Estimating Uncertainty in Cash Flow Projections

Roger M. Hayne, FCAS, MAAA

ESTIMATING UNCERTAINTY IN CASH FLOW PROJECTIONS

by

Roger M. Hayne

Abstract

In order to be complete dynamic financial analysis (DFA) models should deal with both the amount and timing of future loss and loss adjustment expense payments. Even more than asset cash flows, these future payments are very uncertain.

This paper begins by estimating both process and parameter uncertainty in reserves for annuity-type benefits such as available in some automobile no-fault states or in workers compensation. Arguably, such reserves have underlying distributions (inherent in the mortality models) that may be more easily understood and treated than many other casualty coverages. We explore the estimation of both process and parameter uncertainty for this example. In the process we derive formulae that can be used to model uncertainty in other applications, once the various parameters are estimated. Many of the estimation methods covered should generalize to non-annuity applications.

There is also a companion of this paper, titled "Modeling Parameter Uncertainty in Cash Flow Projections" that provides motivation for the estimates contained in this paper. In that paper we discuss approaches to modeling future cash flows and argue for separation of parameter and process uncertainty as well as describing methods to model them both.

Biography

Roger is a Fellow of the Casualty Actuarial Society, a Member of the American Academy of Actuaries, and Consulting Actuary in the Pasadena, California office of Milliman & Robertson, Inc. with over twenty-one years of casualty actuarial consulting experience. Roger is a frequent speaker on reserve and DFA related topics and has authored several papers dealing with considerations and estimates of uncertainty in reserve projections. Roger is currently the chair of the CAS Research Policy and Management Committee and has served as chair of both the CAS Committee on Theory of Risk and the CAS/AAA Joint Committee on the Casualty Loss Reserve Seminar.

1. Introduction

There have been a number of papers and articles dealing with uncertainty in loss reserve estimates. However, dynamic financial analysis for risk bearing entities requires more than simply the distribution of reserves. Also of critical importance is the timing of those future payments and their distribution.

A simple example may clarify the point. Suppose two insurers, *Short Tail Insurance Company* and *Long Tail Insurance Company* are identical in all aspects except for the timing of future payments. Both companies are in runoff, both have $1 million in assets invested in the bank yielding 3% interest, and both will settle all losses in a single payment according to the following distribution:

Table 1: Hypothetical Distribution of Payments

Probability	Amount
20%	$ 500,000
20%	750,000
20%	1,000,000
20%	1,250,000
20%	1,500,000

The only difference is that *Long Tail* will not pay this amount for 10 years, while *Short Tail* must pay it at the end of this year. Even though both insurers have the same assets and face the same distribution of reserves, *Short Tail* would face insolvency 40% of the time while *Long Tail* will only be insolvent 20% of the time (since 1,000,000 x 1.03 = 1,030,000 and 1,000,000 x 1.03^{10}=1,343,916). Though timing may not be everything, it is substantial.

Thus knowing the distribution of the reserves is necessary to model the financial condition of a risk bearing entity, but it is not sufficient. Rather, to appropriately model the future cash flows we need to know the distribution of payments in each future year.

In addition, economic conditions and unanticipated changes in cost inflation often impact reserves and contribute to the variability in both reserves and future payments as well as on assets. Thus, in dynamic financial analysis (DFA) applications where economic assumptions may be used as a "linkage" between asset and liability models, it will probably be necessary to separate the contributions of these economic factors from others in modeling liabilities.

In this paper we will begin with an example of how estimates of the means and variances of payment distributions by year can be made. This first example will focus on claims involving lifetime payments, such as for certain workers compensation claims or unlimited no-fault medical claims. Unlike many casualty claims, the fact that payments are contingent on survival actually provides us with an underlying probability structure for the payments on individual claims and makes discussion of many of the topics we will address more accessible. However, unlike many life coverages, the future payments are contingent not only on the claimant's survival, but on uncertain future costs.

We will then consider how to carry these concepts over to other coverages. These concepts also can be useful in constructing models for use in dynamic financial analysis.

2. *A Relatively Simple Example*

Suppose our insurer only has a fixed book of life pension workers' compensation indemnity claims and does not need to fund for the medical portion of these losses.

Further, to keep this first example relatively simple, we also assume:

2.1 We have mortality tables that appropriately reflect survival probabilities for these claimants.

2.2 There is no escalation of benefits for individual claimants due to inflation or some other index.

2.3 Future annual payments for each claimant are fixed and known.

2.4 We are not currently interested in the time value of money (i.e. no discounting).

2.5 The various claimants are statistically independent.

Here the expected future payments for any individual claim can easily be calculated using a life annuity. Not only can we use the mortality tables to obtain expected costs, but we can also use them to review the expected distribution of payments for our population in any particular future year.

To see this we let:

a_{xt} denote the payment for claimant x in year t in current dollars,

p_{xt} denote the probability that claimant x lives for t years and then dies or otherwise exits the claim population.

It is easy to see the distribution of payments in any future year s is given by:

Table 2: Payment Distribution for a Single Claim

Probability	Amount
$\sum_{t=s}^{\infty} p_{xt}$	a_{xs}
$1-\sum_{t=s}^{\infty} p_{xt}$	0

From this it is easy to see the payments in year s, have expected value

(2.1) $$E(X_s) = a_{xs}\sum_{t=s}^{\infty} p_{xt}$$

and variance

$$\begin{aligned} \mathrm{Var}(X_s) &= E\left(X_s^2\right) - E(X_s)^2 \\ &= a_{xs}^2\sum_{t=s}^{\infty} p_{xt} - a_{xs}^2\left(\sum_{t=s}^{\infty} p_{xt}\right)^2 \\ &= a_{xs}^2\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(1-\sum_{t=s}^{\infty} p_{xt}\right) \end{aligned} \quad (2.2)$$

This is the result we would expect from the binomial distribution for the payments in year s.

In addition, from our assumptions we see that the future payments for this claimant will have a discrete distribution with payments totaling $\sum_{s=1}^{t} a_{xs}$, occurring with probability p_{xt}. Thus the total expected future payment for this claimant is given by:

$$E(X)=\sum_{t=1}^{\infty} p_{xt}\sum_{s=1}^{t} a_{xs}=\sum_{s=1}^{\infty} a_{xs}\sum_{t=s}^{\infty} p_{xt} \tag{2.3}$$

The second is simply the total expected payments in each future year.

Similarly we can also calculate the variance.

$$\operatorname{Var}(X)=\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=\max(r,s)}^{\infty} p_{xt}\right)\left(1-\left(\sum_{t=\min(r,s)}^{\infty} p_{xt}\right)\right) \tag{2.4}$$

Although this formula may not be immediately obvious it is not difficult to derive. We show the derivation in Appendix A.

Thus for a single claimant we can easily obtain the distribution of future payments, its mean and variance as well as the distribution of payments in any future year. We can still explicitly determine the distributions for multiple claimants, however, the calculations become more complex (such calculations may be necessary if, for example, reinsurance attaches on a per incident not per claimant level). For example, for two independent claimants, x and y, the payments in year s have the following discrete distribution:

Table 3: Payment Distribution for Two Claims

Probability	Amount
$\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=s}^{\infty} p_{yt}\right)$	$a_{xs}+a_{ys}$
$\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(1-\sum_{t=s}^{\infty} p_{yt}\right)$	a_{xs}
$\left(1-\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=s}^{\infty} p_{yt}\right)$	a_{ys}
$\left(1-\sum_{t=s}^{\infty} p_{xt}\right)\left(1-\sum_{t=s}^{\infty} p_{yt}\right)$	0

We could derive a similar table for the distribution of total future payments for two claimants. Rather than having simply four separate points, the resulting table would have nxm points where n denotes the number of future years having non-zero probabilities for claimant x and m the number for claimant y. Although we can exactly calculate the resulting distributions for many claimants, the resulting exponential growth in size makes such calculations prohibitive.

On a practical level, however, the problem of combining two distributions is simply one of calculating the aggregate loss distribution for two distributions. Heckman & Meyers[1] provide one means of performing these calculations, Robertson[2] gives another.

We can also approximate the aggregate distribution of the discrete distributions iteratively. We first calculate the aggregate distribution of two distributions exactly, resulting in mxn cells. We then compress this large distribution to, say, m cells and repeat the process with the next distribution. Straightforward combination of cells will usually result in a reduction in the variance in the final distribution while maintaining the mean. The following is an example of this approach.

Consider the two distributions:

Table 4: Distributions for Convolution Example

Variable 1		Variable 2	
Probability	Amount	Probability	Amount
0.60	100	0.20	250
0.40	300	0.80	500

The resulting aggregate distribution is:

Table 5: Distribution of the Sum of Variables

Probability	Amount
0.12	350
0.08	550
0.48	600
0.32	800

A possible compression of this aggregate distribution is:

Table 6: Collapsed Distribution of Sum

Probability	Amount
0.20	430
0.80	680

Here 0.20=0.12+.0.08, 430=(0.12x350+0.08x550)/0.20, and so forth. Note the expected value of 630 is preserved in the compressed distribution but the variance of the exact distribution is 22,240 while that of the compressed distribution is 10,000. There is some flexibility in this method, however, in that the algorithm used to combine the cells could take into account the purpose of the modeling. For example, if the interest is in probabilities of high loss amounts, then we could maintain more detail in the "tail" of the distribution by combining more cells with smaller loss amounts with less combination of higher loss cells. In the above example, the following is another compression:

Table 7: Alternative Collapsed Distribution

Probability	Amount
0.68	550
0.32	800

The mean is again preserved but the variance is now 13,600, closer to that of the exact distribution.

Another possible approximation would be to assume that the aggregate distribution follows a smooth distribution with a limited number of parameters. We could then "back into" the aggregate distribution making use of moments of the true aggregate distribution. For this, however, we need to be able to calculate those moments. For our simple example, however, the calculations follow very simply from (2.3) and (2.4) if we assume that individual claims are independent from one another. Given the fact that the distributions are based on survival probabilities, and our assumption that the probabilities themselves are correct, this is probably not too restrictive in practice.

In this case, letting T denote the random variable corresponding to the aggregate distribution, we see that, assuming we have N claims, the expected aggregate loss is given by:

$$
\begin{aligned}
\mathrm{E}(T) &= \mathrm{E}\left(\sum_{i=1}^{N} X_i\right) \\
&= \sum_{i=1}^{N} \mathrm{E}(X_i) \qquad (2.5) \\
&= \sum_{i=1}^{N} \sum_{s=1}^{\infty} a_{x,s} \sum_{t=s}^{\infty} p_{x,t}
\end{aligned}
$$

Similarly, because we assumed the claims are independent, we can calculate the variance for the aggregate distribution as:

$$
\begin{aligned}
\mathrm{Var}(T) &= \mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) \\
&= \sum_{i=1}^{N} \mathrm{Var}(X_i) \qquad (2.6) \\
&= \sum_{i=1}^{N} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} a_{x,s} a_{x,r} \left(\sum_{t=\max(r,s)}^{\infty} p_{x,t} \right) \left(1 - \left(\sum_{t=\min(r,s)}^{\infty} p_{x,t} \right) \right)
\end{aligned}
$$

Similar calculations based on (2.1) and (2.2) will give us the mean and variance of the total expected annual payments:

$$
\begin{aligned}
\mathrm{E}(T_s) &= \mathrm{E}\left(\sum_{i=1}^{N} X_{si}\right) \\
&= \sum_{i=1}^{N} \mathrm{E}(X_{si}) \qquad (2.7) \\
&= \sum_{i=1}^{N} a_{x,s} \sum_{t=s}^{\infty} p_{x,t}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(T_s) &= \mathrm{Var}\left(\sum_{i=1}^{N} X_{si}\right) \\
&= \sum_{i=1}^{N} \mathrm{Var}(X_{si}) \qquad (2.8) \\
&= \sum_{i=1}^{N} a_{x,s}^{2} \left(\sum_{t=s}^{\infty} p_{x,t} \right) \left(1 - \sum_{t=s}^{\infty} p_{x,t} \right)
\end{aligned}
$$

We note we can calculate the exact distribution for payments in any particular year as with the aggregate distribution for the total. However, in this case, there will "only" be 2^N cells in the distribution. Again, we could use a compression algorithm to obtain approximate distributions.

3. Introducing Some Uncertainty

The problem thus far considers only random fluctuations due to the fact that the exact time of exit from the claimant population is unknown. We have assumed that all other aspects of the problem are known. In short, we have only discussed process uncertainty thus far, i.e., that uncertainty remaining in the situation even if the process itself is known with certainty.

In the real world models used are generally approximations of the underlying process, subject to uncertainty either in their parameters or even whether or not they are appropriate. In this section we begin to introduce uncertainty into the assumptions from section 2.

The first restriction we will relax will be the assumption that underlying survival probabilities for individual claimants are known. In reality payments will often be contingent on the survival of an individual who is already injured and whose injuries may significantly impair chances for continued survival. Thus it may not be appropriate to use standard mortality tables to determine the survival probabilities. It is possible that the tables that are used will be modified or based in some way on populations of injured claimants and thus subject to estimation error.

In addition, it is possible that a claimant will sufficiently recover from his or her injuries so as not to require additional payments from the insurer. Thus exit from the population could occur for reasons other than death. We may need additional modeling to study the effects of such recoveries on exits from the population by claimants.

Since most such analyses focus on the mortality in a year, we let

q_{xt} denote the probability that claimant x will die in year t, given survival through year t-1.

These are the standard mortality probabilities. In terms of the p_{xt} variables defined above we have (possibly mixing notation somewhat):

$$p_{xt} = q_{x+t}\prod_{i=0}^{t-1}(1-q_{x+i})$$

$$(3.1) \qquad = (1-(1-q_{x+t}))\prod_{i=0}^{t-1}(1-q_{x+i})$$

$$= \prod_{i=0}^{t-1}(1-q_{x+i}) - \prod_{i=0}^{t}(1-q_{x+i})$$

Very conveniently, these collapse in the sum to yield:

$$\sum_{t=m}^{\infty} p_{xt} = \sum_{t=m}^{\infty}\left(\prod_{i=0}^{t-1}(1-q_{x+i}) - \prod_{i=0}^{t}(1-q_{x+i})\right)$$

$$(3.2) \qquad = \prod_{i=0}^{m-1}(1-q_{x+i}) - \prod_{i=0}^{\infty}(1-q_{x+i})$$

$$= \prod_{i=0}^{m-1}(1-q_{x+i})$$

In addition to allowing uncertainty in the survival probabilities we will also allow the annual benefits to change over time with economic conditions and allow for discounting of the reserves, as would be the case for the medical portion of workers' compensation or certain automobile no-fault benefits. We will allow the combined economic effect of inflation and discounting to be uncertain. Finally we will allow for some uncertainty in the annual payment estimates for individual claimants. Specifically we will relax our various assumptions to the following:

3.1 The relative survival probabilities among various claimants are known, however, the absolute probabilities are based on an analysis of n exposures. Analytically, we assume that there is a random variable y and constants q^{*}_{xt}, such that for all x and t values:

$$(3.3) \qquad 1-q_{xt} = (1-q^{*}_{xt})y$$

3.2 The a_{xt} values are stated in current dollars. There is escalation in those amounts between time *t-1* and time t in the amount of $1+f_t$. This escalation will be the same for all claimants but may vary from year to year. The $1+f_t$ amounts are not known with certainty.

3.3 The present value of 1 at time *t-1* is $1+v_t$ at time *t*. The $1+v_t$ amounts are not known with certainty.

3.4 There is a random variable *u* and constants a^{*}_{xt} such that for all claimants *x* and time *t*, the following holds:

$$(3.4) \qquad a_{xt} = a^{*}_{xt} u$$

3.5 The various claimants are statistically independent.

3.6 There are random variables w_t and constants f^{*}_t, and v^{*}_t such that, for all *t* values:

$$(3.5) \qquad \frac{1+f_t}{1+v_t} = \frac{1+f^{*}_t}{1+v^{*}_t} w_t$$

The variable *y* in 3.1 could be considered as a global load, reflecting the uncertainty in estimating the overall closure rate from experience. We recognize that this does not consider the uncertainty regarding the relative closure probabilities. For example, it is likely that younger claimants will experience a greater reduction in survival chances due to the injury causing the claim than older claimants will. Thus, except in the simplest situations, the variable *y* probably should not be considered as a mortality load, but rather a global uncertainty parameter.

We can estimate the degree of uncertainty arising from the sample size of *n* life-years used to estimate the survival or closure probabilities. For this we use sample theory and an application of Bayes' Theorem. In fact, if we assume:

1. The random variable *y* has a binomial distribution with expected value θ.
2. The random variable θ itself has a uniform distribution between 0 and 1 (i.e. we have no prior knowledge of the appropriate value of θ).
3. Our sample size is *n*.
4. We observe *z* claims remaining open after one year from our sample.

If we make the more general assumption in 2 above that θ has a beta distribution with parameters α and β it turns out that θ given the observations has a beta distribution with parameters $z+\alpha$ and $n-z+\beta$. We show this in Appendix B. In particular, then,

$$
\begin{aligned}
\mathrm{E}(\theta^r|z) &= \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(z+\alpha)\Gamma(n-z+\beta)}\theta^{z+r+\alpha-1}(1-\theta)^{n-z+\beta-1} \\
&= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(z+\alpha)\Gamma(n-z+\beta)}\int_0^1 \theta^{z+r+\alpha-1}(1-\theta)^{n-z+\beta-1} \\
&= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(z+\alpha)\Gamma(n-z+\beta)}\frac{\Gamma(z+r+\alpha)\Gamma(n-z+\beta)}{\Gamma(\alpha+r+\beta+n)} \\
&= \frac{\Gamma(\alpha+\beta+n)\Gamma(z+r+\alpha)}{\Gamma(z+\alpha)\Gamma(\alpha+r+\beta+n)}
\end{aligned}
\tag{3.6}
$$

Thus, in particular,

$$
\begin{aligned}
\mathrm{E}(\theta|z) &= \frac{\Gamma(\alpha+\beta+n)\Gamma(z+1+\alpha)}{\Gamma(z+\alpha)\Gamma(\alpha+1+\beta+n)} \\
&= \frac{\Gamma(\alpha+\beta+n)\Gamma(z+\alpha)(z+\alpha)}{\Gamma(z+\alpha)\Gamma(\alpha+\beta+n)(\alpha+\beta+n)} \\
&= \frac{z+\alpha}{\alpha+\beta+n}
\end{aligned}
\tag{3.7}
$$

Thus we have:

$$
\begin{aligned}
\mathrm{E}\left(\left(\frac{\theta}{\mathrm{E}(\theta)}\right)^r\right) &= \frac{\mathrm{E}(\theta^r|z)}{\mathrm{E}(\theta|z)^r} \\
&= \left(\frac{\alpha+\beta+n}{z+\alpha}\right)^r \frac{\Gamma(\alpha+\beta+n)\Gamma(z+r+\alpha)}{\Gamma(z+\alpha)\Gamma(\alpha+r+\beta+n)}
\end{aligned}
\tag{3.8}
$$

Now, the special case we will consider is no preference in the prior distribution for θ. This is simply a special case of the beta distribution with $\alpha = \beta = 1$. In this case we have:

$$
\begin{aligned}
\mathrm{E}(y^r) &= \mathrm{E}\left(\left(\frac{\theta}{\mathrm{E}(\theta)}\right)^r\right) \\
&= \left(\frac{n+2}{z+1}\right)^r \frac{\Gamma(n+2)\Gamma(z+r+1)}{\Gamma(z+1)\Gamma(r+n+2)} \qquad (3.9) \\
&= \prod_{i=0}^{r-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}
$$

The last equation follows from the recursive properties of the gamma function and makes calculation easier in practice. In terms of the survival probabilities we have:

$$
\begin{aligned}
\mathrm{E}_\theta\left(\sum_{t=m}^{\infty} p_{xt}\right) &= \mathrm{E}_\theta\left(\prod_{i=0}^{m-1}(1-q_{x+i})\right) \\
&= \mathrm{E}_\theta\left(\prod_{i=0}^{m-1}(1-q^{*}_{x+i})y\right) \\
&= \mathrm{E}_\theta\left(y^m \prod_{i=0}^{m-1}(1-q^{*}_{x+i})\right) \\
&= \mathrm{E}_\theta(y^m)\prod_{i=0}^{m-1}(1-q^{*}_{x+i}) \qquad (3.10) \\
&= \mathrm{E}_\theta\left(\left(\frac{\theta}{\mathrm{E}_\theta(\theta)}\right)^m\right)\prod_{i=0}^{m-1}(1-q^{*}_{x+i}) \\
&= \left(\prod_{i=0}^{r-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}\right)\prod_{i=0}^{m-1}(1-q^{*}_{x+i})
\end{aligned}
$$

As one would expect, the first term in the last product tends to unity as the sample size n becomes large if

$$
\lim_{n\to\infty}\frac{z}{n} = 0 \qquad (3.11)
$$

for some value θ. The proof is shown in Appendix B.

Assumptions 3.2 and 3.4 deal with cost escalation and discounting and 3.6 relates the two. We assume that the combined impact of inflation and discounting is uncertain with the variables w_t providing that uncertainty.

Finally we will modify the assumption that all future payments (at current cost levels) are known to one wherein there is "global" uncertainty regarding future payments. This is reflected in the variable u.

For simplicity we will assume that the variables w_t and u all have independent lognormal distributions, and that the distribution for the various w_t have the same means and variances. In particular we will assume that all these variables are independent and:

$$\begin{aligned} &u \sim \text{lognormal}\left(-\tfrac{1}{2}\sigma^2, \sigma^2\right) \text{ and} \\ &w_t \sim \text{lognormal}\left(-\tfrac{1}{2}\tau^2, \tau^2\right) \text{ for all t.} \end{aligned} \tag{3.12}$$

Here and throughout this paper we will use the normal-transformed parameterization of the lognormal distribution. For example, (3.12) assumes that the normal variable $\ln u$ has a normal distribution with mean $-\frac{1}{2}\sigma^2$ and variance σ^2. More generally when we say

$$x \sim \text{lognormal}\left(\mu, \sigma^2\right) \tag{3.13}$$

we mean that the random variable x has the probability density function

$$f(x) = \frac{\exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)}{x\sigma\sqrt{2\pi}} \tag{3.14}$$

With this parameterization, then we have:

$$\begin{aligned} &E(X) = \exp\left(\mu + \tfrac{1}{2}\sigma^2\right) \\ &\text{Var}(X) = \exp\left(2\mu + \sigma^2\right)\left(\exp\left(\sigma^2\right) - 1\right) \\ &c.v.(X) = \sqrt{\frac{\text{Var}(X)}{E(X)^2}} = \sqrt{\exp\left(\sigma^2\right) - 1} \end{aligned} \tag{3.15}$$

This last relationship shows that, with this parameterization, the coefficient of variation (ratio of standard deviation to the mean) depends only on the σ^2 parameter.

It could be argued quite convincingly that u would not be the same for all claimants or for all years. That is clearly a refinement to the methodology we present here. However, to keep the calculations to a manageable level, we have elected to make this simplifying assumption here. However, the assumption of lognormality for the economic variables is probably much more plausible, although the assumption of constant variance may be somewhat restrictive. In both cases, here, we note that the expected values of both distributions are unity, that is both u and the w_t variables are assumed to represent random shocks to our overall expectations.

We are now ready to calculate the mean and variance of the total population reserve. The calculation makes repeated applications of the following relationships that hold for independent conditional distributions:

$$\begin{aligned} \mathrm{E}(Z) &= \mathrm{E}_{\xi}\left(\mathrm{E}(Z|\xi)\right) \\ \mathrm{Var}(Z) &= \mathrm{E}_{\xi}\left(\mathrm{Var}(Z|\xi)\right) + \mathrm{Var}_{\xi}\left(\mathrm{E}(Z|\xi)\right) \end{aligned} \tag{3.16}$$

In this case we assume that the distribution of the random variable Z with probability density function $f(z,\xi)$ that depends on a parameter ξ which itself is a random variable with probability density function $g(\xi)$. These assumptions result in the following formulae for the mean and variance of the total distribution:

$$\begin{aligned} \mathrm{E}(T) &= \sum_x \sum_{s=1}^{\infty} b^{*}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} \\ \mathrm{Var}(T) &= \left(\sum_x \sum_{s=1}^{\infty} b^{*}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)\mathrm{E}(y^{s})\right)^2\left(\exp(\sigma^2)-1\right) \\ &\quad + \exp(\sigma^2)\sum_{s=1}^{\infty}\left(\sum_x b^{*}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})\exp(s\tau^2)-\mathrm{E}(y^{s})^2\right) \\ &\quad + \exp(\sigma^2)\sum_x \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^{*}_{xs}b^{*}_{xr}\exp(\min(r,s)\tau^2)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^{*}_{xt})\right) \\ &\quad \times\left(\mathrm{E}(y^{\max(r,s)})-\mathrm{E}(y^{r+s})\left(\prod_{t=0}^{\min(r,s)-1}(1-q^{*}_{xt})\right)\right) \end{aligned} \tag{3.17}$$

In these formulae we have taken

(3.18) $$b^*_{xt} = a^*_{xt} \prod_{s=1}^{t} \frac{1+f^*_s}{1+v^*_s}$$

These are the present value of future payments without consideration of uncertainty or the probability of payment. As a practical matter, the value of σ^2 is not needed in the detailed calculations. We can calculate the various terms in (3.17) that involve individual claim information separately, and then include the value of σ^2 in a fairly simple calculation.

If, now, we assume that there is no uncertainty in any of the estimates then $\sigma = \tau = 0$ and the expectations of all powers of y are 1 (infinite sample size) the first three terms in the variance sum vanish leaving:

(3.19) $$\mathrm{Var}(T|\mathrm{Certainty}) = \sum_x \sum_{s=1} \sum_{r=1} b^*_{xs} b^*_{xr} \left(\prod_{t=0}^{\max(r,s)-1} (1-q^*_{xt}) \right) \left(1 - \prod_{t=0}^{\min(r,s)-1} (1-q^*_{xt}) \right)$$

Here, and throughout this paper, we use the term "Certainty" in the formulae to denote the situation where there is no parameter uncertainty. We use this shorthand to help keep the formulae as simple as possible.

Thus incorporating uncertainty regarding the closure rates adds to the expected value of the total. With this we see that $E(T)$ will equal the reserve estimates calculated by the model if the survival rate were based on an infinite population, otherwise said, if we are certain about the annual survival rate.

If we define

(3.20) $$p^*_{xt} = \prod_{s=0}^{t-1} (1-q^*_{xs})$$

Then this last formula becomes the standard variance formula.

$$
\begin{aligned}
\mathrm{Var}(T|\text{Certainty}) &= \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} \left(\sum_{t=\max(r,s)}^{\infty} p^*_{xt} \right) \left(1 - \sum_{t=\min(r,s)}^{\infty} p^*_{xt} \right) \\
&= \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} \left(\left(\sum_{t=\max(r,s)}^{\infty} p^*_{xt} \right) - \left(\sum_{t=r}^{\infty} p^*_{xt} \right) \left(\sum_{t=s}^{\infty} p^*_{xt} \right) \right) \\
(3.21) \qquad &= \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} \left(\sum_{t=\max(r,s)}^{\infty} p^*_{xt} \right) - \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} \left(\sum_{t=r}^{\infty} p^*_{xt} \right) \left(\sum_{t=s}^{\infty} p^*_{xt} \right) \\
&= \sum_x \left(\sum_{t=1}^{\infty} p^*_{xt} \sum_{s=1}^{t} \sum_{r=1}^{t} b^*_{xs} b^*_{xr} - \left(\sum_{s=1}^{\infty} b^*_{xs} \left(\sum_{t=s}^{\infty} p^*_{xt} \right) \right)^2 \right) \\
&= \sum_x \left(\sum_{t=1}^{\infty} p^*_{xt} \left(\sum_{s=1}^{t} b^*_{xs} \right)^2 - \left(\sum_{t=1}^{\infty} p^*_{xt} \left(\sum_{s=1}^{t} b^*_{xs} \right) \right)^2 \right)
\end{aligned}
$$

The actual calculations in deriving (3.17) are quite lengthy and are contained in Appendix C. Similarly we have the following formulae for the mean and variance of payments in year s, as shown in detail in Appendix D.

$$
\mathrm{E}(T_s) = \sum_x b^*_{xs} \left(\prod_{t=0}^{s-1} (1 - q^*_{xt}) \right) \prod_{i=0}^{s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
$$

$$
\begin{aligned}
(3.22) \quad \mathrm{Var}(T_s) &= \left(\sum_x b^*_{xs} \left(\prod_{t=0}^{s-1} (1 - q^*_{xt}) \right) \right)^2 \left(\exp(\sigma^2 + s\tau^2) \mathrm{E}(y^{2s}) - \mathrm{E}(y^s)^2 \right) \\
&+ \exp(\sigma^2 + s\tau^2) \sum_x b^{*2}_{xs} \left(\prod_{t=0}^{s-1} (1 - q^*_{xt}) \right) \left(\mathrm{E}(y^s) - \mathrm{E}(y^{2s}) \left(\prod_{t=0}^{s-1} (1 - q^*_{xt}) \right) \right)
\end{aligned}
$$

Although, to maintain some simplicity we have not substituted from formula (3.9) in the variance formula in either (3.17) or (3.22), both formulae, with this substitution, no longer depend on the conditional variables. It can be easily seen that in the case of no uncertainty (i.e. $\sigma = \tau = 0, \mathrm{E}(y^s) = 1$) the formulae in (3.22) reduce to (2.7) and (2.8). We also see that the expected total reserve in (3.17) is simply the sum of the expected payments by future year from (3.22). However, as we would expect, the variance terms are much less comparable. This is due to the nature of the dependencies we introduced with some of the uncertainty variables.

Thus, for the relatively simple case of known lifetime care claimants we can calculate the mean and variance for both the total reserves and the payments in each future year. We can incorporate at least some parameter uncertainty in these calculations.

In short, these calculations provide a way to estimate the mean and variance of case reserves, including a potential provision for uncertainty in the case estimates as evidenced by the parameter u, but do not consider uncertainty regarding claims that are incurred but not reported. It also does not consider reported claims that are not yet recognized as potential lifetime care claimants or for which there is not sufficient information available to establish estimates of future claim and medical costs.

4. Additional Areas of Uncertainty

We consider three categories of claims.

1. Those having annual cost estimates with case reserves calculated using the annuity model described in sections 2 and 3 above.

2. Claims reported but for which annual cost information is not yet available, and

3. Claims incurred but not reported (true IBNR).

Continuing with our development we have implicitly incorporated additional development in case reserves, along with its corresponding uncertainty, in the estimates in section 3. Thus there is increasing uncertainty as we move through these categories of claims. In the first instance we have information regarding individual claims with uncertainty regarding inflation, investment, exit from the population, and some uncertainty regarding the accuracy of the annual cost estimates. All these elements of uncertainty are present in the second category along with additional uncertainty as to the overall average for the claims themselves. Finally the third category incorporates all this uncertainty as well as uncertainty as to the number of claims to ultimately be reported.

In order to reflect this uncertainty we will use the following notation. Let:

N_R denote the number of claims having annual cost estimates

N_B denote the number of reported claims without specific annual cost estimates

λ denote the expected number of IBNR claims

χ denote a random variable with $E(\chi)=1$ and $Var(\chi)=c$

β denote a random variable with $E(\beta)=1$ and $Var(\beta)=b$

γ denote a random variable with E(γ)=a and Var(γ)=d

ζ denote a random variable with E(ζ)=r and Var(ζ)=z

With this notation, we will use a modification of Algorithm 3.3 from the Heckman & Meyers[1] paper:

1. Select claims with case reserves, $X_1, X_2, \ldots, X_{N_R}$
2. Randomly select a value for χ.
3. Randomly select N from a Poisson distribution with expected value $\lambda\chi$.
4. Randomly select independent claims $X_{N_R+1}, X_{N_R+2}, \ldots, X_{N_R+N_B+N}$ from the same distribution having the mean and variance equal to that of the case reserved claims.
5. Randomly select values for β, ζ, and γ.
6. Calculate the aggregate reserve as

$$(4.1) \qquad T = \beta\left(\sum_{i=1}^{N_R} X_i + \zeta \sum_{i=N_R+1}^{N_R+N_B} X_i + \gamma \sum_{i=N_R+N_B+1}^{N_R+N_B+N} X_i\right).$$

Here χ incorporates uncertainty regarding the claim count estimate, β global uncertainty regarding the overall estimates, ζ additional uncertainty and scaling for known but not-case-reserved claims, and γ additional uncertainty and scaling for IBNR claims. We will assume in the following, that claims other than those with case reserves, except for the scaling values a and r, will have the same mean and variance as those with individual case reserves.

If we consider the case where there are no IBNR claims and that we have case reserve estimates for all claims, (4.1) becomes:

$$(4.2) \qquad T = \beta\sum_{i=1}^{N_R} X_i$$

From this we can calculate the mean as:

$$\text{(4.3)}\qquad \begin{aligned} \mathrm{E}(T_R) &= \mathrm{E}_\beta\left(\mathrm{E}\left(\sum_{i=1}^{N_R} X_i|\beta\right)\right) \\ &= \mathrm{E}_\beta\left(\beta\sum_{i=1}^{N_R}\mathrm{E}(X_i)\right) \\ &= \mathrm{E}_\beta(\beta)\sum_{i=1}^{N_R}\mathrm{E}(X_i) = \sum_{i=1}^{N_R}\mathrm{E}(X_i) = N_R\,\mathrm{E}(X) \end{aligned}$$

Since we are assuming all the claims are independent, this last term denotes the expected claim costs <u>with no parameter uncertainty</u>. This can be calculated using (3.17) by letting the sample size tend to infinity. Now if we let all the uncertainty above be expressed in the parameter β, then we have

$$\text{(4.4)}\qquad \begin{aligned} \mathrm{Var}(T_R) &= \mathrm{E}_\beta\left(\mathrm{Var}(T_R|\beta)\right) + \mathrm{Var}_\beta\left(\mathrm{E}(T_R|\beta)\right) \\ &= \mathrm{E}_\beta\left(\mathrm{Var}\left(\beta\sum_{i=1}^{N_R} X_i\right)\right) + \mathrm{Var}_\beta\left(\mathrm{E}\left(\beta\sum_{i=1}^{N_R} X_i\right)\right) \\ &= \mathrm{E}_\beta\left(\beta^2\,\mathrm{Var}\left(\sum_{i=1}^{N_R} X_i\right)\right) + \mathrm{Var}_\beta\left(\beta\mathrm{E}\left(\sum_{i=1}^{N_R} X_i\right)\right) \\ &= \mathrm{E}_\beta(\beta^2)\,\mathrm{Var}\left(\sum_{i=1}^{N_R} X_i\right) + \mathrm{Var}_\beta(\beta)\mathrm{E}\left(\sum_{i=1}^{N_R} X_i\right)^2 \\ &= \left(\mathrm{Var}_\beta(\beta) + \mathrm{E}_\beta(\beta)^2\right)\mathrm{Var}\left(\sum_{i=1}^{N_R} X_i\right) + \mathrm{Var}_\beta(\beta)\mathrm{E}\left(\sum_{i=1}^{N_R} X_i\right)^2 \\ &= (b+1)\,\mathrm{Var}(T_R|\text{Certainty}) + b\mathrm{E}(T_R|\text{Certainty})^2 \end{aligned}$$

Solving for b we obtain:

$$\text{(4.5)}\qquad b = \frac{\mathrm{Var}(T_R) - \mathrm{Var}(T_R|\text{Certainty})}{\mathrm{Var}(T_R|\text{Certainty}) + \mathrm{E}(T_R|\text{Certainty})^2}$$

We can then use (3.17) or (3.22) to derive a value for b that will explicitly incorporate parameter uncertainty into this algorithm. Assuming, in addition, that estimates for the second and third claim categories depend on case reserves, we are able to quantify a level of global uncertainty inherent in the estimates.

We use calculations similar to those led us to the mean and variance estimates in Appendices C and D to obtain the following:

$$E(T) = (N_R + rN_B + a\lambda)E(T_R|\text{Certainty})/N_R$$

$$(4.6)\ \text{Var}(T) = (b+1)\left(1+\frac{N_B(z+r^2)+(d+a^2)\lambda}{N_R}\right)\text{Var}(T_R|\text{Certainty})$$
$$+\left(\frac{(b+1)\left((d+a^2)(\lambda+c\lambda^2)+\lambda^2 d+zN_B^2\right)}{N_R{}^2}+b\left(1+\frac{rN_B+a\lambda}{N_R}\right)^2\right)E(T_R|\text{Certainty})^2$$

These are shown in detail in Appendix E.

Thus, under the above assumptions, we can express the mean and variance of the distribution of total claims in terms of the mean and variance of the distribution of case reserved claims, without parameter uncertainty, and the various parameters specified above.

On review of that analysis we see that we did not specifically assume that the calculations were for total reserves. Thus a similar formula holds for payments in a particular year:

$$E(T_s) = (N_R + rN_B + a\lambda)E(T_{Rs}|\text{Certainty})/N_R$$

$$(4.7)\ \text{Var}(T_s) = (b+1)\left(1+\frac{N_B(z+r^2)+(d+a^2)\lambda}{N_R}\right)\text{Var}(T_{Rs}|\text{Certainty})$$
$$+\left(\frac{(b+1)\left((d+a^2)(\lambda+c\lambda^2)+\lambda^2 d+zN_B^2\right)}{N_R{}^2}+b\left(1+\frac{rN_B+a\lambda}{N_R}\right)^2\right)E(T_{Rs}|\text{Certainty})^2$$

Inherent in these calculations is that we can use the same uncertainty variables for both the aggregate reserves and for the payments in each year.

We note that, although the genesis of (4.6) and (4.7) were based on a book of life-pension claims, there is nothing in the derivation that requires such a book. If we can separate our reserving problem into the three categories above and are willing to make the assumptions indicated above, we can calculate the variance of the aggregate distribution.

5. Estimating the Parameters

We will consider parameter estimation in two phases, we will first address the ***b*** parameter and then the remaining ones. Again, the discussion will begin with the life annuity model and then move to potential for generalization.

5.1 Estimating the b Parameter

We have already hinted at an approach that we could use to estimate the *b* parameter. Using (4.5) all we need are estimates of the variance of reserved claims with and without parameter uncertainty. The estimate without parameter uncertainty follows directly from the annuity calculations as given in (3.19) or (3.21). Using (3.17) and the assumptions going into that estimate we can derive an estimate of the variance for claims having case reserves if we can estimate:

$E(y^s)$	Uncertainty regarding the mortality assumptions
τ^2	Uncertainty regarding (composite) economic estimates
σ^2	Uncertainty regarding the annual cost estimates

5.1.1 Mortality Considerations

There are other practical issues in the use of mortality assumptions, especially in usual applications in property and casualty insurance. In almost every situation property and casualty claimants eligible for lifetime care will be physically impaired in some manner, either by trauma or disease. Often one may expect the impairment to affect the claimant's future survival chances as compared to the general population. In addition, we could expect different injuries to have different effects on survival probabilities.

There has been substantial research on the effect of spinal cord injuries on survival rates. As opposed to head trauma, spinal cord injuries are relatively easy to categorize and are relatively uniform from patient to patient, and generally do not change during a claimant's life. For example, the following table, attributed to the National Spinal Cord Injury Statistical Center, University of Alabama at Birmingham, shows differences in life expectancies for various levels of spinal cord injury[3]:

Table 8: Life Expectancies by Age and Spinal Cord Injury

	Life Expectancy					
Current Age	Normal	Ventilator Dependent	High Tetraplegic	Low Tetraplegic	Paraplegic	Motor Function at Any Level
20	56.3	19.9	32.8	38.6	44.8	49.0
30	46.9	15.9	26.8	30.7	36.7	40.5
40	37.6	12.4	20.9	23.6	28.8	31.7
50	28.6	9.3	15.5	17.0	21.2	23.4

We have not been able to locate similar statistics for traumatic head injuries. Analysis for such injuries are complicated by the fact that head injuries are more difficult to categorize than spinal cord injuries and, in contrast to spinal cord injuries usually identified by the location and degree of lesion in the spinal column. In addition, the level of severity of a head injury can change substantially during the course of treatment.

Other property and casualty claimants could have still different mortality profiles. For example, a back injury, though disabling a person from employment, may have little or no effect on that person's future life expectancy. Conversely, heart conditions or stress related illnesses could have a substantial impact on future survival chances. Compounding difficulties are the effects of medical treatment on the claimant's survival chances, especially in situations where there is no limit on the amount that can be expended for medical treatment. Thus, unlike many situations where mortality is a consideration, the appropriate survival functions are often uncertain.

For this reason, it may be useful to consider construction or modification of mortality tables to reflect the injured population. In this case the table could be based on a fairly small sample, though could still produce reasonable results. In this case formula (3.9) gives an estimate of $E(y^s)$ under the assumption that uncertainty in the mortality table is uniform across all claimants and ages and depends only on the sample size used in estimating the mortality table and the overall average mortality for the population. However, the considerations above would seem to indicate that (3.9) may only produce a lower bound on the level of uncertainty inherent in the selection of mortality assumptions.

5.1.2 Uncertainty in Economic Assumptions

We note in (3.5) and (3.12) we have made the simplifying assumptions that the net discount rates (ratio of annual cost inflation to annual interest rate) are independent from year to year. In addition, we assumed that the distributions of the rates in each year all have the same coefficient of variation.

There has been much attention recently devoted to modeling economic scenarios in conjunction with dynamic financial analysis, for example Daykin et.al.[4] If we were using such models one could estimate the value of τ^2 using the results of those models.

Although the models can be quite complex, actual economic conditions have experienced some rather spectacular swings, even over the past twenty to thirty years. For example, the hospital room component of the U.S. Consumer Price Index for Urban Wage Earners (CPI-W) increased by 15.7% during 1981 and by only 3.5% during 1996. Interest rates also experienced similar swings during that same time with the average 1 year United States Treasury Bill moving from 14.8% in 1981 to 5.5% in 1996.

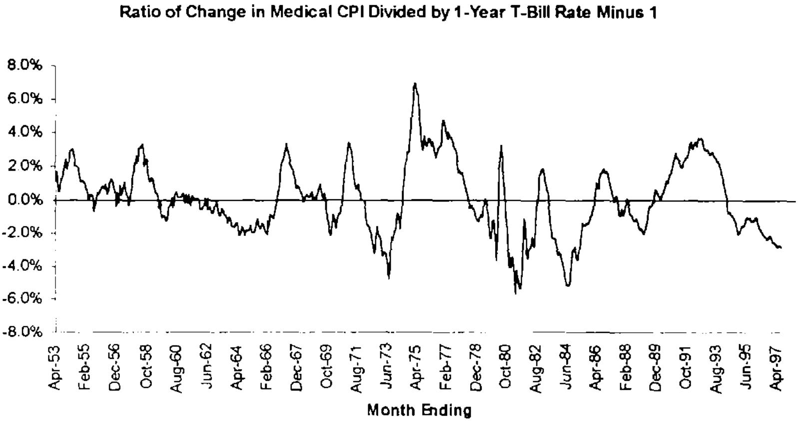
We could also use this historic volatility to estimate τ^2. For this we review the historical volatility in the quantity:

$$(5.1) \qquad \frac{1+f_t^*}{1+v_t^*}-1$$

Here we use f_t^* to denote the annual change in the medical care cost component of the U.S. Consumer Price Index for Urban Wage Earners measured from month t-12 to month t and v_t^* to denote the average yield for 1 Year U.S. Treasury Bills during month t. Of course, if we assume that claim costs would experience a different market basket than medical costs in general then we would re-weight them accordingly.

We also somewhat randomly selected the 1-Year U.S. Treasury Bill rate for this example. Again, unique characteristics of the company's investment portfolio may dictate a different measure for investment return. These values should be illustrative of the degree of variation we could expect in our applications. The following graph shows values of (5.1) for each month from April 1953 through July 1997.

Figure 1: *Relative Real Returns*



Since we have assumed that uncertainty in future net discount will show a lognormal distortion we can estimate the τ^2 parameter as the variance of the natural logarithms of the amounts in (5.1), plus 1. In this case the result is $\tau^2 = 0.000457$.

5.1.3 Uncertainty in Cost Assumptions

The third area of uncertainty reflected in (3.17) deals with the fact that a_{xt}, the annual payment amounts in current dollars, may themselves be uncertain. In workers compensation claims the indemnity amounts are often specified by statute, so the amounts of those payments for life pension cases may not be subject to change. However, one would probably not expect the same degree of certainty in medical payments either for workers compensation or no-fault benefits.

As noted in Section *3. Introducing Some Uncertainty*, we have assumed that claim annual cost estimates in current dollars have the same uncertainty distribution as reflected by the random variable u. In addition to u, the (present value of) annual payments are also affected by the w_t random variables. From a practical viewpoint, this effectively separates two factors that affect the accuracy of estimates of future costs; unexpected levels of inflation (and/or investment return) and actual costs (or services) differing from what had been expected for reasons other than economic conditions.

This dichotomy suggests a way to estimate the parameter σ^2. We could compare actual annual payments with the forecasts made in previous analyses, after adjustment for trend in the form of some index reflecting underlying cost changes. The following table provides an example of such an approach.

Table 9: Actual vs. Expected Payments

Payment	Forecast	Annual Payment		
Year	Year	Actual	Estimated	ln(A/E)
1	0	$ 50,000	$ 45,000	0.1054
2	0	40,000	35,000	0.1335
2	1	40,000	45,000	-0.1178
3	0	30,000	25,000	0.1823
3	1	30,000	35,000	-0.1542
3	2	30,000	30,000	0.0000
			Average	0.0249

In this example, for a single claim, we have actual payments of $50,000, $40,000, and $30,000 in each of the first three years of a claim. In the first analysis (at the beginning of year 1) we estimated payments of $45,000, $35,000, and $25,000 trended to future levels using the selected cost index. The second analysis we adjusted the forecasts for years 2 and 3 to $45,000 and $35,000 respectively, while for the third analysis we estimated $30,000 for the third year.

Since the sample mean of the natural logarithms is the maximum likelihood estimator for the first parameter of a lognormal distribution in our parameterization, and the sample variance is a minimum variance estimator for the second parameter, we could use the sample variance for as an estimator of the σ^2 parameter. We note here that the average does not satisfy the relationship assumed in (3.12). In particular the expected value of the resulting lognormal variable is not unity. Hence our estimates are biased and we should adjust the forecast estimates to remove this indicated bias. Such an adjustment would leave the σ^2 parameter unchanged.

This approach ignores any "aging" considerations. For example, one would expect short-term forecasts to be more accurate than long term ones, all other things being equal. In addition, the longer-term estimates carry less weight in the reserve forecasts due to discounting for mortality if not for investment income.

Also, for medical payments on seriously injured claimants, one would often expect payments in the first years after the accident to be much higher than those in later years after the claimant has medically stabilized. In addition, it could be argued that payments rise during the time just before a claimant's death. The approach we outlined gives equal weight to all forecast errors in estimating the σ^2 parameter. It does, however, have the appeal of a direct comparison of actual versus expected results.

An alternative approach would be to consider the development of claim estimates over time. In such an approach, as in usual incurred loss development, annual cost estimates are gradually replaced by actual payments over the development period. If we take this approach we must keep in mind that we want to separate economic influences from the measurement of movement of claim costs over time.

One such approach would involve recalculating all expected incurred losses each year, replacing expected future payments with actual payments in the annuity calculations and reviewing the development. This would be the most consistent way to handle changes in economic assumptions in the valuations. However, it could be quite time-consuming, especially in situations where there are many claims evaluated over many different development periods, not to mention the need to maintain records of past annual cost estimates for individual claims.

There is an approximation, however, that would allow for the separation of changes in economic assumptions from development in estimates from other causes. At this point we only consider claims having annual cost estimates, since we are trying to quantify the uncertainty in those annual cost estimates. Thus we do not want the development patterns we obtain to be influenced by emergence of new claims, hence aggregation by accident period would not be useful.

This may suggest grouping by report period. However, in that grouping there could be claims reported but which do not yet have individual annual cost estimates attached. The manner in which reserves are set on those claims could influence the review of development on claims having annual cost estimates. Hence report period grouping also seems to be lacking for this purpose.

We thus consider a third alternative, akin to report period. For this we group claims by the period in which they are first case reserved, calling this a reserve period grouping. In the case that there are no "formula" reserves for known claims, this alternative would be equivalent to a report period grouping.

Once claims are grouped in this fashion, we can consider the development of expected incurred losses (calculated using the annuity approach of (2.3)) on fixed groups of claims using a development array format. However, we are faced with several additional difficulties if we wish to focus on the movement and variability in the individual annual cost estimates (the focus of the σ^2 parameter). Those difficulties arise because our reserve estimates may be discounted and because changes in economic or mortality assumptions will cause changes in the expected amounts during the calendar period containing the change and should not be considered when evaluating the variability inherent in the individual annual cost estimates.

Even without changes in underlying assumptions, we are faced with the "unwinding of the discount" phenomenon. By this we mean the fact that incurred losses calculated with discounted reserves will continue to develop upward due to a decreasing effect of discounting, even if all underlying assumptions prove exactly correct. To deal with the unwinding of the discount we discount all amounts to the beginning of the reserve period. This discounting includes the discounting of all payments made to date, as well as discounting of reserves. For convenience we discount to the beginning of the reserve period we are evaluating.

An obvious alternative at this juncture would be to not discount at all. The appeal of discounting at this point, however, is the decreasing influence of remote payments have on the final reserve calculated. As noted above, these remote amounts are probably subject to greater uncertainty. The author recognizes at this point the current discussions regarding the appropriateness of calculating reserves on a discounted basis. None of the methods or results presented here rely on the discount rate being positive. Thus if reserves are carried on a undiscounted basis all the above analysis will apply. However, if the discount rate is negative (implying a significant risk-adjustment due to uncertainty) later payments are given increasing weight in the final expected value calculations.

In any event, however, if we were to discount all amounts to the beginning of the reserve period and if all estimates were exactly correct we would see no development in these amounts over time. In addition, if economic conditions (and assumptions regarding future conditions) remain unchanged all movement in total incurred amounts would reflect changes in future annual cost estimates making up the case reserve estimates. Hence we could quantify variation in those estimates over time, using, for example, techniques developed in Hayne[5], Mack[6] or others.

A practical consideration still remains, however. In reality, assessments of future economic conditions change over time. For example, in the 1980's it may not have been unreasonable to assume that medical cost inflation would remain quite high over a fairly long period of time. However, given the situation in the late 1990's, we may be hard pressed to justify estimates of future inflation at levels experienced in the 1980's. As noted above, such changes would appear as calendar period effects in the development patterns and could mislead estimate of uncertainty in claim cost estimates.

Specific changes such as those in assumed future economic conditions will affect reserve estimates similar to those of currency fluctuations on losses denominated in more than one currency. Borrowing techniques developed to handle such changes, as presented in Duncan and Hayne[7] we can consider a type of two-step development array.

Table 10: Example Two-Stage Development

	Months of Development				
Reserve	12	24		36	
Year	Current	Prior	Current	Prior	Current
1995	$100,000	$110,000	$105,000	$107,100	$109,500
1996	125,000	143,750	137,500		
1997	175,000				

	Development Factors	
	24/12	36/24
1995	1.10	1.02

In this two-stage approach we use "Current" to denote the assumptions inherent in the final selected analysis at the indicated valuation date. For example, $105,000 indicates the total incurred (discounted to the beginning of 1995) using the economic assumptions at the 1996 valuation. Similarly $109,500 represents the discounted incurred (again to the beginning of 1995) using the 1997 economic assumptions.

The "Prior" amounts denote the calculations using the <u>economic</u> assumptions from the prior analysis. For example, the $110,000 represents the forecasts for 1995 claims, using 1996 claim information, but using the economic assumptions inherent in the 1995 (prior) analysis. Thus the difference between $100,000 (1995 at 12 months) and $110,000 is due to the evaluation of the individual claims and not due to different economic assumptions used in calculating the losses. The development factors are then comparisons between the "Prior" at one stage of development with the "Current" at the previous stage. In effect, then, the development isolates changes in economic assumptions from development in underlying cost estimates.

From this point we could use the variation inherent in these development factors to estimate uncertainty in annual cost estimates, and thus the σ^2 parameter.

5.2 Estimating the r and z Parameters

The next portion of total reserves in our consideration is that for known but not-case-reserved claims. If we assume that there is no inherent difference between these claims and those already reported, we could assume their distribution is the same as that for known claims and take $r = 1$ and $z = 0$.

However, there may be other factors considered in setting the formula reserves for these claims. The r and z parameters can then be used to account for these factors and resulting additional uncertainty. For example, assume the formula reserves are set only during the first three years after claim occurrence, using only the most recent three accident years, without any adjustment for trend or differences by report lag. The following then shows one approach to estimating r and z in this case:

Table 11: Estimate of *r* and *z* Parameters

Accident Year	Report Year	Reported Losses	Reported Claims	Loss Average	Loss Standard Deviation
1995	1995	$ 5,000	200	$ 25,000	$ 27,500
1995	1996	5,100	300	17,000	15,300
1995	1997	5,500	250	22,000	23,100
1996	1996	9,800	350	28,000	22,400
1996	1997	4,180	220	19,000	20,900
1997	1997	10,500	350	30,000	31,500
Total		$ 40,080	1,670	$ 24,000	$ 24,635
Expected Without Uncertainty				$ 20,000	$ 18,000
Parameter Estimates:					
r				1.20	
z				0.19	

The estimate for *r* is simply the ratio of the average for the "formula" reserved claims to the expected average (without parameter uncertainty). The estimate of *z* follows from the assumptions regarding the form of uncertainty for these formula reserves. In particular, assuming the random variable *Y* is defined using the notation in Section *4. Additional Areas of Uncertainty* as:

$$(5.2) \qquad Y = \zeta X$$

We then have the following formula for the variance of Y:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}_\zeta\left(\mathrm{Var}(X|\zeta)\right) + \mathrm{Var}_\zeta\left(\mathrm{E}(X|\zeta)\right) \\ &= \mathrm{E}_\zeta\left(\mathrm{Var}(X\zeta)\right) + \mathrm{Var}_\zeta\left(\mathrm{E}(X\zeta)\right) \\ &= \mathrm{E}_\zeta\left(\zeta^2\,\mathrm{Var}(X)\right) + \mathrm{Var}_\zeta\left(\zeta\,\mathrm{E}(X)\right) \\ &= \mathrm{Var}(X)\mathrm{E}\left(\zeta^2\right) + \mathrm{E}(X)^2\,\mathrm{Var}(\zeta) \\ &= \mathrm{Var}(X)\left(\mathrm{Var}(\zeta) + \mathrm{E}(\zeta)^2\right) + \mathrm{E}(X)^2\,\mathrm{Var}(\zeta) \\ &= \mathrm{Var}(X)\left(z + r^2\right) + \mathrm{E}(X)^2 z \end{aligned}$$

Solving for *z* we obtain:

$$z = \frac{\mathrm{Var}(Y) - r^2\,\mathrm{Var}(X)}{\mathrm{Var}(X) + \mathrm{E}(X)^2} \tag{5.3}$$

5.3 Estimating the c, a, and d Parameters

The final portion of total reserves is for claims that are incurred but not reported. As with known claims with formula reserves, if IBNR reserves are estimated using averages for known claims we could estimate the *a* and *d* parameters similar to the way we estimated the *r* and *z* parameters as described in Section 5.2, *Estimating the r and z Parameters*.

We could estimate the *c* parameter in several ways. One approach starts with the assumption that the number of IBNR claims has a Poisson distribution with a "contagion" parameter similar to that used by Heckman and Meyers.[1] With that assumption we see from Appendix E that with our notation above if *N* denotes the number of IBNR claims:

(5.4) $\mathrm{E}(N) = \lambda$, and

(5.5) $\mathrm{Var}(N) = \lambda + c\lambda^2$

Solving (5.5) for *c* we obtain:

$$c = \frac{\mathrm{Var}(N) - \lambda}{\lambda^2} \tag{5.6}$$

If we estimated the number of IBNR claims using development of reported claims then Hayne[5] provides an approach we could use to estimate total variance in the IBNR estimates, if we are willing to assume independence among the various accident (or exposure) years. Consider the following example:

Table 12: Example Reported Count Development

Accident	Months of Development					
Year	12	24	36	48	60	72
1989	176	363	417	477	500	500
1990	314	384	519	524	550	550
1991	178	294	382	405	425	425
1992	323	472	535	590	620	620
1993	264	492	506	572	600	
1994	253	419	441	495		
1995	137	324	410			
1996	304	415				

Following Hayne, and assuming independence of the age-to-age factors (to keep the calculations simple) we calculate the natural logarithms of the age-to-age factors, their means and standard deviations as parameter estimates for the lognormal distributions of the age-to-age factors. Also, given independence the parameters for the age-to-ultimate factors can then be determined from the parameters of the age-to-age factors by simply summing the means and variances. The following shows these calculations:

Table 13: Logarithms of Claim Age-to-Age Factors

Accident	Months of Development				
Year	24/12	36/24	48/36	60/48	72/60
1989	0.7239	0.1387	0.1344	0.0471	0.0000
1990	0.2012	0.3013	0.0096	0.0484	0.0000
1991	0.5018	0.2618	0.0585	0.0482	0.0000
1992	0.3793	0.1253	0.0979	0.0496	0.0000
1993	0.6225	0.0281	0.1226	0.0478	
1994	0.5045	0.0512	0.1155		
1995	0.8608	0.2354			
1996	0.3113				
Mean	0.5132	0.1631	0.0897	0.0482	0.0000
Std.Dev.	0.2182	0.1056	0.0473	0.0009	0.0000
Cumulative:					
Mean	0.8142	0.3011	0.1380	0.0482	0.0000

Finally, using standard formulae for the lognormal we obtain the following projected number of claims and their corresponding variance:

Table 14: Estimate of *c* Parameter

Accident Year	Cumulative Parameters Mean	Cumulative Parameters Std.Dev.	Reported Claims	Forecast Mean	Forecast Std.Dev.
1989	0.0000	0.0000	500	500	-
1990	0.0000	0.0000	550	550	-
1991	0.0000	0.0000	425	425	-
1992	0.0000	0.0000	620	620	-
1993	0.0000	0.0000	600	600	-
1994	0.0482	0.0009	495	519	0.5
1995	0.1380	0.0474	410	471	22.3
1996	0.3011	0.1157	415	565	65.5
1997	0.8142	0.2470	282	656	164.6
Total			4,297	4,906	178.6
Indicated IBNR				609	178.6
Indicated *c* Value:					0.084

6. Conclusion

In this paper we have set out one approach that can be used to systematically estimate variation in both total reserve estimates and in payments in individual future years. In explicitly accounting for various components of uncertainty the actuary can adapt these estimates to be used in DFA applications. In such applications economic conditions can form a link between asset and liability models. Explicit recognition of the influence of such factors on loss reserve and payment uncertainty in the liability models will prevent "double counting" of its effect and result in potentially more realistic DFA models.

We have presented this as a first step. There are obviously many simplifying assumptions even in this rather complex presentation. We hope this framework can provide a useful starting point to build and parameterize models of the amount and timing of insured liabilities.

7. *Bibliography*

[1] Heckman, P.E. and Meyers, G.G., "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," *Proceedings of the Casualty Actuarial Society*, vol. LXX, 1983, pp. 22-61, with additional exhibits presented in vol LXXI, 1984.

[2] Robertson, J.P., "The Computation of Aggregate Loss Distributions," *Proceedings of the Casualty Actuarial Society*, vol. LXXIX, 1992, pp. 57-133.

[3] See the website http://www.neurolaw.com, attributed to the National Spinal Cord Injury Statistical Center, University of Alabama at Birmingham.

[4] Daykin, C.D.; Pentikäinen, T.; and Pesonen, M.; *Practical Risk Theory for Actuaries*, First Edition, 1994, Chapman and Hall, pp. 211-276.

[5] Hayne, R.M.; "An Estimate of Statistical Variation in Development Factor Methods," *Proceedings of the Casualty Actuarial Society*, vol. LXXII, pp. 25-43.

[6] Mack, T.; "Distribution-free Calculation of the Standard Error of Chain Ladder Reserve Estimates," *ASTIN Bulletin*, Volume 23, No. 2, November 1993, pp. 213-226.

[7] Duncan, F. and Hayne, R.; "Some Aspects of Currencies and Exchange Rates in the London Market," *International Topics, Global Insurance Pricing, Reserving and Coverage Issues,* Casualty Actuarial Society, 1991 Discussion Paper Program, pp. 87-144.

APPENDIX A

In this appendix we derive the formula for the variance for an individual life pension claimant, formula (2.4). From our definitions we have:

$$
\begin{aligned}
\mathrm{E}\left(X^2\right) &= \sum_{t=1}^{\infty} p_{xt}\left(\sum_{s=1}^{t} a_{xs}\right)^2 \\
&= \sum_{t=1}^{\infty}\sum_{s=1}^{t}\sum_{r=1}^{t} p_{xt} a_{xs} a_{xr} \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\sum_{t=\max(s,r)}^{\infty} p_{xt} a_{xs} a_{xr} \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{xs} a_{xr} \sum_{t=\max(s,r)}^{\infty} p_{xt} \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs} a_{xr} \sum_{t=s}^{\infty} p_{xt}\right) + \left(\sum_{r=s+1}^{\infty} a_{xs} a_{xr} \sum_{t=r}^{\infty} p_{xt}\right)
\end{aligned}
$$

We also have:

$$
\begin{aligned}
\mathrm{E}(X)^2 &= \left(\sum_{t=1}^{\infty} p_{xt} \sum_{s=1}^{t} a_{xs}\right)^2 \\
&= \left(\sum_{s=1}^{\infty}\sum_{t=s}^{\infty} p_{xt} a_{xs}\right)^2 \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\left(\sum_{t=s}^{\infty} p_{xt} a_{xs}\right)\left(\sum_{t=r}^{\infty} p_{xt} a_{xr}\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\left(a_{xs}\sum_{t=s}^{\infty} p_{xt}\right)\left(a_{xr}\sum_{t=r}^{\infty} p_{xt}\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{xs} a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs} a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) + \left(\sum_{r=s+1}^{\infty} a_{xs} a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right)
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
\operatorname{Var}(X) &= \mathrm{E}\left(X^2\right) - \mathrm{E}(X)^2 \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs}a_{xr}\sum_{t=s}^{\infty} p_{xt}\right) + \left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\sum_{t=r}^{\infty} p_{xt}\right) \\
&\quad -\sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) + \left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs}a_{xr}\sum_{t=s}^{\infty} p_{xt}\right) + \left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\sum_{t=r}^{\infty} p_{xt}\right) - \left(\sum_{r=1}^{s} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) \\
&\quad -\left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt} - \left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right)\right) \\
&\quad +\left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=r}^{\infty} p_{xt} - \left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right)\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} a_{xs}a_{xr}\left(\sum_{t=s}^{\infty} p_{xt}\right)\left(1-\left(\sum_{t=r}^{\infty} p_{xt}\right)\right)\right) + \left(\sum_{r=s+1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=r}^{\infty} p_{xt}\right)\left(1-\left(\sum_{t=s}^{\infty} p_{xt}\right)\right)\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{xs}a_{xr}\left(\sum_{t=\max(r,s)}^{\infty} p_{xt}\right)\left(1-\left(\sum_{t=\min(r,s)}^{\infty} p_{xt}\right)\right)
\end{aligned}
$$

APPENDIX B

In this appendix we derive the conditional distribution of θ given z observed open claims from our population of n claims. We also review the asymptotic behavior of this distribution.

1. Conditional Distribution of θ

We first assume that the number of claims remaining open from one year to the next has a binomial distribution with parameter θ. Although we will assume that θ will be uniformly distributed between 0 and 1, the following result holds in the more general case when θ has with a beta distribution with parameters α and β. In this case z, the number of "successes" (or claims remaining open) is given by:

$$f(z) = \binom{n}{z} \theta^z (1-\theta)^{n-z}$$

The *parameter θ then has the distribution:*

$$h(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

The joint distribution for z and θ is then given by:

$$\begin{aligned} k(z,\theta) &= g(z)h(\theta) \\ &= \binom{n}{z} \theta^z (1-\theta)^{n-z} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \\ &= \binom{n}{z} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{z+\alpha-1} (1-\theta)^{n-z+\beta-1} \end{aligned}$$

We will take $y = \theta / E(\theta)$. Now we need to get the distribution of θ given our observed annual closure rate, or conversely, rate of claims that remain open. From Bayes Theorem we obtain:

$$k(\theta|z) = \frac{k(\theta,z)}{\int_0^1 k(\theta,z)d\theta}$$

The integral in the denominator becomes:

$$\begin{aligned}\int_0^1 \mathrm{k}(\theta, z)d\theta &= \int_0^1 \binom{n}{z}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{z+\alpha-1}(1-\theta)^{n-z+\beta-1}d\theta \\ &= \binom{n}{z}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{z+\alpha-1}(1-\theta)^{n-z+\beta-1}d\theta \\ &= \binom{n}{z}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(z+\alpha)\Gamma(n-z+\beta)}{\Gamma(\alpha+\beta+n)}\end{aligned}$$

This then gives:

$$\begin{aligned}\mathrm{k}(\theta|z) &= \frac{\binom{n}{z}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{z+\alpha-1}(1-\theta)^{n-z+\beta-1}}{\binom{n}{z}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(z+\alpha)\Gamma(n-z+\beta)}{\Gamma(\alpha+\beta+n)}} \\ &= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(z+\alpha)\Gamma(n-z+\beta)}\theta^{z+\alpha-1}(1-\theta)^{n-z+\beta-1}\end{aligned}$$

That is, $\mathrm{k}(\theta|z)$ has a beta distribution with parameters $z+\alpha$ and $n-z+\beta$.

2. Asymptotic Behavior

We first assume that if the portion of claims remaining open tends to a finite limit as the sample size increases then the expected adjustment in (3.16) tends to unity. With this assumption, then, we consider

$$\lim_{n\to\infty}\left(\frac{n+2}{z+1}\right)^s \frac{\Gamma(n+2)\Gamma(z+s+1)}{\Gamma(z+1)\Gamma(s+n+2)}$$

For this evaluation we will use Stirling's approximation for the gamma function for large values of n:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Using this approximation we have:

$$\left(\frac{n+2}{z+1}\right)^s \frac{\Gamma(n+2)\Gamma(z+s+1)}{\Gamma(z+1)\Gamma(s+n+2)} \approx \left(\frac{n+2}{z+1}\right)^s \frac{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(z+s)}\left(\frac{z+s}{e}\right)^{z+s}}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z\sqrt{2\pi(s+n+1)}\left(\frac{s+n+1}{e}\right)^{s+n+1}}$$

$$= \left(\frac{n+2}{z+1}\right)^s \frac{(n+1)^{n+\frac{3}{2}}(z+s)^{z+s+\frac{1}{2}}e^{-(n+z+s+1)}}{z^{z+\frac{1}{2}}(s+n+1)^{s+n+\frac{3}{2}}e^{-(z+s+n+1)}}$$

$$= \left(\frac{n+2}{z+1}\right)^s \left(\frac{n+1}{s+n+1}\right)^{n+\frac{3}{2}} \left(\frac{1}{s+n+1}\right)^s \left(\frac{z+s}{z}\right)^{z+\frac{1}{2}} (z+s)^s$$

$$= \left(\frac{(n+2)(z+s)}{(z+1)(s+n+1)}\right)^s \left(\frac{n+1}{s+n+1}\right)^{n+\frac{3}{2}} \left(\frac{z+s}{z}\right)^{z+\frac{1}{2}}$$

As n gets large we have:

$$\lim_{n\to\infty}\left(\frac{(n+2)(z+s)}{(z+1)(s+n+1)}\right)^s = \lim_{n\to\infty}\left(\frac{\left(1+\frac{2}{n}\right)\left(\frac{z}{n}+\frac{s}{n}\right)}{\left(\frac{z}{n}+\frac{1}{n}\right)\left(1+\frac{s+1}{n}\right)}\right)^s$$

$$= \left(\frac{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)\lim_{n\to\infty}\left(\frac{z}{n}+\frac{s}{n}\right)}{\lim_{n\to\infty}\left(\frac{z}{n}+\frac{1}{n}\right)\lim_{n\to\infty}\left(1+\frac{s+1}{n}\right)}\right)^s$$

$$= \left(\frac{1\times\theta}{\theta\times 1}\right)^s = 1$$

The limits for the other two terms follow from an alternative definition for the exponential function:

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

We thus have:

$$\lim_{n\to\infty}\left(\frac{n+1}{s+n+1}\right)^{n+\frac{3}{2}} = \lim_{n\to\infty}\left(\frac{n+1}{s+n+1}\right)^{\frac{1}{2}-s}\left(1-\frac{s}{s+n+1}\right)^{s+n+1}$$

$$= \left(\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{s+1}{n}}\right)^{\frac{1}{2}-s}\right)\lim_{n\to\infty}\left(1-\frac{s}{s+n+1}\right)^{s+n+1} = e^{-s}$$

Similarly we have:

$$\lim_{n\to\infty}\left(\frac{z+s}{z}\right)^{z+\frac{1}{2}} = \lim_{z\to\infty}\sqrt{\frac{z+s}{z}}\left(1+\frac{s}{z}\right)^{z}$$

$$= \sqrt{\lim_{z\to\infty}\left(1+\frac{s}{z}\right)}\lim_{z\to\infty}\left(1+\frac{s}{z}\right)^{z} = e^{s}$$

Thus we obtain:

$$\lim_{n\to\infty}\left(\frac{n+2}{z+1}\right)^{s}\frac{\Gamma(n+2)\Gamma(z+s+1)}{\Gamma(z+1)\Gamma(s+n+2)} = e^{-s}e^{s} = 1$$

APPENDIX C

In this appendix we derive formulae (3.17), using repeated application of the relationships in (3.16). First, we consider (3.16). From the definitions of the conditional distributions it is clear that

$$\begin{aligned} \mathrm{E}(Z) &= \int\int z\, \mathrm{f}(z|\xi)\, \mathrm{g}(\xi) dz d\xi \\ &= \mathrm{E}_\xi(\mathrm{E}(Z|\xi)) \end{aligned}$$

As for the variance we have

$$\begin{aligned} \mathrm{Var}(Z) &= \int\int z^2\, \mathrm{f}(z|\xi)\, \mathrm{g}(\xi) dz d\xi - \mathrm{E}(Z)^2 \\ &= \mathrm{E}_\xi(\mathrm{E}(Z^2|\xi)) - \mathrm{E}(Z)^2 \\ &= \mathrm{E}_\xi\left(\mathrm{Var}(Z|\xi) + \mathrm{E}(Z|\xi)^2\right) - \mathrm{E}_\xi(\mathrm{E}(Z|\xi))^2 \\ &= \mathrm{E}_\xi\left(\mathrm{Var}(Z|\xi)\right) + \mathrm{E}_\xi\left(\mathrm{E}(Z|\xi)^2 - \mathrm{E}_\xi(\mathrm{E}(Z|\xi))^2\right) \\ &= \mathrm{E}_\xi\left(\mathrm{Var}(Z|\xi)\right) + \mathrm{Var}_\xi(\mathrm{E}(Z|\xi)) \end{aligned}$$

From our assumptions we have

$$\mathrm{E}(T|\theta, u, w_t) = \sum_x \sum_{s=1}^{\infty} b^*_{xs} \left(\prod_{l=0}^{s-1} (1 - q^*_{xl}) \right) \left(u y^s \prod_{r=1}^{s} w_r \right)$$

Similarly, we can compute

$$\begin{aligned} \mathrm{Var}(T|\theta, u, w_t) &= \mathrm{Var}\left(\sum_x X|\theta, u, w_t \right) \\ &= \sum_x \mathrm{Var}(X|\theta, u, w_t) \end{aligned}$$

The last sum holds since we assumed the claims are independent for <u>fixed</u> θ, u, and w_t. We thus need only consider the variance for a single claim. We thus have:

$$\mathrm{Var}(X|\theta, u, w_t) = \mathrm{E}(X^2|\theta, u, w_t) - \mathrm{E}(X|\theta, u, w_t)^2$$

From this we have

$$
\begin{aligned}
E\left(X^2|\theta,u,w_t\right) &= \sum_{t=1}^{\infty} p_{xt}\left(\sum_{s=1}^{t} b^*_{xs} u \prod_{r=1}^{s} w_r\right)^2 \\
&= \sum_{t=1}^{\infty}\sum_{s=1}^{t}\sum_{r=1}^{t} p_{xt} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\sum_{t=\max(s,r)}^{\infty} p_{xt} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z \sum_{t=\max(s,r)}^{\infty} p_{xt} \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z \sum_{t=s}^{\infty} p_{xt}\right) + \left(\sum_{r=s+1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z \sum_{t=r}^{\infty} p_{xt}\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{r} w_q^2 \prod_{z=r+1}^{s} w_z \sum_{t=s}^{\infty} p_{xt}\right) + \left(\sum_{r=s+1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q^2 \prod_{z=s+1}^{r} w_z \sum_{t=r}^{\infty} p_{xt}\right)
\end{aligned}
$$

We also have

$$
\begin{aligned}
E\left(X|\theta,u,w_t\right)^2 &= \left(\sum_{t=1}^{\infty} p_{xt} \sum_{s=1}^{t} b^*_{xs} u \prod_{r=1}^{s} w_r\right)^2 \\
&= \left(\sum_{s=1}^{\infty} u \prod_{r=1}^{s} w_r \sum_{t=s}^{\infty} p_{xt} b^*_{xs}\right)^2 \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\left(u \prod_{q=1}^{s} w_q \sum_{t=s}^{\infty} p_{xt} b^*_{xs}\right)\left(u \prod_{z=1}^{r} w_z \sum_{t=r}^{\infty} p_{xt} b^*_{xr}\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\left(u^2 \prod_{q=1}^{s} w_q \prod_{z=1}^{r} w_z\right)\left(b^*_{xs} \sum_{t=s}^{\infty} p_{xt}\right)\left(b^*_{xr} \sum_{t=r}^{\infty} p_{xt}\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{r} w_q^2 \prod_{z=r+1}^{s} w_z \left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right) + \left(\sum_{r=s+1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q^2 \prod_{z=s+1}^{r} w_z \left(\sum_{t=s}^{\infty} p_{xt}\right)\left(\sum_{t=r}^{\infty} p_{xt}\right)\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
E\left(X^2|\theta,u,w_t\right) - E\left(X|\theta,u,w_t\right)^2 &= \sum_{s=1}^{\infty}\left(\sum_{r=1}^{s} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{r} w_q^2 \prod_{z=r+1}^{s} w_z \left(\sum_{t=s}^{\infty} p_{xt}\right)\left(1-\sum_{t=r}^{\infty} p_{xt}\right)\right) \\
&\quad + \left(\sum_{r=s+1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{s} w_q^2 \prod_{z=s+1}^{r} w_z \left(\sum_{t=r}^{\infty} p_{xt}\right)\left(1-\sum_{t=s}^{\infty} p_{xt}\right)\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{\min(r,s)} w_q^2 \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \left(\sum_{t=\max(r,s)}^{\infty} p_{xt}\right)\left(1-\sum_{t=\min(r,s)}^{\infty} p_{xt}\right) \\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^*_{xs} b^*_{xr} u^2 \prod_{q=1}^{\min(r,s)} w_q^2 \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \left(y^{\max(r,s)} \prod_{t=0}^{\max(r,s)-1} \left(1-q^*_{xt}\right)\right)\left(1-\left(y^{\min(r,s)} \prod_{t=0}^{\min(r,s)-1} \left(1-q^*_{xt}\right)\right)\right)
\end{aligned}
$$

Which gives:

$$\mathrm{Var}(T|\theta,u,w_t)=\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^{*}_{xs}b^{*}_{xr}u^2\prod_{q=1}^{\min(r,s)}w_q^2\prod_{z=\min(r,s)+1}^{\max(r,s)}w_z\left(y^{\max(r,s)}\prod_{l=0}^{\max(r,s)-1}\left(1-q^{*}_{xl}\right)\right)\left(1-\left(y^{\min(r,s)}\prod_{l=0}^{\min(r,s)-1}\left(1-q^{*}_{xl}\right)\right)\right)$$

We will apply the Bayesian relationships above for each variable in succession so to as to appropriately track the various dependencies. First we remove the θ dependence:

$$\begin{aligned}
\mathrm{E}(T|u,w_t)&=\mathrm{E}_\theta\left(\mathrm{E}(T|\theta,u,w_t)\right)\\
&=\mathrm{E}_\theta\left(\sum_x\sum_{s=1}^{\infty}b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\left(uy^s\prod_{r=1}^{s}w_r\right)\right)\\
&=\sum_x\sum_{s=1}^{\infty}b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\left(u\prod_{r=1}^{s}w_r\right)\mathrm{E}(y^s)\\
&=\sum_x\sum_{s=1}^{\infty}b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\left(u\prod_{r=1}^{s}w_r\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}$$

In calculations, the second-to-last representation is probably easier to manage. The variance estimate follows too:

$$\mathrm{Var}(T|u,w_t)=\mathrm{Var}_\theta\left(\mathrm{E}(T|\theta,u,w_t)\right)+\mathrm{E}_\theta\left(\mathrm{Var}(T|\theta,u,w_t)\right)$$

From the above relationships.

$$\begin{aligned}
\mathrm{Var}_\theta\left(\mathrm{E}(T|\theta,u,w_t)\right)&=\mathrm{Var}_\theta\left(\sum_x\sum_{s=1}^{\infty}b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\left(uy^s\prod_{r=1}^{s}w_r\right)\right)\\
&=\mathrm{Var}_\theta\left(\sum_{s=1}^{\infty}\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\left(uy^s\prod_{r=1}^{s}w_r\right)\right)\\
&=\mathrm{Var}_\theta\left(\sum_{s=1}^{\infty}y^s\left(u\prod_{r=1}^{s}w_r\right)\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\right)\\
&=\sum_{s=1}^{\infty}\mathrm{Var}_\theta\left(y^s\left(u\prod_{r=1}^{s}w_r\right)\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\right)\\
&=\sum_{s=1}^{\infty}\left(u\prod_{r=1}^{s}w_r\right)^2\left(\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}\left(1-q^{*}_{xl}\right)\right)\right)^2\mathrm{Var}(y^s)
\end{aligned}$$

From the definitions of y we have

$$
\begin{aligned}
\mathrm{Var}\left(y^{s}\right) &= \mathrm{E}\left(y^{2s}\right) - \mathrm{E}\left(y^{s}\right)^{2} \\
&= \prod_{i=0}^{2s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} - \left(\prod_{i=0}^{s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} \right)^{2} \\
&= \left(\prod_{i=0}^{s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} \right) \left(\prod_{i=s}^{2s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} - \prod_{i=0}^{s-1} \frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} \right)
\end{aligned}
$$

Again, the first representation will probably be easier from a coding point of view. This then gives:

$$
\mathrm{Var}_{\theta}\left(\mathrm{E}(T|\theta,u,w_t)\right) = \sum_{s=1}^{\infty} \left(u \prod_{r=1}^{s} w_r \right)^{2} \left(\sum_{x} b^{*}_{xs} \left(\prod_{t=0}^{s-1} (1 - q^{*}_{xt}) \right) \right)^{2} \left(\mathrm{E}\left(y^{2s}\right) - \mathrm{E}\left(y^{s}\right)^{2} \right)
$$

As for the other term,

$$
\begin{aligned}
\mathrm{E}_{\theta}\left(\mathrm{Var}(T|\theta,u,w_t)\right) &= \mathrm{E}_{\theta}\left(\sum_{x} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^{*}_{xs} b^{*}_{xr} u^{2} \prod_{q=1}^{\min(r,s)} w_q^{2} \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \left(y^{\max(r,s)} \prod_{t=0}^{\max(r,s)-1} (1 - q^{*}_{xt}) \right) \left(1 - \left(y^{\min(r,s)} \prod_{t=0}^{\min(r,s)-1} (1 - q^{*}_{xt}) \right) \right) \right) \\
&= \sum_{x} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^{*}_{xs} b^{*}_{xr} u^{2} \prod_{q=1}^{\min(r,s)} w_q^{2} \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \, \mathrm{E}_{\theta}\left(\left(y^{\max(r,s)} \prod_{t=0}^{\max(r,s)-1} (1 - q^{*}_{xt}) \right) \left(1 - \left(y^{\min(r,s)} \prod_{t=0}^{\min(r,s)-1} (1 - q^{*}_{xt}) \right) \right) \right) \\
&= \sum_{x} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^{*}_{xs} b^{*}_{xr} u^{2} \prod_{q=1}^{\min(r,s)} w_q^{2} \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \left(\prod_{t=0}^{\max(r,s)-1} (1 - q^{*}_{xt}) \right) \times \left(\mathrm{E}\left(y^{\max(r,s)}\right) - \mathrm{E}\left(y^{r+s}\right) \left(\prod_{t=0}^{\min(r,s)-1} (1 - q^{*}_{xt}) \right) \right)
\end{aligned}
$$

Combining these two terms we have:

$$
\begin{aligned}
\mathrm{Var}(T|u,w_t) = & \sum_{s=1}^{\infty} \left(u \prod_{r=1}^{s} w_r \right)^{2} \left(\sum_{x} b^{*}_{xs} \left(\prod_{t=0}^{s-1} (1 - q^{*}_{xt}) \right) \right)^{2} \left(\mathrm{E}\left(y^{2s}\right) - \mathrm{E}\left(y^{s}\right)^{2} \right) \\
& + \sum_{x} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b^{*}_{xs} b^{*}_{xr} u^{2} \prod_{q=1}^{\min(r,s)} w_q^{2} \prod_{z=\min(r,s)+1}^{\max(r,s)} w_z \left(\prod_{t=0}^{\max(r,s)-1} (1 - q^{*}_{xt}) \right) \left(\mathrm{E}_{\theta}\left(y^{\max(r,s)}\right) - \mathrm{E}_{\theta}\left(y^{r+s}\right) \left(\prod_{t=0}^{\min(r,s)-1} (1 - q^{*}_{xt}) \right) \right)
\end{aligned}
$$

Now removing w_t from the terms:

$$
\begin{aligned}
E(T|u) &= E_{w_t}\left(E(T|u,w_t)\right)\\
&= E_{w_t}\left(\sum_x\sum_{s=1}^{\infty} b^{*}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)\left(u\prod_{r=1}^{s} w_r\right)E(y^s)\right)\\
&= \sum_x\sum_{s=1}^{\infty} b^{*}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)E(y^s)E\left(\prod_{r=1}^{s} w_r\right)\\
&= \sum_x\sum_{s=1}^{\infty} b^{*}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)E(y^s)\\
&= \sum_x\sum_{s=1}^{\infty} b^{*}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{*}_{xt})\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}
$$

Again the second-to-last term may be easier to work with computationally. This last item follows from the lognormal assumptions regarding w_t. In particular these assumptions imply that:

$$\left(\prod_{r=1}^{s} w_r\right)^{p} \sim \text{lognormal}\left((-\tfrac{1}{2})(ps\tau^2), p^2 s\tau^2\right)$$

Thus the expectation:

$$E\left(\left(\prod_{r=1}^{s} w_r\right)^{p}\right) = \exp\left((-\tfrac{1}{2})(sp\tau^2) + \tfrac{1}{2}sp^2\tau^2\right)$$

For p = 1 this gives an expectation of unity, giving the above formula. Similarly for the variance term:

$$\text{Var}(T|u) = \text{Var}_{w_t}\left(E(T|u,w_t)\right) + E_{w_t}\left(\text{Var}(T|u,w_t)\right)$$

Taking the terms one at a time:

$$
\begin{aligned}
\mathrm{Var}_{w_t}\left(\mathrm{E}(T|u,w_t)\right) &= \mathrm{Var}_{w_t}\left(\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\left(u\prod_{r=1}^{s} w_r\right)\mathrm{E}(y^s)\right) \\
&= \mathrm{Var}_{w_t}\left(\sum_{s=1}^{\infty}\left(\prod_{r=1}^{s} w_r\right)\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)u\,\mathrm{E}(y^s)\right) \\
&= \sum_{s=1}^{\infty}\mathrm{Var}_{w_t}\left(\left(\prod_{r=1}^{s} w_r\right)\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)u\,\mathrm{E}(y^s)\right) \\
&= \sum_{s=1}^{\infty}\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)u\,\mathrm{E}(y^s)\right)^2 \mathrm{Var}\left(\prod_{r=1}^{s} w_r\right)
\end{aligned}
$$

Again, from the lognormal assumptions:

$$
\begin{aligned}
\mathrm{Var}\left(\prod_{r=1}^{s} w_r\right) &= \mathrm{E}\left(\left(\prod_{r=1}^{s} w_r\right)^2\right) - \mathrm{E}\left(\prod_{r=1}^{s} w_r\right)^2 \\
&= \exp\left(\left(-\tfrac{1}{2}\right)\left(2s\tau^2\right) + \tfrac{1}{2}\left(4s\tau^2\right)\right) - \exp\left(2\left(\left(-\tfrac{1}{2}\right)\left(s\tau^2\right) + \tfrac{1}{2}s\tau^2\right)\right) \\
&= \exp\left(s\tau^2\right) - 1
\end{aligned}
$$

We thus obtain:

$$
\mathrm{Var}_{w_t}\left(\mathrm{E}(T|u,w_t)\right) = \sum_{s=1}^{\infty}\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)u\,\mathrm{E}(y^s)\right)^2\left(\exp\left(s\tau^2\right)-1\right)
$$

As for the second term we have:

$$
\begin{aligned}
\mathrm{E}_{w_t}\left(\mathrm{Var}(T|u,w_t)\right) &= \mathrm{E}_{w_t}\left(\begin{aligned}&\sum_{s=1}^{\infty}\left(u\prod_{r=1}^{s} w_r\right)^2\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&+\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^{\bullet}_{xs}b^{\bullet}_{xr}u^2\prod_{q=1}^{\min(r,s)} w_q^2\prod_{z=\min(r,s)+1}^{\max(r,s)} w_z\left(\prod_{t=0}^{\max(r,s)-1}(1-q^{\bullet}_{xt})\right)\left(\mathrm{E}\left(y^{\max(r,s)}\right)-\mathrm{E}\left(y^{r+s}\right)\left(\prod_{t=0}^{\min(r,s)-1}(1-q^{\bullet}_{xt})\right)\right)\end{aligned}\right) \\
&= \sum_{s=1}^{\infty}\mathrm{E}\left(\left(\prod_{r=1}^{s} w_r\right)^2\right)u^2\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&\quad+\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} b^{\bullet}_{xs}b^{\bullet}_{xr}u^2\,\mathrm{E}\left(\prod_{q=1}^{\min(r,s)} w_q^2\prod_{z=\min(r,s)+1}^{\max(r,s)} w_z\right)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^{\bullet}_{xt})\right)\left(\mathrm{E}\left(y^{\max(r,s)}\right)-\mathrm{E}\left(y^{r+s}\right)\left(\prod_{t=0}^{\min(r,s)-1}(1-q^{\bullet}_{xt})\right)\right)
\end{aligned}
$$

Now from the lognormality assumptions

$$\prod_{q=1}^{m} w_q^2 \prod_{z=m+1}^{n} w_z \sim \text{lognormal}\left(\left(-\tfrac{1}{2}\right)\left(2m\tau^2 + (n-m)\tau^2\right), 4m\tau^2 + (n-m)\tau^2\right), \text{ thus}$$

$$\prod_{q=1}^{m} w_q^2 \prod_{z=m+1}^{n} w_z \sim \text{lognormal}\left(\left(-\tfrac{1}{2}\right)\left((n+m)\tau^2\right), (n+3m)\tau^2\right)$$

This then gives

$$\begin{aligned} \mathrm{E}\left(\prod_{q=1}^{m} w_q^2 \prod_{z=m+1}^{n} w_z\right) &= \exp\left(\left(-\tfrac{1}{2}\right)\left((n+m)\tau^2\right) + \tfrac{1}{2}\left((n+3m)\tau^2\right)\right) \\ &= \exp\left(m\tau^2\right) \end{aligned}$$

This results in:

$$\begin{aligned} \mathrm{E}_{w_1}\left(\mathrm{Var}(T|u, w_1)\right) = &\sum_{s=1}^{\infty} \exp\left(s\tau^2\right) u^2 \left(\sum_x b_{xs}^{*} \left(\prod_{t=0}^{s-1}\left(1 - q_{xt}^{*}\right)\right)\right)^2 \left(\mathrm{E}\left(y^{2s}\right) - \mathrm{E}\left(y^{s}\right)^2\right) \\ &+ \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b_{xs}^{*} b_{xr}^{*} u^2 \exp\left(\min(r,s)\tau^2\right) \left(\prod_{t=0}^{\max(r,s)-1} \left(1 - q_{xt}^{*}\right)\right) \left(\mathrm{E}\left(y^{\max(r,s)}\right) - \mathrm{E}\left(y^{r+s}\right) \left(\prod_{t=0}^{\min(r,s)-1} \left(1 - q_{xt}^{*}\right)\right)\right) \end{aligned}$$

Adding these two terms together we obtain:

$$\begin{aligned} \mathrm{Var}(T|u) = &\, u^2 \sum_{s=1}^{\infty} \left(\sum_x b_{xs}^{*} \left(\prod_{t=0}^{s-1}\left(1 - q_{xt}^{*}\right)\right) \mathrm{E}\left(y^{s}\right)\right)^2 \left(\exp\left(s\tau^2\right) - 1\right) \\ &+ u^2 \sum_{s=1}^{\infty} \exp\left(s\tau^2\right) \left(\sum_x b_{xs}^{*} \left(\prod_{t=0}^{s-1}\left(1 - q_{xt}^{*}\right)\right)\right)^2 \left(\mathrm{E}\left(y^{2s}\right) - \mathrm{E}\left(y^{s}\right)^2\right) \\ &+ u^2 \sum_x \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} b_{xs}^{*} b_{xr}^{*} \exp\left(\min(r,s)\tau^2\right) \left(\prod_{t=0}^{\max(r,s)-1} \left(1 - q_{xt}^{*}\right)\right) \left(\mathrm{E}\left(y^{\max(r,s)}\right) - \mathrm{E}\left(y^{r+s}\right) \left(\prod_{t=0}^{\min(r,s)-1} \left(1 - q_{xt}^{*}\right)\right)\right) \end{aligned}$$

Finally we eliminate u from the formulae. First

$$
\begin{aligned}
\mathrm{E}(T) &= \mathrm{E}_u\left(\mathrm{E}(T|u)\right) \\
&= \mathrm{E}_u\left(\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} u \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right) \\
&= \sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\mathrm{E}(u) \\
&= \sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right) \\
&= \sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}
$$

As for the variance formula we have:

$$
\mathrm{Var}(T) = \mathrm{Var}_u\left(\mathrm{E}(T|u)\right) + \mathrm{E}_u\left(\mathrm{Var}(T|u)\right)
$$

Again we consider the two portions separately.

$$
\begin{aligned}
\mathrm{Var}_u\left(\mathrm{E}(T|u)\right) &= \mathrm{Var}_u\left(\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} u \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right) \\
&= \mathrm{Var}_u\left(u\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right) \\
&= \left(\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right)^2 \mathrm{Var}(u)
\end{aligned}
$$

Since u is lognormal with mean 1 we have

$$
\mathrm{Var}(u) = \exp\left(\sigma^2\right) - 1
$$

We thus obtain:

$$
\mathrm{Var}_u\left(\mathrm{E}(T|u)\right) = \left(\sum_x \sum_{s=1}^{\infty} b^{\bullet}_{xs} \left(\prod_{t=0}^{s-1}\left(1-q^{\bullet}_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right)^2 \left(\exp\left(\sigma^2\right) - 1\right)
$$

As for the second term we have:

$$
\begin{aligned}
\mathrm{E}_u\left(\mathrm{Var}(T|u)\right) &= \mathrm{E}_u\left(\begin{array}{l}
u^2\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}\left(1-q^*_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right)^2\left(\exp\left(s\tau^2\right)-1\right)\\
+u^2\sum_{s=1}^{\infty}\exp\left(s\tau^2\right)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}\left(1-q^*_{xt}\right)\right)\right)^2\left(\mathrm{E}\left(y^{2s}\right)-\mathrm{E}\left(y^s\right)^2\right)\\
+u^2\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp\left(\min(r,s)\tau^2\right)\left(\prod_{t=0}^{\max(r,s)-1}\left(1-q^*_{xt}\right)\right)\left(\mathrm{E}\left(y^{\max(r,s)}\right)-\mathrm{E}\left(y^{r+s}\right)\left(\prod_{t=0}^{\min(r,s)-1}\left(1-q^*_{xt}\right)\right)\right)
\end{array}\right)\\
&= \mathrm{E}\left(u^2\right)\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}\left(1-q^*_{xt}\right)\right)\mathrm{E}\left(y^s\right)\right)^2\left(\exp\left(s\tau^2\right)-1\right)\\
&\quad+\mathrm{E}\left(u^2\right)\sum_{s=1}^{\infty}\exp\left(s\tau^2\right)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}\left(1-q^*_{xt}\right)\right)\right)^2\left(\mathrm{E}\left(y^{2s}\right)-\mathrm{E}\left(y^s\right)^2\right)\\
&\quad+\mathrm{E}\left(u^2\right)\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp\left(\min(r,s)\tau^2\right)\left(\prod_{t=0}^{\max(r,s)-1}\left(1-q^*_{xt}\right)\right)\left(\mathrm{E}\left(y^{\max(r,s)}\right)-\mathrm{E}\left(y^{r+s}\right)\left(\prod_{t=0}^{\min(r,s)-1}\left(1-q^*_{xt}\right)\right)\right)
\end{aligned}
$$

Since u is lognormal, i.e.

$$u \sim \mathrm{lognormal}\left(-\tfrac{1}{2}\sigma^2, \sigma^2\right)$$

then u^2 is also lognormal and

$$u^2 \sim \mathrm{lognormal}\left(-\sigma^2, 4\sigma^2\right)$$

Thus we have

$$\mathrm{E}\left(u^2\right) = \exp\left(\left(-\sigma^2\right) + \tfrac{1}{2}\left(4\sigma^2\right)\right) = \exp\left(\sigma^2\right)$$

This then gives:

$$\mathrm{E}_u\big(\mathrm{Var}(T|u)\big) = \exp(\sigma^2)\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\mathrm{E}(y^s)\right)^2\left(\exp(s\tau^2)-1\right)$$

$$+\exp(\sigma^2)\sum_{s=1}^{\infty}\exp(s\tau^2)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)$$

$$+\exp(\sigma^2)\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp(\min(r,s)\tau^2)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^{\max(r,s)})-\mathrm{E}(y^{r+s})\left(\prod_{t=0}^{\min(r,s)-1}(1-q^*_{xt})\right)\right)$$

$$=\exp(\sigma^2)\sum_{s=1}^{\infty}\exp(s\tau^2)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\mathrm{E}(y^s)^2-\exp(\sigma^2)\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\mathrm{E}(y^s)^2$$

$$+\exp(\sigma^2)\sum_{s=1}^{\infty}\exp(s\tau^2)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\mathrm{E}(y^{2s})-\exp(\sigma^2)\sum_{s=1}^{\infty}\exp(s\tau^2)\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\mathrm{E}(y^s)^2$$

$$+\exp(\sigma^2)\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp(\min(r,s)\tau^2)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^{\max(r,s)})-\mathrm{E}(y^{r+s})\left(\prod_{t=0}^{\min(r,s)-1}(1-q^*_{xt})\right)\right)$$

$$=\exp(\sigma^2)\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\exp(s\tau^2)\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)$$

$$+\exp(\sigma^2)\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp(\min(r,s)\tau^2)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^{\max(r,s)})-\mathrm{E}(y^{r+s})\left(\prod_{t=0}^{\min(r,s)-1}(1-q^*_{xt})\right)\right)$$

Finally putting the two terms together we obtain:

$$\mathrm{Var}(T)=\left(\sum_x\sum_{s=1}^{\infty}b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\mathrm{E}(y^s)\right)^2\left(\exp(\sigma^2)-1\right)$$

$$+\exp(\sigma^2)\sum_{s=1}^{\infty}\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})\exp(s\tau^2)-\mathrm{E}(y^s)^2\right)$$

$$+\exp(\sigma^2)\sum_x\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}b^*_{xs}b^*_{xr}\exp(\min(r,s)\tau^2)\left(\prod_{t=0}^{\max(r,s)-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^{\max(r,s)})-\mathrm{E}(y^{r+s})\left(\prod_{t=0}^{\min(r,s)-1}(1-q^*_{xt})\right)\right)$$

APPENDIX D

In this appendix we show the derivation of formulae (3.19) for payments in a particular year. As with the total mean and variance we begin with the mean and variance for fixed parameter values and then, step, by step, remove dependency on the various uncertainty parameters. Without any uncertainty and dropping the explicit i subscript, (2.7) and (2.8) give:

$$E(T_s|\text{Certainty}) = \sum_x a_{xs} \sum_{t=s}^{\tau} p_{xt}$$
$$\text{Var}(T_s|\text{Certainty}) = \sum_x a_{xs}^2 \left(\sum_{t=s}^{\tau} p_{xt} \right)\left(1 - \sum_{t=s}^{\tau} p_{xt} \right)$$

Thus, incorporating cost escalation, discounting, and our uncertainty variables, we have:

$$E(T_s|\theta, u, w_s) = \sum_x b^{*}_{xs} \left(\prod_{t=0}^{s-1} (1 - q^{*}_{xt}) \right)\left(uy^s \prod_{t=1}^{s} w_t \right)$$
$$= \left(uy^s \prod_{t=1}^{s} w_t \right) \sum_x b^{*}_{xs} \left(\prod_{t=0}^{s-1} (1 - q^{*}_{xt}) \right)$$

As with the aggregate,

$$\text{Var}(T_s|\theta, u, w_t) = \text{Var}\left(\sum_x X_s|\theta, u, w_t \right)$$
$$= \sum_x \text{Var}(X_s|\theta, u, w_t)$$

The last sum holds since we assumed the claims are independent for <u>fixed</u> θ, u, and w_t. We thus need only consider the variance for a single claim. We thus have:

$$\text{Var}(X_s|\theta, u, w_t) = E(X_s^2|\theta, u, w_t) - E(X_s|\theta, u, w_t)^2$$

Breaking this into parts then we have:

$$E(X_s^2|\theta, u, w_t) = \left(b^{*}_{xs} u \prod_{t=1}^{s} w_s \right)^2 \left(y^s \prod_{t=0}^{s-1} (1 - q^{*}_{xt}) \right)$$

and

$$E(X_s|\theta,u,w_t)^2 = \left(b^*_{xs}u\prod_{r=1}^{s} w_s\right)^2\left(y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)^2$$

Hence we have

$$\begin{aligned}\mathrm{Var}(X_s|\theta,u,w_t) &= \left(b^*_{xs}u\prod_{r=1}^{s} w_s\right)^2\left(y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right) - \left(b^*_{xs}u\prod_{r=1}^{s} w_s\right)^2\left(y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)^2 \\ &= \left(b^*_{xs}u\prod_{r=1}^{s} w_s\right)^2\left(y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(1-y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\end{aligned}$$

This then gives:

$$\mathrm{Var}(T_s|\theta,u,w_t) = \sum_x\left(b^*_{xs}u\prod_{r=1}^{s} w_s\right)^2\left(y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(1-y^s\prod_{t=0}^{s-1}(1-q^*_{xt})\right)$$

We now use the Bayesian relationships to work down the conditional variables. First we remove the θ dependence.

$$\begin{aligned}E(T_s|u,w_s) &= E_\theta\left(\left(uy^s\prod_{r=1}^{s} w_r\right)\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right) \\ &= \left(u\prod_{r=1}^{s} w_r\right)\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)E(y^s) \\ &= \sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(u\prod_{r=1}^{s} w_r\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}\end{aligned}$$

In calculations, the second-to-last representation is probably easier to manage. The variance estimate follows too:

$$\mathrm{Var}(T_s|u,w_t) = \mathrm{Var}_\theta\left(E(T_s|\theta,u,w_t)\right) + E_\theta\left(\mathrm{Var}(T_s|\theta,u,w_t)\right)$$

From the above relationships:

$$\mathrm{Var}_\theta\left(\mathrm{E}(T_s|\theta,u,w_t)\right) = \mathrm{Var}_\theta\left(\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\left(uy^s\prod_{r=1}^{s}w_r\right)\right)$$
$$= \mathrm{Var}_\theta\left(y^s\left(u\prod_{r=1}^{s}w_r\right)\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\right)$$
$$= \left(u\prod_{r=1}^{s}w_r\right)^2\left(\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\right)^2\mathrm{Var}(y^s)$$

From Appendix C we have:

$$\mathrm{Var}(y^s) = \left(\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}\right)\left(\prod_{i=s}^{2s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)} - \prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}\right)$$

Again, the first representation will probably be easier from a coding point of view. This then gives:

$$\mathrm{Var}_\theta\left(\mathrm{E}(T_s|\theta,u,w_t)\right) = \left(u\prod_{r=1}^{s}w_r\right)^2\left(\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)$$

As for the other term,

$$\mathrm{E}_\theta\left(\mathrm{Var}(T_s|\theta,u,w_t)\right) = \mathrm{E}_\theta\left(\sum_x\left(b^{*}_{xs}u\prod_{r=1}^{s}w_s\right)^2\left(y^s\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\left(1-y^s\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\right)$$
$$= \sum_x\left(b^{*}_{xs}u\prod_{r=1}^{s}w_s\right)^2\mathrm{E}_\theta\left(\left(y^s\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)-\left(y^{2s}\prod_{l=0}^{s-1}(1-q^{*}_{xl})^2\right)\right)$$
$$= \sum_x\left(b^{*}_{xs}u\prod_{r=1}^{s}w_s\right)^2\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})^2\right)\right)$$

Combining these two terms we have:

$$\mathrm{Var}(T_s|u,w_t) = \left(u\prod_{r=1}^{s}w_r\right)^2\left(\sum_x b^{*}_{xs}\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)$$
$$+\sum_x\left(b^{*}_{xs}u\prod_{r=1}^{s}w_s\right)^2\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^{*}_{xl})^2\right)\right)$$

Now removing w_t from the terms:

$$
\begin{aligned}
E(T_s|u) &= E_{w_t}\left(E(T_s|u,w_t)\right)\\
&= E_{w_t}\left(\left(u\prod_{r=1}^{s} w_r\right)\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)E_\theta(y^s)\right)\\
&= \sum_x b^{\bullet}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)E(y^s)E\left(\prod_{r=1}^{s} w_r\right)\\
&= \sum_x b^{\bullet}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)E(y^s)\\
&= \sum_x b^{\bullet}_{xs}u\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}
$$

Again the second-to-last term may be easier to work with computationally. This last item follows from the lognormal assumptions regarding w_t as in Appendix C.

Similarly for the variance term:

$$
\mathrm{Var}(T_s|u) = \mathrm{Var}_{w_t}\left(E(T_s|u,w_t)\right) + E_{w_t}\left(\mathrm{Var}(T_s|u,w_t)\right)
$$

Taking the terms one at a time, using the lognormal relationships in Appendix C:

$$
\begin{aligned}
\mathrm{Var}_{w_t}\left(E(T_s|u,w_t)\right) &= \mathrm{Var}_{w_t}\left(\left(u\prod_{r=1}^{s} w_r\right)\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)E(y^s)\right)\\
&= \mathrm{Var}_{w_t}\left(\left(\prod_{r=1}^{s} w_r\right)\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)uE(y^s)\right)\\
&= \left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)uE(y^s)\right)^2 \mathrm{Var}\left(\prod_{r=1}^{s} w_r\right)\\
&= \left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)uE(y^s)\right)^2\left(\exp(s\tau^2)-1\right)
\end{aligned}
$$

As for the second term we have:

$$
\begin{aligned}
\mathrm{E}_{w_t}\left(\mathrm{Var}(T_s|u,w_t)\right) &= \mathrm{E}_{w_t}\left(\begin{array}{l}\left(u\prod_{t=1}^{s} w_t\right)^2\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
+\sum_x\left(b^*_{xs}u\prod_{t=1}^{s} w_s\right)^2\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^*_{xt})^2\right)\right)\end{array}\right)\\
&= \mathrm{E}\left(\left(\prod_{t=1}^{s} w_t\right)^2\right)u^2\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
&\quad+\sum_x b^{*2}_{xs}u^2\,\mathrm{E}\left(\left(\prod_{t=1}^{s} w_t\right)^2\right)\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)\\
&= \exp(s\tau^2)u^2\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
&\quad+\sum_x b^{*2}_{xs}u^2\exp(s\tau^2)\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)
\end{aligned}
$$

Adding these two terms together we obtain:

$$
\begin{aligned}
\mathrm{Var}(T_s|u) &= \left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)u\,\mathrm{E}(y^s)\right)^2\left(\exp(s\tau^2)-1\right)\\
&\quad+\exp(s\tau^2)u^2\left(\sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
&\quad+\sum_x b^{*2}_{xs}u^2\exp(s\tau^2)\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\right)
\end{aligned}
$$

Finally we eliminate u from the formulae. First

$$
\begin{aligned}
\mathrm{E}(T_s) &= \mathrm{E}_u\left(\mathrm{E}(T_s|u)\right)\\
&= \mathrm{E}_u\left(\sum_x b^*_{xs}u\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\mathrm{E}(y^s)\right)\\
&= \sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\mathrm{E}(y^s)\mathrm{E}(u)\\
&= \sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\mathrm{E}(y^s)\\
&= \sum_x b^*_{xs}\left(\prod_{t=0}^{s-1}(1-q^*_{xt})\right)\prod_{i=0}^{s-1}\frac{(n+2)(z+i+1)}{(z+1)(n+i+2)}
\end{aligned}
$$

As for the variance formula we have:

$$\mathrm{Var}(T_s) = \mathrm{Var}_u\big(\mathrm{E}(T_s|u)\big) + \mathrm{E}_u\big(\mathrm{Var}(T_s|u)\big)$$

Again we consider the two portions separately.

$$\begin{aligned}
\mathrm{Var}_u\big(\mathrm{E}(T_s|u)\big) &= \mathrm{Var}_u\left(\sum_x b^{\bullet}_{xs} u\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\mathrm{E}(y^s)\right)\\
&= \mathrm{Var}_u\left(u\sum_x b^{\bullet}_{xs} u\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\mathrm{E}(y^s)\right)\\
&= \left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\mathrm{E}(y^s)\right)^2 \mathrm{Var}(u)\\
&= \left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\mathrm{E}(y^s)\right)^2 \left(\exp(\sigma^2)-1\right)
\end{aligned}$$

As for the second term we have:

$$\mathrm{E}_u\big(\mathrm{Var}(T|u)\big) = \mathrm{E}_u\left(\begin{array}{l}
\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)u\mathrm{E}(y^s)\right)^2\left(\exp(s\tau^2)-1\right)\\
+\exp(s\tau^2)u^2\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
+\sum_x b^{\bullet 2}_{xs}u^2\exp(s\tau^2)\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)
\end{array}\right)$$

$$\begin{aligned}
&= \mathrm{E}(u^2)\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\mathrm{E}(y^s)\right)^2\left(\exp(s\tau^2)-1\right)\\
&\quad+\mathrm{E}(u^2)\exp(s\tau^2)\left(\sum_x b^{\bullet}_{xs}\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right)\\
&\quad+\mathrm{E}(u^2)\sum_x b^{\bullet 2}_{xs}\exp(s\tau^2)\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{t=0}^{s-1}(1-q^{\bullet}_{xt})\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp(\sigma^2)\left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\mathrm{E}(y^s)\right)^2\left(\exp(s\tau^2)-1\right) \\
&\quad + \exp(\sigma^2)\exp(s\tau^2)\left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)^2\left(\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&\quad + \exp(\sigma^2)\sum_x b^{*2}_{xs}\exp(s\tau^2)\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right) \\
&= \exp(\sigma^2)\left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)^2\left(\exp(s\tau^2)\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&\quad + \exp(\sigma^2)\sum_x b^{*2}_{xs}\exp(s\tau^2)\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)
\end{aligned}
$$

Finally putting the two terms together we obtain:

$$
\begin{aligned}
\mathrm{Var}(T_s) &= \left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\mathrm{E}(y^s)\right)^2\left(\exp(\sigma^2)-1\right) \\
&\quad + \exp(\sigma^2)\left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)^2\left(\exp(s\tau^2)\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&\quad + \exp(\sigma^2)\sum_x b^{*2}_{xs}\exp(s\tau^2)\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right) \\
&= \left(\sum_x b^*_{xs}\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)^2\left(\exp(\sigma^2)\exp(s\tau^2)\mathrm{E}(y^{2s})-\mathrm{E}(y^s)^2\right) \\
&\quad + \exp(\sigma^2)\sum_x b^{*2}_{xs}\exp(s\tau^2)\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\left(\mathrm{E}(y^s)-\mathrm{E}(y^{2s})\left(\prod_{l=0}^{s-1}(1-q^*_{xl})\right)\right)
\end{aligned}
$$

APPENDIX E

We will calculate the mean and variance of T in stages. We first consider IBNR claims. As with Heckman & Meyers, Algorithm 3.3 gives:

$$E(N) = E_\chi\big(E(N|\chi)\big) = E_\chi(\lambda\chi) = \lambda\, E_\chi(\chi) = \lambda$$

and we also have:

$$\begin{aligned}
\mathrm{Var}(N) &= E_\chi\big(\mathrm{Var}(N|\chi)\big) + \mathrm{Var}_\chi\big(E(N|\chi)\big) \\
&= E_\chi(\lambda\chi) + \mathrm{Var}_\chi(\lambda\chi) \\
&= \lambda\, E_\chi(\chi) + \lambda^2\, \mathrm{Var}_\chi(\chi) \\
&= \lambda + c\lambda^2
\end{aligned}$$

To ease the notation in what follows we will assume that the claims $X_{N_R+1}, X_{N_R+2}, \ldots, X_{N_R+N_B+N}$ are independently selected from a distribution with mean

$$E(X) = E(T_R|\text{Certainty})/N_R$$

and variance

$$\mathrm{Var}(X) = \mathrm{Var}(T_R|\text{Certainty})/N_R$$

This last relationship follows since

$$\mathrm{Var}(T_R|\text{Certainty}) = \mathrm{Var}\left(\sum_{i=1}^{N_R} X_i|\text{Certainty}\right) = \sum_{i=1}^{N_R} \mathrm{Var}(X_i|\text{Certainty})$$

Now fixing β, ζ, and γ we have:

$$\begin{aligned}
E(T|\beta,\gamma,\zeta) &= E_N\Big(E\big(\beta\big(X_1 + X_2 + \ldots + X_{N_R} + \zeta(X_{N_R+1} + \ldots + X_{N_R+N_B}) + \gamma(X_{N_R+N_B+1} + \ldots + X_{N_R+N_B+N})\big)\big)\Big) \\
&= E_N\left(\beta\left(\sum_{i=1}^{N_R} E(X_i) + \zeta\sum_{i=1}^{N_B} E(X_{N_R+i}) + \gamma\sum_{i=1}^{N} E(X_{N_R+N_B+i})\right)\right) \\
&= E_N\big(\beta(N_R\, E(X) + \zeta N_B\, E(X) + \gamma N\, E(X))\big) \\
&= \beta(N_R + \zeta N_B + \gamma\lambda)E(X)
\end{aligned}$$

For the variance in this case we have:

$$\begin{aligned}
\operatorname{Var}(T|\beta,\gamma,\zeta) &= \mathrm{E}_N\left(\operatorname{Var}(T|\beta,\gamma,\zeta,N)\right) + \operatorname{Var}_N\left(\mathrm{E}(T|\beta,\gamma,\zeta,N)\right) \\
&= \mathrm{E}_N\left(\operatorname{Var}\left(\beta\left(\sum_{i=1}^{N_R} X_i + \zeta\sum_{i=1}^{N_B} X_{N_R+i} + \gamma\sum_{i=1}^{N} X_{N_R+N_B+i}\right)\right)\right) \\
&\quad + \operatorname{Var}_N\left(\mathrm{E}\left(\beta\left(\sum_{i=1}^{N_R} X_i + \zeta\sum_{i=1}^{N_B} X_{N_R+i} + \gamma\sum_{i=1}^{N} X_{N_R+N_B+i}\right)\right)\right) \\
&= \mathrm{E}_N\left(\beta^2\left(\operatorname{Var}\left(\sum_{i=1}^{N_R} X_i\right) + \operatorname{Var}\left(\zeta\sum_{i=1}^{N_B} X_{N_R+i}\right) + \operatorname{Var}\left(\gamma\sum_{i=1}^{N} X_{N_R+N_B+i}\right)\right)\right) \\
&\quad + \operatorname{Var}_N\left(\beta\left(\mathrm{E}\left(\sum_{i=1}^{N_R} X_i\right) + \mathrm{E}\left(\zeta\sum_{i=1}^{N_B} X_{N_R+i}\right) + \mathrm{E}\left(\gamma\sum_{i=1}^{N} X_{N_R+N_B+i}\right)\right)\right) \\
&= \mathrm{E}_N\left(\beta^2\left(\operatorname{Var}(T_R|\text{Certainty}) + N_B\zeta^2\operatorname{Var}(X) + N\gamma^2\operatorname{Var}(X)\right)\right) \\
&\quad + \operatorname{Var}_N\left(\beta\left(N_R\,\mathrm{E}(X) + N_B\zeta\,\mathrm{E}(X) + N\gamma\,\mathrm{E}(X)\right)\right) \\
&= \beta^2\left(N_R\operatorname{Var}(X) + N_B\zeta^2\operatorname{Var}(X) + \mathrm{E}_N(N)\gamma^2\operatorname{Var}(X)\right) + \beta^2\gamma^2\,\mathrm{E}(X)^2\operatorname{Var}_N(N) \\
&= \beta^2\left(N_R + N_B\zeta^2 + \lambda\gamma^2\right)\operatorname{Var}(X) + \beta^2\gamma^2\,\mathrm{E}(X)^2\left(\lambda + c\lambda^2\right)
\end{aligned}$$

Similarly we have, for a fixed values of β and ζ we have:

$$\begin{aligned}
\mathrm{E}(T|\beta,\zeta) &= \mathrm{E}_\gamma\left(\mathrm{E}(T|\beta,\gamma,\zeta)\right) \\
&= \mathrm{E}_\gamma\left(\beta(N_R + \zeta N_B + \gamma\lambda)\mathrm{E}(X)\right) \\
&= \beta(N_R + \zeta N_B + a\lambda)\mathrm{E}(X)
\end{aligned}$$

For the variance in this case we have:

$$\begin{aligned}
\operatorname{Var}(T|\beta,\zeta) &= \mathrm{E}_\gamma\left(\operatorname{Var}(T|\beta,\gamma,\zeta)\right) + \operatorname{Var}_\gamma\left(\mathrm{E}(T|\beta,\gamma,\zeta)\right) \\
&= \mathrm{E}_\gamma\left(\beta^2\left(N_R + N_B\zeta^2 + \lambda\gamma^2\right)\operatorname{Var}(X) + \beta^2\gamma^2\,\mathrm{E}(X)^2\left(\lambda + c\lambda^2\right)\right) + \operatorname{Var}_\gamma\left(\beta(N_R + \zeta N_B + \gamma\lambda)\mathrm{E}(X)\right) \\
&= \mathrm{E}_\gamma\left(\beta^2\left(N_R + N_B\zeta^2\right)\operatorname{Var}(X) + \gamma^2\beta^2\left(\mathrm{E}(X)^2\left(\lambda + c\lambda^2\right) + \lambda\operatorname{Var}(X)\right)\right) + \operatorname{Var}_\gamma\left(\beta(N_R + \zeta N_B + \gamma\lambda)\mathrm{E}(X)\right) \\
&= \beta^2\left(N_R + N_B\zeta^2\right)\operatorname{Var}(X) + \mathrm{E}_\gamma\left(\gamma^2\right)\beta^2\left(\mathrm{E}(X)^2\left(\lambda + c\lambda^2\right) + \lambda\operatorname{Var}(X)\right) + \beta^2\lambda^2\,\mathrm{E}(X)^2\operatorname{Var}_\gamma(\gamma) \\
&= \beta^2\left(N_R + N_B\zeta^2\right)\operatorname{Var}(X) + \left(d + a^2\right)\beta^2\left(\mathrm{E}(X)^2\left(\lambda + c\lambda^2\right) + \lambda\operatorname{Var}(X)\right) + \beta^2\lambda^2\,\mathrm{E}(X)^2 d \\
&= \beta^2\left(\left(N_R + N_B\zeta^2 + \left(d + a^2\right)\lambda\right)\operatorname{Var}(X) + \left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d\right)\mathrm{E}(X)^2\right)
\end{aligned}$$

Now for a fixed β we have:

$$\begin{aligned}\mathrm{E}(T|\beta) &= \mathrm{E}_{\zeta}\big(\mathrm{E}(T|\beta,\zeta)\big)\\ &= \mathrm{E}_{\zeta}(\beta(N_R + \zeta N_B + \gamma\lambda)\mathrm{E}(X))\\ &= \beta(N_R + rN_B + a\lambda)\mathrm{E}(X)\end{aligned}$$

The variance calculation also follows:

$$\begin{aligned}\mathrm{Var}(T|\beta) &= \mathrm{E}_{\zeta}\big(\mathrm{Var}(T|\beta,\zeta)\big) + \mathrm{Var}_{\zeta}\big(\mathrm{E}(T|\beta,\zeta)\big)\\ &= \mathrm{E}_{\zeta}\Big(\beta^2\big(\big(N_R + N_B\zeta^2 + (d+a^2)\lambda\big)\mathrm{Var}(X) + \big((d+a^2)(\lambda + c\lambda^2) + \lambda^2 d\big)\mathrm{E}(X)^2\big)\Big) + \mathrm{Var}_{\zeta}(\beta(N_R + \zeta N_B + a\lambda)\mathrm{E}(X\\ &= \beta^2\big(\big(N_R + N_B\mathrm{E}_{\zeta}(\zeta^2) + (d+a^2)\lambda\big)\mathrm{Var}(X) + \big((d+a^2)(\lambda + c\lambda^2) + \lambda^2 d\big)\mathrm{E}(X)^2\big) + \mathrm{E}(X)^2 N_B^2\beta^2\,\mathrm{Var}_{\zeta}(\zeta)\\ &= \beta^2\big(\big(N_R + N_B(z + r^2) + (d+a^2)\lambda\big)\mathrm{Var}(X) + \big((d+a^2)(\lambda + c\lambda^2) + \lambda^2 d\big)\mathrm{E}(X)^2\big) + z\beta^2 N_B^2\mathrm{E}(X)^2\\ &= \beta^2\big(\big(N_R + N_B(z + r^2) + (d+a^2)\lambda\big)\mathrm{Var}(X) + \big((d+a^2)(\lambda + c\lambda^2) + \lambda^2 d + zN_B^2\big)\mathrm{E}(X)^2\big)\end{aligned}$$

Thus, combining these results, we have:

$$\begin{aligned}\mathrm{E}(T) &= \mathrm{E}_{\beta}\big(\mathrm{E}(T|\beta)\big)\\ &= \mathrm{E}_{\beta}\big(\beta(N_R + rN_B + a\lambda)\mathrm{E}(X)\big)\\ &= \mathrm{E}_{\beta}(\beta)(N_R + rN_B + a\lambda)\mathrm{E}(X)\\ &= (N_R + rN_B + a\lambda)\mathrm{E}(X)\end{aligned}$$

Finally we have:

$$
\begin{aligned}
\mathrm{Var}(T) &= \mathrm{E}_\beta\left(\mathrm{Var}(T|\beta)\right) + \mathrm{Var}_\beta\left(\mathrm{E}(T|\beta)\right)\\
&= \mathrm{E}_\beta\left(\beta^2\left(\left(N_R + N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda\right)\mathrm{Var}(X) + \left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right)\mathrm{E}(X)^2\right)\right)\\
&\quad + \mathrm{Var}_\beta(\beta(N_R + rN_B + a\lambda)\mathrm{E}(X))\\
&= \mathrm{E}_\beta(\beta^2)\left(\left(N_R + N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda\right)\mathrm{Var}(X) + \left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right)\mathrm{E}(X)^2\right)\\
&\quad + \mathrm{Var}_\beta(\beta)(N_R + rN_B + a\lambda)^2\,\mathrm{E}(X)^2\\
&= (b+1)\left(\left(N_R + N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda\right)\mathrm{Var}(X) + \left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right)\mathrm{E}(X)^2\right)\\
&\quad + b(N_R + rN_B + a\lambda)^2\,\mathrm{E}(X)^2\\
&= (b+1)\left(N_R + N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda\right)\mathrm{Var}(X)\\
&\quad + \left((b+1)\left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right) + b\left(N_R + rN_B + a\lambda\right)^2\right)\mathrm{E}(X)^2
\end{aligned}
$$

Thus, in terms of estimates for case reserved claims without parameter uncertainty:

$$
\begin{aligned}
\mathrm{Var}(T) &= (b+1)\left(N_R + N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda\right)\frac{\mathrm{Var}(T_R|\text{No uncertainty})}{N_R}\\
&\quad + \left((b+1)\left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right) + b\left(N_R + rN_B + a\lambda\right)^2\right)\left(\frac{\mathrm{E}(T_R|\text{No uncertainty})}{N_R}\right)^2\\
&= (b+1)\left(1 + \frac{N_B\left(z + r^2\right) + \left(d + a^2\right)\lambda}{N_R}\right)\mathrm{Var}(T_R|\text{No uncertainty})\\
&\quad + \left(\frac{(b+1)\left(\left(d + a^2\right)\left(\lambda + c\lambda^2\right) + \lambda^2 d + zN_B^2\right)}{{N_R}^2} + b\left(1 + \frac{rN_B + a\lambda}{N_R}\right)^2\right)\mathrm{E}(T_R|\text{No uncertainty})^2
\end{aligned}
$$