Parameterizing Interest Rate Models

Kevin C. Ahlgrim, ASA,
Stephen P. D'Arcy, FCAS, MAAA, and
Richard W. Gorvett, FCAS, MAAA

Biographical Information

Kevin C. Ahlgrim is a graduate student in the Department of Finance at the University of Illinois at Urbana-Champaign. His research interests include fixed income derivatives and financial risk management of life insurers. Kevin has taught a course on financial markets and has assisted in developing a course for delivery over the internet.

Kevin received his B.S. in Actuarial Science from the University of Illinois. He is an Associate of the Society of Actuaries and a member of the American Academy of Actuaries. He has worked as a consulting actuary at Aon in Chicago and as a pricing actuary at CIGNA in Hartford.

Stephen P. D'Arcy is a Professor of Finance at the University of Illinois at Urbana-Champaign. He is a Fellow of the Casualty Actuarial Society, a member of the American Academy of Actuaries, and is currently President of the American Risk and Insurance Association. He received his B.A. in Applied Mathematics from Harvard College and his Ph.D. in Finance from the University of Illinois.

Prior to his academic career, Steve was the Actuary at CUMIS Insurance Society. His research interests include dynamic financial analysis, financial pricing models for property-liability insurers, catastrophe insurance futures, pension funding and insurance regulation.

Richard W. Gorvett is currently an Assistant Professor at The College of Insurance in New York City. In August, 1999, he will become an Assistant Professor in the Actuarial Science Program at the University of Illinois at Urbana-Champaign. He is a Fellow of the Casualty Actuarial Society, a Member of the American Academy of Actuaries, and an Associate in Risk Management. He received a B.S. in Mathematics from the University of Illinois at Chicago, an MBA in Finance and Econometrics from the University of Chicago, and a Ph.D. in Finance from the University of Illinois at Urbana-Champaign, where his dissertation was entitled "Dynamic Financial Analysis of Property-Liability Insurance Companies." He has co-authored a paper entitled "A Comparison of Property-Liability Insurance Financial Pricing Models" for the *Proceedings of the Casualty Actuarial Society* and two papers on Dynamic Financial Analysis for Call Paper Programs. He is continuing research into the application of financial theory to insurance. Rick has taught courses in property-liability insurance, corporate risk management, investments, corporate finance, and financial risk management. Prior to entering academia, he worked as an actuary and consultant with several firms in the Chicago area.

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ABSTRACT

Actuaries are now being called upon to incorporate interest rate models in a variety of applications, including dynamic financial analysis (DFA), ratemaking, and valuation. Although there are many articles and texts on interest rate models, most of these presume an understanding of financial terminology and mathematical techniques that makes it difficult to begin to learn this material. This paper provides an overview, at a level aimed at actuaries, of some common interest rate models used by financial economists. The purpose of this paper is to explain the basics of interest rate modeling by demonstrating the different models both graphically and empirically, and by showing how changing the various model parameters affects the results. Several of the more popular interest rate models are simulated, and the results are compared with historical interest rate movements.

Introduction

The volatility of interest rates has become an important feature of the modern financial environment. Changes in interest rates can impact the way in which companies compete and can even impact the ultimate survival of the firm. Financial intermediaries, such as banks and insurance companies, may be especially exposed to interest rate fluctuations because both their assets and liabilities are correlated with interest rate movements. Mismatches of interest rate sensitivities (or durations) of assets and liabilities can have a magnified effect on surplus. A popular example of the potential vulnerability of financial intermediaries is based on the experience of the savings and loan industry in the 1980s. Rapidly increasing interest rates quickly turned profits into billions of dollars of losses and numerous insolvencies. The assets of S&Ls were primarily long-term, fixed-rate mortgages; their liabilities were mostly short-term demand deposits. When the interest rates paid on those short-term deposits increased, the normal differential between the interest rate they were receiving on their assets and that which they were paying on their liabilities disappeared or even reversed. Given such potential effects of interest rate volatility, it has become important to develop models of interest rate changes so that risk management tools can be used to insulate the firm from financial disaster.

Traditionally, insurance companies have not incorporated interest rate models into the product development and pricing processes. Pricing actuaries typically used "conservative," fixed interest rates when developing products. By crediting policyholders with a low interest rate, or ignoring investment income when setting property-liability insurance rates, insurers had some assurance that they could ultimately earn the assumed rate of return used in pricing. Any excess interest earnings served as a cushion to protect surplus against adverse experience, as well as being a source of insurer profits.

The assumption of fixed interest rates was an acceptable practice during periods when interest rates were low and relatively stable. In fact, such an environment existed in the U.S. into the 1970s. The fixed interest rate assumption used by most insurers seemed innocuous.

However, in 1979, the U.S. Federal Reserve altered its policy from one that targeted interest rates to a policy that now targets inflation via the money supply. As a result, interest rates became significantly more volatile. During the transition of the early 1980s, interest rates spiked upward to unprecedented levels. It was clear that the interest rate environment had shifted dramatically.

The change in the Fed policy affected insurers in several ways. First, the underlying value of insurance products changed due to the change in interest rate volatility. Insurance products typically include embedded options that give specific rights to policyholder and, in some cases, to the insurer. An example of these options is the right to renew the policy on terms set at the beginning of the coverage period. The value of these embedded options is highly sensitive to the underlying interest rate assumption and, more importantly, to the volatility of future interest rates. Insurers that used a fixed interest rate assumption ignored the interest-sensitive option values in their policies.

It has long been recognized that life insurers are exposed to interest rate risk. The life insurance industry experienced heavy disintermediation when interest rates increased in the 1980s. Before the rapid increase in rates, life insurers believed that high interest rate scenarios were in their favor because they implied additional income. However, they failed to understand the risks in their liabilities. The policy loan feature of ordinary life insurance policies capped the interest rate that could be charged to the policyholder. Once interest rates exceeded that cap, policyholders were able to borrow at the policy loan rate, and then turn around and invest the proceeds at higher yields. The result was an outflow of cash from the life insurance industry that caused many insurers to sell bonds at depressed prices due to the high yield environment.

It is also becoming evident that property-liability insurers are exposed to interest rate risk on both sides of the balance sheet. Fixed income assets of property-liability insurers have the same exposure to interest rate risk that life insurer assets have, with market values declining as interest rates increase. On the other hand, the liabilities of property-liability insurers are not fixed values. Since inflation is correlated with interest rates, and future claim payments on loss reserves will increase with inflation, the statutory values of liabilities will tend to increase as interest rates increase. Thus, an increase in interest rates leads to a decline in asset value and an increase in the value of liabilities, creating a magnified effect on the surplus of property-liability insurers.

In most DFA models for property-liability insurers, interest rates are the driving factor in the model, affecting investment income, loss severity, asset returns, and target underwriting profit margins (see, for example, D'Arcy, Gorvett, et al, 1997 and 1998). DFA models are being used for analyzing insurer solvency, in valuing insurers in mergers and acquisitions, and as a business planning tool. The results from DFA applications are heavily dependent upon the particular interest rate model used, as well as the parameters chosen for the models.

These examples help illustrate how critical the underlying interest rate assumption is to the evaluation of insurance company assets and liabilities. Insurers must incorporate the new interest rate paradigm into their pricing and asset/liability management (ALM) processes by using assumptions that reflect the stochastic nature of interest rates. Fortunately, within the field of finance, extensive effort has been devoted to developing stochastic interest rate models.

Financial researchers have long been concerned with the dynamics of interest rates. Models have been formulated using two approaches: (1) a general equilibrium framework, where interest rate changes are derived from economic agents who maximize expected utility; and (2) the no-arbitrage approach, which assumes that financial markets have no arbitrage opportunities. Examples of the general equilibrium approach include the models of Vasicek (1977), Dothan (1978), Cox, Ingersoll, and Ross (CIR) (1985), Brennan and Schwartz (1979) and Longstaff and Schwartz (1992). Two models based on arbitrage arguments are Ho and Lee (1986) and Heath, Jarrow, and Morton (HJM) (1992).

The choice of interest rate model is not a trivial decision. The form of the model used in the pricing or ALM process depends on the characteristics of the insurance products being reviewed. Choosing a model is always a tradeoff between perfectly describing the actual interest rate process and having a tractable model that can be used to value a variety of financial instruments. One consideration in selecting an interest rate assumption is to compare modeled prices of financial assets with market prices, if a market exists. When using a model for a specific application, one should compare market prices of assets that are similar in terms of interest rate sensitivity. Another consideration is choosing which interest rate to model. The spot rate is today's interest rate for a specific maturity. A forward rate is an interest rate that is applicable to future periods[1]. After deciding on which interest rates to model, one must determine how many parameters to include. Using more parameters obviously increases the complexity of a model, so one must consider whether the added complexity yields sufficient benefits. Finally, choosing the values of the parameters in an interest rate model can be the most important, as well as the most challenging, factor in implementation.

This paper aims to illustrate how various models operate and to show how well the models fit historical data. Through descriptions and illustrations of the models, it is hoped that this paper will increase the comfort level of casualty actuaries with these new tools and encourage them to begin to apply them in pricing and asset/liability management functions.

The estimates used in this presentation are based on previous work in the area. Chan, Karolyi, Longstaff, and Sanders (1992) (hereafter CKLS), empirically estimate and compare several popular interest rate processes used in the literature. Their most important finding is that the interest rate volatility is sensitive to the level of the interest rate. Also, Amin and Morton (1994) estimate parameters for six forms of the HJM model. They find that two-parameter models fit market price data better, but that the resulting estimates are less stable.

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[1] Example: The expected forward rate from year one to year two can be implied from the current spot rates based on the following formula: $(1+i_1)(1+f) = (1+i_2)^2$, where f is the forward rate and i_t is the t-year spot rate. If the one-year spot rate is 3% and the two-year spot rate is 4% (expressed as an annual rate), this implies that the forward rate is $1.04^2/1.03 - 1 = 5.01\%$

Introduction to Notation

The various interest rate models will be presented here in the mathematics of continuous time. The finance literature is based on continuous time because functions of continuous time processes (e.g., options that are dependent on the interest rate) have desirable features including continuity and differentiability. This allows many of these functions to have closed form solutions without the need for numerical procedures. The mathematics behind discrete time processes is not always as elegant. Later in this paper, we discuss how to translate the continuous time processes into discrete time for use in insurance applications (see the "Simulations" section).

The interest rate models that are presented in this paper are either models of the short-term rate or the forward rate. The short-term rate (also called the short rate or instantaneous rate) is the (annualized) rate of return expected over the next instant. For example, the return (r) over the next instant (dt) on an initial wealth level (W) earns

$$dW = rWdt$$

The time t prices of bonds (P) that pay \$1 at time T are determined by expectations of investors regarding the evolution of the short rate from time t until maturity.

$$P(t,T) = E\left[\exp\left(-\int_t^T r(u)du\right)\right]$$

This formula shows that the price of a bond is simply the discounted value *over every instant* from time t until maturity at T. Instead of modeling interest rates explicitly, many financial economists (e.g., Vasicek (1977), Dothan (1978), Cox-Ingersoll-Ross (1985) and Brennan-Schwartz (1979)) model the changes in the short-term rate using the following generic, stochastic form:

$$dr_t = a(r_t,t)dt + \sigma(r_t,t)dB_t$$

To understand the changes in interest rates, consider individually the two terms on the right-hand side of the equation. The first term represents the predictable, deterministic portion of changes in the interest rate Thus, $a(r_t,t)$ is the expected change in the short-term rate and is called the instantaneous drift. The second term represents the uncertainty in interest rate changes; B_t represents a standard Brownian motion so that dB_t is essentially a random draw from the standard normal distribution, which is then scaled by the magnitude $\sigma(r_t,t)$. The second term in the stochastic equation thus denotes the volatility of interest rate changes. Most interest rate models begin with this form but differ in their specifications of the terms $a(r_t,t)$ and $\sigma(r_t,t)$.

Instead of modeling the short-term rate, other authors (Ho-Lee (1986) and Heath-Jarrow-Morton (1992)) use a process for forward rates. The instantaneous forward rate (f) represents the interest rate available now for an investment to be made at a future time. It is implicitly defined by a difference in bond prices, which reflects the expected instantaneous interest rate T-t periods in the future:

$$\frac{P(t,T+dt)}{P(t,T)} = \exp(-f(t,T)dt)$$

By rearranging and integrating, we can obtain the bond price in terms of the existing instantaneous forward rates:

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

One can interpret this formula in the same manner as in footnote 1. We are "constructing" a spot rate which applies from time t to T by including consecutive instantaneous forward rates. Ho-Lee and HJM model the entire term structure by using a process for forward rates of all maturities:

$$df(t,T) = a(t,T,f(t,T))dt + \sigma(t,T,f(t,T))dB_t$$

Here, the terms $a(t,T,f(t,T))$ and $\sigma(t,T,f(t,T))$ are the drift and the volatility, respectively, of the forward rate and are analogous to the short-rate drift and volatility discussed above.

Having defined the notation and general stochastic process used to model interest rates, we turn to describing desirable features of an interest rate model and then present alternative models that have been used in the literature.

Characteristics of Interest Rate Movements

Before presenting the interest rate models, we discuss some general features of interest rate movements. Our attempt is to provide some intuitive form for an interest rate model.

1. The volatility of yields at different maturities varies. In particular, long-term rates do not vary as much as shorter term rates.
2. Interest rates are mean-reverting. Interest rate increases tend to be followed by rate decreases; conversely, when rates drop, they tend to be followed by rate increases.
3. Rates of different maturities are positively correlated. Rates for maturities that are closer together have higher correlations than maturities that are farther apart.
4. Interest rates should not be allowed to become negative.
5. Based on the results reported in CKLS, the volatility of interest rates should be proportional to the level of the rate.

No known model captures all of the features mentioned above. Therefore, one of the first steps in choosing an interest rate model is to understand which of these characteristics are important based on the use of the model. One should resist the urge to rank models based on the number of listed conditions that are satisfied. Instead, it is imperative that the modeler understand the limitations of alternative models and their impact on the desired application.

Equilibrium vs. Arbitrage

The first distinction of interest rate models is between those that are derived from equilibrium models of the economy, and those that are based on arbitrage arguments. Equilibrium interest rate models are based on the assumption that bond prices, and yields, are determined by the market's assessment of the evolution of the short-term interest rate. In the models discussed here, the short rate is assumed to follow a diffusion (a continuous time stochastic) process. The general form for these models is described in terms of changes in the short rate, as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t^{\gamma} dB_t$$

r_t = current level of the instantaneous rate
κ = speed of the mean reversion
θ = rate to which the short rate reverts
σ = volatility of the short rate
γ = proportional conditional volatility exponent
B_t = standard Brownian motion

The first important feature of this type of model is mean reversion of the short-term rate. This feature is appealing since it presumes that when rates become very high or very low, they will tend to revert to "normal" levels. The speed of reversion is determined by the parameter κ. This parameter ultimately affects the shape of the yield curve. If κ is high, the yield curve quickly trends toward the long-run yield rate θ. If κ is low, the yield curve slowly trends toward θ. (See Figure 1 versus Figure 2.)

The difference between the Vasicek, CIR, and Dothan models (see below) primarily revolves around the parameter γ (the exponent). Vasicek assumes it to be 0, CIR assumes it to be 0.5, and Dothan assumes it to be 1.0. The basic question distinguishing the models is whether the conditional volatility of changes in interest rates is proportional to the level of the rate. This subsequently determines the parameter γ. CKLS (1992) have provided empirical estimates of the exponent. Their main finding is that the conditional volatility of interest rates is significantly related to the level of the short rate. In fact, their estimate of γ is around 1.5. Although their work has been the subject of some criticism due to their estimation period, it nonetheless illustrates the strong relationship between the level of interest rates and volatility. Throughout most periods, γ has been estimated between 0.5 and 1.0 (Phoa 1997). The exponent of individual models will be discussed more fully when we look at the individual models in the next section.

Equilibrium models are criticized because they do not fit the existing term structure. Although parameters can be chosen to minimize errors from today's yield curve, the fit will not be perfect. Whereas this is a valid criticism for models being used to value financial assets for trading purposes, it may not be a problem when the models are being used for long-term financial modeling, such as in DFA.

Arbitrage-free models take the entire yield curve as given and model the dynamics of the entire curve. The only constraint of such an approach is that yield curve movements do not produce any arbitrage opportunities. Heath, Jarrow, Morton (1992, hereafter HJM) generalize the arbitrage-free framework by modeling the forward rates derived from the current yield curve. The continuous time model of Ho and Lee (1986, hereafter Ho-Lee) is the simplest case of the HJM framework.

In the next section we look at several of the popular interest rate models used today.

The Vasicek Model

Vasicek formulates the interest rate model in terms of changes in the short-term (or instantaneous) interest rate:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dB_t$$

The price of a bond, $P(t,T)$, is then dependent on the expected path of future interest rates. Vasicek shows that bond prices have the following form:

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)}$$

where $A(t,T)$ and $B(t,T)$ are functions of only κ, θ, and σ and independent of the current spot rate, $r(t)$. Bond yields, $R(t,T)$ are then related to prices by:

$$P(t,T) = e^{-R(t,T)(T-t)}$$

These two equations determine the entire term structure of interest rates. Since bond prices and yields have closed-form solutions, the Vasicek model is very easy to implement in practice, with no need for complicated simulation techniques. Also, there are closed form solutions for certain interest rate-dependent claims such as options.

The Vasicek model assumes that (absolute) changes in the interest rate are normally distributed, due to the inclusion of the Wiener process. From the normality assumption it follows that bond prices are lognormally distributed. One weakness of the model is that normality in interest rate changes results in a (small) positive probability of negative interest rates [2].

Another feature of the Vasicek model is that all bond prices are related to the same factor, the instantaneous interest rate. Consequently, all bond price movements are derived from movements in the same factor. This implies that all bond prices are perfectly correlated. Thus, another shortcoming of the Vasicek model is that the dynamics of the term structure are severely limited.

Note that, from the general case above, the Vasicek model assumes $\gamma=0$. The conditional volatility of interest rate changes is constant and equal to σ. The results of CKLS (1992) illustrate that the assumption of constant volatility is questionable. The link between interest rate volatility and the level of the rate implies that the Vasicek model may provide unrealistic interest rate forecasts. When rates are low, volatility is overstated, and when interest rates are high, volatility is understated.

[2] It may be argued that this is not necessarily an implausible scenario. There have been some periods in the U.S. that real interest rates have been negative.

In summary, the Vasicek model is very tractable and provides convenient closed-form solutions for many interest rate-dependent instruments. However, the model has some serious drawbacks including restricted dynamics of the term structure and constant conditional volatility.

The Vasicek model is illustrated in Figures 1 through 3. Each exhibit illustrates the yield curves based on three different realizations of the modeled instantaneous rate. Figure 1 is based on parameter estimates from CKLS. However, it should be pointed out that, in their tests, they reject the model of Vasicek (1977) because of its homoskedastic feature. Note that when the instantaneous yield is high, the curve is inverted, and when the short rate is low, the curve is normal. In all cases, the long-term yields tend toward the parameter θ, the long-run average. In Figure 2, the mean reversion parameter (κ) has been increased. Note that longer yields revert back to the long-run average more quickly. In Figure 3, the long-run average is decreased and all yield curves tend to the lower long-run curve.

The Cox, Ingersoll, Ross Model

Another model of interest rates was formulated by Cox, Ingersoll, and Ross (1985) (CIR). The model is as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t$$

The CIR model is also known as the "square-root" process because the volatility is related to the square root of the current level of the interest rate. Unlike the Vasicek model, the CIR model relates the conditional volatility to the level of the short rate. A second improvement of the CIR model over the Vasicek model is that interest rates cannot be negative [3]. Although CKLS find that interest rate volatility is more sensitive to the level of interest rates than proposed by the CIR

[3] Note that negative interest rates are ruled out in the continuous time case. However, it is possible that interest rates become negative if a discrete process is used in simulations.

specification, other researchers defend the model by commenting on the estimation approaches employed by CKLS (see Eom, 1994).

The CIR model can also be used to determine bond prices analytically. CIR show that bond prices are determined by the following (the "hats" simply indicate that the equations for $A(t,T)$ and $B(t,T)$ are different under CIR vs. the Vasicek model).

$$P(t,T) = \hat{A}(t,T)e^{-r(t)B(t,T)}$$

Thus, the equation above can be used to derive the yield curve for the CIR model. Because the driving factor of bond prices and yields is still the short-term rate, the CIR specification again assumes perfect correlation among all bonds and therefore restricts term structure dynamics.

The resulting yield curves of the CIR model are very similar to the Vasicek curves presented in Figures 1 through 3. The difference between the models relate more to the dynamics of yield curve fluctuations than to the shape of a particular curve given the instantaneous rate.

The Dothan Model

The model of Dothan (1978) increases the volatility exponent to 1.0:

$$dr_t = \sigma r_t dB_t$$

Because of the higher exponent, the model relates the volatility of interest rate movements more strongly to the level of interest rates. Courtadon (1982) extends Dothan's model to include mean reversion in the drift. Dothan's model is more difficult to implement in practice because there are no closed form, analytic solutions as in the Vasicek and CIR models. The user must resort to simulation to implement the model. Given the lack of closed form solutions and the inability of general equilibrium models to match the existing yield curve, the Dothan model has not been a popular model for use in evaluating interest rate securities.

Multi-Factor Models

To alleviate the problem of correlated bond prices, a model can incorporate two or more stochastic factors. In the two-factor model as described in Brennan and Schwartz (1979, 1982), one factor is used to represent the short-term rate while the other factor is the rate θ on a perpetuity (i.e., the long-term rate).

$$dr_t = \kappa(\theta - r_t)dt + \sigma_1 r_t dB_1$$
$$d\theta = \sigma_2 \theta dB_2$$
$$dB_1 dB_2 = \rho dt$$

σ_1, σ_2 = volatility of the short- and long-rate processes, respectively
dB_1, dB_2 = standard Brownian motions
ρ = correlation between short- and long-rate processes

Another popular two-factor model is presented in Longstaff and Schwartz (1992), where the second factor is stochastic volatility of the short-term rate. By explicitly modeling these factors separately, the potential range of yield curve dynamics is enhanced.

Heath, Jarrow, Morton Framework

The restrictions on yield curve movements of the one-factor models make them less exact, which in some cases, such as investment banking, represents a serious drawback. The main limitation is that yields of all maturities are perfectly correlated. However, history shows that different parts of the yield curve can shift in different directions and this can wreak havoc on an insurer's surplus. The interdependence across all maturities is most critical for insurers where assets and liabilities have unequal sensitivities at different points on the yield curve (see Reitano, 1990 and 1992).

Litterman and Scheinkman (1991) show that there are two additional factors, aside from parallel shifts in the yield curve, that have affected bond returns. The first factor, called steepening, reflects the fact that short-term rates may move in the opposite direction of long-term rates. The Brennan and Schwartz (1979) model above addresses the potential for a steepening term structure. The second factor affecting bond returns in Litterman and Scheinkman (1991) is a curvature component. This factor addresses the potential for intermediate yields to be more or less volatile than extreme maturities.

As mentioned above, a criticism of equilibrium models is that they are not arbitrage-free in the sense that the yield curves produced by the models do not match the existing term structure. This makes these models unsatisfactory for pricing option-embedded securities. If the model cannot accurately portray the existing term structure, there is little confidence that it will accurately imitate the dynamics of the curve (Hull, 1993).

Heath, Jarrow, and Morton (1992) use the no-arbitrage argument to develop the process for the *forward* rate implied by the relationship of bond prices.

$$df(t,T) = \mu(t,T,f(t,T))dt + \sigma(t,T,f(t,T))dB_t$$

$f(t,T)$ = instantaneous forward rate at time t with maturity T
$\mu(t,T,f(t,T))$ = drift of the forward rate process
$\sigma(t,T,f(t,T))$ = volatility of the forward rate process
B_t = standard Brownian motion

HJM find that by imposing the no-arbitrage argument to term structure movements, the drift of the forward rate process can be stated in terms of volatilities. Thus, the structure of the volatility becomes the most important element of the HJM model. Different functional forms of the volatility reveal an entire family of HJM models. In particular, a simple functional form is of the following type:

$$\sigma(t,T,f(t,T)) = \sigma_0 f(t,T)^\gamma$$

Amin and Morton (1994) look at a more general form and estimate the parameters of several specifications.

Historical Data

The choice of interest rate model can have an enormous impact on the resulting interest rate risk of any financial instrument. Although determining a perfect model of interest rates is beyond the scope of this paper, understanding the impact of the choice of interest rate model will assist insurers in analyzing the inherent risks of the embedded options in their liabilities and in choosing the appropriate model for their analyses. Any individual who wishes to use a model to simulate interest rate movements must first get a feel for historical changes. This section illustrates the historical movements in Treasury yields over the last 45 years. For ease of presentation, the focus will be on four critical points on the yield curve: (1) the one-year rate, (2) the three-year rate, (3) the five-year rate, and (4) the ten-year rate. Historical rates will then be compared with the theoretical models at these points. The data is taken from the St. Louis Federal Reserve web site.

The time series of the four yields is illustrated in Figure 4. A casual inspection of Figure 4 shows that interest rates increased from 1953 through 1979. Then, interest rates spiked in the early 1980s during the transition of the Federal Reserve policy mentioned above. Finally, since the peak in 1981, yields have exhibited a general downward trend.

Table 1 presents some summary statistics on the levels of yields over the 45 year period. These statistics help illustrate several features of historical interest rates. The first result relates to the shape of the yield curve on any particular date. The yield curve is a graphical representation of the relationship of the yield on bonds and their maturities. Figures 5 through 7 show three yield curves that have been observed historically. Typically, long-term yields are higher than short-term yields. When this occurs, the yield curve is upward sloping. The upward sloping yield curve is common enough that it is characterized as a "normal" curve as depicted in Figure 5. Occasionally, yield curves become inverted – short-term rates exceed long-term rates (see Figure 6). Inverted curves are typically observed in periods of high interest rates and the yield inversion is usually short-lived. Finally, humped yield curves are characterized by increasing yields at the short end of the curve. Eventually, as the term to maturity increases more, the yields begin to fall slightly (see Figure 7). Many humped yield curves occur during the transition from an inverted yield curve to a normal curve.

In Table 1, the yield curve is categorized according to its shape: normal, inverted, humped, or other. These classifications are made strictly on the four yield points, 1 year, 3 year, 5 year and 10 year. The precise definition of yield curve shape, as it is used here, is based on yield curve slope. The slope is the difference between two adjacent yields. A normal curve has positive slope everywhere, while an inverted curve has negative slope everywhere. A humped yield curve initially has positive slope and eventually has negative slope. If the yield curve does not fit into one of these profiles, it is classified as "other." Note that the yield curve classifications are based on end-of-the-month yields, so that a monthly observation is based on only one moment in time. If the yield curve is normal at that time even though it was inverted at all other times during the month, the curve is nonetheless classified as normal.

It should be noted that the magnitude of the slope does not impact our classification of yield curve slope. In particular, we do not use a "flat" yield classification. A flat yield curve exists if the yields on bonds of all maturities are equal. At no time in the 45 year history is the yield curve exactly flat. However, differences in yields of various maturities may be negligible. Rather than define the term negligible, the approach used here amounts to distributing almost-flat yield curves into the other categories.

Several statistics in Table 1 illustrate how often the normal yield curve occurs. First, the yield curve has been upward sloping over two-thirds of the time. Inversions occurred in only 11.6% of the months and a humped curve occurred 13.4% of the time. In addition to the frequencies of the various shapes, other statistical information points to the tendency of rates to be increasing with maturity. The mean of each of the four yields increases with maturity, and the yield percentiles seem to imply that the typical shape of the yield curve is normal, except when yields are high.

The next group of results illustrates the relationship of yield volatility and maturity. Long-term rates have lower standard deviations, lower skewness, a smaller range of outcomes, and higher autocorrelations than short rates. Thus, our earlier conjecture that long-term yields are less volatile than short-term rates seems to have statistical support.

Two other results are worth pointing out. First, short-term rates appear more positively skewed than longer yields. This could mean that changes in the long-term rate are more symmetric, or it could indicate that large, positive changes in the interest rates are more common in the short-term rate. The second point is that correlations between yields decrease as the rates are further apart.

Instead of looking at yield levels, Figures 8 through 11 look at changes in interest rates. Figure 8 looks at the monthly time series of absolute changes in the one-year yield rate. The Fed policy transition period stands out as an extremely volatile period for short-term rates. To gain further perspective on the transition period, Figure 9 looks at these changes on a relative basis. The extreme volatility of the early 1980s loses some of its distinction when viewed on a relative basis. The implication of Figures 8 and 9 provide some intuitive support for the CKLS result that interest rate volatility is related to the level of the interest rate. When interest rates were high, the percentage changes in yields were about the same as the percentage changes when rates were low. Figures 10 and 11 present a similar story for the 10-year yield.

To get a feel for the volatility of interest rate movements, we computed the standard deviations of the one-year and ten-year changes in yield. For the one-year yield, the standard deviation of absolute changes in the monthly rate over the entire period is 0.47 and the standard deviation of relative changes is 0.07. As expected, the volatility of changes in the ten-year yield is significantly less. The standard deviation of absolute changes in the monthly ten-year yield rate over the entire period is 0.29 and of percentage changes, the standard deviation is 0.03.

Simulations

The interest rate models presented in this paper have been introduced in a continuous time framework. Although some continuous time models may lead to closed form solutions for simple cash flows such as non-callable bonds, insurance liabilities are more complicated. To use the model's dynamics in insurance applications, such as in DFA, one must use discrete time intervals for the interest rate process. This section discusses how to translate the continuous time process into a discrete process and then illustrates the interest rate models presented in this paper through simulations.

As an example of discretization, consider the Vasicek model. Other models follow directly from the Vasicek results. Recall the Vasicek model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dB_t$$

By using short time intervals, the discrete process approximates the continuous process. More precise estimates will be obtained through the use of short time intervals (hours or minutes) which is most appropriate for trading activities. However, with insurance applications, long-term modeling is required and the use of longer intervals (such as monthly) is more appropriate.

All models presented here include a standard Brownian motion. Random changes in the Brownian motion are based on draws from the standard normal distribution scaled by the time interval. There are two popular approaches for generating standard normal distribution random variables. The first method is to take the average of twelve uniform random variables on the interval [0,1]. The second method is to translate two uniform random variables (u_1, u_2) according to the following:

$$\varepsilon = \sqrt{-2\ln(u_1)} \times \cos(2\pi u_2)$$

The monthly interest rate process then becomes (ε is the standard normal random variable):

$$\Delta r_t = \kappa(\theta - r_t) \times \frac{1}{12} + \sigma\varepsilon\sqrt{\frac{1}{12}}$$

We use this discrete approach to perform monthly simulations of several interest rate models. Our goal is to get a feel for how the models operate and to compare the resulting simulated yield distributions with the historical distribution. The Vasicek and CIR models are the most straightforward to simulate because yields have closed form solutions that depend on only the short rate. Also, the simulations are stable due to the mean reversion drift term. We simulate the yield curve for 10,000 months using the parameter estimates of CKLS:

Parameter	***Vasicek***	***CIR***
Mean reversion strength (κ)	0.1779	0.2339
Long-term rate (θ)	0.0866	0.0808
Volatility (σ)	0.0200	0.0854

The results of the Vasicek simulations are shown in Table 2. The shape of the yield curve is more frequently inverted than in the historical experience. In fact, the statistics show that the "average" yield curve is actually slightly inverted, but close to being flat. An inspection of the percentile statistics reveals that at low percentiles (when the one-year yield is low), the yield curve appears to be upward sloping. As the short rate increases, the curve is inverted. Another note from the shape frequencies illustrates the restrictions of the Vasicek model on the shape of the yield curve. The yield curve is normal, inverted, or humped. No other shape is seen under the Vasicek model. The standard deviation and percentile statistics show that the long yields are less volatile in the Vasicek model. All yields are perfectly correlated, as expected based on the fact that all yields are derived from the same instantaneous (short) rate. As explained in the presentation of the model, interest rates can become negative with the Vasicek model. In fact, the first percentile is negative.

Compared to the historical rates, the Vasicek model is negatively skewed and less peaked. This can be seen in the skewness and excess kurtosis statistics as well as by looking at the distributions of the one-year and ten-year yields. The historical distributions are shown in Figures 12 and 13 while the Vasicek simulation distributions are shown in Figures 14 and 15.

The CIR simulation results are presented in Table 3, and the distributions of one- and ten-year yields are illustrated in Figures 16 and 17. As in the Vasicek case, the CIR model is more frequently inverted than in historical data (47.6% inversions in the CIR simulation vs. 11.6% historically). The average yield curve is inverted but is close to being flat. The percentiles reveal a pattern similar to the Vasicek results. When the short rate is low, the curve appears normal. As the one-year yield increases, the yield curve inverts. One difference from the Vasicek results is that the median yield curve is almost perfectly flat. The yield curve shape is never other than normal, inverted, or humped.

The volatility of the ten-year yield is lower than the one-year yield volatility as measured by the standard deviation and interquartile range. Also, note that interest rates in the CIR model remain positive. The correlations among yields of all maturities are all 1.0. Finally, there is positive skewness for all rates, and the value is closer to the historical statistics than the Vasicek model. The distribution of longer maturities appears more peaked relative to historical numbers (see the excess kurtosis numbers and Figures 13 vs. 17).

Given the popularity of arbitrage free models, we present some short simulations of 100 months to see how these models function. Because the Ho-Lee model is the constant volatility case of the HJM model, we present a simulation on the more general HJM framework. Recall that the drift in an HJM framework is a function of the volatilities. Thus, unlike the Vasicek and CIR models, the drift is positive and the interest rate is not mean-reverting. Using long simulations to generate smooth distributions of yields is not possible because the curve will (on average) continue to increase. Rates quickly begin to drift to "unrealistic" levels. The arbitrage-free models are used to assure that the interest rate process does not generate arbitrage opportunities in the short term. As the interest rates are observed, the model is recalibrated and another simulation is performed. Thus, the simulation performed here uses only 100 months. In that simulation, the ending yield curve is near 13%, demonstrating the drift in these types of models.

Another difficulty when comparing HJM models to others is in calculating the shape statistics given the drifting problem. The shape of the curve becomes too dependent on the initial curve given the short simulation period. If the curve starts out as normal, most of the subsequent curves remain normal. Similarly, just the opposite occurs when the yield curve is initially inverted. In the simulations presented, the initial curve was based on year-end 1998 yields.

Results of the HJM simulation are presented in Table 4 and in Figures 18 and 19. The important feature of these results is that yields of different maturities are not driven by the same factor. Therefore, statistics such as skewness, excess kurtosis, and correlations are not exactly the same for all yields (although they are close). Contrast these differences with the results of Vasicek and CIR models, where these statistics are identical for all maturities.

Caveats

The results illustrated here use the entire historical period of April 1953 to July 1998 as a benchmark for comparing alternative models. This choice was based on obtaining a larger amount of data (compared with other studies) to generate smoother yield distributions, as well as to provide some perspective on interest rates over longer periods. However, the change in Fed policy in 1979 presents an important question regarding whether comparisons among interest rate models are robust to the Fed's shift in focus. To look at these effects, a similar analysis could be performed across different subperiods. One possible breakdown would look at results under the two different Federal Reserve policies. Yield statistics can be generated under the "interest rate target policy" and also under the "inflation target policy." Another subperiod analysis could attempt to isolate the transition period and compare the pre- and post-transition periods to determine if the new Fed policy has affected the underlying interest rate dynamics. It should be pointed out that other factors may be contributing to the dynamics of the curve across any subperiod analysis. For example, the post-transition economy has been very strong with only one short recessionary period. Using only post-transition statistics may not completely embody the true potential for interest rate movements. The main point is that when using yield curve statistics to parameterize an interest rate model, one should be aware of any underlying factors that may be affecting the dynamics of yields and incorporate judgment in choosing specific models.

As mentioned above, the parameters used in the simulations were based on estimates reported by CKLS (1992) and Amin and Morton (1994). The CKLS study looks at the period from June 1964 through December 1989. Amin and Morton look at the period from 1987 through 1992. Estimation over different time periods will more than likely generate different parameters. Thus, one must keep in mind the interest rate environment when estimating parameters from past data for use in future periods. Care should be taken to ensure that the potential interest rate dynamics are consistent with the parameter assumptions.

Conclusion

Interest rate volatility now requires that actuaries incorporate stochastic interest rate assumptions into the pricing, forecasting, and valuation processes. The goal of this paper has been to provide a simplified introduction to and illustration of these models. The focus has been on comparing the results of simulations based on a variety of stochastic interest rate models with historical interest rate statistics. It is hoped that this work helps casualty actuaries begin the process of incorporating these modeling skills into their actuarial toolkits.

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TABLE 1

Historical Yield Statistics
Entire Period (4/53 - 7/98)

Yield Curve Shape

Normal	68.8%
Inverted	11.6%
Humped	13.4%
Other	6.3%

Yield Statistics

	1 Yr	3 Yr	5 Yr	10 Yr
Mean	6.08	6.47	6.64	6.81
Std Dev	3.01	2.88	2.84	2.81
Skewness	0.97	0.84	0.77	0.68
Excess Kurtosis	1.10	0.69	0.48	0.16

Percentiles

	1 Yr	3 Yr	5 Yr	10 Yr
1%	1.07	1.59	1.94	2.38
5%	2.05	2.52	2.72	2.90
10%	2.94	3.38	3.47	3.48
25%	3.81	4.17	4.24	4.25
50%	5.61	6.20	6.44	6.68
75%	7.71	8.01	8.04	8.20
90%	9.97	10.47	10.63	10.78
95%	12.08	12.48	12.59	12.56
99%	15.17	14.69	14.59	14.29

Correlations

		1 Yr	3 Yr	5 Yr	10 Yr
1 Yr		1.000	0.985	0.969	0.944
3 Yr			1.000	0.997	0.984
5 Yr				1.000	0.995
10 Yr					1.000
Auto					
	1	0.988	0.991	0.993	0.995
	2	0.967	0.976	0.980	0.986
	3	0.948	0.963	0.970	0.979
	4	0.932	0.951	0.960	0.972
	5	0.918	0.940	0.951	0.964

TABLE 2

Vasicek Simulation Statistics
(10,000 Simulations)

Yield Curve Shape

Normal	41.6%
Inverted	54.8%
Humped	3.6%
Other	0.0%

Yield Statistics

	1 Yr	3 Yr	5 Yr	10 Yr
Mean	8.81	8.75	8.68	8.52
Std Dev	3.83	3.24	2.77	1.95
Skewness	-0.16	-0.16	-0.16	-0.16
Excess Kurtosis	-0.19	-0.19	-0.19	-0.19

Percentiles

	1 Yr	3 Yr	5 Yr	10 Yr
1%	-0.38	0.97	2.04	3.84
5%	2.33	3.27	4.00	5.22
10%	3.69	4.42	4.98	5.92
25%	6.26	6.60	6.84	7.23
50%	8.94	8.86	8.77	8.59
75%	11.62	11.13	10.72	9.96
90%	13.60	12.80	12.14	10.97
95%	14.69	13.73	12.94	11.53
99%	17.22	15.87	14.76	12.82

Correlations

		1 Yr	3 Yr	5 Yr	10 Yr
1 Yr		1.000	1.000	1.000	1.000
3 Yr			1.000	1.000	1.000
5 Yr				1.000	1.000
10 Yr					1.000
Auto					
	1	0.991	0.991	0.991	0.991
	2	0.982	0.982	0.982	0.982
	3	0.973	0.973	0.973	0.973
	4	0.965	0.965	0.965	0.965
	5	0.956	0.956	0.956	0.956

Note: Model parameters from CKLS estimates: $\kappa = 0.1779$, $\theta = 0.0866$, $\sigma = 0.0200$

TABLE 3

CIR Simulation Statistics
(10,000 Simulations)

Yield Curve Shape

Normal	47.7%
Inverted	47.6%
Humped	4.7%
Other	0.0%

Yield Statistics

	1 Yr	3 Yr	5 Yr	10 Yr
Mean	8.08	8.04	7.98	7.86
Std Dev	2.89	2.31	1.88	1.20
Skewness	0.92	0.92	0.92	0.92
Excess Kurtosis	1.49	1.49	1.49	1.49

Percentiles

	1 Yr	3 Yr	5 Yr	10 Yr
1%	2.92	3.90	4.62	5.71
5%	3.95	4.73	5.29	6.14
10%	4.73	5.35	5.80	6.46
25%	6.14	6.48	6.71	7.05
50%	7.71	7.73	7.73	7.70
75%	9.57	9.23	8.95	8.48
90%	11.80	11.01	10.40	9.41
95%	13.42	12.31	11.45	10.09
99%	17.19	15.33	13.90	11.66

Correlations

		1 Yr	3 Yr	5 Yr	10 Yr
1 Yr		1.000	1.000	1.000	1.000
3 Yr			1.000	1.000	1.000
5 Yr				1.000	1.000
10 Yr					1.000
Auto					
	1	0.976	0.976	0.976	0.976
	2	0.955	0.955	0.955	0.955
	3	0.934	0.934	0.934	0.934
	4	0.914	0.914	0.914	0.914
	5	0.894	0.894	0.894	0.894

Note: Model parameters from CKLS estimates: $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$

TABLE 4

HJM Simulation Statistics
(100 Simulations)

Yield Statistics

	1 Yr	3 Yr	5 Yr	10 Yr
Mean	7.39	7.51	7.60	7.80
Std Dev	2.26	2.27	2.31	2.44
Skewness	0.51	0.53	0.54	0.54
Excess Kurtosis	-0.88	-0.85	-0.85	-0.86

Percentiles

	1 Yr	3 Yr	5 Yr	10 Yr
1%	4.45	4.48	4.52	4.59
5%	4.79	4.85	4.90	4.99
10%	5.00	5.10	5.13	5.21
25%	5.25	5.45	5.53	5.63
50%	7.48	7.58	7.65	7.83
75%	8.65	8.75	8.85	9.10
90%	11.02	11.16	11.30	11.68
95%	11.57	11.74	11.92	12.38
99%	12.09	12.26	12.44	12.89

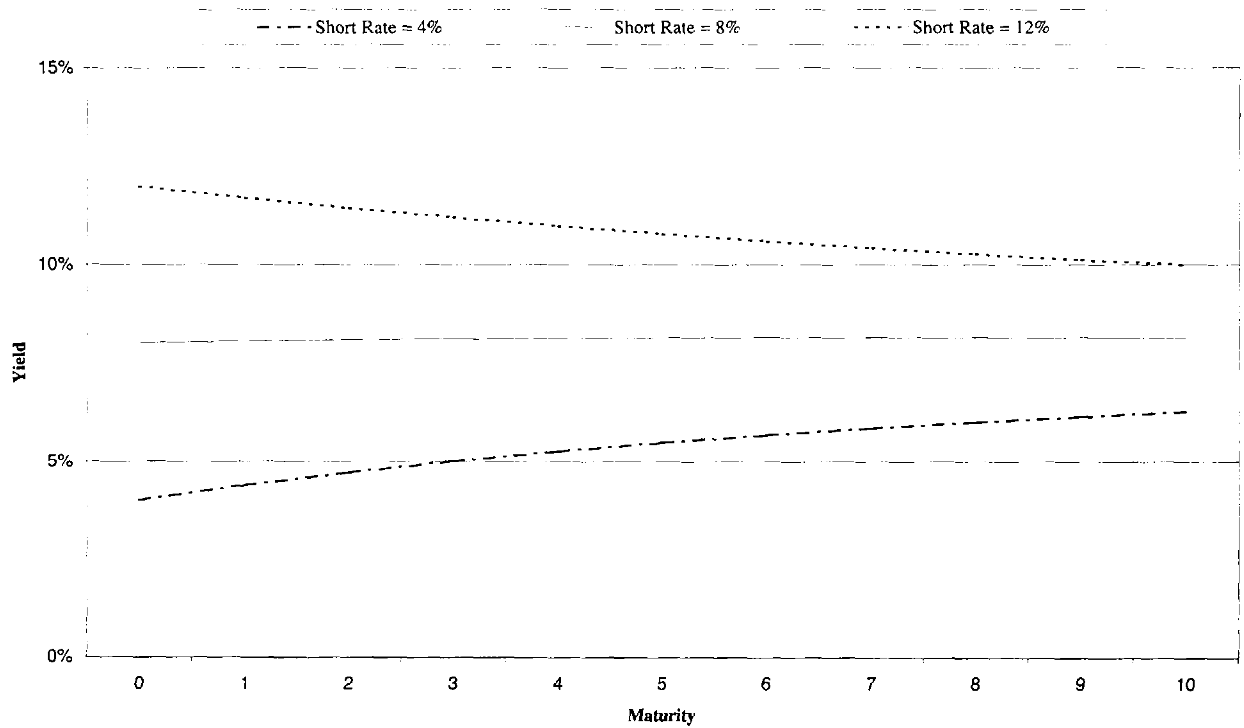
Correlations

		1 Yr	3 Yr	5 Yr	10 Yr
1 Yr		1.000	0.999	0.999	0.999
3 Yr			1.000	1.000	1.000
5 Yr				1.000	1.000
10 Yr					1.000
Auto					
	1	0.986	0.986	0.987	0.987
	2	0.969	0.969	0.969	0.972
	3	0.954	0.953	0.954	0.957
	4	0.938	0.938	0.939	0.943
	5	0.925	0.923	0.925	0.929

Note: Model parameters from Amin and Morton: $\sigma = 0.0485$, $\gamma = 0.5$

FIGURE 1

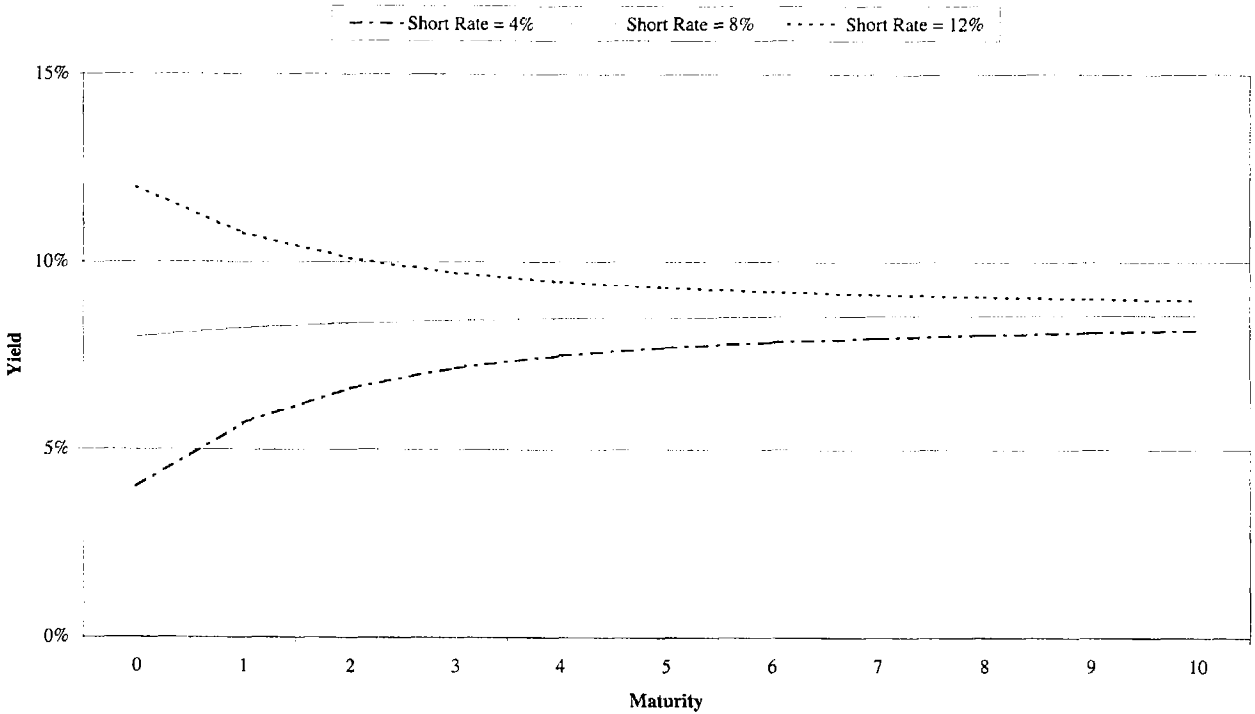
Vasicek Model Yield Curves
CKLS Parameters



Parameters: $\kappa = 0.1779$, $\theta = 0.0866$, $\sigma = 0.0200$

FIGURE 2

Vasicek Model Yield Curves
CKLS Estimates - Change in Mean Reversion

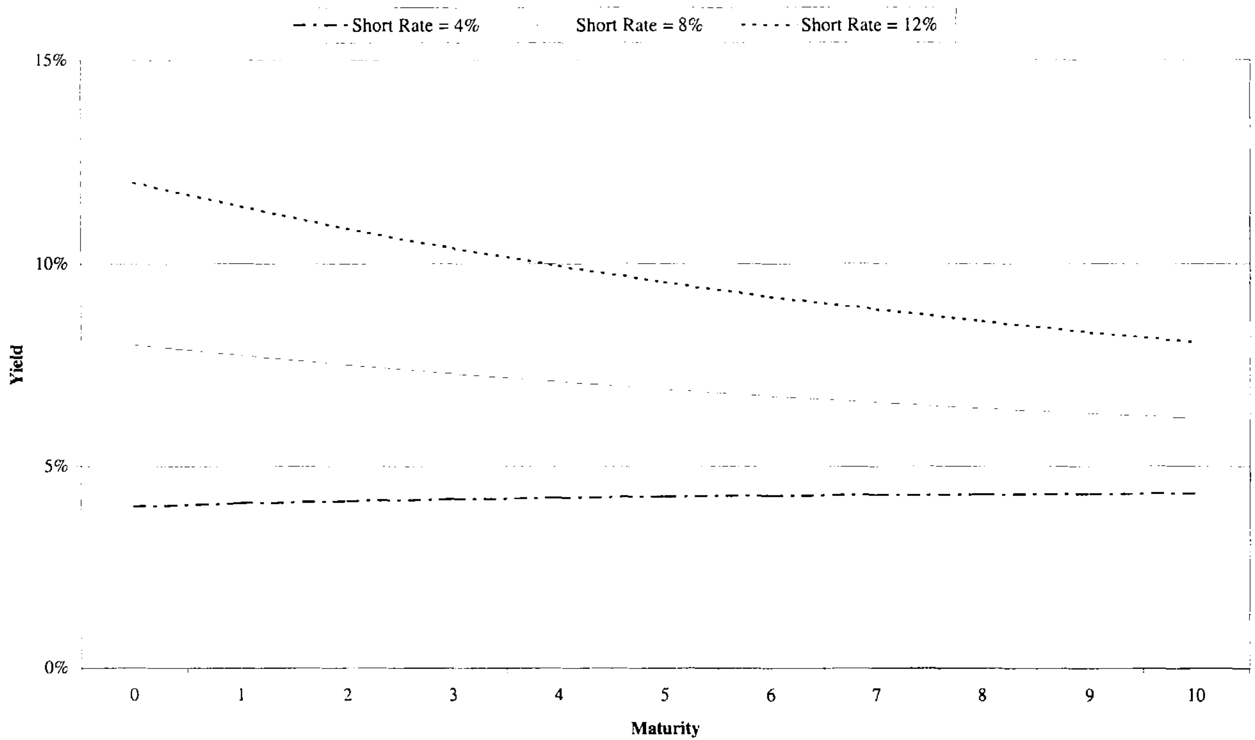


Parameters: κ = 1.0000, θ = 0.0866, σ = 0.0200

FIGURE 3

Vasicek Model Yield Curves

CKLS Parameters - Change in Long-Term Rate



Parameters: $\kappa = 0.1779$, $\theta = 0.0500$, $\sigma = 0.0200$

FIGURE 4

Time Series of Yields

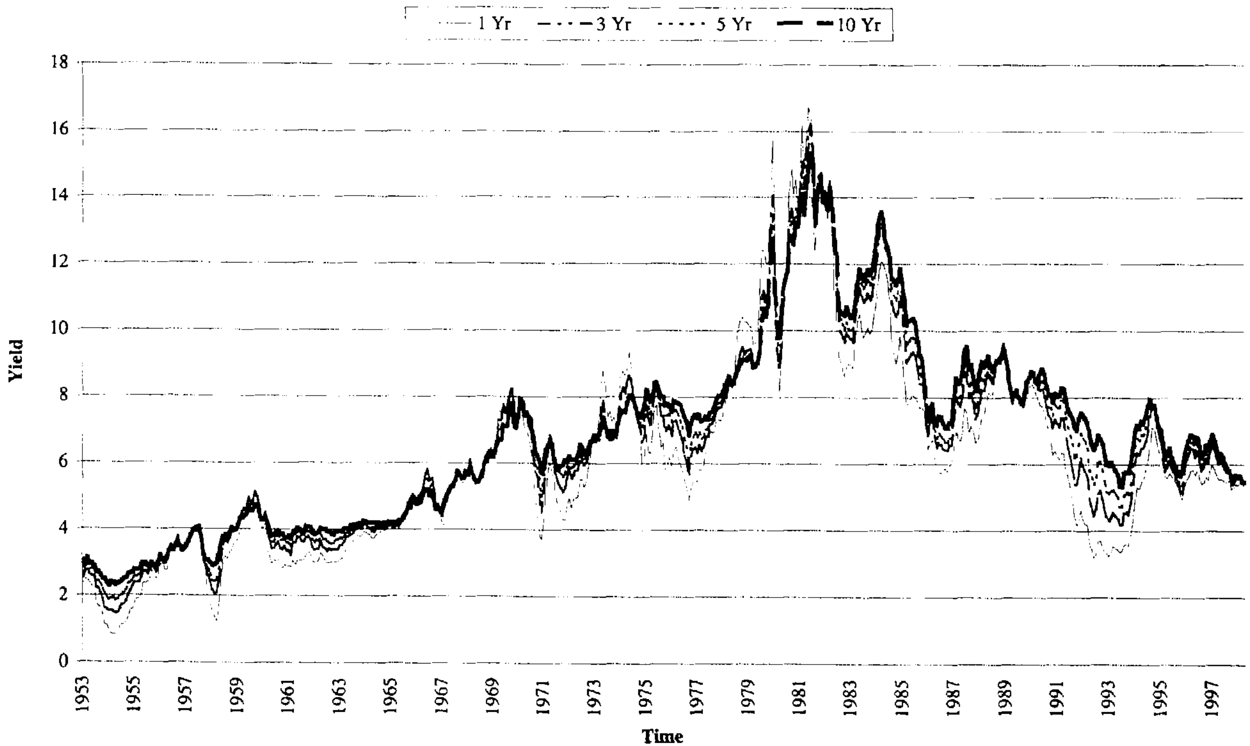


FIGURE 5
Normal Yield Curve
June 1991

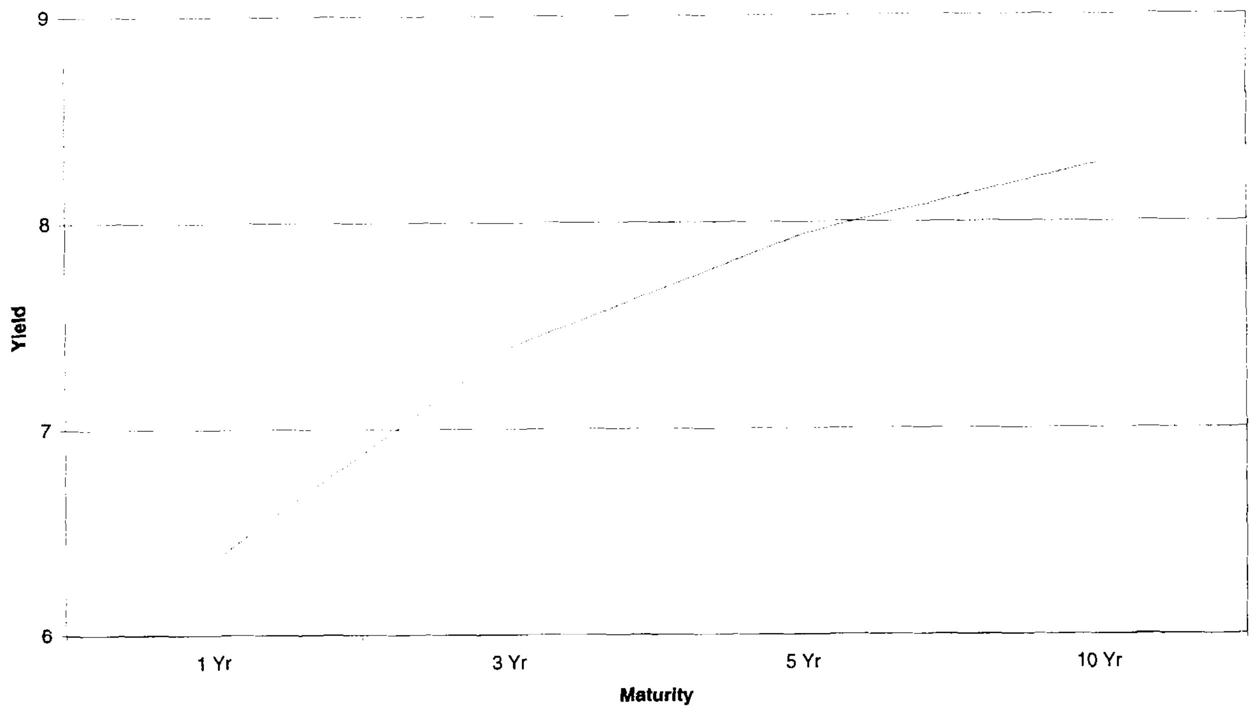


FIGURE 6
Inverted Yield Curve
August 1973

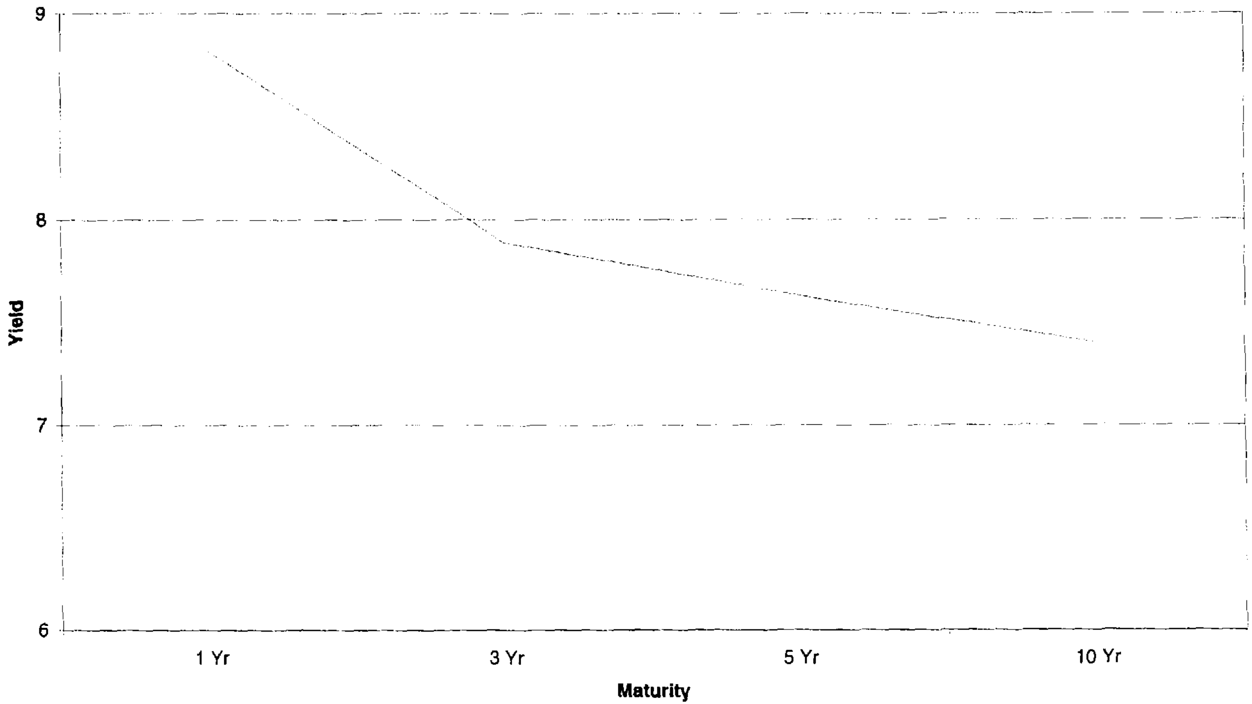


FIGURE 7
Humped Yield Curve
July 1970

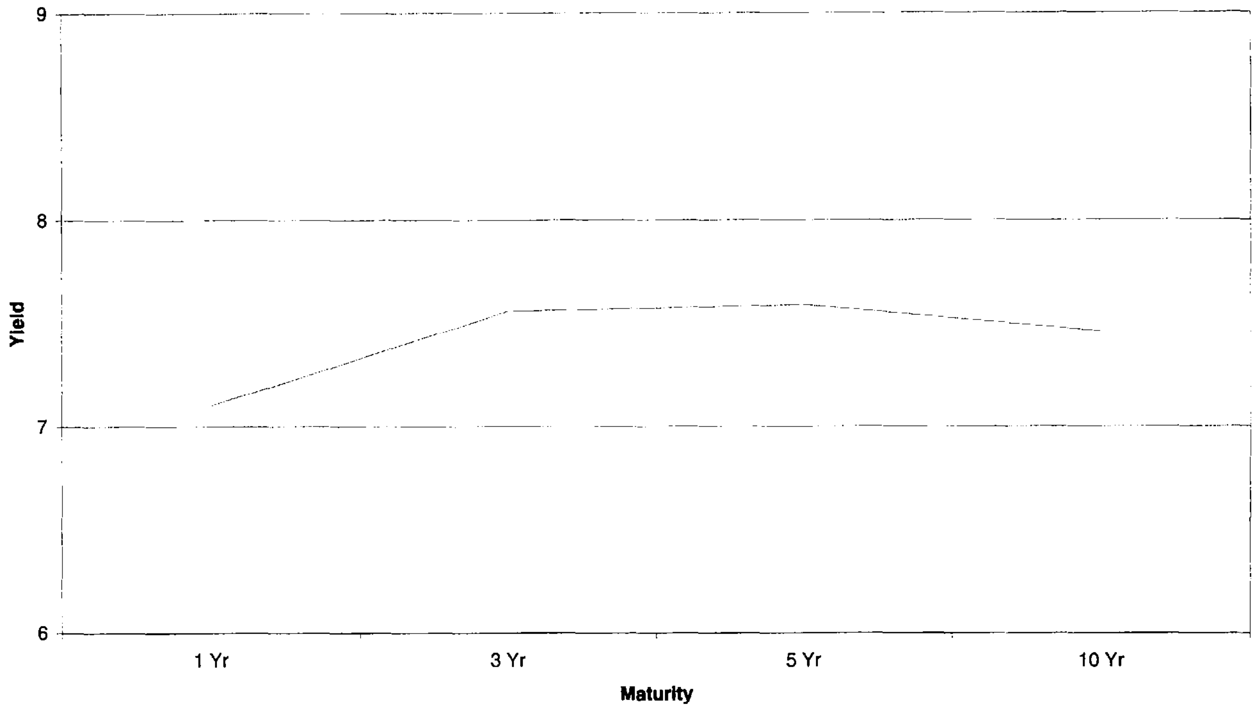


FIGURE 8

Time Series of Monthly Absolute Change in 1 Year Yield

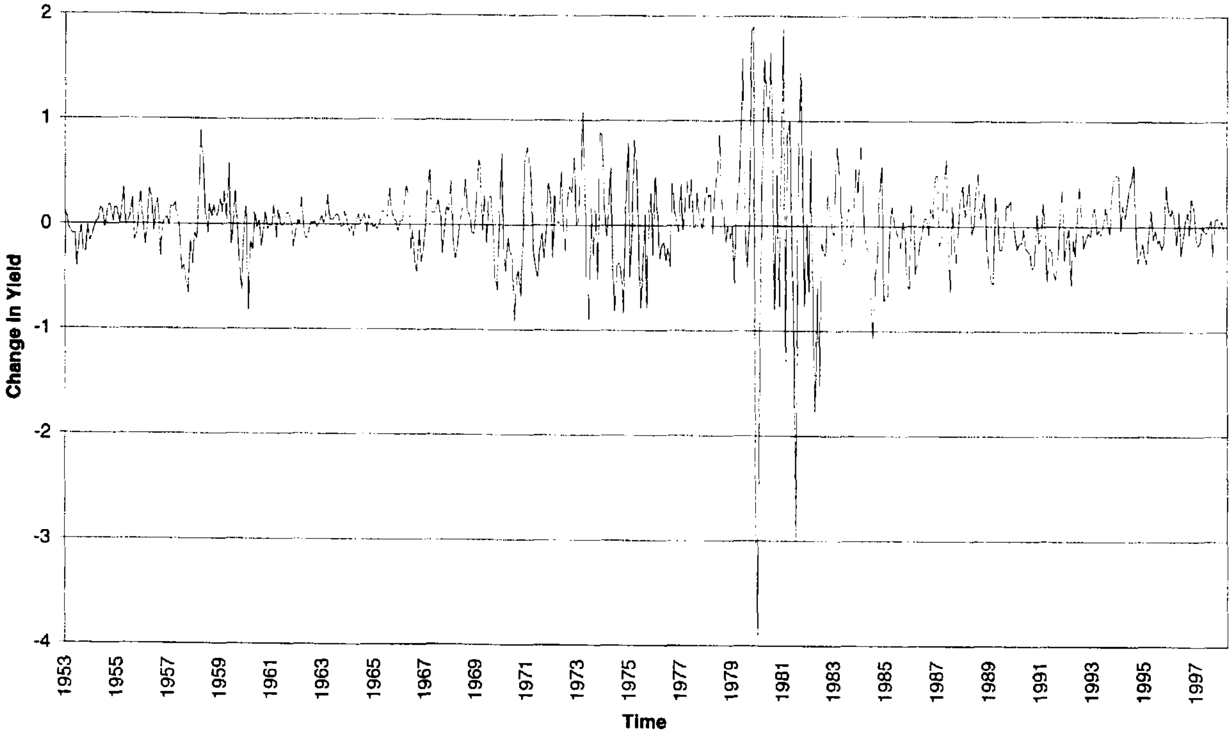


FIGURE 9

Time Series of Monthly Percentage Change in 1-Year Yields

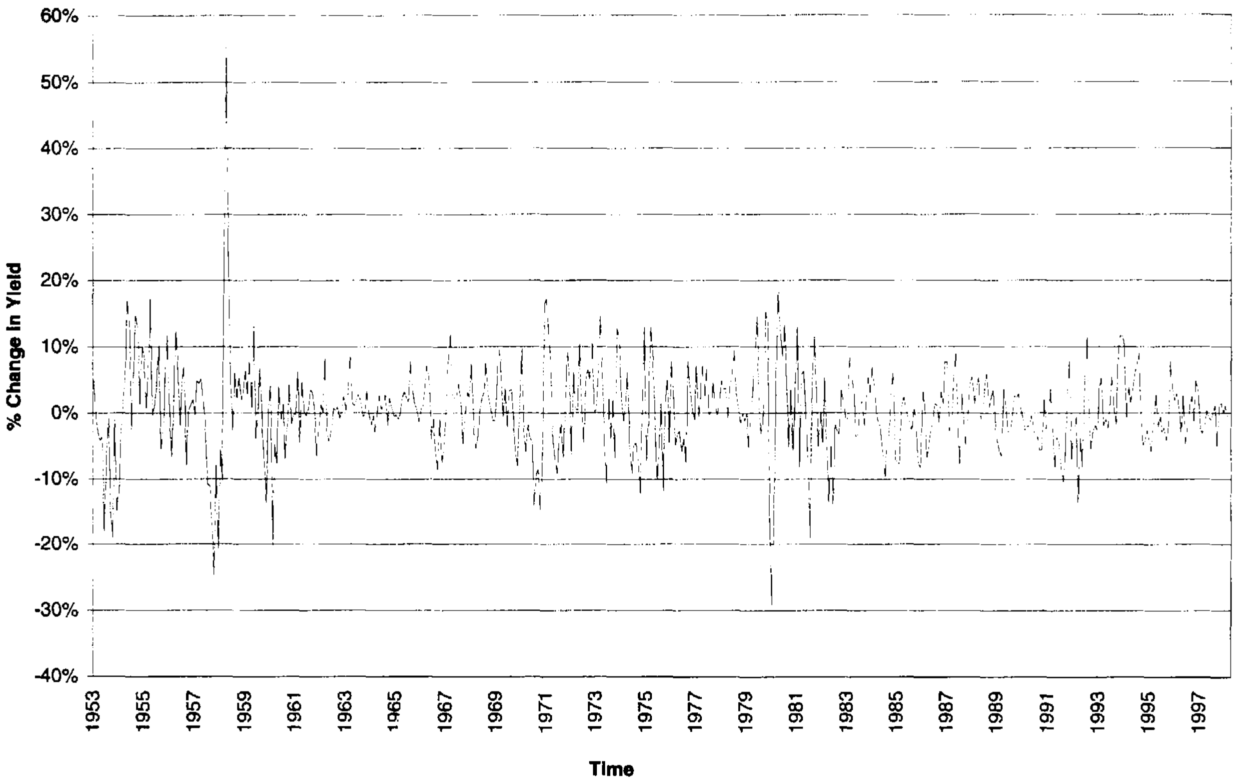


FIGURE 10

Time Series of Monthly Absolute Change in 10-Year Yield

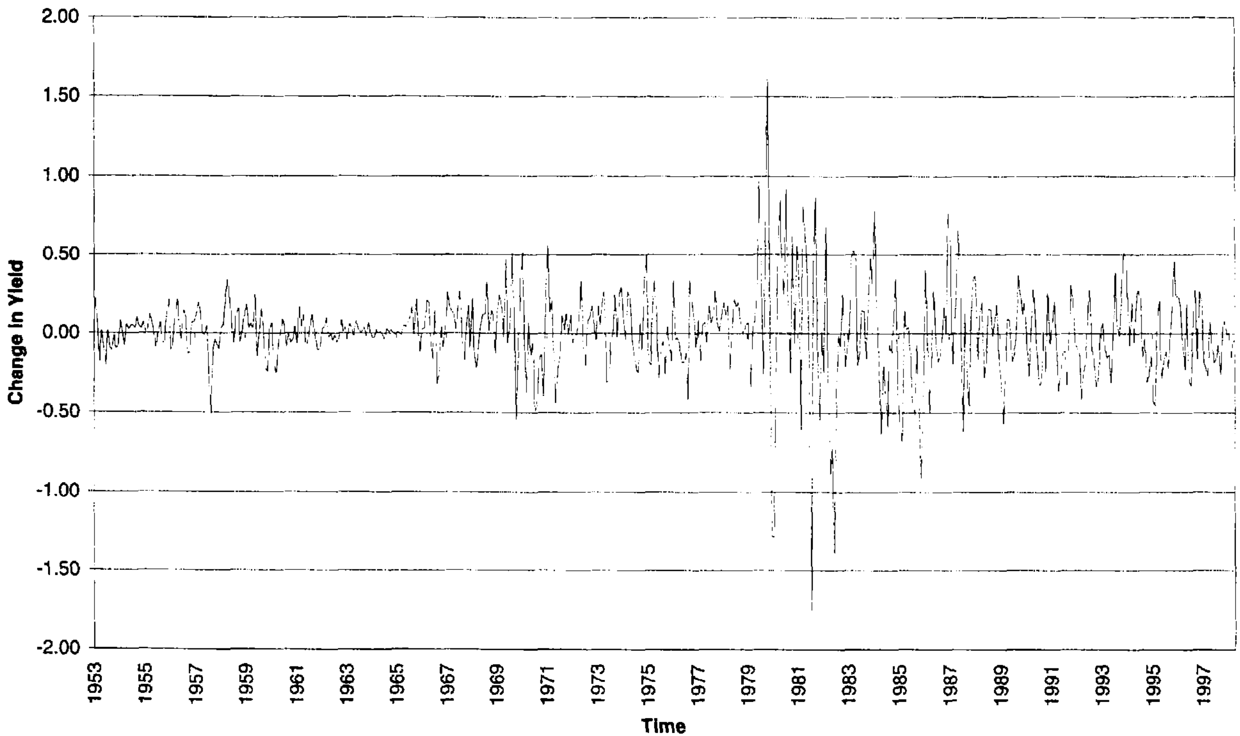


FIGURE 11

Time Series of Monthly Percentage Change in 10-Year Yield (Historical)

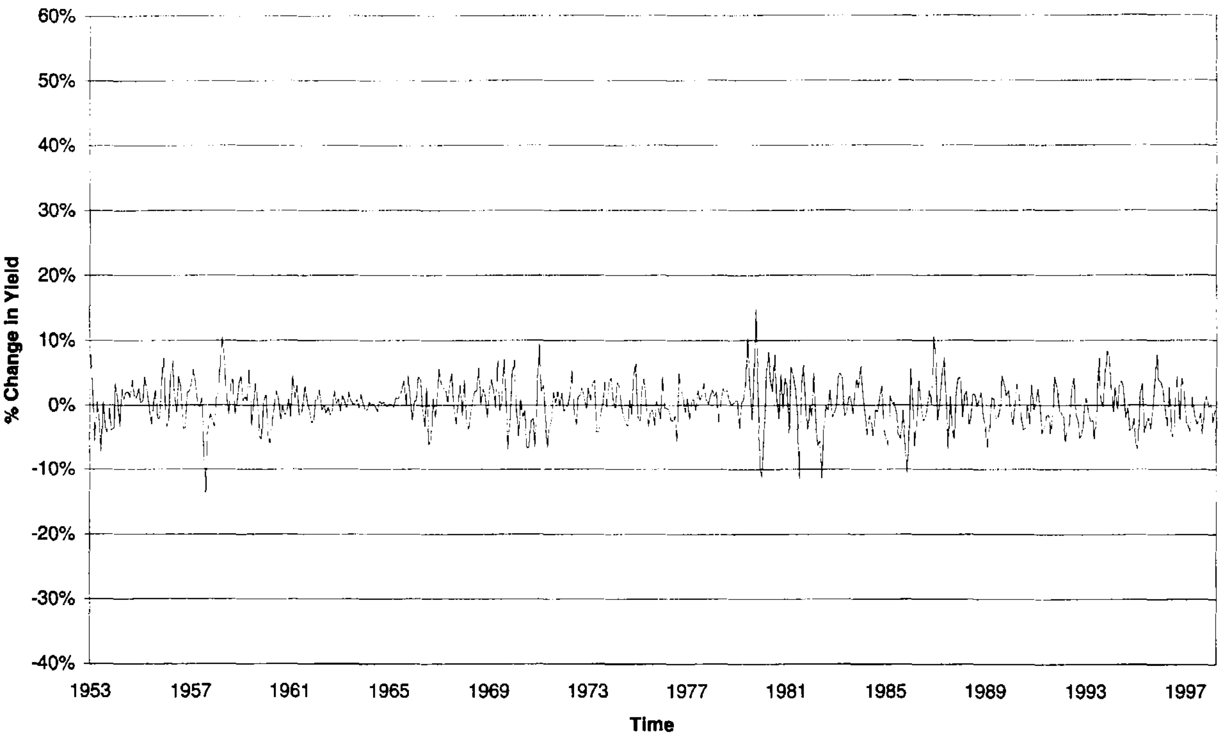


FIGURE 12

Historical 1 Year Yield Distribution

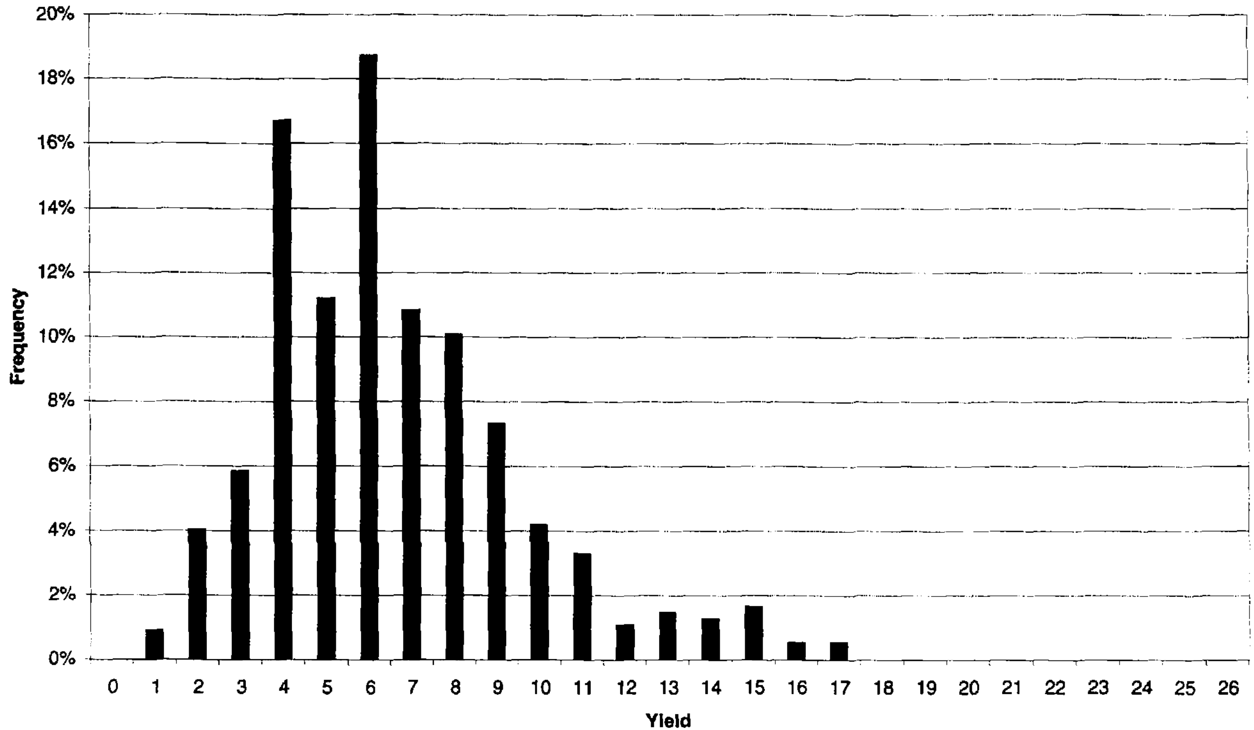


FIGURE 13
Historical 10 Year Yield Distribution

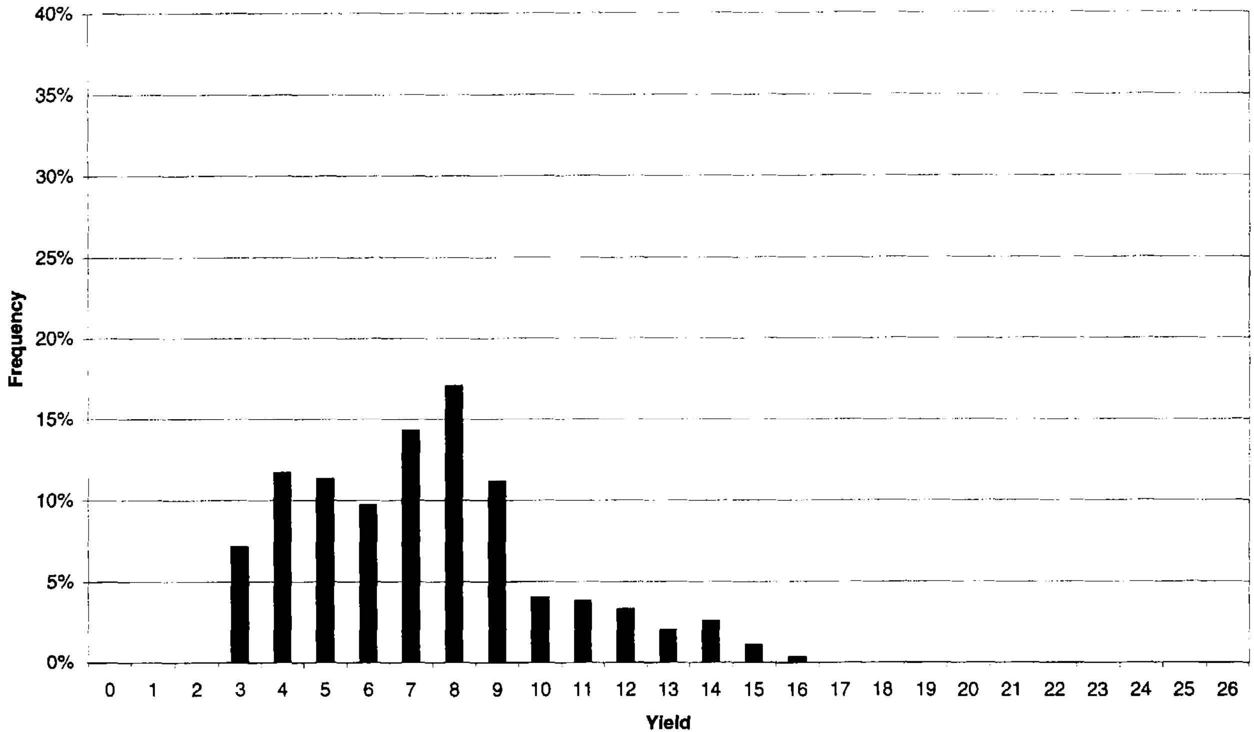
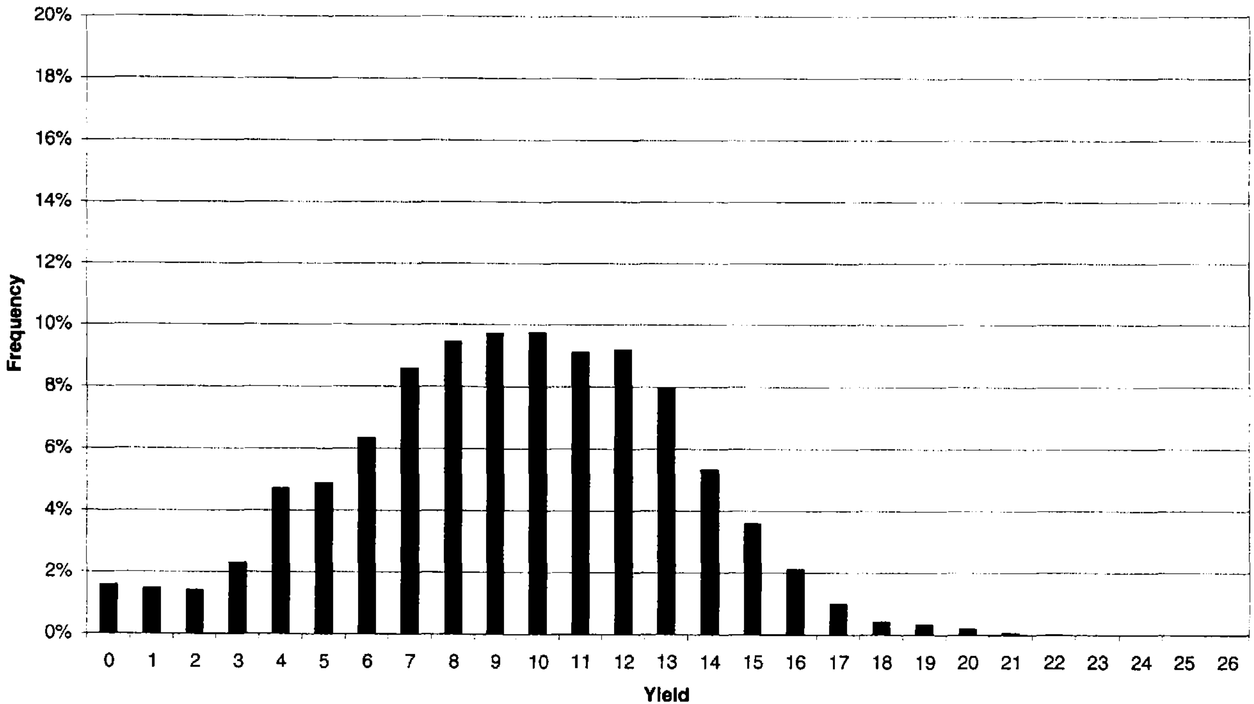


FIGURE 14

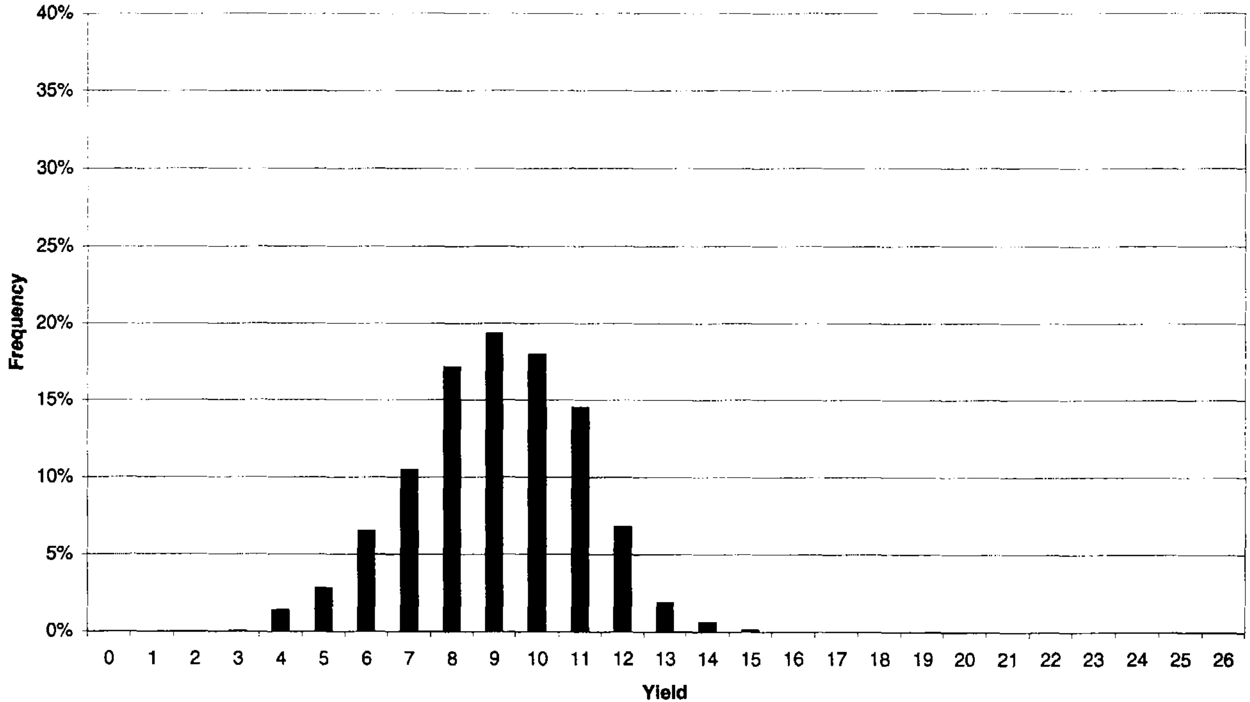
Vasicek Simulation

1 Year Yield Distribution



Note: Model parameters from CKLS estimates: κ = 0.1779, θ = 0.0866, σ = 0.0200

FIGURE 15
Vasicek Simulation
10 Year Yield Distribution

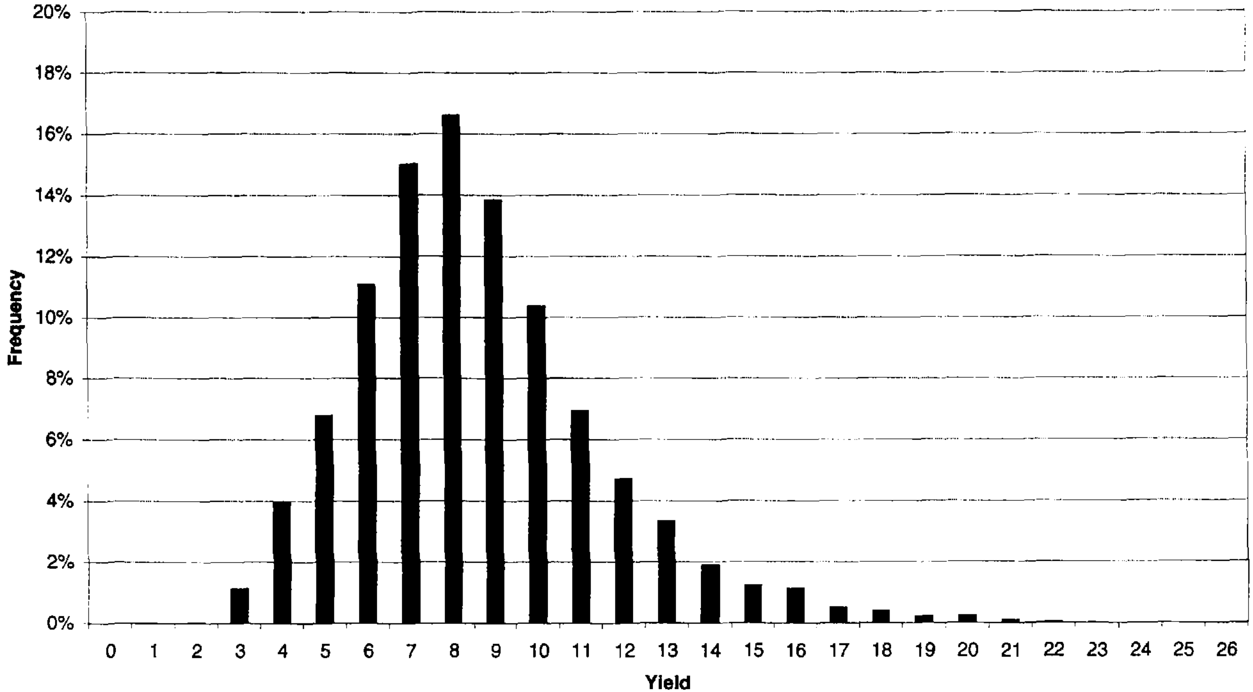


Note: Model parameters from CKLS estimates: $\kappa = 0.1779$, $\theta = 0.0866$, $\sigma = 0.0200$

FIGURE 16

CIR Simulation

1 Year Yield Distribution

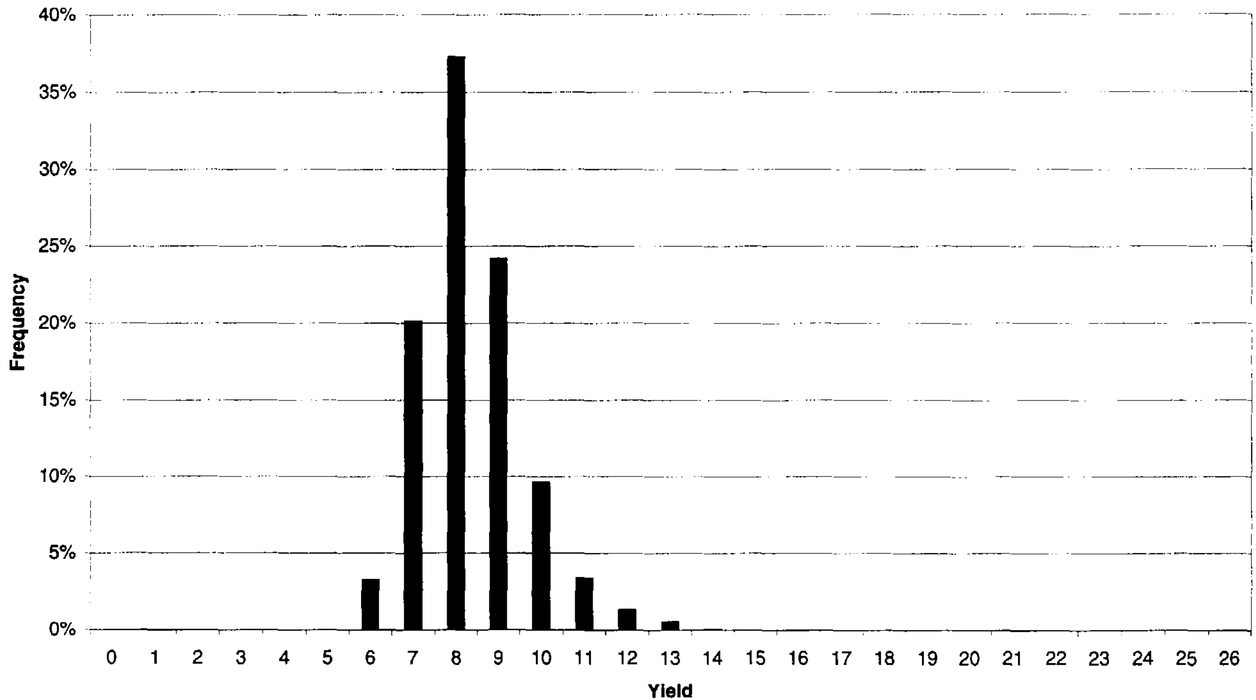


Note: Model parameters from CKLS estimates: $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$

FIGURE 17

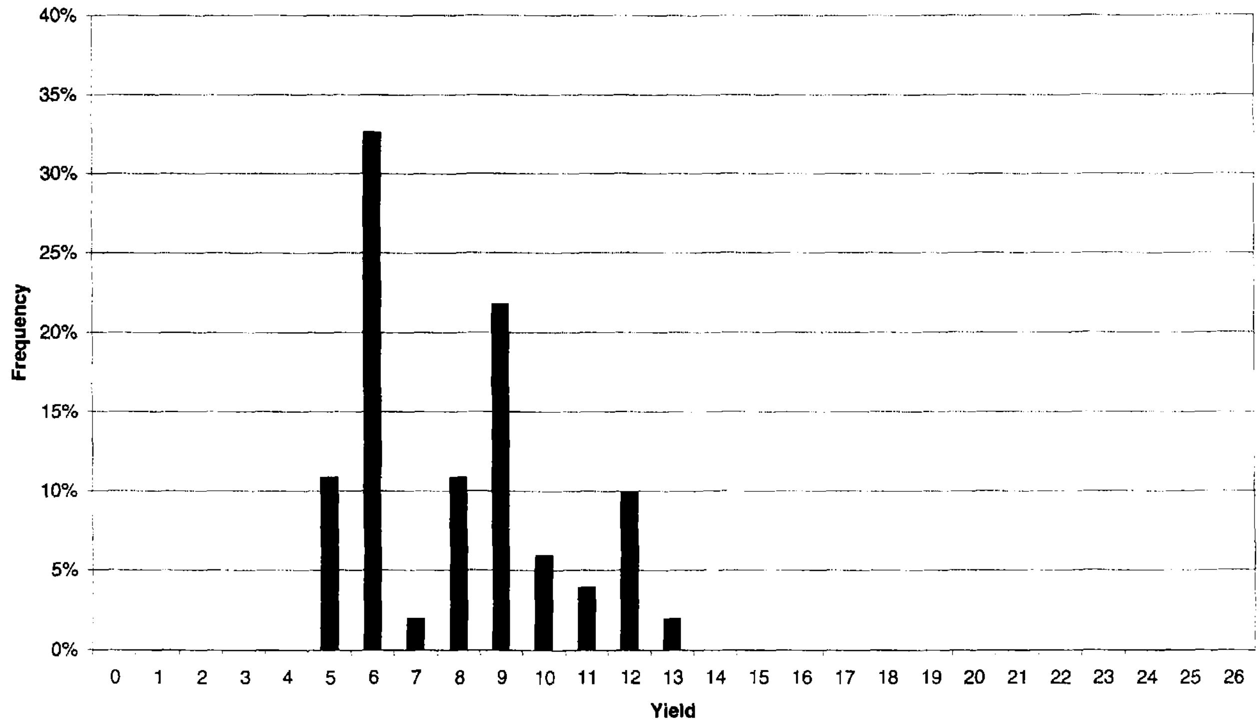
CIR Simulation

10 Year Yield Distribution



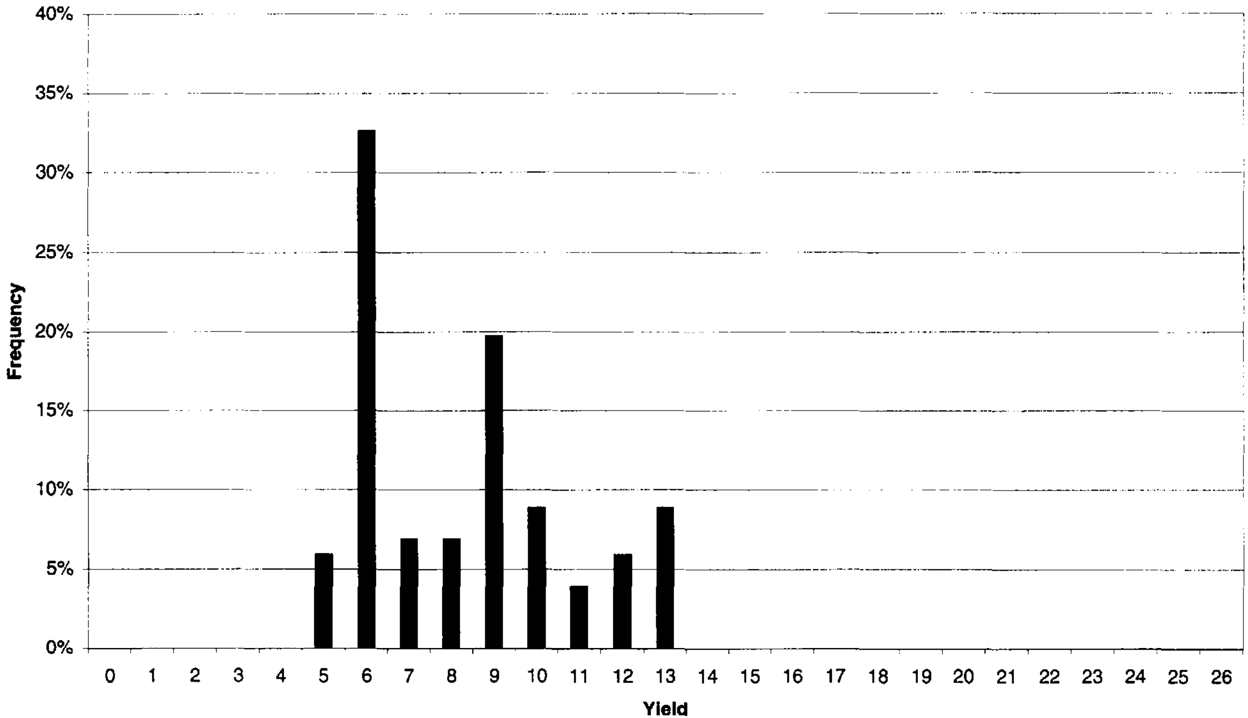
Note: Model parameters from CKLS estimates: $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$

FIGURE 18
HJM Simulation
1 Year Yield Distribution



Note: Model parameters from Amin and Morton: σ =0.0485, γ=0.5

FIGURE 19
HJM Simulation
10 Year Yield Distribution



Note: Model parameters from Amin and Morton: σ=0.0485, γ=0.5