Surplus, Profit and Conditional Expectation

David R. Clark, FCAS

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Abstract

This paper presents an approach to the allocation of surplus and profit loads to different product lines or policies within a single insurance company. The method presented calculates the allocation for each line based on the expected value conditional on the full surplus being exhausted. The use of conditional expectations produces a familiar variance-based allocation for a number of specific cases, and variance is the least squares approximation in the general case. The formula is easily extended to include correlations between policies.

It is the intention of this paper merely to highlight the mathematical connection between conditional expectations and variances. Setting profit loads based on variance is already a popular and practical technique, and this additional theoretical support should encourage actuaries to keep it as one method in their "toolbox".

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Background and Statement of the Problem

One of the most important goals of a stock insurance company is to provide its stockholders with an attractive return on their investment. In order to operate, the company must have sufficient surplus to cover the possibility that actual losses are worse than the original projections, and therefore have the ability to make good on the monetary promise of the insurance policies sold. The total amount of surplus required may be determined by regulatory formulas such as risk-based capital or by "probability of ruin" models.

Roughly speaking, this surplus is the investment made by the stockholders. In theory, the stockholders will demand a return on this investment based on their evaluation of the variability of results and the correlation with their other investments. The stockholder's primary concern is the size and variability of the overall company return; the relative performance of individual lines of business or policies within the company's book of business is largely a matter of indifference.

The challenge to insurance company management is how to set profit targets for individual lines of business and policies such that the overall company target can be met.

For purposes of this paper, we will assume that the overall dollars of profit load and surplus are given, and we will focus on the allocation problem only. For simplification, we will also ignore such real-world complications as differing investment income streams and federal income taxes.

Outline of the Conditional Expectation Approach

We begin by assuming that the company has only two lines of business, for which the losses will be denoted "x" and "y". The pure premium for each of these lines is the mathematical expected value: E[x] or E[y].

The total amount of funds required to be available to cover these losses is given as "T". This may be considered the total assets available to pay claims, and is made up of the pure premiums, profit loads and overall surplus.

$$T = E[x] + P_x + E[y] + P_y + \text{Surplus}$$

$$\text{where } P_x = \text{profit on line x}$$
$$P_y = \text{profit on line y}$$

For internal company analysis, the surplus may also be distributed to individual lines such that $S = S_x+S_y$. As a simplifying assumption, we will let the ratio of profit load to allocated surplus be a constant for all lines of business; that is, $P_x/S_x = P_y/S_y$.

The allocation problem is to determine how to split the excess of "T" over E[x]+E[y] in a reasonable manner. This can be done based on the conditional expectation of each line of business, which answers the question: "what is the expected value of each line given that total losses for all lines equal T"?

The notation is:

$E[x|x+y=T]$ = the expected value of x, given that x+y=T

From this expression, it is obvious that

$$T = E[x|x+y=T] + E[y|x+y=T] = E[x]+P_x+S_x + E[y]+P_y+S_y .$$

The profit and allocated surplus for line "x" is then

$$P_x + S_x = E[x|x+y=T] - E[x].$$

The general form of the conditional expectation is given by:

$$E[x \mid x+y=T] = \frac{\int_0^T x\, f_x(x) f_y(T-x)dx}{\int_0^T f_x(x) f_y(T-x)dx}$$

The Normal Distribution

As a first case, we will assume that each of the two lines of business are normally distributed with means and standard deviations μ_x , μ_y , σ_x and σ_y, and that the two lines are independent.

The expression for the conditional expectation is:

$$E[x \mid x+y=T] = \mu_x\left(\frac{\sigma_y^2}{\sigma_x^2+\sigma_y^2}\right)+(T-\mu_y)\left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)$$

Based on a rearrangement of terms, the allocated profit and surplus for line x is therefore given in proportion to its variance:

$$E[x \mid x+y=T]-E[x] = (T-E[x]-E[y])\left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)$$

For the Normal distribution, this expression is exact and can be easily generalized to include additional lines of business and correlations between lines:

$$E[x \mid x+y+z=T]-E[x] \quad = \quad (T-E[x]-E[y]-E[z])\, Z_x$$

$$\textit{where} \qquad Z_x = \left(\frac{\sigma_x^2 + \sigma_x\sigma_y\rho_{xy} + \sigma_x\sigma_z\rho_{xz}}{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\sigma_x\sigma_y\rho_{xy} + 2\sigma_x\sigma_z\rho_{xz} + 2\sigma_y\sigma_z\rho_{yz}} \right)$$

From this expression, the two extreme cases of perfect independence and perfect correlation result in an allocation by variance and standard deviation, respectively:

Perfect Independence	Perfect Correlation
$\rho_{xy} = \rho_{xz} = \rho_{yz} = 0$	$\rho_{xy} = \rho_{xz} = \rho_{xz} = 1$
$Z_x = \dfrac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}$	$Z_x = \dfrac{\sigma_x}{\sigma_x + \sigma_y + \sigma_z}$

Other Distributions: Gamma and Exponential

The results above are derived from the Normal distribution, which does have some problems when applied to typical insurance operations. First, the Normal distribution allows for negative values, which cannot take place with insurance claims. Second, the distribution is symmetrical about the average value whereas many insurance distributions are highly skewed.

Another example of conditional expectation is the use of two gamma distributions with a common scale parameter.

$$f_x(x) \quad = \quad \frac{\beta^{\alpha_x} x^{\alpha_x - 1} e^{-x\beta}}{\Gamma(\alpha_x)} \qquad\qquad f_y(y) \quad = \quad \frac{\beta^{\alpha_y} y^{\alpha_y - 1} e^{-y\beta}}{\Gamma(\alpha_y)}$$

$$E[x] = \frac{\alpha_x}{\beta} \qquad\qquad E[y] = \frac{\alpha_y}{\beta}$$

$$Var(x) = \frac{\alpha_x}{\beta^2} \qquad\qquad Var(y) = \frac{\alpha_y}{\beta^2}$$

From these distributions, the conditional expectation is expressed as:

$$E[x \mid x+y=T]-E[x] \quad = \quad (T-E[x]-E[y])\left(\frac{\alpha_x}{\alpha_x + \alpha_y} \right)$$

Under this example, the profit and surplus is distributed in proportion to variance, as well as to the expected values of each line. This is based on the assumption of a common scale parameter for both lines. What we are really assuming is that the two lines are made up of similar risks, but that one line writes more than the other.

An expansion of this approach would allow for different scale parameters for the two lines. Unfortunately, the mathematics quickly becomes intractable. A simpler example is to allow each line to have an exponential distribution.

$$f_x(x) = \frac{e^{-x/\mu_x}}{\mu_x} \qquad\qquad f_y(y) = \frac{e^{-y/\mu_y}}{\mu_y}$$

$$E[x] = \mu_x \qquad\qquad E[y] = \mu_y$$

$$Var(x) = \mu_x^2 \qquad\qquad Var(y) = \mu_y^2$$

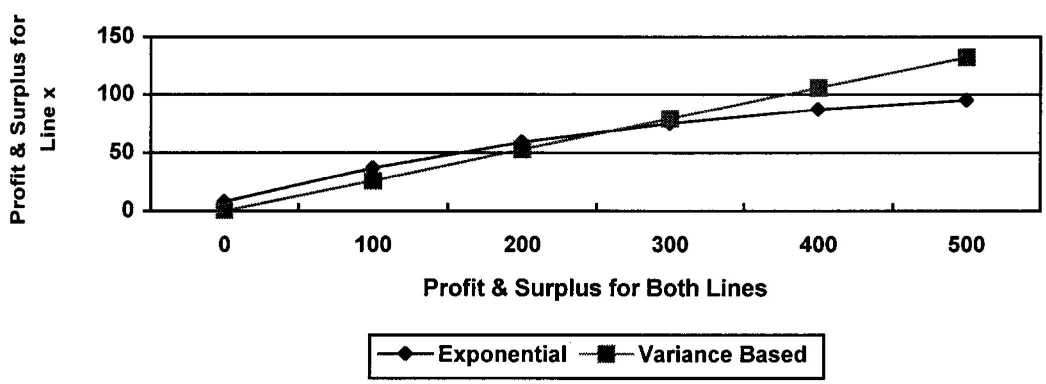
In this case, the conditional expected value is expressed as:

$$E[x \mid x + y = T] \quad = \quad \frac{\mu_x \mu_y}{\mu_x - \mu_y} - T\left(\frac{e^{-T/\mu_x}}{e^{-T/\mu_y} - e^{-T/\mu_x}}\right)$$

The allocation of profit and surplus is no longer seen to be in proportion to the variances by line of business. However, as the table below shows, the variance-based measure is not a terrible approximation. In fact, it represents a regression line fit to the conditional expectation (at all possible T values), as the next section will show.

Line x $\mu_x = 75$
Line y $\mu_y = 125$

	Scenarios for Available Surplus					
T =	200	300	400	500	600	700
Total Surplus =	0	100	200	300	400	500
Estimated Profit & Surplus for Line x						
Exponential	8	37	59	75	87	95
Variance-based	0	26	53	79	106	132



A Linear Least Squares Approximation

The mathematics for most distributional forms rapidly becomes too complex for the conditional expectations to be calculated directly. However, it is easy to estimate the linear least squares approximation. This is illustrated by the case in which there are two independent lines of business.

$$LS = \int_0^\infty \int_0^\infty [(x - E[x]) - (x + y - E[x] - E[y])Z]^2 f_x(x) f_y(y)\,dx\,dy$$

The portion of profit and surplus allocated to line x is represented by "Z". This number can be found by setting

$$\frac{\partial LS}{\partial Z} = 0$$

The resulting "Z" reproduces the variance formula.

$$E[x \mid x + y = T] \approx E[x]\left(\frac{\sigma_y^2}{\sigma_x^2 + \sigma_y^2}\right) + (T - E[y])\left(\frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}\right)$$

Finally, it is interesting to note that the form of this expression is very similar to the Bayesian credibility formula, familiar to most actuaries:

$$E[x_2 \mid x_1] \approx E[x]\left(\frac{\sigma_W^2}{\sigma_B^2 + \sigma_W^2}\right) + x_1\left(\frac{\sigma_B^2}{\sigma_B^2 + \sigma_W^2}\right)$$

where σ_B^2 = variance "between", or variance of hypothetical means
σ_W^2 = variance "within", or process variance
$E[x]$ = *a priori*, or original estimate of expected value
x_1 = actual loss in the first period
$E[x_2|x_1]$ = Bayesian estimate for the second period

The Bayesian credibility formula is likewise an exact expression for a limited number of distributional forms, and is the linear least squares approximation in the more general case.

Conclusion

This paper suggests an alternative way of viewing the allocation of profit and surplus based on expected losses conditional on total losses for the insurance company exactly equaling the available assets. This model leads directly to a variance-based allocation method.

The overall surplus and needed profit are given in the original balancing equation.

$$P+S = T-\sum_j E[x_j]$$

where

P	total needed profit for the company
S	total surplus for the company
T	total assets available to pay claims
x_j	losses for line of business "j"
$\sum_j$	summation over all lines

For a given line of business "i", the profit and surplus are allocated based on the conditional expected value.

$$P_i+S_i = E[x_i \mid T=\sum_j x_j]-E[x_i]$$

The least squares approximation to the conditional expected value is calculated based on variances and correlations between lines.

$$P_i+S_i \approx (P+S)\left(\frac{\sigma_i \sum_j \sigma_j \rho_{ij}}{\sum_k \sigma_k \sum_j \sigma_j \rho_{kj}}\right)$$

Finally, this general formula has two special cases:

$$P_i+S_i \approx \begin{cases} (P+S)\left(\dfrac{\sigma_i^2}{\sum_j \sigma_j^2}\right) & \textit{perfect independence} \\ \\ (P+S)\left(\dfrac{\sigma_i}{\sum_j \sigma_j}\right) & \textit{perfect correlation} \end{cases}$$