25000	313,337,705	0.89849	21,497,863
0.9444	329,341,831	0.30000	317,878,662	0.91151	23,625,415
0.9591	334,466,024	0.35000	321,340,633	0.92144	25,358,748
0.9722	339,041,527	0.40000	324,201,892	0.92964	26,886,924
0.9846	343,355,345	0.45000	326,682,998	0.93676	28,300,816
0.9967	347,593,543	0.50000	328,908,037	0.94314	29,655,678
1.0091	351,911,788	0.55000	330,958,545	0.94902	30,993,278
1.0222	356,477,451	0.60000	332,897,518	0.95458	32,353,637
1.0366	361,516,871	0.65000	334,784,712	0.95999	33,786,019
1.0535	367,405,105	0.70000	336,693,717	0.96546	35,367,585
1.0750	374,898,054	0.75000	338,745,055	0.97135	37,249,963
1.1063	385,798,096	0.80000	341,177,138	0.97832	39,787,220
1.1546	402,646,697	0.85000	344,102,571	0.98671	43,387,666
1.2074	421,069,012	0.90000	346,410,127	0.99333	46,765,737
1.2337	430,237,994	0.92500	347,211,396	0.99562	48,093,992
1.2631	440,507,642	0.95000	347,849,380	0.99745	49,240,996
1.2699	442,853,224	0.95500	347,960,736	0.99777	49,452,565
1.2771	445,358,595	0.96000	348,067,139	0.99808	49,658,835
1.2848	448,069,818	0.96500	348,168,712	0.99837	49,860,037
1.2934	451,052,995	0.97000	348,265,535	0.99865	50,056,374
1.3030	454,408,648	0.97500	348,357,632	0.99891	50,248,035
1.3142	458,301,948	0.98000	348,444,960	0.99916	50,435,172
1.3277	463,036,205	0.98500	348,527,363	0.99940	50,617,947
1.3456	469,276,259	0.99000	348,604,500	0.99962	50,796,509
1.3737	479,076,895	0.99500	348,675,584	0.99982	50,971,099
1.4314	499,191,423	0.99900	348,726,512	0.99997	51,108,659
1.4542	507,127,002	0.99950	348,732,250	0.99998	51,125,941
1.5040	524,496,702	0.99990	348,736,608	1.00000	51,139,995

XYZ Insurance Company – Correlation Matrix for Lines of Insurance

	L1 Yr 3	L2 Yr3 Lag0	L2 Yr3 Lag1	L2 Yr3 Lag2	L2 Yr2 Lag1	L2 Yr2 Lag2	L2 Yr1 Lag2	L3 Yr3 Lag0	L3 Yr3 Lag1	L3 Yr3 Lag2	L3 Yr2 Lag1	L3 Yr2 Lag 2	L3 Yr1 Lag2
L1 Yr 3	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
L2 Yr3 Lag0	0.0000	1.0000	0.3511	0.3245	0.3511	0.3245	0.3245	0.4528	0.4165	0.3541	0.4165	0.3541	0.3541
L2 Yr3 Lag1	0.0000	0.3511	1.0000	0.3097	0.3351	0.3097	0.3097	0.4322	0.3975	0.3379	0.3975	0.3379	0.3379
L2 Yr3 Lag2	0.0000	0.3245	0.3097	1.0000	0.3097	0.2863	0.2863	0.3995	0.3675	0.3124	0.3675	0.3124	0.3124
L2 Yr2 Lag1	0.0000	0.3511	0.3351	0.3097	1.0000	0.3097	0.3097	0.4322	0.3975	0.3379	0.3975	0.3379	0.3379
L2 Yr2 Lag2	0.0000	0.3245	0.3097	0.2863	0.3097	1.0000	0.2863	0.3995	0.3675	0.3124	0.3675	0.3124	0.3124
L2 Yr1 Lag2	0.0000	0.3245	0.3097	0.2863	0.3097	0.2863	1.0000	0.3995	0.3675	0.3124	0.3675	0.3124	0.3124
L3 Yr3 Lag0	0.0000	0.4528	0.4322	0.3995	0.4322	0.3995	0.3995	1.0000	0.5127	0.4359	0.5127	0.4359	0.4359
L3 Yr3 Lag1	0.0000	0.4165	0.3975	0.3675	0.3975	0.3675	0.3675	0.5127	1.0000	0.4009	0.4716	0.4009	0.4009
L3 Yr3 Lag2	0.0000	0.3541	0.3379	0.3124	0.3379	0.3124	0.3124	0.4359	0.4009	1.0000	0.4009	0.3408	0.3408
L3 Yr2 Lag1	0.0000	0.4165	0.3975	0.3675	0.3975	0.3675	0.3675	0.5127	0.4716	0.4009	1.0000	0.4009	0.4009
L3 Yr2 Lag 2	0.0000	0.3541	0.3379	0.3124	0.3379	0.3124	0.3124	0.4359	0.4009	0.3408	0.4009	1.0000	0.3408
L3 Yr1 Lag2	0.0000	0.3541	0.3379	0.3124	0.3379	0.3124	0.3124	0.4359	0.4009	0.3408	0.4009	0.3408	1.0000