CASUALTY ACTUARIAL SOCIETY FORUM

Winter 1998
Including the Ratemaking Call Papers

CASUALTY ACTUARIAL SOCIETY
ORGANIZED 1914

Printed for the Society by
Colortone Press
Landover, Maryland

NOTICE

The Casualty Actuarial Society is not responsbile for statements or opinions expressed in the papers in this publication. These papers have not been reviewed by the CAS Committee on Review of Papers.

The Casualty Actuarial Society Forum
Winter 1998 Edition
Including the Ratemaking Call Papers

To CAS Members:

This is the Winter 1998 Edition of the Casualty Actuarial Society *Forum*. It contains eight Call Papers on the topic of ratemaking, which are to be presented at the Ratemaking Seminar, March 12-13, in Chicago, Illinois.

The Casualty Actuarial Society Forum is a non-refereed journal printed by the Casualty Actuarial Society. The viewpoints published herein do not necessarily reflect those of the Casualty Actuarial Society.

The CAS *Forum* is edited by the CAS Committee for the Casualty Actuarial Society *Forum*. Members of the committee invite all interested persons to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but the paper's content must be relevant to the interests of the CAS membership. Members of the Committee for the Casualty Actuarial Society *Forum* request that the following procedures be followed when submitting an article for publication in the *Forum*:

1. Authors should submit a camera-ready original paper, and two copies.
2. Authors should not number their pages.
3. All exhibits, tables, charts, and graphs should be in original format and camera ready.
4. Authors should avoid using gray-shaded graphs, tables, or exhibits. Text and exhibits should be in solid black and white.
5. Authors should submit an electronic file of their paper using a popular word processing software (eg., Microsoft Word and WordPerfect) for inclusion on the CAS Web Site.

The CAS *Forum* is printed periodically based on the number of call paper programs and articles submitted. The committee publishes two to four editions during each calendar year.

All comments or questions may be directed to the Committee for the Casualty Actuarial Society *Forum*.

Sincerely,

Robert G. Blanco, CAS *Forum* Chairperson

The 1998 Ratemaking Call Papers
Presented at the
1998 Ratemaking Seminar
March 12-13, 1998
Chicago Hilton and Towers
Chicago, Illinois

The Winter 1998 Edition of the CAS *Forum* is a cooperative effort of the CAS Continuing Education Committee on the CAS *Forum* and the Research and Development Committee on Ratemaking.

The CAS Committee on Ratemaking is pleased to present for discussion eight papers prepared in response to its 1998 Ratemaking Call Paper Program. These papers include papers that will be discussed by the authors at the 1998 CAS Seminar on Ratemaking, March 12-13, in Chicago, Illinois.

Table of Contents

Ratemaking Call Papers

The Balancing of Ratemaking Assumptions and Annual Financial Planning Assumptions

by Scott Anderson, FCAS

"THE BALANCING OF RATEMAKING ASSUMPTIONS AND ANNUAL FINANCIAL PLANNING ASSUMPTIONS."

Submitted by Scott Anderson, FCAS
with Prologue by Martha Winslow, FCAS

Abstract

When an elaborate operational and financial plan is prepared for the following year, including assumptions regarding prospective rate changes, goals are made with regard to premium levels and profitability. If certain assumptions such as catastrophe loads, loss trends and the effects of variability are not explicitly linked to the assumptions used for ratemaking on the product and state level, a built-in bias may be created for either rate inadequacy or rate redundancy that does not deliver the results as shown in a financial plan for a business segment. The goal of this paper is to show some of the pitfalls and provide basic ideas for balancing the ongoing ratemaking effort to the annual financial plan. This is particularly important in the current environment of changing catastrophe expectations and the increasing involvement of actuaries in financial planning.

Prologue

The premise of this paper is that for a given segment of business the assumptions that underlie the ratemaking calculations should be reconciled with the assumptions underlying the financial projections that are a part of the annual operational plan. As actuaries get more and more involved in the running of the business they need to become more than just purveyors of actuarial technique. They need to think like business people and understand the implications of their various work products and how they tie together.

A company's annual operational plan will consist of objectives for the year, initiatives designed to help the company achieve those objectives and a translation of all that into premium, loss and expense projections for the upcoming year. It is highly likely that the company actuary will be asked to do this translation. The work will consist of taking current experience and projecting it forward making various assumptions about rate and value changes, loss trend, cost of the operation, etc. Reflected in those assumptions will be the expected effect of the various initiatives on the specific actuarial assumptions.

At another time of the year the company actuary will be asked to calculate the indicated rate need for the business. Again, the actuary will use actuarial assumptions to project historical experience into the future. In this exercise the goal is to determine the rate level needed to attain the profit levels required by the company. Business executives will use these indications to make decisions about what rates to file for the product in the states that it is offered.

Using the loss trend as an example, how might the loss trend used to develop the operational plan and the loss trend used to develop the rate indications compare? Presumably since the two work products are done at different times in the year, would not the actuary want to reflect the very most recent information available for each? Would the actuary reflect all the same estimated effects of the planned initiatives in the rate indication even before there was enough experience to determine whether the action had the intended effect? How should the loss trend be handled if ratemaking is done at a finer level of detail than the financial plan? These and other questions would all have to be answered situationally by the actuary doing the work. The point here is not that the assumptions used in the financial plan and ratemaking need to be identical, but that the actuary needs to understand why the assumptions are either the same or different.

The operational plan and its attendant financial objectives are intended to be met each year. In order to make that happen business executives need to make decisions consistent with that plan. When the actuary promulgates an indicted rate need, the company executives need to understand how that rate indication relates back to the operational plan. It is with that understanding that they will be able to meet their financial goals. The actuary needs to understand this link. S/he can play a vital role in meeting the company's objectives by providing the analysis that allows the operational plan to be reconciled with the rate indications.

The key assumptions that need to be reconciled include the expected level of profitability, the loss trend, the load for catastrophes, any large losses that are smoothed, and expenses. These are all assumptions that will either change from one work product to another either because of the time period used or the analytical technique used.

The paper that follows walks through the specifics of how this reconciliation can be done for these key assumptions. Being cognizant of the need for the reconciliation is one thing and executing it is another. This paper addresses the execution of the premise described in this prologue.

"THE BALANCING OF RATEMAKING ASSUMPTIONS AND ANNUAL FINANCIAL PLANNING ASSUMPTIONS."

When an elaborate operational and financial plan is prepared for the following year, including assumptions regarding prospective rate changes, goals are made with regard to premium levels and profitability. If certain assumptions such as catastrophe loads, loss trends and the effects of variability are not explicitly linked to the assumptions used for ratemaking on the product and state level, a built-in bias may be created for either rate inadequacy or rate redundancy that does not deliver the results as shown in a financial plan for a business segment. My goal is to show some of the pitfalls and provide basic ideas for balancing the ongoing ratemaking effort to the annual financial plan. This is particularly important in the current environment of changing catastrophe expectations and the increasing involvement of actuaries in financial planning. The following is an actual project, some of the details have been changed to protect confidentiality.

Introduction

This paper is based around a generic model for calculating a rate indication. The model selected uses the loss ratio method and is fairly standard among mid-sized personal insurance carriers. All segments of the book are analyzed at the state/product/coverage level and certain elements are aggregated to similar levels as the financial plan. If done at the appropriate time of year, this allows for comparison to the annual financial plan as opposed to the typical state by state analysis done throughout the year. Excluded from this discussion are any specific comments regarding the calculation of the permissible loss ratio and any other issues not related to the development of expected losses and their effects. I will discuss some of the specific elements that we found to be at issue. Many elements such as Loss Development are not discussed but are assumed to be in agreement with financial planning assumptions. The specific elements would vary based on the type of products and the size of book that is analyzed. The products we are looking at are all considered personal lines therefore we can immediately exclude such issues as retro premiums and any analysis of actual premiums versus manual premium. Any issues concerning actual versus projected premiums are considered exposure equivalents and should not have an effect on the projected loss level, although premium plans do have an effect on expenses and profit projections. We are looking at as many as one million policies in a medium sized book, so the view that we are taking is high level and only as detailed as state/product/coverage group.

Following are the specific elements discussed in this paper: Selected Trends, Complement of Credibility, Catastrophe Loading and/or Excess Wind and Water Loads, Large Loss Loading and Indicated Rate Need. These elements are aggregated to match the same level of detail as used in the financial planning process to allow for comparison.

Selected Trends

The information includes: industry trends by coverage for state and countrywide, internal company trends by coverage and program for state and countrywide, selected trends by state and program.

The programs include: non-standard auto, standard auto, preferred auto, standard homeowners, preferred homeowners, packaged policies with all personal lines coverages offered.

All of the indications were trended to a common new business effective date, this allows the mathematics to be straight forward when comparing to a financial plan on an annual basis. Additional trend will accrue on changes taken at later dates, this can be easily adjusted on a state by state basis.

TABLE 1

	Preferred		Standard		Industry
Coverage	Observed	Selected	Observed	Selected	Trends
BI	2.8%	2.3%	2.5%	2.3%	-1.9%
PD	8.0%	7.5%	8.5%	8.2%	7.5%
MED	3.2%	2.3%	2.9%	2.3%	see BI
UM	3.1%	2.3%	3.3%	2.3%	see BI
PIP	3.5%	2.3%	3.8%	2.3%	3.2%
LIABILITY	4.1%	3.3%	4.0%	3.5%	1.6%
COMP	5.1%	4.9%	4.8%	4.6%	4.1%
COLL	7.7%	7.5%	7.8%	7.4%	7.5%
PHY DAM	6.7%	6.5%	6.7%	6.4%	7.5%
TOTAL	4.8%	4.2%	4.7%	4.3%	3.2%

The SELECTED above in Table 1 is the weighted totals of the selected trends used in the calculation of the indications in each program, state and coverage. The OBSERVED above is the observed countrywide trend determined on an aggregate countrywide basis with the effects of large losses and catastrophes removed. The observed trend on aggregate data is often not the weighted average of trends that are determined at a more homogenous level.

A significant difference may exist between the indicated rate need as projected versus the financial "plan". The financial plan includes anticipated changes in claims and underwriting processes, these changes are only included in the historic trend as those effects become part of the experience. For that reason, additional analysis is needed to adjust for planned and expected future changes to the loss trend. In order to explicitly separate these discretionary internal forces from the projection of profitability we calculated the indications such that the "pure" indication does not include anticipated internal effects. An adjustment is then needed that allows for these anticipated effects to be explicitly demonstrated to management. The prospective rate change decision can then be made intelligently as part of the entire product management process.

There is a significant level of uncertainty in calculating the effect of underwriting and claim actions. The needed effect is more often known, while the actions are created to meet those needed effects. Action plans usually include a significant amount of negotiation, management accountabilities should be set targeting the desired effects. The difference between projection and optimistic planning needs to be understood and facts need to be separated from wishful thinking during the estimation process.

To explicitly determine the adjustment to the indication for a prospective change in the trend, a minor modification to the model that allows for the selection of separate historic and prospective trends was made. Sensitivity testing with time periods held constant, varying levels of loss and varying selected historic trends indicated that there is a very robust relationship between the change in the indication and the difference between the two selected trends. Given our specific policy terms and implementation lags, we found this relationship to be a 1 to 1.6 ratio. The following is an example using numbers:

The selected historic trend is 4%.

The resulting indication is +3%.

The selected prospective trend is +5.5%

The resulting change in the indication is: (5.5% - 4.0%) * 1.6 = +2.4%

The indication adjusted for this differing planned prospective trend is now 5.4%, 3% + 2.4%, due to the expectation of a higher trend in the future versus the empirical trend. These adjustments can be used to account for expected changes in the book of business, claims handling practices or industry aggregate information.

It should be noted that the ratio stated above, 1 to 1.6, is dependent on the permissible loss ratio and issues regarding fixed expense versus variable expense as well as time lags and policy terms. The ratio for a particular product should be determined as explained above with varying inputs.

We found that the weighted averages of the selected trends were significantly lower than the aggregate trends. The aggregate trends were more stable, and were considered more applicable from a financial planning viewpoint. This indicated that our bias was toward assuming that there has been and would be an overall trend in the future that was less than actually projected. This is often due to a bias in the selection of a trend based on many different sources but rarely ever selecting from the high end of the range. It must be decided if the average of the selected trends is appropriate given our actual experience and the plan for the following years. If a difference is appropriate, documentation should support the reasons.

Complement of Credibility

Credibility weighted indications are used when, due to the amount of variability, the data analyzed will not give a significant answer. A credibility weighted indication will be an answer that falls between the actual indication and a complement of credibility. How close this final answer is to the initial indication depends on the volume and variability of data used in the analysis.

A common practice of using the annual trend as the complement of credibility assumes that rates are currently adequate. This assumes that the current rate, increased by trend, would be a reasonable default if credibility was found to be zero. This may be faulty and is biased if rates were not adequate.

In our previous methodology, the selected annual trend was used for the complement of credibility. If the total indication is greater than the selected trends, the following holds true.

(Total before Credibility > Total after Credibility > Total of Trends)

Adjusting for this bias caused issues when discussing with non-actuaries. Many states with small business volume and low levels of actual loss activity received significant swings due to this change.

This entire book of products analyzed over five years is considered well above the standard of credibility. Therefore, the total indication after credibility standards are applied should not be less than the total indication before credibility standards are applied. (Total before Credibility ≅ Total after Credibility)

For this reason, the complement of credibility selected is the countrywide indication for that program and coverage. If the countrywide indication is still not considered credible, the total across all programs for that coverage is used. If that total is still not considered credible, the total of all coverages is used. In any case, a credibility complement is available that allows the total indication for the book of products to remain the same.

This choice of the complement of credibility was not used in the past due to the lack of availability of the countrywide totals with consistent loss periods and effective dates. We believe this new choice removes the bias inherent in other choices of the credibility complement.

Catastrophe Loading and/or Excess Wind and Water Loads

The following detail is offered to explain the difference in indications and the financial plan that is due to the varied methods of smoothing and handling weather related losses. An explicit number should be developed that compares the net difference of using the two different loading procedures.

TABLE 2

Process	Losses Selected	Detail
Financial Plan	Excess Wind and Water	State Specific
Rate Indications	Defined by Catastrophe #	Countrywide

Our indication model uses the ISO Excess Wind and Water methodology. Our financial plan separates losses using the presence or absence of a Catastrophe number on the claim record. We had decided that due to the changing dollar threshold on the assignment of a Catastrophe number, we would plan catastrophe along with certain weather related causes of loss. While these two methodologies are not in perfect synchronization, we can attempt to balance the two and determine if the two different smoothing methodologies are both setting equivalent smoothed loads.

The financial plan for catastrophe and weather related losses is determined on a countrywide basis. This high level of detail created issues when reviewing a state with a higher probability of this type of loss. In the current indication analysis, the ISO Excess Wind and Water Loads by state are used. This differentiates between the different loss potential in the different states and product lines. The ISO loads used are as published in the appropriate Circular.

The effect of smoothing will either have a net effect of removing loss dollars from the analysis or adding loss dollars to the analysis. To define the differences between the two methodologies, the net effects of each of the two smoothing methods were calculated. Loss dollars used for this calculation are undeveloped losses valued at 12/31/96. The calculation was done separately for coverages and products.

The catastrophe loading, or smoothing, should not significantly change the level of loss on a sizable book of business when looked at in total over time. Any bias should be understood and adjusted.

Large Loss Loading

The large loss loading, or smoothing, should not significantly change the level of loss on a sizable book of business when looked at in total. Given the size of our book, we wanted to determine if the large loss load actually balanced with the total of our large losses for the previous years. Then we needed to determine if this level of loss is what would be expected in the coming years that are shown in the financial plan.

The Large Loss Loads in the past were calculated countrywide. This high level of detail created issues when reviewing a state with a possibility of large loss less than countrywide. In the current analysis, regional loads are determined separately for each program. This analysis differentiates between the different large loss potential in the different regions of the country and product lines.

TABLE 3

Region	Standard	Preferred	Package
Great Lakes	1.07	1.09	1.18
South	1.02	1.04	1.16
Coastal	1.04	1.08	1.14
North	1.02	1.04	1.12

The Large Loss Loads are equivalent to our actual large losses over the five year period. This ensures integrity with our financial plan and our view that our total large losses are considered credible over a five year period.

Indicated Rate Need

The financial plan includes a planned rate and value change over each of the following years. Both the written and the earned effects of the rate changes are explicit in the plan. These rate change plans are based on the countrywide line of business data used in the financial planning process. We need to know if the rate making model is now giving us different rate indications when determined at the program/state/coverage level.

The rate changes are totaled and compared to the plan. If we have done the exercises above and know that we have removed any biases from our methodologies, the more detailed view should be providing us with the more credible answers. If we then compare these new indicated rate actions with our financial plan we should be able to tie together rate actions, claim actions, underwriting actions and expected profitability.

Summary of Findings

As we went through this process for the first time, we found significant differences between the definitions and applications of our assumptions. Of significant note were the catastrophe smoothing, loss trend and the complement of credibility.

The differing methods of handling catastrophes are based on the different uses of data. One view is to explain past experience and the expected future effects on the following year's finances. The other view is the expected values used in the pricing models for the existing book and mix of business. Both views need to be used, but an understanding and method of translating must be determined.

The trend is critical in the calculation of the indication, it is all too easy to insert expectations into the selection process. Any planned expectations different from projections should be documented and the underlying actions understood. None of us want to project a large trend that is not realized as well as vice versa. Selected trends were adjusted in the final output to reflect the overall trend level, this was done to remove bias.

The complement of credibility was determinable after all of these indications were completed. Other choices are definitely available, but the financial plan must link to the final selection.

A note to data integrity, many small data issues can leverage themselves into significant issues. Determining certain ratios without ALAE and then using those numbers against losses including ALAE can have a noticeable effect on the final indication. Care must be taken to think through, test and document assumptions to determine if material differences could arise.

The gains from this exercise were significant:

- We have a better understanding of our trend and factor selection methods.

- We are able to show specific opportunities for attainment of the financial plan.

- We are much more prepared to explain the differences as viewed by underwriting professionals and financial professionals.

Copyrighted Material

Acknowledgments

Staff members for their diligent work on the project.

The author of the Prologue for her inspiration, guidance and freedom.

Pricing the Hurricane Peril—Change is Overdue

by David R. Chernick, FCAS

Pricing the Hurricane Peril - Change is Overdue

by David R. Chernick, FCAS

Introduction

The hurricane peril is currently a very hot topic at Casualty Actuarial Society meetings and seminars. The advent of this interest occurred in the aftermath of Hurricane Andrew, which made landfall in Homestead, Florida on August 24, 1992. Hurricane Andrew damaged or destroyed thousands of buildings and caused an estimated $16 billion in insured losses. Insured damage of this proportion was unprecedented, and could have been much greater had the hurricane taken a slightly different but equally likely track. In response, actuaries began to seriously reevaluate their ratemaking procedures for this peril.

In this paper I will document the history of ratemaking techniques used for the hurricane peril. Non-insurance data will be presented to show that historical techniques and typical insurance incurred loss data are inappropriate to properly price this peril. I will concentrate on expected loss costs for hurricanes, or in other words the mean of the potential loss distribution. The concept of risk load will be left to other authors in our society.

History

The hurricane peril has historically been covered under various property insurance products, including but not limited to extended coverage, commercial multi-peril and homeowners. The first reference to wind ratemaking that I found in the Casualty Actuarial Society journals was in 1951. Mr. M.H. McConnell wrote: "Similar exposure to catastrophic losses exists with respect to other coverages written by Fire Insurance Companies such as Extended Coverage. The November 25, 1950 windstorm affecting thousands of policyholders in New England and the Middle Atlantic States is a recent example of such a catastrophe. The estimated losses for this storm are almost $200,000,000 and the number of claims may reach 500,000. Because of low frequency, slavish adherence to indicated rate levels might result in violent fluctuations in rates as well as violent fluctuations in relativity. To achieve a desirable degree of stability, exercise of underwriting judgment is required in selecting rate levels." [1]

Hurricanes are definitely low frequency, potentially high severity events. Even though the November 25, 1950 storm was not officially a hurricane, there is evidence that members of the society were concerned with the impact this type of event could have on ratemaking. Although it is difficult to determine how many years of ratemaking data were used to generate rate level indications for extended coverage policies at that time, it appears that the number of years used to price the wind peril was small. Mr. McConnell's solution is that underwriting judgment be used in selecting rate levels to account for the low frequency of severe storms.

In 1949, Mr. J. H. Finnegan documents the beginning of catastrophe coding. "For the purpose of obtaining information on the losses paid for the various tornadoes, hurricanes and similar catastrophes which occur each year, the National Board began in April, 1949 the practice of assigning a catastrophe serial number for all such occurrences. Such numbers are assigned whenever preliminary estimates indicate that the loss will amount to $1,000,000 or more in any state." [2] Clearly, insurance data for hurricanes is not available prior to 1949. Even after 1949, it has been my experience that detailed company data for individual hurricanes has not been kept until recently. In any case, historical ratemaking data for the hurricane peril is limited.

In 1959, Laurence H. Longley-Cook documents for the first time in the records of our society the number of years used in pricing the "windstorm" peril for extended coverage policies. Ten years of historical experience was used. "Rate making for extended coverage abounds with interesting actuarial problems many of which have received little attention. Since windstorm is by far the major peril, it is important to realize that owing to the correlation between losses - one storm involving many thousands

[1] M. H. McConnell - "A Casualty Man Looks at Fire Insurance Rate Making" PCAS Volume XXXVIII, 1951, pp. 103-104.

[2] J. H. Finnegan - "Statistics of the National Board of Fire Underwriters" PCAS Volume XLIII, 1956, pp. 93.

of losses - normal standards of credibility do not apply. This is being recognized by using 10 years rather than 5 years loss experience for rate adjustment. However, in states exposed to hurricanes, the 10-year loss experience may have an abnormal or subnormal number of such storms, and even longer term weather studies make it difficult to establish the normal frequency of hurricanes. The problem is further complicated by the conflicting views of weather men on the relative bearing on trends of sunspot cycles and longer term climatic changes." [3] Mr. Longley-Cook cautions that a 10 year experience period for hurricanes is not long enough for ratemaking, but does not offer a solution.

In 1960 Ernest T. Berkley wrote, "The seminar concentrated on a Homeowners policy on an indivisible premium basis as a prime example of a multiple peril policy.... The removal of the restrictions of the Appleton Rule in 1949 made it possible to combine fire and extended coverage, theft and liability coverages in a single policy which could be written by either a casualty or a fire company.... After covering the foregoing historical aspects the seminar proceeded with a discussion of the principal points brought out in the paper and review, which may be summarized as follows: 1. ... ... 5. Several miscellaneous points including the variation in loss frequency for windstorm versus other coverages and the associated windstorm catastrophe hazard." [4]

In this paper we learn that prior to 1949 the Appleton rule prevented combining coverages, and removal of restrictions led to the creation of multi-peril policies which included the hurricane peril. Mr. Berkeley writes about a seminar that concentrated on homeowners multi-peril policies and stated that there were concerns regarding windstorm frequency and windstorm catastrophe potential. Again, the issue of the wind peril was discussed, but no solutions were offered.

Prior to 1957, rates for multi-peril policies were developed by combining rates for the component coverages. Beginning in 1957, at least one company began using its own homeowners only data for ratemaking. Today, many companies use company specific data for homeowners ratemaking.

---

[3] Laurence H. Longley-Cook - "Notes on Some Actuarial Problems of Property Insurance" PCAS Volume XLVI, 1959, pp. 80.

[4] Ernest T. Berkeley - "Rate Making and Statistics for Multiple Peril Policies" PCAS Volume XLVII, 1960, pp. 231-233.

A cite of LeRoy J. Simon from his 1961 paper follows: "Referring to Homeowners rating history, it started as a sum of components and remained this way for some time. As component rates changed, so did the Homeowners rate change. In 1957 at least one company swung over to using Homeowners experience to set the Homeowners rates.... Two important features that couldn't be discussed too thoroughly were reinsurance problems and the catastrophe problem. The latter question arose in connection with rate making for all the property coverages as a single unit. The presence of a hurricane in two years would distort the figures, so would the absence of a hurricane in two years distort the figures." [5] Again frequency variation for the hurricane peril was a major concern. Yet again, no solution was offered.

In 1962 Edward S. Allen described another seminar on package policy ratemaking. "A discussion of principles for package policy ratemaking at the present stage of package policy development will obviously produce more questions than answers....Since discussions in the two sessions of the seminar developed in quite different directions, it might be of interest to the participants as well as others, to list some of the comments and opinions expressed incidental to the general conclusions as summarized above. An abbreviated list is as follows:

1. ...

8. Catastrophe coverage and small loss coverage should be treated differently." [6]

Consistent with prior authors Mr. Allen suggested that catastrophe coverage be treated differently.

Frederic J. Hunt, Jr.'s paper "Homeowners - The First Decade" was published in the Proceedings in 1962. This paper gives an excellent overview of the actuarial perspective of the first ten years of the homeowners policy. A relevant section follows: "The question of credibility and the treatment of catastrophes in Homeowners rate-making, together with some related problems, need actuarial study

[5] LeRoy J. Simon - "Rate Making for Package Policies" PCAS Volume XLVIII, 1961, pp. 205-206.
[6] Edward S. Allen - "Package Policy Ratemaking" PCAS Volume XLIX, 1962, pp. 66-67.

and I am hopeful that, at least when the history of the second decade of Homeowners is written, it will include an account of the satisfactory disposition of these items." [7]

The challenge of Mr. Hunt to find solutions to the problems of credibility and treatment of catastrophes was not answered. Maybe the lack of major hurricanes or other catastrophes caused this or maybe our Society had more pressing issues to address. Surprisingly, in the twenty five years between 1963 and 1989, only one property insurance paper was published in the journals of the CAS. That was Michael A. Walters' paper, "Homeowners Insurance Ratemaking", published in 1974. This paper is near and dear to actuaries of my generation since it was the major property insurance article on the principles of ratemaking exam syllabus. "By the same token, if no hurricanes or other catastrophes have occurred during the experience period under review (now five years in Homeowners insurance), it would also be a mistake to assume that the potential for catastrophe has vanished. Therefore, an averaging process is utilized whereby the actual incurred losses from catastrophic events during the experience period are removed and substituted by the expected value of such losses based upon a long range view of at least twenty years experience for that state." [8]

Mr. Walters continued the caution from the 1960's. He articulated the hurricane frequency problem quite well. In 1974 the standard homeowners ratemaking base was 5 years of data. However, Mr. Walters stressed that for catastrophes, at least 20 years of ratemaking type data should be used.

An attempt to address the ratemaking problems of the hurricane peril was ISO's excess wind procedure. This procedure was developed by ISO and first used in ratemaking sometime prior to 1990. Simply described, the ISO excess wind procedure developed an expected wind pure premium by splitting actual data into basic wind and excess wind components. The expected basic wind component is derived by a long term average (Non excess wind losses / Non wind losses). The expected excess wind component is derived by taking the ratio of excess wind to non-excess wind losses over a longer period

---

[7] Frederic J. Hunt, Jr. - "Homeowners - The First Decade" PCAS Volume XLIX, 1962, pp. 39.
[8] Michael A. Walters - "Homeowners Insurance Ratemaking" PCAS Volume LXI, 1974, pp. 23-24.

of time and supplementing state data with regional data. The ISO excess wind procedure is just a slightly more sophisticated technique that still uses a limited historical period of time.

The following quote is from Mark Homan's 1990 paper, "Homeowner Insurance Pricing." "The first adjustment made to these losses is for catastrophic losses. Catastrophe losses are relatively infrequent and do not affect each year similarly. The indicated rate level should include a provision for expected catastrophes, instead of those that happened to occur in the experience period. To make this adjustment, a longer time period, and possibly a larger body of data, is used to compensate for the infrequent nature of these losses. The procedure described here is very similar to the ISO excess wind procedure." [9]

Mr. Homan, although not directly referring to the hurricane peril, again warns a longer period of time is needed in the development of a ratemaking provision for catastrophes. He goes on to state for the first time in our actuarial literature that "a larger body of data" is "possibly" a solution. I believe it is self evident that a larger body of data (i.e. non-insurance data) is necessary to properly price the hurricane peril. Note that even after Hurricane Hugo in 1989, Mr. Homan advocated using the ISO excess wind procedure to price the hurricane peril.

Also in 1990, David H. Hays and W. Scott Farris directly addressed the hurricane peril in their paper "Pricing the Catastrophe Exposure in Property Insurance Ratemaking". A specific adjustment is suggested to bring the actual hurricane frequency to the frequency level indicated by 120 years of meteorological data and to bring the recorded severity to current cost and exposure levels.

"A company's hurricane data may be sparse. Therefore, it may be appropriate to modify company data or to substitute data from other sources. External data can be either historical or simulated.... One easy adjustment to a company's hurricane data that can be made is to adjust the frequencies of the various hurricanes in the company sample to reflect known historical frequencies over a longer period. The number of hurricane occurrences by wind speed and landfall is available from

[9] Mark J. Homan - "Homeowners Insurance Pricing" Casualty Actuarial Society 1990 Discussion Paper Program, pp. 727.

various sources for at least 122 years. If a company can identify the wind speed and the landfall for the hurricanes in its data, the adjustment to known frequencies can be accomplished by the following formula:

$$E(h) = H * \frac{F * Y}{N * 100}$$

Where,

E(h) = Expected Dollars of loss for an individual hurricane

H = Dollars of loss for the hurricane adjusted to current inflation and exposure distribution

Y = Number of years in the sample data

N = Observed number of occurrences by intensity and windspeed

F = Expected 100 year frequency from external sources."[10]

I will comment on this procedure more specifically in the frequency section of this paper.

In 1992, John Bradshaw and Mark Homan in their paper "Homeowners Excess Wind Loads" wrote: "The ISO procedure has its flaws. However, due to the difficulty in obtaining a sufficient volume of credible data for any other method, it remains the most widely used method. The adjustment outlined in this paper allows for the elimination of one of the major flaws in the ISO procedure, namely its reliance on past history as a representative sample of possible losses....

An additional shortcoming of the ISO procedure is that it fails to adjust for demographic shifts. In particular, it does not consider the increase in coastal exposures. The adjustment of the model

---

[10] David H. Hays & W. Scott Farris - "Pricing the Catastrophe Exposure in Property Insurance Ratemaking," Casualty Actuarial Society 1990 Discussion Paper Program, pp. 491-492.

reflects the current distribution of a company's book and can be updated periodically to reflect any shifts. This does not eliminate the ISO shortfalls since many of the years are still based purely on history. However, the additional year from the model will dampen this problem with the ISO procedure."[11] Messers. Bradshaw and Homan's contribution to CAS ratemaking procedures is basically that the ISO excess wind procedure can be improved by adding a year that represents a one in 50 year storm. The authors point our many flaws in the ISO wind procedure and there are other limitations not mentioned in this paper. Even the authors admit the adjustment will only "dampen" the "problem" with the ISO procedure. Simply put, the ISO excess wind procedure is not an appropriate tool for pricing the hurricane peril.

In 1996 Burger, Fitzgerald, White and Woods published a paper titled "Incorporating a Hurricane Model into Property Ratemaking," where they explain that ISO had decided to replace their excess wind procedure with data from a computer simulation model. They concluded: "After evaluating the limitations of the traditional loss smoothing approaches, ISO decided to use a computer simulation modeling approach for measuring the hurricane catastrophe peril."[12]

Also in 1996, Michael A. Walters and Francois Morin published "Catastrophe Ratemaking Revisited." They endorse using computer simulation models as a ratemaking tool, and conclude: "In summary, computer models are now capable of simulating catastrophic events and creating probabilistic models of reality that can be used to generated expected loss costs for catastrophe perils."[13] In the next sections of this paper I will expound on the limitations of using traditional insurance data to price the hurricane peril.

---

[11] John Bradshaw & Mark J. Homan - "Homeowners Excess Wind Loads: Augmenting the ISO Wind Procedure," Casualty Actuarial Society Forum, Spring 1992, pp. 49.
[12] Burger, Fitzgerald, White and Woods - "Incorporating a Hurricane Model into Property Ratemaking," Casualty Actuarial Society Forum Winter 1996, pp. 141.
[13] Michael A. Walters & Francois Morin - "Catastrophe Ratemaking Revisited (Use of Computer Models to Estimate Loss Costs)," Casualty Actuarial Society Forum Winter 1996, p.364.

Non-Insurance Data

Various meteorological data exists on Atlantic hurricanes since the late 1800's. The primary source of this meteorological data is the National Weather Service, specifically, publications NOAA Technical Report NWS23[14] and NOAA Technical Report NWS38[15]. In addition, "Tropical Cyclones of the North Atlantic Ocean 1871-1980"[16] was valuable. The quality and amount of data available is more extensive and more accurate for recent storms. Messers. Hays and Farris in their paper referred to 122 years of data, implying back to 1871. My analysis requires accurate landfall locations and identification of Saffir/Simpson category. Hurricanes prior to 1899 are not covered in the NWS reports, and thus I have decided to use the 98 years from 1899 to 1996 for this paper. Using National Weather Service reports and several other sources, I compiled Exhibit 1.

Exhibit 1 is a chart of the number of hurricanes that made landfall on the Gulf or Atlantic coasts of the United States for each year since 1899, broken down by Saffir/Simpson category. Some hurricanes made landfall more than once. For the purpose of this exhibit, a hurricane is counted each time it made landfall at hurricane strength. For example, Hurricane Andrew was counted twice, once in Florida and once in Louisiana. The assignment of a Saffir/Simpson category at landfall cannot be determined precisely and often requires some judgment. In addition, two storms listed in the National Weather Service publications were not counted in this list because it was determined that one actually made landfall in Mexico and the other in Canada.

In many years there were no hurricanes making landfall in the United States. In 1985, the most landfalls occurred (seven). In the 98 years listed, there were 176 landfalls or an average of 1.8 landfalls per

[14] NOAA Technical Report NWS 23, "Meteorological Criteria for Standard Project Hurricane and Probable Maximum Hurricane Windfields, Gulf and East Coasts of the United States," Washington, DC, September 1979, U.S. Department of Commerce, National Oceanic and Atmospheric Administration, National Weather Service.
[15] NOAA Technical Report NWS 38, "Hurricane Climatology for the Atlantic and Gulf Coasts of the United States," Silver Spring, MD, April 1987, U.S. Department of Commerce, National Oceanic and Atmospheric Administration, National Weather Service.

year. Only two storms were categorized as 5 on the Saffir/Simpson scale. These were a 1935 storm that made landfall in Monroe County, Florida and Camille in 1969 which made landfall in Hancock County, Mississippi.

Frequency

Is 10 or 20 or 30 years of typical ratemaking data enough to accurately price the hurricane peril? To test this, the data on Exhibit 1 was analyzed. Exhibit 2 was created from the data on Exhibit 1 and shows the number of landfalling hurricanes by decade. The 1990's are not yet a full decade and the two 1899 storms were not included in Exhibit 2. Even though we would not directly use this data for individual state ratemaking since it is for all states combined, it clearly demonstrates the variability of hurricane frequency. The number of hurricane landfalls in a decade varies from a high of 27 to a low of 14. Most experts agree that more intense storms cause proportionally more damage than less intense storms. Thus, from a ratemaking perspective a large portion of the loss cost will be attributable to the more intense storms. For the purpose of categorization, a major hurricane is defined as one of category 3 or higher on the Saffir/Simpson scale. The variation in hurricane landfall frequency is even more pronounced for major hurricanes, ranging from a low of 4 to a high of 10.

Turning now to state data, the variation in hurricane frequency is even greater. In the United States, rates are regulated by state. Ideally, from a ratemaking perspective rates should be made for homogeneous subsets of a state, (i.e. territories). Exhibit 3 is included for reference, and is a consolidation of all storms listed in Exhibit 1 by state of landfall. Hurricane landfall frequency differs significantly by state. To analyze this further I have selected Texas and South Carolina. Exhibits 4 and 5 show the hurricane landfall data for these states in the same format as Exhibit 2. For my simple analysis I have not counted hurricanes making landfall outside of Texas or South Carolina but causing damage to

---

[16] "Tropical Cyclones of the North Atlantic Ocean", NOAA, Asheville, NC, June 1978, Revised July 1981, Prepared by the National Climatic Center, Asheville, NC in cooperation with the National Hurricane Center and National Hurricane Research Laboratory, Coral Gables, FL.

properties located within those states. However, the potential for hurricanes making landfall outside the state being priced but causing damage in that state should be considered in determining rates.

I have added several rows of summary data to Exhibits 4 and 5. I have shown the total number of hurricanes making landfall in the latest 27 years (1970-1996). Also shown are rows labeled high and low. These are the sum of the three consecutive decades that had the highest and lowest number of hurricane landfalls, respectively. In order to more easily compare frequency I have added a row showing the annual frequency for the 97 year total and each of the three time periods just described. In 97 years there have been 32 hurricanes making landfall in Texas for an annual frequency of .330, or just less than 1 every 3 years. If we were to use historical insurance data from 1920 to 1949 the underlying frequency was .433 or 31.2% greater than the 97 year history. From 1950 to 1979 the underlying frequency was .200 or 39.3% less than the 97 year history.

Exhibit 5 displays the same type of data for South Carolina. The variation in frequency is similar to Texas, but the overall frequency is much lower. On average, a hurricane makes landfall in South Carolina once every eight years, and a major hurricane occurs about once every twenty years. As in Texas, when the shorter time periods are compared, there is significant variation in hurricane frequency.

On a statewide basis the hurricane frequency in a single 20 or 30 year period of data can differ significantly from the longer term mean. If the data for hurricane frequency is refined further to county or rating territory, the variation is even greater. There are many areas that had devastating damage from a hurricane in one year and long periods of no storms. This variation in landfall frequency is shown graphically on Exhibits 6 through 10, which display the tracks of major hurricanes by decade, beginning with the 1940's.

In the 1940's, 5 of the 8 major hurricanes made landfall in Florida (Exhibit 6). In the 1950's most of the activity was on the East Coast with only two storms making landfall in Florida (Exhibit 7). In the 1960's the activity moved to the Gulf of Mexico, with only Donna moving up the east coast after an initial landfall in the Florida Gulf (Exhibit 8). All four major hurricane landfalls occurred in the Gulf of Mexico during the 1970's (Exhibit 9). In fact, between 1961 and 1983 no hurricane made landfall on the eastern coast of the United States north of Monroe County, Florida. Finally, in the 1980's the six major storms were well dispersed (Exhibit 10).

Where will the next Atlantic hurricane make landfall? Going back to the hurricane history of South Carolina, Exhibit 11 displays the tracks of the 3 major hurricanes prior to 1989. No major hurricane on record made landfall near Charleston, SC. Exhibit 12 shows what the South Carolina major hurricane landfalls look like after 1989. This demonstrates that new and unique landfalls are possible, presenting an exposure to loss which historical ratemaking data will never capture.

Clearly, hurricane landfall frequency varies widely over time. The smaller the geographic area being considered, the greater the variation. Ten or twenty or even thirty years of historical data will not adequately capture the true underlying probability of a hurricane making landfall. In addition, for smaller geographical areas such as rating territories, 98 or even 122 years will not capture the true underlying frequency potential.

In their paper, Messers. Hays and Farris state that we can adjust for hurricane frequency. Essentially, their method adjusts the observed frequency for a finite number of years of rate making data to a long term frequency. The "adjusted" frequency is then applied to "current level" losses for each hurricane in the experience period. This procedure is clearly better than blindly using ratemaking data, yet it is still inadequate. Strictly from a frequency perspective, this adjustment method may produce appropriate frequency estimates for large geographic regions. However, if used for smaller geographic areas such as rating territories, even 122 years of data is not enough to capture the true underlying frequency. This method will also fail to account for new and unique landfalls. More importantly, this frequency adjustment does not account for the even greater variation in storm severity and the impact of a changing exposure base.

Severity

Reliance on historical ratemaking data to price the hurricane peril fails to accurately reflect expected severity for two major reasons. These are a changing exposure base and the large variation in severity of hurricanes. Several authors have presented possible techniques to adjust for the changing exposure base. I will not specifically comment on the adjustments suggested. However, in general if historical traditional ratemaking data is used in pricing the hurricane peril, the issue of a changing exposure base requires attention by the ratemaking actuary.

Traditional ratemaking techniques developed a catastrophe provision by using historical ratios of catastrophe losses to non-catastrophe losses. More recently I have seen the catastrophe provision calculated by comparing catastrophe losses to amount of insurance years. The second method is more responsive to one aspect of a changing exposure base (i.e. total amount of insurance). However, neither of these methods can properly capture the expected loss of the hurricane peril.

No book of business stays the same over a 10 year period, let alone 20 or 30 years. For illustrative purposes, assume you are using 25 years of actual insurance ratemaking data to price the hurricane peril. Assume further that the only hurricane to produce losses in this period in the state being priced was Zelda, a category 3 storm 20 years ago. Would the exact same storm today cause the same insured losses relative to either non-catastrophe losses or amount of insurance years? The answer is no.

A company's distribution of business by distance to the coast changes over time. The amount of insured damage Zelda caused twenty years ago is known, and it is related to the amount of business that was in areas of high winds. If a greater percent of the total business is closer to the coast today than it was when Zelda made landfall, then the loss per exposure will be greater (all other things equal). The loss per exposure will be less, if a lower percent of the total business is closer to the coast than it was at the time of Zelda.

Population density in coastal areas is increasing. Since windspeeds of a hurricane are greater closer to the coast, and the number of dwellings closer to the coast is increasing, it follows that solely because of this factor Zelda will cause more damage today than it did twenty years ago.

The type and quality of construction change over time. This can have both positive and negative effects on the amount of damage Zelda will cause today relative to 20 years ago. Building materials are different today, some of which are more wind resistant and some are less. Building codes change over time, as does their enforcement. If Zelda were to make landfall today it would have a different effect on any dwelling built in the last twenty years than is captured in the loss data from twenty years ago.

The amount and type of coverage provided in a policy change over time. Recent examples include guaranteed replacement cost, law and ordinance coverage, and exclusions to non-attached structures. There has also been a movement to higher wind-only deductibles or hurricane-only deductibles. These include both percentage options and higher dollar deductibles. Any of these changes to coverage will make the losses caused by Zelda less predictive of the potential loss for today's book of business. The true exposure to the hurricane peril in a current book of business can be far different than it was twenty years ago. While adding more years of experience may improve the ability to estimate hurricane frequency, it will also introduce significant exposure changes.

The changing exposure base issues are important reasons historical ratemaking data is inappropriate for pricing the hurricane peril. Just as important is the potential variation in the strength of a hurricane and how much damage a single storm will cause. History tells us that hurricanes making landfalls vary in strength from Category 1 storms with sustained wind speeds of 74 mph to Category 5 storms like Camille with sustained wind speeds in excess of 150 mph. At any given landfall, a full spectrum of possible storm strengths exists, which translate into a tremendous range of possible damage to property. Even within a given Saffir/Simpson category of storms, other factors also introduce variability into the potential total damage to property. These include the radius of maximum winds, track direction, forward speed and surrounding meteorological conditions. Additionally, similar storms can cause significantly different damage to property when they make landfall at different locations. This is where factors such as population density, building codes, construction quality, terrain, and other geographic features come into play. In any given state it would take thousands of storm observations to begin to approach a sample of storms that reflected the true potential distribution of storm severity over all potential landfalls.

Each hurricane is unique. No two storms, no matter how similar, will cause the same amount of damage relative to the exposure base. In a 25 year traditional ratemaking data base, will one storm such as Zelda be representative of potential future hurricane damage in the state? Can one or two, or even ten storms in a given experience period ever truly reflect the complete spectrum of possible event severity? Absolutely not.

South Carolina history is a good example of the problems with using historical data and methods to price the hurricane peril. If property insurance rates made in 1988 in South Carolina included a catastrophe provision based on 25 years of insurance data, the only hurricane reflected in the rates would have been Bob, a small category 1 storm in 1985. The next year Hurricane Hugo made landfall just north of Charleston as a category 4 storm resulting in unprecedented property damage.

Loss Costs

Before addressing the solution, there is one other problem with using traditional ratemaking techniques and data. Using historical insurance ratemaking data to price the hurricane peril can cause large swings in rates simply because a significant event occurred in the recent past. The best example of this is Hurricane Andrew. Using data from a 1991 Allstate rate filing, I have estimated the impact on rates the year before and after Andrew. The average premium for homeowners insurance in Florida prior to Andrew was approximately $260. Of this, $4 was the provision for the hurricane peril. The catastrophe provision was based on twenty years of data, a period when there were no major hurricane losses. The catastrophe provision was calculated by using a ratio of catastrophe losses to non-catastrophe losses. If the same rate level indication methodology was used, only updating for one additional year of catastrophe data, the indicated average rate would be $434, including a hurricane provision of $170. The true underlying loss cost for a given exposure does not change when a hurricane makes landfall. Our actuarial techniques need to change.

More History

Before concluding, there are two more references I would like to make. In 1981, David A. Arata wrote, "This paper argues that computer simulation is an underappreciated and, therefore, underutilized casualty actuarial resource".[17] Further in his paper Mr. Arata wrote, "Computer simulation can also be used to improve pricing of exposures for which historical information is unavailable or not indicative of future experience."[18] The second published paper, "A Formal Approach to Catastrophe Risk Assessment in Management", written by Karen M. Clark makes the following conclusion: "The model-generated expected loss estimates can be used to calculate Catastrophe premium loadings."[19] As early as 1981 the concept of using models to help ratemakers price insurance products was contained in the Proceedings of the CAS. For the next decade actuaries continued to rely on the historical techniques using historical ratemaking data to price the hurricane peril.

Conclusion

After considering the techniques currently used to price the hurricane peril, I conclude that the only tool available that captures a reasonable estimate of average annual costs is a computer simulation model. From a frequency perspective, short periods of historical data do not give accurate estimates of

---

[17] David A. Arata, FCAS, "Computer Simulation and the Actuary: A study in Realizable potential," PCAS LXVIII, 1981, Page 24.
[18] Ibid., page 43.
[19] Karen M. Clark, "A Formal Approach to Catastrophe Risk Assessment in Management," PCAS LXXIII, 1986, page 88.

the true underlying storm frequency. This problem exists on a statewide basis, but is even more acute for rating territories.

Also, new and unique landfalls are not captured by using historical experience. Computer simulation models can adequately address these issues.

The 98 year history of storm frequency displayed in Exhibits 1 through 10 demonstrates that there is great year to year variation in hurricane landfall frequency at all levels of geographic detail. I conclude that all available hurricane data should be used to compute hurricane frequency. This is easily accomplished in a hurricane simulation model. Any good model will incorporate a probability distribution at many landfall locations that is derived from the available history. The models can easily reflect the fact that new and unique landfalls are possible. Estimates for geographic areas as small as rating territories will be accurate if enough iterations are accounted for in the model.

From a severity perspective, the major problems with using a limited period of historical data to price the hurricane peril are a changing exposure base and the almost infinite possible severity of storms. Under the category of changing exposure base are the issues of distance to coast, density, coverage in force, type of construction, building codes, enforcement of building codes and policy provisions. Computer simulation models are able to eliminate or account for all of the problems associated with a changing exposure base.

Exposure changes over time become moot because the current distribution of business is the input for any model. Thus, the model output is reflective of the current distribution of business. The issues of distance to coast, density and coverage in force changing over the experience period become non-issues because all model output is reflective of the current book of business.

The models can account for type of construction, the effect of new construction and building codes, and the enforcement of building codes. These factors impact damage ratios for individual buildings in different ways. As an input to computer models, geo-coding of a company's current book of business will allow the impact of these factors on individual buildings to be reflected . Changes in policy provisions are also easily handled by models. The models can be run for current policy provisions and can also be run to estimate the impact of changes in policy provisions by comparing the output of different input assumptions. In fact, the models can be used to approximate the value of any type of mitigation effort.

Most importantly the problems of variation in severity are easily overcome by computer simulation models. A whole spectrum of possible storms with a full range of severities can be generated at any landfall. There is no longer a need to base a rate on only one or two observations.

The problem of rate instability discussed in the loss cost section is solved by using computer simulation models. If properly incorporated into base rates, the hurricane portion of individual rates based on computer simulation models will be stable. The occurrence of a major storm will not cause large rate increases, as it would if actual data were used to make rates.

Our profession has been extremely slow to react to a problem first documented in our literature in 1951. An analogy comes to mind between any ratemaker that continues to rely on historical ratemaking data and techniques and the ostrich that sticks its head in the sand. The time to change our methodology is now.

Ref: 9197draftpaper

EXHIBIT 1

LANDFALLING HURRICANES 1899-1996
EASTERN AND GULF COASTS OF THE UNITED STATES

YEAR	SAFFIR/SIMPSON CATEGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
1899		1	1			2	1
1900				1		1	1
1901	1	1				2	0
1902						0	0
1903	2	1				3	0
1904	1					1	0
1905						0	0
1906	1	1	2			4	2
1907						0	0
1908	1					1	0
1909			2	1		3	3
1910		1	1			2	1
1911	1	1				2	0
1912	2					2	0
1913	2					2	0
1914						0	0
1915	1			2		3	2
1916	3	1	2			6	2
1917			1			1	1
1918			1			1	1
1919				1		1	1
1920	1	1				2	0
1921		1	1			2	1
1922						0	0
1923	1					1	0
1924	2					2	0
1925	1					1	0
1926		1	2	1		4	3
1927						0	0
1928		1		1		2	1
1929	1	1	1			3	1
1930						0	0
1931						0	0
1932	1			1		2	1
1933	1	1	3			5	3
1934		1	1			2	1
1935		2			1	3	1
1936	1	1	1			3	1
1937						0	0
1938	1		1			2	1
1939	2					2	0
1940		2				2	0
1941		2	1			3	1
1942	1		1			2	1
1943		1				1	0
1944	1		3			4	3
1945	1	1	1			3	1
1946	1					1	0
1947	2	1	1	1		5	2
1948	1	1	1			3	1
1949	1	1	1			3	1
1950	1		2			3	2
1951						0	0
1952	1					1	0
1953	2					2	0
1954	1	1	2	1		5	3

LANDFALLING HURRICANES 1899-1996
EASTERN AND GULF COASTS OF THE UNITED STATES

YEAR	SAFFIR/SIMPSON CATEGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
1955	1		2			3	2
1956		1				1	0
1957				1		1	1
1958						0	0
1959	2		1			3	1
1960	1		2	1		4	3
1961				1		1	1
1962						0	0
1963	1					1	0
1964		3	1			4	1
1965			2			2	2
1966	1	1				2	0
1967			1			1	1
1968		1				1	0
1969	1				1	2	1
1970			1			1	1
1971	2	1				3	0
1972	2					2	0
1973						0	0
1974			1			1	1
1975			1			1	1
1976	1					1	0
1977	1					1	0
1978						0	0
1979	1	2	1			4	1
1980			1			1	1
1981						0	0
1982						0	0
1983			1			1	1
1984			1			1	1
1985	3	1	3			7	3
1986	2					2	0
1987	1					1	0
1988	1					1	0
1989	2			1		3	1
1990						0	0
1991		1				1	0
1992			1	1		2	2
1993						0	0
1994						0	0
1995	1	1	1			3	1
1996		1	1			2	1
TOTAL	64	40	55	15	2	176	72

U.S. HURRICANE LANDFALLS BY DECADE
1900 THROUGH 1996

DECADE	SAFFIR/SIMPSON CATEGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
1900's	6	3	4	2	0	15	6
1910's	9	3	5	3	0	20	8
1920's	6	5	4	2	0	17	6
1930's	6	5	6	1	1	19	8
1940's	8	9	9	1	0	27	10
1950's	8	2	7	2	0	19	9
1960's	4	5	6	2	1	18	9
1970's	7	3	4	0	0	14	4
1980's	9	1	6	1	0	17	7
1990's	1	3	3	1	0	8	4
TOTAL	64	39	54	15	2	174	71

U.S. HURRICANE LANDFALLS BY STATE
1899 THROUGH 1996

STATE	SAFFIR/SIMPSON CATAGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
TEXAS	12	6	9	5		32	14
LOUISIANA	9	4	9	3		25	12
MISSISSIPPI			1		1	2	2
ALABAMA	3	1	2			6	2
FLORIDA	19	19	17	5	1	61	23
GEORGIA		2				2	0
SOUTH CAROLINA	5	3	3	2		13	5
NORTH CAROLINA	10	4	8			22	8
VIRGINIA						0	0
MARYLAND						0	0
DELAWARE						0	0
NEW JERSEY	1					1	0
NEW YORK	2		5			7	5
CONNECTICUT						0	0
RHODE ISLAND		1				1	0
MASSACHUSETTS	1		1			2	1
NEW HAMPSHIRE						0	0
MAINE	2					2	0
TOTAL	64	40	55	15	2	176	72

TEXAS HURRICANE LANDFALLS BY DECADE 1900 THROUGH 1996

DECADE	SAFFIR/SIMPSON CATAGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
1900's	0	0	1	1	0	2	2
1910's	2	1	1	2	0	6	3
1920's	1	1	0	0	0	2	0
1930's	1	1	1	1	0	4	2
1940's	2	3	2	0	0	7	2
1950's	1	0	0	0	0	1	0
1960's	1	0	1	1	0	3	2
1970's	1	0	1	0	0	2	1
1980's	3	0	2	0	0	5	2
1990's	0	0	0	0	0	0	0
TOTAL	12	6	9	5	0	32	14
AVERAGE	0.124	0.062	0.093	0.052	0.000	0.330	0.144
1970-1996	4	0	3	0	0	7	3
AVERAGE	0.148	0.000	0.111	0.000	0.000	0.259	0.111
HIGH*	4	5	3	1	0	13	4
AVERAGE	0.133	0.167	0.100	0.033	0.000	0.433	0.133
LOW*	3	0	2	1	0	6	3
AVERAGE	0.100	0.000	0.067	0.033	0.000	0.200	0.100

*Based on total hurricane landfalls for three consecutive decades

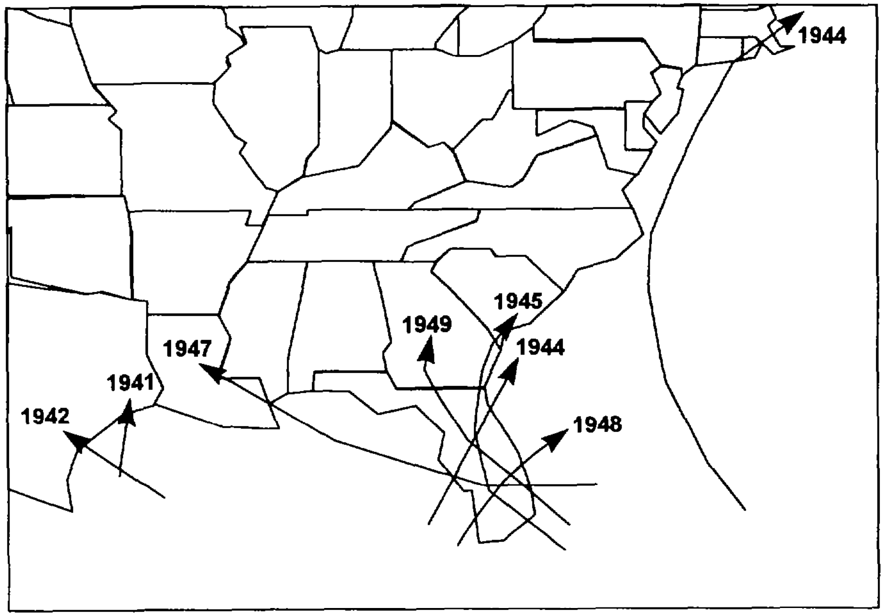
SOUTH CAROLINA HURRICANE LANDFALLS BY DECADE
1900 THROUGH 1996

DECADE	SAFFIR/SIMPSON CATAGORY 1	2	3	4	5	TOTAL HURRICANES	MAJOR HURRICANES
1900's	1	0	1	0	0	2	1
1910's	1	1	0	0	0	2	0
1920's	0	0	0	0	0	0	0
1930's	0	0	0	0	0	0	0
1940's	0	1	0	0	0	1	0
1950's	2	0	1	1	0	4	2
1960's	0	0	0	0	0	0	0
1970's	0	0	0	0	0	0	0
1980's	1	0	0	1	0	2	1
1990's	0	0	1	0	0	1	1
TOTAL	5	2	3	2	0	12	5
AVERAGE	0.052	0.021	0.031	0.021	0.000	0.124	0.052
1970-1996	1	0	1	1	0	3	2
AVERAGE	0.037	0.000	0.037	0.037	0.000	0.111	0.074
HIGH*	2	1	1	1	0	5	2
AVERAGE	0.067	0.033	0.033	0.033	0.000	0.167	0.067
LOW*	0	1	0	0	0	1	0
AVERAGE	0.000	0.033	0.000	0.000	0.000	0.033	0.000

*Based on total hurricane landfalls for three consecutive decades

EXHIBIT 6

MAJOR U.S. HURRICANES

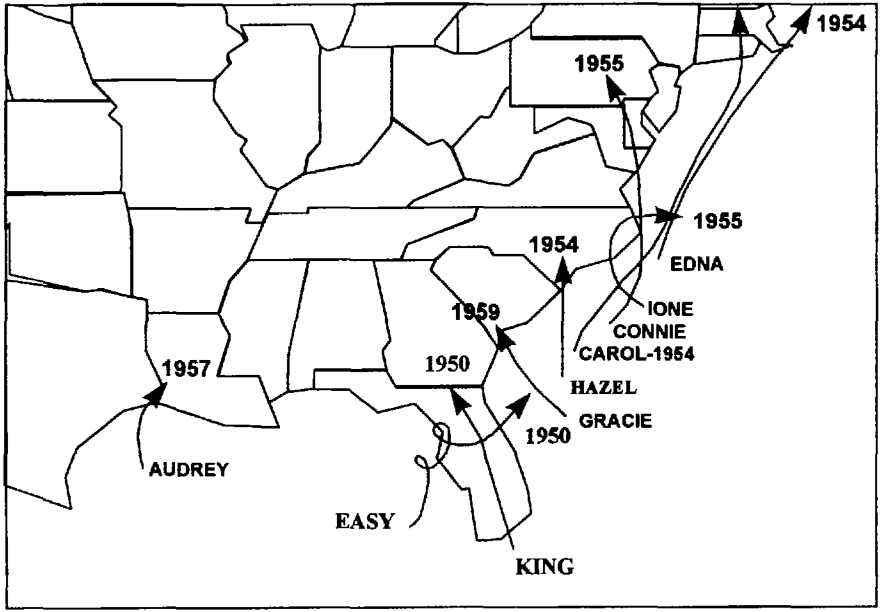


1940-1949

Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 7

MAJOR U.S. HURRICANES

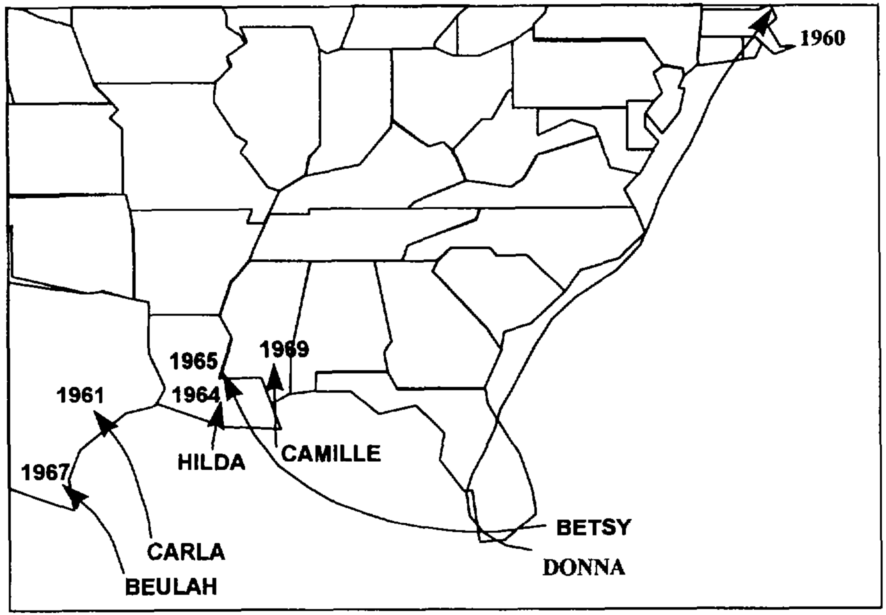


1950-1959

Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 8

MAJOR U.S. HURRICANES

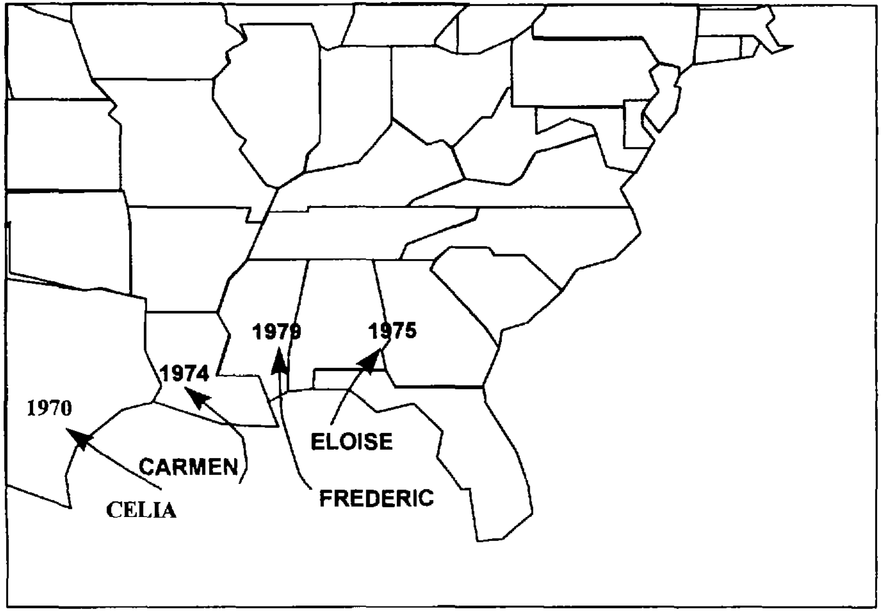


1960-1969

Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 9

MAJOR U.S. HURRICANES

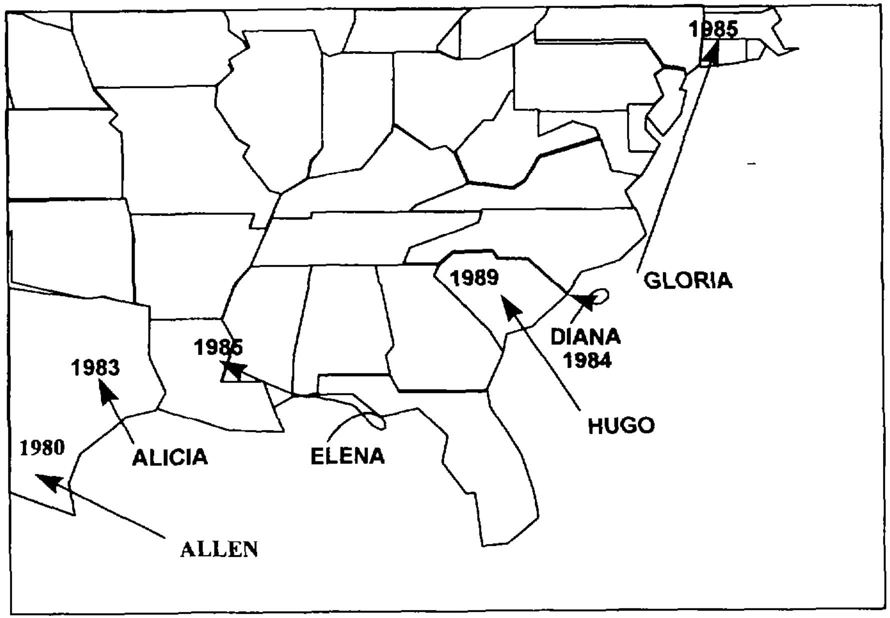


1970-1979

Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 10

MAJOR U.S. HURRICANES



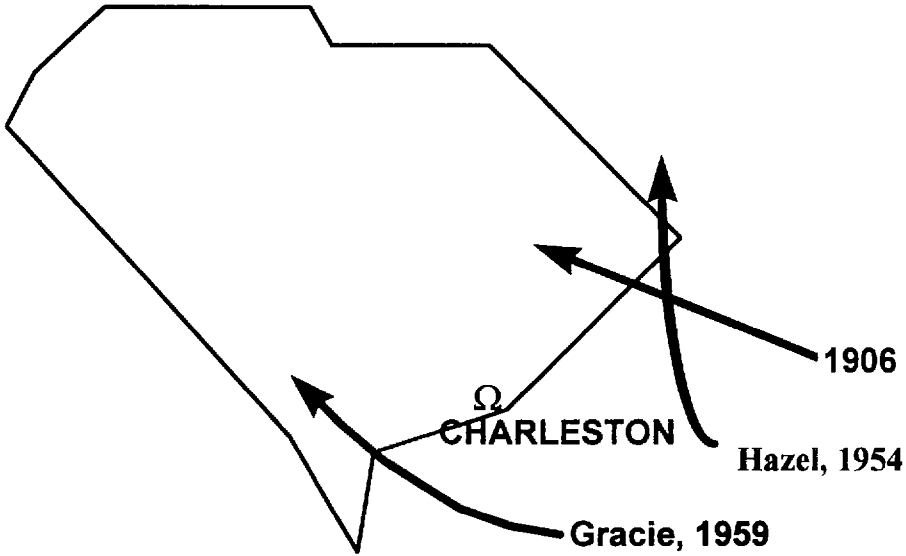
1980-1989

Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 11

LANDFALL FREQUENCY MAJOR SOUTH CAROLINA HURRICANES

AS OF 1988

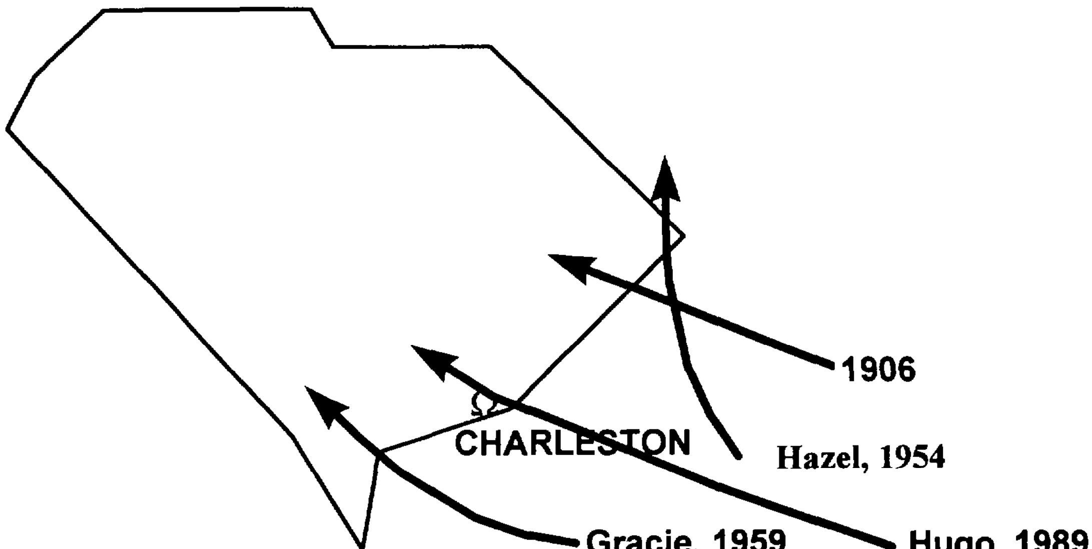


Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

EXHIBIT 12

LANDFALL FREQUENCY MAJOR SOUTH CAROLINA HURRICANES

AS OF 1989



Major Hurricanes Defined As Saffir-Simpson Categories 3, 4 and 5

The Usefulness of the R^2 Statistic

by Ross Fonticella, ACAS

The Usefulness of the R^2 Statistic

Introduction:

Almost every Actuarial Department uses least square regression to fit frequency, severity, or pure premium data to determine loss trends. Many actuaries use the R^2 statistic to measure the goodness-of-fit of the trend. Actually, the R^2 statistic measures how significantly the slope of the fitted line differs from zero, which is not the same as a good fit.

In the Fall, 1991 Casualty Actuarial Society Forum, D. Lee Barclay wrote A Statistical Note On Trend Factors: The Meaning of R-Squared. Through simple graphical examples, Barclay showed that the coefficient of variation (R^2) is, by itself, a poor measure of goodness-of-fit. Barclay's numerical examples provide additional support for this argument. But, his paper did not analyze the formulas used in regression analysis.

By understanding the formulas and what they describe, we can further understand why the R^2 statistic is not a reliable measure of a good fit. This paper will analyze these formulas important to regression analysis: (1) the basic linear regression model, (2) the Analysis of Variance sum of squares formulas, and (3) the R^2 formula in terms of the sum of squares. With an understanding of these formulas and what they measure, actuaries can properly use the R^2 value to best determine the forecasted trend.

Formulas:

The Analysis of Variance (ANOVA) approach to regression analysis is based on partitioning the Total Sum of Squares into the Error Sum of Squares and Regression Sum of Squares.

(1) The basic linear regression model is stated as: $Y_i = B_0 + B_1 X_i$

where Y_i = the observed dependent variable

X_i = the independent variable in the ith trial

$\hat{Y}_i$ = the fitted dependent variable for the independent variable X_i

$\overline{Y}$ = mean $Y_i = \sum Y_i / n$

(2) Analysis of Variance (ANOVA) Approach to Regression Analysis

SSTO = Total Sum of Squares = $\sum (Y_i - \overline{Y})^2$

= Measure of the variation of the observed values around the mean.

SSE = Error Sum of Squares = $\sum(Y_i - \hat{Y}_i)^2$

= Measure of the variation of the observed values around the regression line.

SSR = Regression Sum of Squares = $\sum (\hat{Y}_i - \overline{Y})^2$

= Measure of the variation of the fitted regression values around the mean

= SSTO - SSE = Difference between Total and Error Sum of Squares.

(3) Coefficient of Determination: R^2 = (SSTO - SSE)/SSTO = SSR/SSTO.

What the ANOVA formulas measure when $R^2=1$ and $R^2=0$:

From the above formulas, we see the relevance of $R^2=1$. If all of the observed values (Y_i) fall on the fitted regression line: then $Y_i = \hat{Y}_i$, $SSE = \sum(Y_i - \hat{Y}_i)^2 = 0$, and $R^2 = 1$. Since there is no variation of the actual observations from the fitted values, the independent variable accounts for all of the variation in the observations Y_i.

Conversely, if the slope of the regression line is $B_1 = 0$, then $\hat{Y}_i = \bar{Y}$, $SSR = \sum(\hat{Y}_i - \bar{Y})^2 = 0$, and $R^2 = 0$. Because the SSR measures the variation in the fitted values around the mean, no variation tells us that all of the variation is explained by the mean. So the linear regression model does not tell us anything additional when the data is completely explained by the mean.

R^2 (SSR/SSTO) measures the proportion of the variation of the observations around the mean that is explained by the fitted regression model. The closer R^2 is to 1, the greater the degree of association between X and Y. Conversely, if all of the variation is explained by the mean, then $R^2=0$, but this should not mean that the data is not useful for forecasting purposes.

Numerical Examples:

We can use the numerical examples from Barclay's paper to examine the ANOVA formula values when $R^2=0$ and $R^2=1$. Example #1 will show that even when $R^2=0$, an appropriate forecast can be made by examining the data from the ANOVA formulas.

Barclay generates data from a normal distribution with a mean of 50 and variance 1 to get the observations in Example #1. The line of best fit has $B_0 = 49.38813$ and $B_1 = .0366667$.

Example #1	Y observed	Y fitted	Error (residuals)	Total	Regression
X	Y_i	$\hat{Y}_i$	$Y_i - \hat{Y}_i$	$Y_i - \bar{Y}$	$\hat{Y}_i - \bar{Y}$
1	48.746	49.425	-.0679	-0.844	-0.165
2	49.914	49.461	0.453	0.324	-0.128
3	49.246	49.498	-0.252	-0.344	-0.092
4	50.297	49.535	0.762	0.707	-0.055
5	48.455	49.571	-1.116	-1.135	-0.018
6	50.088	49.608	0.480	0.498	0.018
7	50.559	49.645	0.914	0.969	0.055
8	50.173	49.681	0.492	0.583	0.092
9	49.336	49.718	-0.382	-0.254	0.128
10	49.084	49.755	-0.671	-0.506	0.165
Sum	495.898	495.898	0.000	0.000	0.000
Mean	49.5898	49.590			
Sum of Squares			(SSE) 4.460	(SSTO) 4.571	(SSR) 0.111
$R^2 =$	0.024				

The ANOVA formulas have these properties for a regression fit with a slope close to zero:

(1) $Y_i \approx \bar{Y}$, note the values in column Y fitted ($\hat{Y}$) are not far from $\bar{Y}$ = 49.590.

(2) SSE ≈ SSTO

The analysis of variance sum of squares are:

$SSTO = \sum (Y_i - \bar{Y})^2 = 4.571$

$SSE = \sum (Y_i - \hat{Y}_i)^2 = 4.460$

$SSR = \sum (\hat{Y}_i - \bar{Y})^2 = 0.111$

The variation around the regression line (SSE) is not much better (smaller) than the total variation (SSTO).

(3) $R^2 = (SSTO - SSE)/ SSTO = SSR / SSTO$
$= (4.571 - 4.460)/ 4.571 = 0.111 / 4.571 = .024$

Because the SSE is not much less than the SSTO, the R^2 value is close to 0. For SSR to be large, there needs to be a lot of variation of the fitted values around the mean. So anytime there is not a lot of variation in the data, the $R^2 \approx 0$. While this means that not much additional is explained by the fitted model, the "fit" may reasonably represent the data. And projecting with a slope of zero may be an appropriate forecast. Of course, you don't need regression to project a slope of zero, you can just forecast the mean.

In Example #2, Barclay adds 0 to the first Y observed, one to the second Y observed, two to the third, etc. The line of best fit has B_0 = 48.38813, and B_1 = 1.036667. This provides an interesting example for comparing the fit and the numerical values in the ANOVA formulas.

Example #2	Y observed	Y fitted	Error (residuals)	Total	Regression
X	Y_i	$\hat{Y}_i$	$Y_i - \hat{Y}_i$	$Y_i - \bar{Y}$	$\hat{Y}_i - \bar{Y}$
1	48.746	49.425	-0.679	-5.344	-4.665
2	50.914	50.461	0.453	-3.176	-3.628
3	51.246	51.498	-0.252	-2.844	-2.592
4	53.297	52.535	0.762	-0.793	-1.555
5	52.455	53.571	-1.116	-1.635	-0.518
6	55.088	54.608	0.480	0.998	0.518
7	56.559	55.645	0.914	2.469	1.555
8	57.173	56.681	0.492	3.083	2.592
9	57.336	57.718	-0.382	3.246	3.628
10	58.084	58.755	-0.671	3.994	4.665
Sum	540.898	540.898	0.000	0.000	0.000
Mean	54.0898	54.090			
Sum of Squares			(SSE) 4.460	(SSTO) 93.121	(SSR) 88.661
R^2 = 0.952					

The interesting part of this example is that the residuals (Y_i - $\hat{Y}_i$) are exactly the same as in Example #1. So the SSE is the same. Recall that Linear Regression minimizes the sum of the squared residuals. Should the lines in Example #1 and Example #2 have the same fit?

Let's look at the ANOVA formulas to see the properties of a "good fit" as measured by $R^2 = 1$:

(1) $Y_i \approx \hat{Y}_i$; the fitted values ($\hat{Y}_i$ column) are close to the observed (Y_i column); a "good fit." Here we decide that $Y_i \approx \hat{Y}_i$, in favor of $Y_i \approx \bar{Y}$, because there is more variation in the observations from the mean. We choose $Y_i \approx \hat{Y}_i$, even though we have the same values for the residuals as in Example #1.

(2) SSE $\approx$ 0.
The analysis of variance sum of squares are:
$SSTO = \sum (Y_i - \bar{Y})^2 = 93.121$
$SSE = \sum (Y_i - \hat{Y}_i)^2 = 4.460$
$SSR = \sum (\hat{Y}_i - \bar{Y})^2 = 88.661$
The variation around the regression line (SSE) is much better (smaller) than the total variation (SSTO).

(3) $R^2 = (SSTO - SSE)/SSTO = SSR / SSTO$
$= (93.121 - 4.460)/93.121 = 88.661/93.121 = .952$

The SSE is much less than the SSTO. So a large proportion of the variation of the actual observations around the mean is being explained by the fitted line. With the SSE close to zero, most of the observations are on the fitted line. However, you will note that this is relative, because we have the same SSE as in Example #1. It is because a large proportion of the SSTO is explained by the fitted line, that we decide there is a good fit.

<u>What does the R^2 statistic measure?</u>
The R^2 statistic is a useful tool to determine whether or not $B_1 = 0$. For in regression, if $B_1 = 0$, there is no good reason to use the fitted line. As actuaries, we are often trying to forecast. If the slope is zero ($B_1 = 0$), then we can use the mean to forecast the fitted value.

In fact, the formula for B_1 can be written as a function of R^2:

$$B_1 = [\Sigma(Y_i - \bar{Y})^2 / \Sigma(X_i - \bar{X})^2]^{1/2} r, \text{ where } r = \pm\sqrt{R^2} \text{ with the sign the same as the slope.}$$

So when $B_1=0$, then $R^2=0$; and when $R^2=0$, then $B_1=0$.

Both Example #1 and Example #2 have the same residuals, or SSE. From one perspective, each line has the same fit. The reason for the difference between the R^2 values was that in Example #2, the fitted slope is much different from zero and explains proportionally more of the larger variation in the SSTO.

In the first example, the low R^2 value would have us reject the fitted line. Should we reject the data, in favor of some other measure, like a medical CPI? I don't think so, because we can reasonably forecast that subsequent observations will be close to 49.5 (the mean). In Example #2, we get a good fit and would use $B_1 = 1.036667$. But, will the forecast of subsequent observations be any better than the forecast in Example #1? Unlikely.

The usefulness of the R^2 statistic is to measure the significance of the slope of the regression line. Since the R^2 is not a good measure of the goodness-of-fit, when the R^2 is not higher than some arbitrary benchmark, we should not just reject the data and look for other information to trend. If the slope is not significant ($R^2 = 0$) there could be a good "fit" as explained by the mean. We can see this by considering the values from the ANOVA formulas (SSE, SSR, and SSTO) which show how much of the variation is explained by the model relative to the mean. There are many other factors to be considered before accepting or rejecting the regression fit, such as patterns in the residuals. It is always useful to graph the fitted line against the observed values to look for these patterns.

Additional Formulas

The method of least squares finds values of B_0 and B_1 that minimize Q, where $Q = \sum (Y_i - \hat{Y}_i)^2 = \sum (Y_i - B_0 - B_1 X_i)^2$

Residuals $e_i = Y_i - \hat{Y}_i = Y_i - B_0 - B_1 X_i$

ANOVA formula relationship:

Note: The sum of the components and the sum of the squared deviations have the same relationship:

	$Y_i - \bar{Y}$	=	$\hat{Y}_i - \bar{Y}$	+	$Y_i - \hat{Y}_i$
	Total deviation	=	Deviation of fitted regression value around the mean	+	Deviation around the regression line
and	SSTO	=	SSR	+	SSE

Bibliography

John Neter and William Wasserman, Applied Linear Statistical Models, 1974.

Abraham, B.; and Ledolter, J., Statistical Methods for Forecasting, 1983.

D. Lee Barclay, A Statistical Note on Trend Factors: The Meaning of "R-Squared", Casualty Actuarial Society Forum, Fall 1991 Edition.

Statistical Models and Credibility

by Leigh J. Halliwell, FCAS, MAAA

Statistical Models and Credibility

Leigh J. Halliwell, FCAS, MAAA

Abstract

The theory of credibility is a cornerstone of actuarial science. Actuaries commonly use it, and with some pride regard it as their own invention, something which surpasses statistical theory and sets actuaries apart from statisticians. Nevertheless, the development of statistical models by statisticians and econometricians in the latter half of this century is very relevant to credibility theory; it can ground as well as generalize much of the theory, particularly the branch thereof known as least-squares credibility. It is the purpose of this paper to show how the theory and practice of credibility can benefit from statistical modeling.

The first half of the paper consists of eleven sections, notes, references, and twenty exhibits. The technical content is subdued, and readers may content themselves with this half. But the technically inclined are invited to study the six appendices (A through F) of the second half. Due to space limitations of the Call Paper Program, some of the appendices may be deleted. If this should happen, the deleted appendices can be obtained by calling the author at (201) 278-8860.

The author is grateful to Kenneth Kasner, FCAS, MAAA, for his thoughtful and kind review of the draft of this paper.

Mr. Halliwell is a Fellow of the Casualty Actuarial Society and a member of the American Academy of Actuaries. In August 1997 he became a consultant at the New York office of Milliman and Robertson. For two years prior to that he lived in Mexico City as the Regional Actuary of Latin America for the Zurich Insurance Group. And prior to that he was the Chief Actuary of the Louisiana Workers' Compensation Corporation, Baton Rouge, LA. His actuarial career began at the National Council on Compensation Insurance in Boca Raton, FL.

1. Introduction

Throughout the twentieth century actuaries have been practicing something that they call credibility. Although acknowledging some connections with statistics, especially with regard to Bayesian credibility, actuaries have tended to regard credibility as transcending statistics. This is illustrated in the historical sketch of the following section. But this paper will proceed to show that advances in statistical modeling during the latter half of this century legitimate and deepen typical uses of credibility. In order not to presume on the readers' knowledge of modern statistics, Sections 3, 4, and 5 will outline and illustrate the linear statistical model. The treatment of credibility *per se* will begin in Section 6, where we will show how to introduce prior (or non-sample) information into the statistical model. It is hoped that the reader will be persuaded that to express credibility in statistical terms is not only possible, but also advantageous. Six appendices at the end of the paper provide mathematical foundations for much of what is glossed over in the sections.

2. An Historical Perspective on Credibility

To Matthew Rodermund was entrusted the formidable task of writing the introduction to the textbook *Foundations of Casualty Actuarial Science*. The task was formidable because it demanded a engaging history of the casualty actuarial profession and a distillation of its essence. Rodermund states, "It is the concept of credibility that has been the casualty actuaries' most important and most enduring contribution to casualty actuarial science."

[11:3].* After recounting the accomplishments of actuaries in experience rating, retrospective rating, merit rating, ratemaking, and reserving – all with an eye on credibility, he asks, "Readers who have come this far may conclude from what they've read that casualty actuarial science is the study and application of the theory of credibility, and that's all. Is it all?" [11:19] An affirmative answer is implied. And almost thirty years earlier L. H. Longley-Cook, although more reserved than Rodermund, prefaced his famous monograph on credibility with the words "Credibility Theory is one of the cornerstones of actuarial science ..." [9:3]

The "Statement of Principles Regarding Property and Casualty Ratemaking," adopted by the Casualty Actuarial Society in 1988, defines credibility to be "a measure of the predictive value that an actuary attaches to a particular body of data."[1] Actuaries often speak equivalently of the "weight" given to a body of data. The language of *attaching* or *giving* credibility to data is suggestive of an important point made by Longley-Cook:

> ... the amount of credibility to be attached to a given body of data is not entirely an intrinsic property of the data. For example, there is always stated or implied in any measure of credibility the purpose to which data are to be used.
>
> ...
>
> Hence, we see that credibility is not a simple property of data which can be calculated by some mathematical formula ... [9: 4]

If credibility is not entirely intrinsic to the data, then it is at least partially extrinsic. In practice, credibility is largely, if not entirely, extrinsic to the data. And what is extrinsic to the data pertains to informed judgment; so it is fitting that Longley-Cook concluded his monograph as follows:

> It is perhaps necessary to stress that credibility procedures are not a substitute for informed judgment, but an aid thereto. Of necessity so many practical considerations must enter into

* In the '[*n*:*p*]' format '*n*' is the reference number and '*p*' gives the page number(s).

any actuarial work that the student cannot substitute the blind application of a credibility formula for the careful consideration of all aspects of an actuarial problem. [9:25] (also quoted in [11:10f.])

Since the credibility of data is the predictive value or weight given to the data, the question arises what to do when the actuary judges the data not to have enough predictive value or weight. The answer is to weight the answer which is based on the data with an answer based on informed judgment; so it is natural for actuaries to speak of credibility-weighting the empirical answer with another source of information.

One great teacher and apologist of credibility was Arthur L. Bailey. Writing between 1945 and 1950, he claimed that certain credibility procedures conflicted with current statistical theory; in fact, statistical training could hinder someone from accepting these procedures:

> The basis for these credibility formulas has been a profound mystery to most people who have come in contact with them. The actuary finds them difficult to explain and, in some cases, even difficult to understand. Paradoxical as it may be, the more contact a person has had with statistical practices in other fields or the more training a person has had in the theory of mathematical statistics, the more difficult it has been to understand these credibility procedures or the validity of their application. [3:7]

Bailey listed as three offending credibility procedures (1) the use of prior hypotheses in estimation, (2) an estimation of groups together which is more accurate than estimating each group separately, and (3) estimating for an individual that belongs to a heterogeneous population [4:59f.]. Speaking from his own experience and with the ardor of a convert, he wrote:

> I personally entered the casualty insurance field from the completely unassociated field of statistical research in the banana business. The first year or so I spent proving to myself that all of the fancy actuarial procedures of the casualty business were mathematically unsound. They are unsound, if one is bound to accept the restrictions implied or specifically placed on the development of the classical statistical methods. Later on I realized that the hard-shelled underwriters were recognizing certain facts of life neglected by the statistical theorists. Now I am convinced that casualty insurance statisticians are a step ahead of those in most fields.

> This is because there has been a truly epistemological review of the basic conditions of which their statistics are measurements. I can only urge a similar review be made by statisticians in other fields. [4:61]

Bailey [3] sought to ground these procedures in what later became known as Bayesian analysis. No doubt, in his day statistical theory could not accommodate certain actuarial ideas. Therefore, he saw the actuarial profession as in "revolt," as for example when he wrote:

> Philosophers have recently discussed the credibilities to be given to various elements of knowledge, thus undermining the accepted philosophy of the statisticians. However, it appears to be only in the actuarial field that there has been an organized revolt against discarding all prior knowledge when an estimate is to be made using newly acquired data. [3:9f.]

But a revolt involving Bayesian analysis was soon to happen among the statisticians, as Allen Mayerson remarked in 1964:

> Statistical theory has now caught up with the actuary's problems. Starting with the 1954 book by Savage, and buttressed by the 1959 volume by Schlaifer and the 1961 book by Raiffa and Schlaifer, there has been, among probabilists and statisticians, an organized revolt against the classical approach and a trend toward the use of prior knowledge for statistical inference.
>
> ...
>
> The relationship between Bayes' theorem and credibility was first noticed by Arthur Bailey, who showed that the formula $ZA+(1-Z)B$ can be derived from Bayes' theorem ...
>
> ...
>
> It seems appropriate, in view of the growing interest among statisticians in the Bayesian point of view, to attempt to continue the work started 15 years ago by Bailey, and, using modern probability concepts, try to develop a theory of credibility which will bridge the gap that now separates the actuarial from the statistical world. [10:85f.]

Bayesian analysis has continued to be a popular basis of credibility theory. It plays a prominent role in Gary Venter's momentous chapter on credibility in the *Foundations* textbook [13]. But Bailey's seminal idea was a "greatest accuracy credibility" [2:20], of which Venter writes:

> The most well developed approach to greatest accuracy credibility is *least squares credibility*, which seeks to minimize the expected value of the square of the estimation error ...
>
> More recent statistical theory, Bayesian analysis for example, also addresses the use of data to update previous estimates, and this will be introduced later below. Credibility theory shares with Bayesian analysis the outlook toward data as strictly a source to update prior knowledge. Credibility, particularly least squares credibility is sometimes labeled Bayesian or empirical Bayesian for this reason. It also gives the same result as Bayesian analysis in some circumstances, although credibility theory can be developed within the frequentist view of probability ...
>
> Frequentist refers to an interpretation of probability as solely an expression of the relative frequency of events, in contrast to a subjectivist view which regards probability as a quantification of opinion. This latter view is a hallmark of Bayesian analysis. [13:384]

This quotation clearly indicates that Bayesian analysis is not the be-all and end-all of credibility theory. Rather, despite some similarities, greatest accuracy credibility is independent from Bayesian analysis, and especially from the on-going philosophical debate between the frequentists and the subjectivists. With all the limelight on Bayesian analysis, actuaries have not realized that statistical theory now has some non-Bayesian things to say about credibility. In particular, modern statistical modeling can accommodate the three "offending" credibility procedures mentioned above; moreover, it provides a richer world of ideas than the one-dimensional formula $ZA+(1-Z)B$.

3. An Overview of the Linear Statistical Model

In an earlier paper [7] the author treated the best linear unbiased estimation (BLUE) of the linear statistical model. That treatment was detailed and self-contained; so the author will assume it, rather than derive it. In Appendix C of that paper the author compared BLUE with Gary Venter's formulation of least-squares credibility [13:418], and concluded:

> Thus Venter is essentially doing best linear unbiased estimation on a linear model. The author hopes that actuaries will begin to see the subject of credibility from the perspective of statistical modeling. [7:335]

It is for the purpose of realizing that hope that the present paper is written.

The form of a linear[2] statistical model is $\mathbf{y} = X\beta + \mathbf{e}$, where $\text{Var}[\mathbf{e}] = \Sigma = \sigma^2\Phi$. In this model $\mathbf{y}$ and $\mathbf{e}$ are $(t\times 1)$ vectors, X is a $(t\times k)$ matrix, β is a $(k\times 1)$ vector, and Σ and Φ are $(t\times t)$ matrices. The design matrix X is known, or posited; $\mathbf{y}$ is observed. Although the parameter vector β is not known, it is not random; an estimator of β is random, but β itself is not. What injects randomness into the vector $\mathbf{y}$ is the error term $\mathbf{e}$. $\mathbf{e}$ is not observable; however, $E[\mathbf{e}] = 0_{(t\times 1)}$, and $\text{Var}[\mathbf{e}]$ is known, or posited, at least to within a proportionality constant, i.e., $\text{Var}[\mathbf{e}] \propto \Phi$. No assumption is made as to the probability distribution of $\mathbf{e}$.

Most presentations of the linear statistical model dwell on how to estimate β, but there is a wider approach. Suppose that the t rows of the $\mathbf{y}$ are of two types, those which have been observed and those which have not. The observed portion of $\mathbf{y}$ we will call $\mathbf{y}_1$ and say that it is $(t_1\times 1)$; the unobserved will be $\mathbf{y}_2$ and $(t_2\times 1)$. Of course, $t_1 + t_2 = t$. We can also arrange the rows of the model so that the observed portion comes first. Similarly partition X and $\mathbf{e}$, so that the model looks like:

$$\begin{matrix} \mathbf{y}_1 = X_1\beta + \mathbf{e}_1 \\ \mathbf{y}_2 = X_2\beta + \mathbf{e}_2 \end{matrix}, \text{ where } \text{Var}\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \sigma^2\Phi = \sigma^2\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

Since variance matrices are symmetric (cf. [7:304] and [8:43]), $\Sigma_{21} = \Sigma_{12}'$ and $\Phi_{21} = \Phi_{12}'$.

Being unobserved, $\mathbf{y}_2$ contains missing values. The problem is to formulate an estimator of $\mathbf{y}_2$ based on $\mathbf{y}_1$, X, and Σ. In particular, we want the estimator to be linear in $\mathbf{y}_1$, to be unbiased, and to be in some way best; i.e., we want the best linear unbiased estimator (BLUE) of $\mathbf{y}_2$. In Appendix C of the earlier paper [7] it is shown that the BLUE of $\mathbf{y}_2$ is:

$$\begin{aligned}
\hat{\mathbf{y}}_2 &= X_2\hat{\beta} + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{y}_1 - X_1\hat{\beta}) \\
\mathrm{Var}[\mathbf{y}_2 - \hat{\mathbf{y}}_2] &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} + (X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1)\mathrm{Var}[\hat{\beta}](X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1)', \text{ where} \\
\hat{\beta} &= (X_1'\Sigma_{11}^{-1}X_1)^{-1}X_1'\Sigma_{11}^{-1}\mathbf{y}_1 \text{ and} \\
\mathrm{Var}[\hat{\beta}] &= (X_1'\Sigma_{11}^{-1}X_1)^{-1}
\end{aligned}$$

This is equivalent to:

$$\begin{aligned}
\hat{\mathbf{y}}_2 &= X_2\hat{\beta} + \Phi_{21}\Phi_{11}^{-1}(\mathbf{y}_1 - X_1\hat{\beta}) \\
\mathrm{Var}[\mathbf{y}_2 - \hat{\mathbf{y}}_2] &= \sigma^2(\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12}) + (X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)\mathrm{Var}[\hat{\beta}](X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)', \text{ where} \\
\hat{\beta} &= (X_1'\Phi_{11}^{-1}X_1)^{-1}X_1'\Phi_{11}^{-1}\mathbf{y}_1 \text{ and} \\
\mathrm{Var}[\hat{\beta}] &= \sigma^2(X_1'\Phi_{11}^{-1}X_1)^{-1}
\end{aligned}$$

If σ^2 is not known, it can be unbiasedly estimated as $\hat{\sigma}^2 = \dfrac{\hat{\mathbf{e}}_1'\Phi_{11}^{-1}\hat{\mathbf{e}}_1}{t_1 - k}$, where $\hat{\mathbf{e}}_1 = \mathbf{y}_1 - X_1\hat{\beta}$ [7:333f.].

What does it mean for $\hat{\mathbf{y}}_2$ to be best? As explained in Appendix A, of two competing linear unbiased estimators the best estimator is the one the variance of whose prediction error is smaller:

$$\begin{aligned}
\mathrm{Var}[\mathbf{y}_2 - \hat{\mathbf{y}}_2] &\le \mathrm{Var}[\mathbf{y}_2 - \tilde{\mathbf{y}}_2], \text{ or} \\
0 &\le \mathrm{Var}[\mathbf{y}_2 - \tilde{\mathbf{y}}_2] - \mathrm{Var}[\mathbf{y}_2 - \hat{\mathbf{y}}_2]
\end{aligned}$$

This means that the right-hand side of the second inequality is a non-negative definite matrix. The estimator with the caret is at least as good as the one with the tilde; and if the expression is non-zero, it is better.

Before applying this overview to credibility, the next two sections will warm the reader up with two simple linear models. Prior to riding a horse it is wise to practice on ponies.

4. The Simplest Statistical Model (Example 1)

Suppose that we have seven non-covarying and identically distributed observations of a random variable: 6.164, 11.103, 9.663, 12.998, 10.329, 9.564, and 9.602. A simple model of the i^{th} observation (i = 1, ... , 7) is $y_i = \beta + e_i$, where $\mathrm{Var}[e_i] = \sigma^2$. The matrix formulation is:

$$\begin{bmatrix} 6.164 \\ 11.103 \\ 9.663 \\ 12.998 \\ 10.329 \\ 9.564 \\ 9.602 \end{bmatrix} = \mathbf{y} = \mathrm{X}\beta + \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\beta + \mathbf{e}$$

Since the observations are non-covarying and identically distributed, $\mathrm{Var}[\mathbf{e}] = \sigma^2 \mathrm{I}_7$. In this simple example $\hat{\beta} = (\mathrm{X}'\mathrm{X})^{-1}(\mathrm{X}'\mathbf{y}) = \frac{\sum 1 * y_i}{\sum 1 * 1} = \bar{y} = 9.917$. So the parameter is the mean of the observations, and the estimator of σ^2 is the sample variance (= 4.240). One might react that this is like using a sledgehammer to crack a walnut: "Why go to all this trouble when the mean and the variance are the obvious solutions from the start?" The answer, however, deserves to be pondered: This model, the simplest of all, undergirds the mean and variance functions; these functions are in reality pre-packaged solutions of the simplest linear model.

Exhibits 1 and 2 present and solve this model. The seven observations are contained in $\mathbf{y}_1$. Since these observations are non-covarying, the off-diagonal elements of Φ_{11} are zero; since they are identically distributed, the diagonal elements of Φ_{11} are equal (ones). Thus, according to the formulas of the previous section (which are repeated in the exhibits), β and its variance may be estimated.

However, in this example we have chosen to estimate, or to predict, a certain (11×1) vector $\mathbf{y}_2 = X_2\beta + \mathbf{e}_2$. What $\mathbf{y}_2$ estimates is determined by X_2, Φ_{21}, and Φ_{22}. The first seven elements of $\mathbf{y}_2$ have the same variance as $\mathbf{e}_1$ and are perfectly correlated with $\mathbf{e}_1$. This means that as far as this statistical model is concerned, these seven elements are indiscernible from $\mathbf{e}_1$, and hence *are* $\mathbf{e}_1$. The eighth element of $\mathbf{y}_2$ models the constant 0. The ninth element models a new error term, i.e., an error term which has the same variance as $\mathbf{e}_1$ but does not covary with $\mathbf{e}_1$. The last two elements of $\mathbf{y}_2$ model β without an error term and with a new error term. Exhibit 2 derives the estimate of $\mathbf{y}_2$ and the variance of its prediction error.

5. Another Simple Statistical Model (Example 2)

Exhibits 3 and 4 concern a slightly less simple example. We have actual utility expenses for thirteen months (Sep95-Sep96). For each of these months there is a suitable utility index. We desire to estimate the expenses for the next three months (Oct96-Dec96), and are comfortable with 160, 162, and 168 as predictions of the utility index.

Many actuaries would simply rescale the last month's expenses. For example, Oct96 expenses are expected to be 2,192*(160/156.779) = 2,237. But this ignores the information from the earlier months. If one were to do a similar calculation for the other twelve months, one would then have thirteen estimates in need of combination. If this combination were performed correctly, one would be doing a statistical model in a roundabout manner.

Exhibit 4 tackles the problem directly. The observed expenses are equal to β times the utility index plus a error term. However, Φ_{11} is not of constant variance. It seems reasonable for the standard deviation of expenses to be proportional to the utility index (e.g., if prices were to double, the expense swings would double). This causes the variances of the expenses to be proportional to the squares of the utility indices, which squares are found along the diagonals of Φ_{11} and Φ_{22}. Each month's error is assumed not to covary with the other months' errors. In this exhibit β and $\mathbf{y}_2$ are estimated in accordance with the formulas already mentioned. One can also take linear combinations of $\mathbf{y}_2$ and of the variance of its prediction error. For example $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\hat{\mathbf{y}}_2 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}2{,}339\\2{,}368\\2{,}456\end{bmatrix} = \begin{bmatrix}7{,}163\end{bmatrix}$ is the estimated expense for the entire fourth quarter. Moreover, the variance of its prediction error is $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\mathrm{Var}\begin{bmatrix}\mathbf{y}_2 - \hat{\mathbf{y}}_2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}40672 & 2941 & 3050\\2941 & 41695 & 3089\\3050 & 3089 & 44841\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}145370\end{bmatrix}$, for a standard deviation of 381.

6. A Simple Example of a Model with Prior Information (Example 3)

Now that we have warmed up on two simple models, let us see how to express credibility in a statistical model. We return again to the seven observations of Example 1 (Exhibit 1). The numbers 6.164, ... , 9.602 were actually generated as random numbers with mean 10 and variance 4. Therefore, the mean and variance estimates of 9.917 and 4.240 are close. Of course, if one knew the true parameters, they would not need to be estimated.

But suppose that prior to observation we believed (for whatever reason) that the mean is 11 and the variance is 3. Could we benefit from combining observation with our prior belief? (We will assume that the prior belief is well-founded, so that it is prior information, rather than prior *mis*information.) The answer is "Yes;" it is possible, even advisable, to combine prior information with observation.

One way of combining is Bayesian inference (Appendix B). But a simpler way is to treat the prior information *as if* it had been observed. Therefore, in Exhibit 5 the prior information is appended to the observations as an eighth row (separated from the genuine observations by a light line). In an earlier paper the author referred to prior information as quasi-observation [7:Section 6 and Appendix E]. Judge [8] refers to observation as sample information and to prior information as non-sample information. Combining the two is called mixed estimation [8:877]. Our formulation of this hybrid model, which differs only slightly from Judge's, is:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} X \\ R \end{bmatrix} \beta + \begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix}, \text{ where } \mathrm{Var}\begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}$$

So the best linear unbiased estimator of β is:

$$\begin{aligned} \hat{\beta} &= \left(\begin{bmatrix} X \\ R \end{bmatrix}' \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}^{-1} \begin{bmatrix} X \\ R \end{bmatrix} \right)^{-1} \begin{bmatrix} X \\ R \end{bmatrix}' \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix} \\ &= \left(\begin{bmatrix} X' & R' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & \\ & V^{-1} \end{bmatrix} \begin{bmatrix} X \\ R \end{bmatrix} \right)^{-1} \begin{bmatrix} X' & R' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & \\ & V^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix} \\ &= \left(X'\Sigma^{-1}X + R'V^{-1}R \right)^{-1} \left(X'\Sigma^{-1}\mathbf{y} + R'V^{-1}\mathbf{r} \right) \end{aligned}$$

Certain properties of this estimator are explored in Appendices A and B. In particular, the estimator is a matrix-weighted average of more familiar estimators and has a smaller variance. These properties depend on the block diagonality of the hybrid variance matrix, i.e., that **e** and **v** do not covary. This is a natural assumption; however, the estimator can accommodate covariance if these properties are surrendered.

Exhibit 5 works out the mixed estimate of β as 10.099. This is equivalent to what actuaries would call a weighted-average of the data with the prior hypothesis, where the weight of the data, 0.832, results from the well-known $n/(n+k)$ formula. It is interesting, perhaps surprising, that the variance of the mixed estimator, 0.904, is less than both the variance from the unmixed model (4.240) and the variance of the prior hypothesis (3.000). This synergy of combination is analyzed in Appendix A.

One complicating detail of this model has to do with the variance matrix. Usually we specify the variance matrix not *absolutely*, but *relatively*, or to within a proportionality constant. In other words, in the model $\mathbf{y} = X\beta+\mathbf{e}$, where $\text{Var}[\mathbf{e}] = \Sigma = \sigma^2\Phi$, the estimator of β is invariant to the scale of Σ. So we usually specify Φ, calculate the estimator of β, and then derive an estimate of σ^2. In the unusual event that V/σ^2 is known (or, V is known to within the same proportionality constant to within which Σ is known), then one can use the mixed estimator with the relative hybrid variance matrix. However, the usual case is that V is known absolutely and Σ is known relatively. In this case the author recommends that σ^2 be estimated in the unmixed model, and that the absolute matrix $\begin{bmatrix} \hat{\sigma}^2\Phi & \\ & V \end{bmatrix}$ be used in the mixed model. (This implies that one should solve the unmixed model as a prelude to solving the mixed.) This was done in Exhibit 5, where the 4.240 down the diagonal of Φ_{11} is the estimate of the σ^2 of Example 1. Using an estimate of the absolute variance for the absolute variance itself disturbs the optimality (the "bestness" of "best linear unbiased") of the estimator; however, statisticians and econometricians feel that this is a small price to pay for the benefit derived from combining observation with prior information. Moreover, the estimate of σ^2 in the mixed model (0.904 in Exhibit 5) will not significantly differ from 1 if the absolute variance matrix is correct. Therefore, one can assume the estimator of σ^2 in the mixed model to be a chi-square random variable with *df* degrees of freedom divided by *df* (i.e., a gamma random variable with mean 1 and variance $2/df$) and can perform a significance test. But seldom is there a problem, and this will not be mentioned again in the following examples.

7. A Statistical Model of Merit Rating (Example 4)

A simple method of merit rating a driver is to make the premium proportional to the expected number of accidents. This ignores differences of severity, e.g., driver A is half as likely to have an accident as driver B, but perhaps his accidents are likely to be twice as severe. However, as with experience rating in workers' compensation, it is natural to suppose that the insured has more control over whether an accident will happen than over how severe it will be. So we wish to estimate a driver's accident frequency, and the problem is to determine how much a driver's accident record should differentiate him from his peers.

Lester Dropkin paved the way for a Bayesian solution, viz., that every driver has his own accident frequency m, and that the number of his claims is Poisson distributed with mean m. Therefore, the probability of x claims is $\frac{m^x}{x!}e^{-m}$.[3] Moreover, the frequencies of the drivers of a certain class are gamma-distributed with parameters r and a [5:392f.]. So the probability density function of the ms is $\frac{a^r}{\Gamma(r)}e^{-am}m^{r-1}$, and the ms are distributed with mean r/a and variance r/a^2. As Dropkin showed [5:399], the claim count distribution of a driver randomly selected from the class is negative binomial with mean r/a and variance $\frac{r}{a}\frac{a+1}{a}$.

But the posterior density of a driver's one-period m given $x_1, \ldots, x_n$ accidents in n previous periods is proportional or equal to:

$$
\begin{aligned}
&\propto \left(\prod \frac{m^{x_i}}{x_i!} e^{-m}\right) \frac{a^r}{\Gamma(r)} e^{-am} m^{r-1} \\
&\propto \left(\prod m^{x_i} e^{-m}\right) e^{-am} m^{r-1} \\
&\propto \left(m^{\sum x_i} e^{-nm}\right) e^{-am} m^{r-1} \\
&\propto e^{-(a+n)m} m^{\left(r+\sum x_i\right)-1} \\
&= \frac{(a+n)^{r+\sum x_i}}{\Gamma\left(r+\sum x_i\right)} e^{-(a+n)m} m^{\left(r+\sum x_i\right)-1}
\end{aligned}
$$

This posterior density is gamma with parameters $r' = r + \sum x_i$ and $a' = a + n$. The posterior mean, to which the merit-rated premium should be proportional, is a weighted average of the prior mean (r/a) and the empirical mean (cf. also [10:99-101]):

$$
\begin{aligned}
\frac{r'}{a'} &= \frac{r + \sum x_i}{a+n} \\
&= \frac{a\frac{r}{a} + n\frac{\sum x_i}{n}}{a+n} \\
&= \frac{a\frac{r}{a} + n\bar{x}}{a+n}
\end{aligned}
$$

The same result is obtained from the following linear model:

$$
\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ r/a \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \beta + \begin{bmatrix} e_1 \\ \vdots \\ e_n \\ v \end{bmatrix}, \text{ where } \operatorname{Var}\begin{bmatrix} e_1 \\ \vdots \\ e_n \\ v \end{bmatrix} = \begin{bmatrix} r/a & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & r/a & 0 \\ 0 & 0 & 0 & r/a^2 \end{bmatrix}
$$

Each x_i is explained as some mean value β plus an error, where the error is like a Poisson random variable (with parameter r/a) centered about zero. But the last row is a quasi-observation: it is as if β had been observed as r/a but obfuscated with an error whose variance is r/a^2. The mixed estimator is:

$$\hat{\beta} = \left(\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}' \begin{bmatrix} r/a & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & r/a & 0 \\ 0 & 0 & 0 & r/a^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}' \begin{bmatrix} r/a & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & r/a & 0 \\ 0 & 0 & 0 & r/a^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ r/a \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} a/r & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & a/r & 0 \\ 0 & 0 & 0 & a^2/r \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} a/r & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & a/r & 0 \\ 0 & 0 & 0 & a^2/r \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ r/a \end{bmatrix}$$

$$= \frac{\frac{a}{r}x_1 + \ldots + \frac{a}{r}x_n + \frac{a^2}{r}\frac{r}{a}}{\frac{a}{r} + \ldots + \frac{a}{r} + \frac{a^2}{r}}$$

$$= \frac{x_1 + \ldots + x_n + r}{1 + \ldots + 1 + a}$$

$$= \frac{r'}{a'}$$

The statistical model reaches the same conclusion without assuming a distributional form.

Exhibit 6 shows another example of merit rating. A driver had one accident in the second of three periods (years). The variance of his yearly accidents is assumed to be 0.0625 (standard deviation of 0.25). But there is prior information that drivers of this class are expected to have 0.25 claims per period with a variance of 0.0225 (standard deviation of 0.15). In Part A of the exhibit the three years are three one-year observations. But in Part B they are summarized into one three-year observation. The estimates are the same in both parts, but their variances differ. This hints that summarization is attended with loss of information about prediction error variance. An amount of 1 over three years could mean 1/3 each year and no apparent variance. Or it could mean widely varying positive and negative amounts by year and an arbitrarily large variance. If actuaries wish to speak of variances, then they should know where to stop summarizing the data.

8. Stochastic and Exact Constraints (Example 5)

The prior information, or the quasi-observation, $\mathbf{r} = R\beta + \mathbf{v}$ is a stochastic constraint since $\mathbf{v}$ does not have to be zero. However, as $V = Var[\mathbf{v}]$ approaches a zero matrix, the constraint behaves more and more like the exact constraint $r = R\beta$. In an earlier paper [6:26] the author filled out a loss triangle by means of estimated pure premiums by payout year. But the pure premiums by year were exactly constrained so that the sum of the first seven of them (the pure premium of payments before 84 months) was 7.213. Exhibit 7 shows that the same result is obtained by adding a quasi-observation that this sum is 7.213 with a error whose variance is 10^{-15} relative to the variances of the observations.[4] Exhibit 8 shows how different the estimate is when the constraint is relaxed. (One should not suppose that the estimates of σ^2 in the two exhibits are equal; they differ by about six million.) Appendix C proves that the mixed model (stochastically constrained model) approaches the (non-stochastically) constrained model as V approaches zero.

9. Credibility and Random Effects (Example 6)

So far, credibility has been statistically modeled by adding quasi-observations to observations, i.e., by mixing non-sample with sample information. The non-sample information is aptly considered to be logically prior to, if not also temporally prior to, the sample information. It too may have been derived from a sample; but if so, its sample is a different sample. If the two samples are grouped into a grand statistical model, such as the first grand model of Appendix A, the submodels are naturally considered as non-simultaneous, or *temporally extensive* or *longitudinal*. For example, if we begin observing the pure premium of State X with the prior opinion that it is 0.10 with a standard deviation of 0.02, we opine thus because in the past we have observed the pure premiums of similar States A, B, … .

But credibility may also involve the simultaneous modeling of similar entities. Each entity has its own model, and the models are grouped into a grand model; however, the (sub)models are simultaneous, or *temporally intensive* or *latitudinal*. Example 6, which begins with Exhibit 9, will illustrate this concept. This example, taken from Venter [13:433], consists of six observations of a pure premium from each of nine states. If the pure premiums were unrelated, then one could do no better than to solve nine independent models (to take nine averages). If the pure premiums had to be equal, then one could do no better than to average the fifty-four observations. But an actuary would rightly feel that the truth lies in between these two extremes: the pure premiums of the states are neither unrelated nor identical. The pure premium of one state is related with those of the other states, but it also has some identity of its own. A natural way of expressing this is to assume that the pure premiums deviate randomly from a common value, e.g., $\boldsymbol{\beta}_i = \boldsymbol{\beta}_0 + \mathbf{v}_i$. $\boldsymbol{\beta}_0$ is the (fixed) effect common to all the states, and $\mathbf{v}_i$ is the (random) effect which differentiates State i from the other states. Each $\mathbf{v}_i$ is distributed with mean zero and some (known or unknown) variance V, and the $\mathbf{v}_i$s do not covary one with another. It is this assumption of being distributed that makes the effect random.

For the moment we will abstract from the example. In general we have n models, each of the form $\mathbf{y}_i = X_i\boldsymbol{\beta}_i + \mathbf{e}_i$, where $\mathrm{Var}[\mathbf{e}_i] = \Sigma_i$ and the $\mathbf{e}_i$s do not covary. At this point we have n independent models. But now we introduce the random-effects linkage, viz., that $\boldsymbol{\beta}_i = \boldsymbol{\beta}_0 + \mathbf{v}_i$. Now each model becomes:

$$\begin{aligned}
\mathbf{y}_i &= X_i\boldsymbol{\beta}_i + \mathbf{e}_i \\
&= X_i(\boldsymbol{\beta}_0 + \mathbf{v}_i) + \mathbf{e}_i \\
&= X_i\boldsymbol{\beta}_0 + (X_i\mathbf{v}_i + \mathbf{e}_i) \\
&= X_i\boldsymbol{\beta}_0 + \boldsymbol{\tau}_i, \\
\text{where } E[\boldsymbol{\tau}_i] &= X_i E[\mathbf{v}_i] + E[\mathbf{e}_i] \\
&= 0 \\
\text{and } \mathrm{Var}[\boldsymbol{\tau}_i] &= X_i \mathrm{Var}[\mathbf{v}_i] X_i' + \mathrm{Var}[\mathbf{e}_i] \\
&= X_i V X_i' + \Sigma_i \\
&= T_i
\end{aligned}$$

The formula for $\mathrm{Var}[\boldsymbol{\tau}_i]$ assumes that $\mathbf{v}_i$ and $\mathbf{e}_i$ do not covary. Moreover, since $\mathbf{v}_i$ and $\mathbf{e}_i$ do not covary across groups, the $\boldsymbol{\tau}_i$s do not covary one with another. Thus we have the grand model in $\boldsymbol{\beta}_0$: $\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \boldsymbol{\beta}_0 + \begin{bmatrix} \boldsymbol{\tau}_1 \\ \vdots \\ \boldsymbol{\tau}_n \end{bmatrix}$, where $\mathrm{Var}\begin{bmatrix} \boldsymbol{\tau}_1 \\ \vdots \\ \boldsymbol{\tau}_n \end{bmatrix} = \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}$. The solution of this model presents no difficulties, as long as V is known.[5] Hence, the only difficulty of this form of credibility is to estimate V, the random-effects variance, if one has no prior information about it.

So the difficult task of Example 6 is to estimate the β_is and their common variance.[6] This involves first solving the model as if it were a fixed-effects model, as in Exhibit 10. The estimate of β in this exhibit contains the nine group means, which are carried over to Exhibit 11. The estimate of the grand mean β_0 is 0.563, and the variance of the group means about β_0 is 0.0662.[7] It would be a mistake to think that this represents the random-effects variance, because we have calculated the variance of the *estimates* of the β_is, rather than the variance of the β_is themselves. Unlike the β_is themselves, the estimates of the β_is are affected by the error terms, the $\mathbf{e}_i$s. So 0.0662 has two variance components, one from the $\mathbf{v}_i$s and one from the $\mathbf{e}_i$s, which is the reason for labeling it Var[**v**+**e**]. Back in Exhibit 10 the variance of β was estimated as if the model were a fixed-effects model. The variance of the grand parameter of a fixed-effects model must be $(k \times k)$ block diagonal (here k equals one); and it is reasonable to attribute these variances to the $\mathbf{e}_i$s. Since the $\mathbf{v}_i$s and the $\mathbf{e}_i$s do not covary, one can estimate V by averaging the differences of these variances from Var[**v**+**e**]; thus V is estimated to be 0.0067.[8] Exhibit 11 goes on to show that we have derived the expected value of the process variance (EVPV) and the variance of the hypothetical means (VHM), which implies to an actuary that the credibility of each group is 10.1%. This will be checked at the end of the example.

But now we can estimate $\mathrm{Var}[X_i \mathbf{v}_i + \mathbf{e}_i] = \mathrm{Var}[\tau_i] = X_i V X_i' + \Sigma_i$, which the exhibit calls the Φ for each group. In Exhibit 12 the random-effects model is solved for the grand parameter β_0, and the variance matrix of this model is block diagonal in Φ. But we are more interested in estimating the β_is (where $\beta_i = \beta_0 + v_i$) than we are in estimating β_0. So in Exhibit 13 we formulate $\mathbf{y}_2$ as an estimator of these β_is (we also leave an estimator of β_0 in its first row): $\mathbf{y}_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \beta_0$, and the covariance matrix Φ_{21} takes the $\mathbf{v}_i$s into account. (See the discussion of the covariance matrix T_{21} in Appendix E for details.) So the estimate of $\mathbf{y}_2$ is obtained from the familiar formula $\hat{\mathbf{y}}_2 = X_2\hat{\beta}_0 + \Phi_{21}\Phi_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\beta}_0\right)$. The exhibit illustrates that this estimator is equivalent to giving the fixed-effects estimators 10.1% credibility against the grand mean, as well as that the simple average of the $\hat{\beta}_i$s is $\hat{\beta}_0$. Appendix E backs up these specific illustrations with general proofs.

The results are the same as Venter's [13:432f.]. One might question whether anything has been gained by the setting up of a statistical model. Venter's discussion of credibility is hard enough for actuaries to understand; statistically modeling credibility may seem even harder. However, after developing some familiarity with best linear unbiased estimation, one will find it to be the more natural and more powerful way of handling credibility. Three reasons for its being more powerful are: 1) statistical modeling preserves two moments (the variance as well as the mean), 2) combinations of the parameter estimates can be estimated, and 3) it allows for multidimensional credibility. The third reason will be illustrated in the following trend model.

10. Random-Effects Credibility and Trend Modeling (Example 7)

A simulation of loss ratios for nine states over a six-year period is shown in Exhibit 14, and is graphed in Exhibit 15. Simulating nine states makes for a cluttered graph; however, as a practical matter, the reliable estimation of a random-effects variance requires enough data to distinguish the groups from one another. This requires a fair number of groups and/or a fair number of observations per group. The author knows of no rule as to what is a "fair" number, but a "fair" number of groups is probably not much less than the nine of this and the previous example. An upward trend is evident in the graph; but the states obviously have different slopes and intercepts. In fact, State G seems to have a negative slope.

Exhibit 16 solves the problem as a fixed-effects model, with the (18×1) $\hat{\beta}$ containing the (2×1) trend parameters of the nine states. The variance of the error matrix (Φ) is I_{54}, which simplifies the formulas. $\mathrm{Var}\left[\hat{\beta}\right]$ is diagonal in the same (2×2) block $\begin{bmatrix} 0.0011 & -0.0003 \\ -0.0003 & 0.0001 \end{bmatrix}$, which, as mentioned in the previous section, means that the model is balanced. The trend parameters vary widely by state, and State G is showing a negative slope. But will the negative slope be credible?

The random-effects variance is estimated in Exhibit 17. The mean state parameter is $\begin{bmatrix} 42.3\% \\ 2.7\% \end{bmatrix}$, and the individual states' parameters vary about it by $\begin{bmatrix} 0.0033 & -0.0005 \\ -0.0005 & 0.0006 \end{bmatrix}$. Removing the effect of the error term, we are left with the estimated random-effects variance $V = \begin{bmatrix} 0.0022 & -0.0003 \\ -0.0003 & 0.0005 \end{bmatrix}$.[9] The variance of the intercept is greater than that of the slope (0.0022 versus 0.0005). Also, the intercept and the slope negatively covary (−0.0003, or a correlation coefficient of −27.0%). This is common in random-effects trend models: there seems to be a centroid through which every group pivots. So a higher than average intercept tends to pair with a lower than average slope, and *vice versa*. The exhibit then derives the Φ matrix for the random-effects model.

The random-effects trend model is set up and solved in Exhibit 18. From the previous section and Appendix E, and because of the balance, it comes as no surprise that the grand parameter $\hat{\beta}_0 = \begin{bmatrix} 42.3\% \\ 2.7\% \end{bmatrix}$, the simple average of the fixed-effects parameters. But we really want to estimate the states' trends, which are sums of the grand parameter and the random effects. This is accomplished in Exhibit 19, in which the most difficult concept is Φ_{21}. The blocks of this matrix represent how $\beta_i = \beta_0 + \mathbf{v}_i$ covaries with $\mathbf{y}_i = X_i\beta_i + \mathbf{e}_i$:

$$\begin{aligned} \mathrm{Cov}[\beta_i, \mathbf{y}_i] &= \mathrm{Cov}[\beta_0 + \mathbf{v}_i, X_i(\beta_0 + \mathbf{v}_i) + \mathbf{e}_i] \\ &= \mathrm{Cov}[\beta_0 + \mathbf{v}_i, X_i\beta_0 + X_i\mathbf{v}_i + \mathbf{e}_i] \\ &= \mathrm{Cov}[\mathbf{v}_i, X_i\mathbf{v}_i + \mathbf{e}_i] \\ &= \mathrm{Cov}[\mathbf{v}_i, X_i\mathbf{v}_i] + \mathrm{Cov}[\mathbf{v}_i, \mathbf{e}_i] \\ &= \mathrm{Cov}[\mathbf{v}_i, X_i\mathbf{v}_i] \\ &= \mathrm{Cov}[\mathbf{v}_i, \mathbf{v}_i]X_i' \\ &= \mathrm{Var}[\mathbf{v}_i]X_i' \\ &= VX_i' \end{aligned}$$

The usual formula for $\hat{\mathbf{y}}_2$ yields the random-effects trend parameters by state. State G remains with a negative slope, though less negative than before.

In Appendix E the relationship between the fixed-effects and random-effects estimators is explored. One result is the discovery of a $(k \times k)$ matrix Z such that:

$$\text{Random-effects } \hat{\beta}_i = \hat{\gamma}_i = Z\left(\text{Fixed-effects } \hat{\beta}_i\right) + \left(I_k - Z\right)\hat{\beta}_0$$

This is the k-dimensional extension of the well-known scalar (or 1-dimensional) credibility formula. Exhibit 20 expresses the random-effects estimates in this Z form. As remarked in Appendix A, a matrix-weighted average of two vectors is usually not collinear with the two vectors. But somewhat surprising is that occasionally the matrix-weighted average can fall outside the range of the two vectors. For example, the posterior slope of State A (5.9%) is outside the range of the prior and empirical slopes (2.7% and 5.7%). This happens also with the intercept of State F and with the slope of State H. Non-zero off-diagonal elements of Z make this possible.

11. Conclusion

Practice precedes theory and systematization. For example, the Egyptians were doing geometry for centuries before Euclid wrote the *Elements*. Euclid didn't discover Geometry; he didn't correct it; he may not even have contributed much in the way of new theorems. But he systematized it, made it rigorous, and enabled centuries of mathematicians to develop it further. So too, actuaries have been practicing beneficial things under the name of credibility largely in ignorance of statistical theory. But just as Euclid made Geometry better, so too the theory of statistical modeling makes credibility better.

How does credibility benefit? As mentioned at the ends of the Introduction and of Section 9, statistical modeling furnishes the actuary with the variances as well as with the means; and from this the actuary can work with combinations of estimates. But perhaps most important, statistical theory and modeling are as at home in n dimensions as in one. When the author began to study statistics and econometrics he erroneously believed that his linear algebra and multivariate calculus were sufficient for statistical work. As his background was typical for an actuary, he can "speak from his own experience and with the ardor of a convert" (as did Arthur Bailey in a quote of Section 2) that most of us actuaries, even the technically inclined, are Flatlanders as regards our statistical skills. As our problems become more complex, as well as the tools with which to solve them, this defect will become more grievous.

Bailey's three offending credibility procedures (cf. Section 2) were statistically ahead of their time. But times have changed, and now it is incumbent upon actuaries to keep up with the times. The examples of this paper show how these procedures are legitimated and generalized by current statistical theory. For the use of prior hypotheses in estimation see Examples 3, 4, and 5. For the estimation of groups together which is more accurate than estimating each separately see Examples 6 and 7. And the estimation of an individual that belongs to a heterogeneous population is in essence a disguised use of a prior hypothesis; but see especially Section 7. The appendices of the paper lay the theoretical groundwork for the examples, a groundwork from which credibility has much to gain.

Notes

[1] Longley-Cook's definition is similar: "The word credibility was originally introduced into actuarial science as a measure of the credence that the actuary believes should be attached to a particular body of experience for ratemaking purposes." [9:3] "Predictive value" in the CAS statement has a more precise meaning that Longley-Cook's noun "credence." One reason for not giving credence to data is that it is suspected of being erroneous. But in credibility theory the quality of the data is not at issue; it is supposed to be valid data. What is at issue is the value of the data for predicting.

[2] The title of this paper is "Statistical Models and Credibility," but only *linear* statistical models will be treated. In the earlier paper [7:325f.] the author argued that due to the multivariate Taylor's expansion, linearity is not much of a restriction. The interested reader can refer to Judge, who devotes a chapter of his book to non-linear statistical models [8:508-511].

[3] There is an easy way to derive the form of the Poisson distribution with parameter m. One need only to remember the Taylor series for e^m:

$$\begin{aligned}1 &= e^m e^{-m} \\ &= \left(\sum_{x=0}^{\infty} \frac{m^x}{x!}\right) e^{-m} \\ &= \left(\sum_{x=0}^{\infty} \frac{m^x}{x!} e^{-m}\right) \\ &= \sum_{x=0}^{\infty} \text{Prob}(x)\end{aligned}$$

[4] Of course, the results are not really the same, only very close (to within the decimals shown in the exhibit). Reducing the variance of the quasi-observation still more will at some point run up against computational problems, and the results will stray. The author recommends that a tight stochastic constraint should not be substituted for an exact constraint.

[5] As in Section 6, either the Σ_is are known absolutely, in which case V must be known absolutely, or the Σ_is are known relatively, in which case V must be known to within the same proportionality constant to within which the Σ_is are known.

[6] Since in this example the β_is are the means of the groups (the hypothetical means), their common variance is what actuaries call the variance of the hypothetical means. But in general, the β_is are $(k\times 1)$ parameters; so their common variance could be called the variance of the hypothetical parameters.

[7] Appendix D derives the formulas for the sample mean and variance of n identically distributed non-covarying ($k\times 1$) random vectors.

[8] The fixed-effects model is an instance of the first grand model of Appendix A, where it is proved that $\operatorname{Var}\left[\hat{\beta}\right]=\operatorname{Var}\begin{bmatrix}\hat{\beta}_1\\ \vdots\\ \hat{\beta}_n\end{bmatrix}=\begin{bmatrix}\operatorname{Var}\left[\hat{\beta}_1\right] & & \\ & \ddots & \\ & & \operatorname{Var}\left[\hat{\beta}_n\right]\end{bmatrix}=\begin{bmatrix}\left(X_1'\Sigma_1^{-1}X_1\right)^{-1} & & \\ & \ddots & \\ & & \left(X_n'\Sigma_n^{-1}X_n\right)^{-1}\end{bmatrix}$.
When the n blocks of this matrix are equal, the submodels are equally influential in the determination of V. In this situation the corresponding random-effects model is said to be balanced. Both Examples 6 and 7 are balanced. The estimation of V by variance components is particularly suited to balanced models. The estimate of the V of an unbalanced model can be thrown off by the more volatile groups, and can easily end up not being non-negative definite. Nothing precludes positing V by prior information, and this recourse is the more recommended according as the model is the more unbalanced. Also, Appendix F mentions that V can be estimated by maximum likelihood, which despite its complexity is sometimes a useful alternative to variance components.

[9] Compare these estimates with the true values used in the simulation: $\beta_0=\begin{bmatrix}40.0\%\\ 3.0\%\end{bmatrix}$ and $V=\begin{bmatrix}0.0025 & -0.0001875\\ -0.0001875 & 0.000225\end{bmatrix}$ (so $\rho=-25\%$). And by generating bivariate normal random vectors with mean β_0 and variance V, the true β_is were:

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\\ \beta_7\\ \beta_8\\ \beta_9\end{bmatrix}=\begin{bmatrix}47.7\%\\ 5.4\%\\ \hdashline 41.5\%\\ 3.8\%\\ \hdashline 47.3\%\\ 3.1\%\\ \hdashline 39.3\%\\ 0.8\%\\ \hdashline 44.0\%\\ 5.1\%\\ \hdashline 37.1\%\\ 2.9\%\\ \hdashline 46.1\%\\ -0.8\%\\ \hdashline 38.8\%\\ 1.5\%\\ \hdashline 38.1\%\\ 3.8\%\end{bmatrix}$$

Normal random variables with a standard deviation of 4.0% were added to the resulting trend lines to form the fifty-four loss ratios of Exhibit 14.

References

1. Amemiya, Takeshi, Advanced Econometrics, Cambridge, MA, Harvard University Press, 1985.

2. Bailey, Arthur, "A Generalized Theory of Credibility," *PCAS* XXXII, 1945, 13-20.

3. Bailey, Arthur, "Credibility Procedures, Laplace's Generalization of Bayes' Rule, and the Combination of Collateral Knowledge with Observed Data," *PCAS* XXXVII, 1950, 7-23.

4. Bailey, Arthur, "Discussion of 'An Introduction to Credibility Theory by L. H. Longley-Cook,'" *PCAS* L, 1963, 59-61. Posthumously reprinted from the Journal of the American Teachers of Insurance (1950), 17-24.

5. Dropkin, Lester, "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records," *PCAS* LXXIV, 1987, 391-405. Reprinted from *PCAS* XLVI, 1959.

6. Halliwell, Leigh J., "Statistical and Financial Aspects of Self-Insurance Funding," Alternative Markets/Self Insurance, 1996 Discussion Paper Program, Casualty Actuarial Society, 1996, 1-46.

7. Halliwell, Leigh J., "Conjoint Prediction of Paid and Incurred Losses," 1997 Loss Reserving Discussion Papers, Casualty Actuarial Society, 1997, 241-379.

8. Judge, George G., Hill, R. C., *et al.*, Introduction to the Theory and Practice of Econometrics (Second Edition), New York, John Wiley & Sons, 1988.

9. Longley-Cook, L.H., "An Introduction to Credibility Theory," Casualty Actuarial Society, 1962.

10. Mayerson, Allen L. "A Bayesian View of Credibility," *PCAS* LI, 1964, 85-104.

11. Rodermund, Matthew, "Introduction," Foundations of Casualty Actuarial Science, Casualty Actuarial Society, 1990, 1-24.

12. SAS/STAT® Software: Changes and Enhancements through Release 6.12, Cary, NC, SAS Institute Inc., 1997.

13. Venter, Gary G., "Credibility," Foundations of Casualty Actuarial Science, Casualty Actuarial Society, 1990, 375-483.

Exhibit 1

Example 1: The Simplest Statistical Model

$\mathbf{y} = X\beta + \mathbf{e}$, where $Var[\mathbf{e}] = \sigma^2\Phi$

y_1

6.164
11.103
9.663
12.998
10.329
9.564
9.602

X_1

1
1
1
1
1
1
1

Φ_{11}

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1

Φ_{12}

1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0

$y_1 - X_1\hat\beta$

-3.754
1.185
-0.255
3.080
0.411
-0.353
-0.315

y_2

-
-
-
-
-
-
-
-
-
-
-

X_2

0
0
0
0
0
0
0
0
0
1
1

Φ_{21}

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Φ_{22}

1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1

$X_1'\Phi_{11}^{-1}y_1$: 69.421

$X_1'\Phi_{11}^{-1}X_1$: 7

$X_1'\Phi_{11}^{-1}$

1	1	1	1	1	1	1

$df = t_1 - k$: 6

$\hat\beta$: 9.917

$(X_1'\Phi_{11}^{-1}X_1)^{-1}$: 0.143

$\hat\sigma^2$: 4.240

$Var[\hat\beta]$: 0.606

$$\hat\beta = \left(X_1'\Phi_{11}^{-1}X_1\right)^{-1} X_1'\Phi_{11}^{-1}y_1$$

$$Var\left[\hat\beta\right] = \hat\sigma \left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}$$

$$\hat\sigma^2 = \left(y_1 - X_1\hat\beta\right)'\Phi_{11}^{-1}\left(y_1 - X_1\hat\beta\right)\Big/df$$

Exhibit 2

Example 1. The Simplest Statistical Model (Cont'd)

$\Phi_{21}\Phi_{11}^{-1}X_1$

1
1
1
1
1
1
1
0
0
0
0

$\Phi_{21}\Phi_{11}^{-1}$

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$\Phi_{21}\Phi_{11}^{-1}\Phi_{12}$

1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$X_2-\Phi_{21}\Phi_{11}^{-1}X_1$

-1
-1
-1
-1
-1
-1
-1
0
0
1
1

$\Phi_{22}-\Phi_{21}\Phi_{11}^{-1}\Phi_{12}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1

$\hat{\mathbf{y}}_2$

-3.754
1.185
-0.255
3.080
0.411
-0.353
-0.315
0.000
0.000
9.917
9.917

$\mathrm{Var}[\mathbf{y}_2-\hat{\mathbf{y}}_2]$

0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0.606	0.606	0.606	0.606	0.606	0.606	0.606	0	0	-0.606	-0.606
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	4.240	0	0
-0.606	-0.606	-0.606	-0.606	-0.606	-0.606	-0.606	0	0	0.606	0.606
-0.606	-0.606	-0.606	-0.606	-0.606	-0.606	-0.606	0	0	0.606	4.846

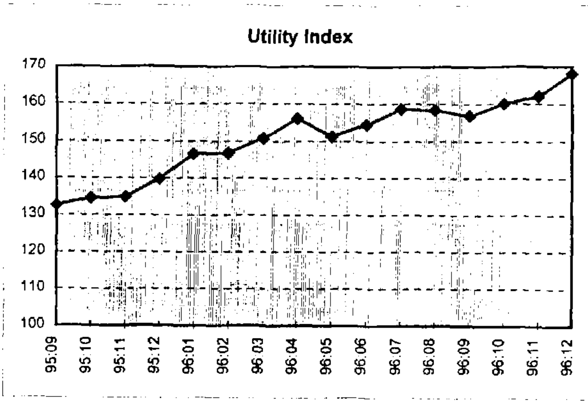
$$\hat{\mathbf{y}}_2 = X_2\hat{\beta} + \Phi_{21}\Phi_{11}^{-1}(\mathbf{y}_1 - X_1\hat{\beta})$$

$$\mathrm{Var}[\mathbf{y}_2 - \hat{\mathbf{y}}_2] = \hat{\sigma}^2(\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12}) + (X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)\mathrm{Var}[\hat{\beta}](X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)'$$

Exhibit 3

Example 2: Expense Model

Month	Index	Expense
95:09	132.545	1,714
95:10	134.440	1,804
95:11	134.820	1,862
95:12	139.690	2,265
96:01	146.572	2,553
96:02	146.745	2,170
96:03	150.687	2,315
96:04	155.983	2,217
96:05	151.240	2,279
96:06	154.417	2,293
96:07	158.616	2,171
96:08	158.302	2,263
96:09	156.779	2,192
96:10	160	
96:11	162	
96:12	168	



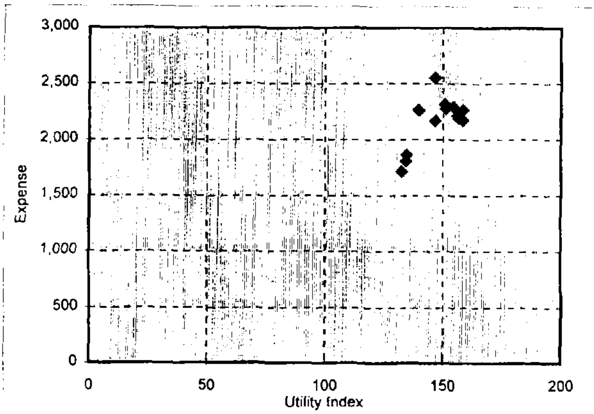


Exhibit 4

Example 2: Expense Model (Cont'd)

$\mathbf{y} = X\beta + \mathbf{e}$, where $Var[\mathbf{e}] = \sigma^2\Phi$

y_1	X_1
1,714	132.545
1,804	134.440
1,862	134.820
2,265	139.690
2,553	146.572
2,170	146.745
2,315	150.687
2,217	155.983
2,279	151.240
2,293	154.417
2,171	158.616
2,263	158.302
2,192	156.779

Φ_{11}

17568	0	0	0	0	0	0	0	0	0	0	0	0
0	18074	0	0	0	0	0	0	0	0	0	0	0
0	0	18176	0	0	0	0	0	0	0	0	0	0
0	0	0	19513	0	0	0	0	0	0	0	0	0
0	0	0	0	21483	0	0	0	0	0	0	0	0
0	0	0	0	0	21534	0	0	0	0	0	0	0
0	0	0	0	0	0	22707	0	0	0	0	0	0
0	0	0	0	0	0	0	24331	0	0	0	0	0
0	0	0	0	0	0	0	0	22874	0	0	0	0
0	0	0	0	0	0	0	0	0	23845	0	0	0
0	0	0	0	0	0	0	0	0	0	25159	0	0
0	0	0	0	0	0	0	0	0	0	0	25060	0
0	0	0	0	0	0	0	0	0	0	0	0	24580

Φ_{12}

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0

$y_1 - X_1\hat{\beta}$
-224
-161
-109
223
410
25
112
-63
68
36
-148
-51
-100

y_2	X_2
-	160
-	162
-	168

Φ_{21}

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

Φ_{22}

25600	0	0
0	26244	0
0	0	28224

$X_1'\Phi_{11}^{-1}y_1$	$X_1'\Phi_{11}^{-1}X_1$	$df = t_1 - k$
190.039	13	12

$X_1'\Phi_{11}^{-1}$

0.008	0.007	0.007	0.007	0.007	0.007	0.007	0.006	0.007	0.006	0.006	0.006	0.006

$\hat{\beta}$	$(X_1'\Phi_{11}^{-1}X_1)^{-1}$	$\hat{\sigma}^2$
14.618	0.077	1.475

$Var[\hat{\beta}]$
0.113

$\hat{y}_2$
2,339
2,368
2,456

$Var[y_2 - \hat{y}_2]$

40672	2941	3050
2941	41695	3089
3050	3089	44841

Exhibit 5

Example 3: Example 1 with Prior Information

$$\mathbf{y} = X\beta + \mathbf{e}, \text{ where } Var[\mathbf{e}] = \sigma^2\Phi$$

$\mathbf{y}_1$	X_1	Φ_{11}								$\mathbf{y}_1 - X_1\hat{\beta}$
6.164	1	4.240	0	0	0	0	0	0	0	-3.936
11.103	1	0	4.240	0	0	0	0	0	0	1.004
9.663	1	0	0	4.240	0	0	0	0	0	-0.437
12.998	1	0	0	0	4.240	0	0	0	0	2.898
10.329	1	0	0	0	0	4.240	0	0	0	0.229
9.564	1	0	0	0	0	0	4.240	0	0	-0.535
9.602	1	0	0	0	0	0	0	4.240	0	-0.497
11.000	1	0	0	0	0	0	0	0	3.000	0.901

$X_1'\Phi_{11}^{-1}\mathbf{y}_1$	$X_1'\Phi_{11}^{-1}X_1$	$X_1'\Phi_{11}^{-1}$								$df = t_1 - k$
20.038	1.984155	0.236	0.236	0.236	0.236	0.236	0.236	0.236	0.333	7

$\hat{\beta}$	$(X_1'\Phi_{11}^{-1}X_1)^{-1}$	$\hat{\sigma}^2$
10.099	0.504	0.904

$Var[\hat{\beta}]$
0.455

Credibility-Weighed Estimate			
Prior	11.000		0.168
Empirical	9.917	$Z =$	0.832
Posterior	10.099		1.000

Credibility Weight	
EVPV (σ^2)	4.240
VHM (τ^2)	3.000
k = EVPV/VHM	1.413
n	7
$Z = n/(n+k)$	0.832

Exhibit 6

Example 4: Merit Rating

A. Separate Observations

y_1

0
1
0
0.25

X_1

1
1
1
1

Φ_{11}

0.0625	0	0	0
0	0.0625	0	0
0	0	0.0625	0
0	0	0	0.0225

Φ_{21}

0
0
0
0

$y_1 - X_1\hat{\beta}$

-0.293
0.707
-0.293
-0.043

y_2: -

X_2: 1

Φ_{12}

0	0	0	0

Φ_{22}: 0.0625

$X_1'\Phi_{11}^{-1}y_1$: 27.111

$X_1'\Phi_{11}^{-1}X_1$: 92.44444

$X_1'\Phi_{11}^{-1}$

16	16	16	44.444

$df = t_1 - k$: 3

$\hat{\beta}$: 0.293

$(X_1'\Phi_{11}^{-1}X_1)^{-1}$: 0.011

$\hat{\sigma}^2$: 3.609

$Var[\hat{\beta}]$: 0.039

$\hat{y}_2$: 0.293

$Var[y_2 - \hat{y}_2]$: 0.265

B. Summarized Observations

y_1

1
0.25

X_1

3
1

Φ_{11}

0.1875	0
0	0.0225

Φ_{21}

0
0

$y_1 - X_1\hat{\beta}$

0.120
-0.043

y_2: -

X_2: 1

Φ_{12}

0	0

Φ_{22}: 0.0625

$X_1'\Phi_{11}^{-1}y_1$: 27.111

$X_1'\Phi_{11}^{-1}X_1$: 92.44444

$X_1'\Phi_{11}^{-1}$

16	44.444

$df = t_1 - k$: 1

$\hat{\beta}$: 0.293

$(X_1'\Phi_{11}^{-1}X_1)^{-1}$: 0.011

$\hat{\sigma}^2$: 0.160

$Var[\hat{\beta}]$: 0.002

$\hat{y}_2$: 0.293

$Var[y_2 - \hat{y}_2]$: 0.012

Exhibit 7

Example 5: Stochastic Constraint for Exact Constraint

Year	Age	y	X							Φ	$y - X\hat{\beta}$
1988	12	268,354	131332							1	32,538
1988	24	166,572		131332						1	-88,469
1988	36	32,329			131332						-133,541
1988	48	53,610				131332					-59,763
1988	60	8,124					131332				-63,069
1988	72	16,924						131332			-44,391
1988	84	39,109							131332		-7,552
1989	12	246,981	141672								-5,244
1989	24	359,380		141672							84,237
1989	36	229,016			141672						50,087
1989	48	89,539				141672					-52,760
1989	60	118,635					141672				41,837
1989	72	100,292						141672			34,150
1990	12	203,178	141677								-49,056
1990	24	375,768		141677							100,615
1990	36	276,617			141677						97,682
1990	48	74,912				141677					-47,392
1990	60	86,428					141677				9,627
1991	12	395,630	142578								141,793
1991	24	260,643		142578							-16,259
1991	36	167,709			142578						-12,364
1991	48	270,692				142578					147,611
1992	12	207,698	143286								-47,399
1992	24	174,615		143286							-103,661
1992	36	162,640			143286						-18,327
1993	12	167,681	138262								-78,472
1993	24	280,178		138262							11,659
1994	12	215,740	121858							1	-1,208
		7.213	1	1	1	1	1	1	1	1E-15	9.92E-07

$X'\Phi^{-1}y$
7.21E+15
7.21E+15
7.21E+15
7.21E+15
7.21E+15
7.21E+15
7.21E+15

$X'\Phi^{-1}X$

1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15
1E+15	1E+15	1E+15	1E+15	1E+15	1E+15	1E+15

$X'\Phi^{-1}$

1E+05		1E+05	1E+15
	1E+05		1E+15
			1E+15
		...	1E+15
			1E+15
			1E+15
			1E+15

$df = t-k$
22

$\hat{\beta}$
1.780
1.942
1.263
0.863
0.542
0.467
0.355
7.213

$(X'\Phi^{-1}X)^{-1}$

7.16E-12	-4.6E-13	-5.4E-13	-6.9E-13	-9.3E-13	-1.4E-12	-3.1E-12
-4.6E-13	8.01E-12	-6.1E-13	-7.8E-13	-1.1E-12	-1.6E-12	-3.5E-12
-5.4E-13	-6.1E-13	9.45E-12	-9.3E-13	-1.3E-12	-1.9E-12	-4.2E-12
-6.9E-13	-7.8E-13	-9.3E-13	1.17E-11	-1.6E-12	-2.4E-12	-5.3E-12
-9.3E-13	-1.1E-12	-1.3E-12	-1.6E-12	1.53E-11	-3.3E-12	-7.1E-12
-1.4E-12	-1.6E-12	-1.9E-12	-2.4E-12	-3.3E-12	2.17E-11	-1.1E-11
-3.1E-12	-3.5E-12	-4.2E-12	-5.3E-12	-7.1E-12	-1.1E-11	3.42E-11

$\hat{\sigma}^2$
6.27E+09

$Var[\hat{\beta}]$

0.04490	-0.00286	-0.00342	-0.00432	-0.00585	-0.00899	-0.01946
-0.00286	0.05021	-0.00385	-0.00488	-0.00659	-0.01013	-0.02192
-0.00342	-0.00385	0.05924	-0.00581	-0.00787	-0.01210	-0.02619
-0.00432	-0.00488	-0.00581	0.07335	-0.00995	-0.01530	-0.03310
-0.00585	-0.00659	-0.00787	-0.00995	0.09581	-0.02072	-0.04483
-0.00899	-0.01013	-0.01210	-0.01530	-0.02072	0.13619	-0.06894
-0.01946	-0.02192	-0.02619	-0.03310	-0.04483	-0.06894	0.21445

Exhibit 8

Example 5: Stochastic Constraint for Exact Constraint (Cont'd)

Year	Age	y	X							Φ				$y - X\hat{\beta}$
1988	12	266,354	131332							1				33,523
1988	24	166,572		131332							1			-87,379
1988	36	32,329			131332									-132,215
1988	48	53,610				131332								-58,087
1988	60	8,124					131332							-60,799
1988	72	16,924						131332						-40,900
1988	84	39,109							131332					-0
1989	12	246,981	141672											-4,181
1989	24	359,380		141672										85,434
1989	36	229,016			141672									51,517
1989	48	69,539				141672								-50,952
1989	60	118,635					141672							44,285
1989	72	100,292						141672						37,915
1990	12	203,178	141677											-47,993
1990	24	375,768		141677										101,813
1990	36	276,617			141677									99,112
1990	48	74,912				141677								-45,584
1990	60	86,428					141677							12,076
1991	12	395,630	142578											142,862
1991	24	260,643		142578										-15,054
1991	36	167,709			142578									-10,925
1991	48	270,692				142578								149,430
1992	12	207,698	143286											-46,324
1992	24	174,615		143286										-102,450
1992	36	162,640			143286									-16,880
1993	12	167,681	138262											-77,435
1993	24	280,176		138262										12,827
1994	12	215,740	121858									1		-294
		7.213	1	1	1	1	1	1	1				1	0 140171

$X'\Phi^{-1}y$	$X'\Phi^{-1}X$							$X'\Phi^{-1}$				$df = t-k$
2 34E+11	1.32E+11	1	1	1	1	1	1	1E+05		1E+05	1	22
2.27E+11	1	1 17E+11	1	1	1	1	1		1E+05		1	
1 23E+11	1	1	9.83E+10	1	1	1	1				1	
6 61E+10	1	1	1	7 77E+10	1	1	1				1	
3.01E+10	1	1	1	1	5 74E+10	1	1				1	
1.64E+10	1	1	1	1	1	3.73E+10	1				1	
5 14E+09	1	1	1	1	1	1	1.72E+10				1	

$\hat{\beta}$	$(X'\Phi^{-1}X)^{-1}$							$\hat{\sigma}^2$
1.773	7.56E-12	-6.4E-23	-7.7E-23	-9.7E-23	-1.3E-22	-2E-22	-4.4E-22	6 27E+09
1.934	-6.4E-23	8 52E-12	-8.7E-23	-1.1E-22	-1.5E-22	-2 3E-22	-4.9E-22	
1.253	-7.7E-23	-8.7E-23	1 02E-11	-1.3E-22	-1.8E-22	-2.7E-22	-5.9E-22	
0.850	-9.7E-23	-1 1E-22	-1.3E-22	1 29E-11	-2.2E-22	-3.4E-22	-7 5E-22	
0 525	-1.3E-22	-1 5E-22	-1.8E-22	-2 2E-22	1 74E-11	-4 7E-22	-1E-21	
0.440	-2E-22	-2 3E-22	-2.7E-22	-3.4E-22	-4 7E-22	2.68E-11	-1.6E-21	
0 298	-4.4E-22	-4.9E-22	-5.9E-22	-7.5E-22	-1E-21	-1.6E-21	5.8E-11	
7 073								

$\mathrm{Var}[\hat{\beta}]$

0 04739	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0 00000	0.05338	0.00000	0 00000	0.00000	0.00000	0.00000
0 00000	0 00000	0 06377	0.00000	0.00000	0 00000	0.00000
0.00000	0 00000	0.00000	0.08061	0.00000	0 00000	0.00000
0.00000	0 00000	0 00000	0 00000	0.10917	0.00000	0 00000
0.00000	0 00000	0.00000	0.00000	0.00000	0.16789	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.36325

Exhibit 9

Example 6: The Simplest Random-Effects Model

Preliminary Fixed-Effects Model

	y	X									$y - X\hat{\beta}$
1	0.430	1									-0.371
1	0.375	1									-0.426
1	2.341	1									1.541
1	0.175	1									-0.626
1	1.016	1									0.216
1	0.466	1									-0.335
2	0.247		1								-0.553
2	1.587		1								0.787
2	1.939		1								1.139
2	0.712		1								-0.088
2	0.054		1								-0.746
2	0.261		1								-0.539
3	0.661			1							0.242
3	0.237			1							-0.182
3	0.063			1							-0.356
3	0.250			1							-0.169
3	0.602			1							0.183
3	0.700			1							0.281
4	0.182				1						0.043
4	0.351				1						0.212
4	0.011				1						-0.129
4	0.022				1						-0.118
4	0.019				1						-0.121
4	0.252				1						0.113
5	0.311					1					-0.504
5	0.664					1					-0.151
5	1.002					1					0.188
5	0.038					1					-0.777
5	0.370					1					-0.445
5	2.502					1					1.688
6	0.301						1				-0.316
6	0.253						1				-0.364
6	0.044						1				-0.573
6	0.109						1				-0.508
6	2.105						1				1.488
6	0.891						1				0.274
7	0.219							1			-0.495
7	1.186							1			0.472
7	0.431							1			-0.283
7	1.405							1			0.691
7	0.241							1			-0.473
7	0.804							1			0.090
8	0.002								1		-0.204
8	0.058								1		-0.148
8	0.235								1		0.029
8	0.018								1		-0.188
8	0.713								1		0.507
8	0.208								1		0.002
9	0.796									1	0.242
9	0.260									1	-0.294
9	0.932									1	0.378
9	0.857									1	0.303
9	0.129									1	-0.425
9	0.349									1	-0.205

Exhibit 10

Example 6: The Simplest Random-Effects Model (Cont'd)

Solution of Fixed Effects

X'y

4.803
4.800
2.513
0.837
4.887
3.703
4.286
1.234
3.323

X'X

6	0	0	0	0	0	0	0	0
0	6	0	0	0	0	0	0	0
0	0	6	0	0	0	0	0	0
0	0	0	6	0	0	0	0	0
0	0	0	0	6	0	0	0	0
0	0	0	0	0	6	0	0	0
0	0	0	0	0	0	6	0	0
0	0	0	0	0	0	0	6	0
0	0	0	0	0	0	0	0	6

$\hat{\beta}$

0.801
0.800
0.419
0.140
0.815
0.617
0.714
0.206
0.554

$(X'X)^{-1}$

0.16667	0	0	0	0	0	0	0	0
0	0.16667	0	0	0	0	0	0	0
0	0	0.16667	0	0	0	0	0	0
0	0	0	0.16667	0	0	0	0	0
0	0	0	0	0.16667	0	0	0	0
0	0	0	0	0	0.16667	0	0	0
0	0	0	0	0	0	0.16667	0	0
0	0	0	0	0	0	0	0.16667	0
0	0	0	0	0	0	0	0	0.16667

t	54
k	9
$df = t - k$	45
$\hat{\sigma}^2$	0.357

$Var[\hat{\beta}]$

0.0595	0	0	0	0	0	0	0	0
0	0.0595	0	0	0	0	0	0	0
0	0	0.0595	0	0	0	0	0	0
0	0	0	0.0595	0	0	0	0	0
0	0	0	0	0.0595	0	0	0	0
0	0	0	0	0	0.0595	0	0	0
0	0	0	0	0	0	0.0595	0	0
0	0	0	0	0	0	0	0.0595	0
0	0	0	0	0	0	0	0	0.0595

Exhibit 11

Example 6: The Simplest Random-Effects Model (Cont'd)

Estimation of the Variance of the Random Effects

$\hat{\beta}$	$\hat{\beta}-\hat{\beta}_0$	$(\hat{\beta}-\hat{\beta}_0)(\hat{\beta}-\hat{\beta}_0)'$	Var[v+e]	- Var[e]	= Var[v]
0.801	0.238	0.0565	0.0662	0.0595	0.0067
0.800	0.237	0.0563	0.0662	0.0595	0.0067
0.419	-0.144	0.0207	0.0662	0.0595	0.0067
0.140	-0.423	0.1791	0.0662	0.0595	0.0067
0.815	0.252	0.0634	0.0662	0.0595	0.0067
0.617	0.054	0.003	0.0662	0.0595	0.0067
0.714	0.152	0.023	0.0662	0.0595	0.0067
0.206	-0.357	0.1275	0.0662	0.0595	0.0067
0.554	-0.009	8E-05	0.0662	0.0595	0.0067

	$\hat{\beta}_0$	Var[v+e]	V = Var[v]
Mean:	0.563	0.0662	0.0067

Credibility Weight	
EVPV (σ^2)	0.3570
VHM (V)	0.0067
k = EVPV/VHM	53.332
n	6
$Z = n/(n+k)$	10.1%

Φ for each group

0.3637	0.0067	0.0067	0.0067	0.0067	0.0067
0.0067	0.3637	0.0067	0.0067	0.0067	0.0067
0.0067	0.0067	0.3637	0.0067	0.0067	0.0067
0.0067	0.0067	0.0067	0.3637	0.0067	0.0067
0.0067	0.0067	0.0067	0.0067	0.3637	0.0067
0.0067	0.0067	0.0067	0.0067	0.0067	0.3637

Φ^{-1} for each group

2.754	-0.047	-0.047	-0.047	-0.047	-0.047
-0.047	2.754	-0.047	-0.047	-0.047	-0.047
-0.047	-0.047	2.754	-0.047	-0.047	-0.047
-0.047	-0.047	-0.047	2.754	-0.047	-0.047
-0.047	-0.047	-0.047	-0.047	2.754	-0.047
-0.047	-0.047	-0.047	-0.047	-0.047	2.754

Exhibit 12

Example 6. The Simplest Random-Effects Model (Cont'd)

Estimation of the General Mean β_0

	y_1	X_1	$y_1 - X_1\hat{\beta}_0$	Φ_{11}^{-1}							
1	0.430	1	-0.133	2.753812	-0.04721	-0.04721	-0.04721	-0.04721	-0.04721	0	0
1	0.375	1	-0.188	-0.04721	2.753812	-0.04721	-0.04721	-0.04721	-0.04721	0	0
1	2.341	1	1.778	-0.04721	-0.04721	2.753812	-0.04721	-0.04721	-0.04721	0	0
1	0.175	1	-0.388	-0.04721	-0.04721	-0.04721	2.753812	-0.04721	-0.04721	0	0
1	1.016	1	0.453	-0.04721	-0.04721	-0.04721	-0.04721	2.753812	-0.04721	0	0
1	0.466	1	-0.097	-0.04721	-0.04721	-0.04721	-0.04721	-0.04721	2.753812	0	0
2	0.247	1	-0.316	0	0	0	0	0	0	2.753812	-0.04721
2	1.587	1	1.024	0	0	0	0	0	0	-0.04721	2.753812
2	1.939	1	1.376	0	0	0	0	0	0	-0.04721	-0.04721
2	0.712	1	0.149	0	0	0	0	0	0	-0.04721	-0.04721
2	0.054	1	-0.509	0	0	0	0	0	0	-0.04721	-0.04721
2	0.261	1	-0.302	0	0	0	0	0	0	-0.04721	-0.04721
3	0.661	1	0.098	0	0	0	0	0	0	0	0
3	0.237	1	-0.326	0	0	0	0	0	0	0	0
3	0.063	1	-0.500	0	0	0	0	0	0	0	0
3	0.250	1	-0.313	0	0	0	0	0	0	0	0
3	0.602	1	0.039	0	0	0	0	0	0	0	0
3	0.700	1	0.137	0	0	0	0	0	0	0	0
4	0.182	1	-0.381	0	0	0	0	0	0	0	0
4	0.351	1	-0.212	0	0	0	0	0	0	0	0
4	0.011	1	-0.552	0	0	0	0	0	0	0	0
4	0.022	1	-0.541	0	0	0	0	0	0	0	0
4	0.019	1	-0.544	0	0	0	0	0	0	0	0
4	0.252	1	-0.311	0	0	0	0	0	0	0	0
5	0.311	1	-0.252	0	0	0	0	0	0	0	0
5	0.664	1	0.101	0	0	0	0	0	0	0	0
5	1.002	1	0.439	0	0	0	0	0	0	0	0
5	0.038	1	-0.525	0	0	0	0	0	0	0	0
5	0.370	1	-0.193	0	0	0	0	0	0	0	0
5	2.502	1	1.939	0	0	0	0	0	0	0	0
6	0.301	1	-0.262	0	0	0	0	0	0	0	0
6	0.253	1	-0.310	0	0	0	0	0	0	0	0
6	0.044	1	-0.519	0	0	0	0	0	0	0	0
6	0.109	1	-0.454	0	0	0	0	0	0	0	0
6	2.105	1	1.542	0	0	0	0	0	0	0	0
6	0.891	1	0.328	0	0	0	0	0	0	0	0
7	0.219	1	-0.344	0	0	0	0	0	0	0	0
7	1.186	1	0.623	0	0	0	0	0	0	0	0
7	0.431	1	-0.132	0	0	0	0	0	0	0	0
7	1.405	1	0.842	0	0	0	0	0	0	0	0
7	0.241	1	-0.322	0	0	0	0	0	0	0	0
7	0.804	1	0.241	0	0	0	0	0	0	0	0
8	0.002	1	-0.561	0	0	0	0	0	0	0	0
8	0.058	1	-0.505	0	0	0	0	0	0	0	0
8	0.235	1	-0.328	0	0	0	0	0	0	0	0
8	0.018	1	-0.545	0	0	0	0	0	0	0	0
8	0.713	1	0.150	0	0	0	0	0	0	0	0
8	0.208	1	-0.355	0	0	0	0	0	0	0	0
9	0.796	1	0.233	0	0	0	0	0	0	0	0
9	0.260	1	-0.303	0	0	0	0	0	0	0	0
9	0.932	1	0.369	0	0	0	0	0	0	0	0
9	0.857	1	0.294	0	0	0	0	0	0	0	0
9	0.129	1	-0.434	0	0	0	0	0	0	0	0
9	0.349	1	-0.214	0	0	0	0	0	0	0	0

$X_1'\Phi_{11}^{-1}y_1$	$X_1'\Phi_{11}^{-1}X_1$	df
76.505	135.96	53

$X_1'\Phi_{11}^{-1}$							
2.517766	2.517766	2.517766	2.517766	2.517766	2.517766	2.517766	2.517766

$\hat{\beta}_0$	$(X_1'\Phi_{11}^{-1}X_1)^{-1}$	$\hat{\sigma}^2$
0.563	0.0074	1.000

$\text{Var}[\hat{\beta}_0]$
0.0074

Exhibit 13

Example 6: The Simplest Random-Effects Model (Cont'd)

Estimation of the Group Means

	y_2	X_2	Φ_{21}							
	-	1	0	0	0	0	0	0	0	0
1	-	1	0.0067	0.0067	0.0067	0.0067	0.0067	0.0067	0	0
2	-	1	0	0	0	0	0	0	0.0067	0.0067
3	-	1	0	0	0	0	0	0	0	0
4	-	1	0	0	0	0	0	0	0	0
5	-	1	0	0	0	0	0	0	0	0
6	-	1	0	0	0	0	0	0	0	0
7	-	1	0	0	0	0	0	0	0	0
8	-	1	0	0	0	0	0	0	0	0
9	-	1	0	0	0	0	0	0	0	0

	$\hat{y}_2$	$\Phi_{21}\Phi_{11}^{-1}$							
	0.563	0	0	0	0	0	0	0	0
1	0.587	0.016854	0.016854	0.016854	0.016854	0.016854	0.016854	0	0
2	0.587	0	0	0	0	0	0	0.016854	0.016854
3	0.548	0	0	0	0	0	0	0	0
4	0.520	0	0	0	0	0	0	0	0
5	0.588	0	0	0	0	0	0	0	0
6	0.568	0	0	0	0	0	0	0	0
7	0.578	0	0	0	0	0	0	0	0
8	0.527	0	0	0	0	0	0	0	0
9	0.562	0	0	0	0	0	0	0	0

Credibility-Weighed Estimates					
Group	1 - Z	Prior	Z	Empirical	Posterior
1	0.899	0.563	0.101	0.801	0.587
2		0.563		0.800	0.587
3		0.563		0.419	0.548
4		0.563		0.140	0.520
5		0.563		0.815	0.588
6		0.563		0.617	0.568
7		0.563		0.714	0.578
8		0.563		0.206	0.527
9		0.563		0.554	0.562
Unweighted Mean					0.563

Exhibit 14

Example 7: Random-Effects Trend Model

Preliminary Fixed-Effects Model

State	Year	Loss Ratio y	X																		$y - X\hat{\beta}$
A	1	54.3%	1	1																	1.8%
A	2	57.2%	1	2																	-1.1%
A	3	64.6%	1	3																	0.6%
A	4	67.6%	1	4																	-2.2%
A	5	73.5%	1	5																	-2.0%
A	6	84.1%	1	6																	2.9%
B	1	44.2%			1	1															0.6%
B	2	48.6%			1	2															1.0%
B	3	54.8%			1	3															3.2%
B	4	48.2%			1	4															-7.4%
B	5	57.7%			1	5															-1.9%
B	6	68.2%			1	6															4.5%
C	1	53.9%					1	1													0.1%
C	2	57.0%					1	2													1.9%
C	3	54.8%					1	3													-1.8%
C	4	59.9%					1	4													2.0%
C	5	52.7%					1	5													-6.6%
C	6	65.2%					1	6													4.5%
D	1	41.8%							1	1											-2.0%
D	2	45.2%							1	2											1.2%
D	3	45.1%							1	3											0.8%
D	4	46.4%							1	4											1.9%
D	5	43.9%							1	5											-0.9%
D	6	44.0%							1	6											-1.0%
E	1	46.3%									1	1									1.5%
E	2	48.6%									1	2									-2.4%
E	3	57.6%									1	3									0.5%
E	4	63.3%									1	4									0.1%
E	5	69.6%									1	5									0.3%
E	6	75.4%									1	6									-0.1%
F	1	46.9%											1	1							3.6%
F	2	38.4%											1	2							-6.3%
F	3	48.1%											1	3							1.9%
F	4	46.0%											1	4							-1.6%
F	5	53.4%											1	5							4.4%
F	6	48.2%											1	6							-2.1%
G	1	45.7%													1	1					-0.6%
G	2	44.5%													1	2					-0.8%
G	3	44.2%													1	3					-0.2%
G	4	46.7%													1	4					3.2%
G	5	43.2%													1	5					0.6%
G	6	39.5%													1	6					-2.2%
H	1	38.2%															1	1			0.8%
H	2	42.0%															1	2			2.3%
H	3	36.8%															1	3			-5.4%
H	4	46.1%															1	4			1.4%
H	5	47.3%															1	5			0.2%
H	6	50.2%															1	6			0.7%
I	1	43.1%																	1	1	3.2%
I	2	44.8%																	1	2	0.8%
I	3	47.3%																	1	3	-0.9%
I	4	46.4%																	1	4	-6.0%
I	5	52.5%																	1	5	-4.1%
I	6	67.9%																	1	6	7.1%

Exhibit 15

Example 7: Random-Effects Trend Model (Cont'd)

Loss Ratios by State and Year

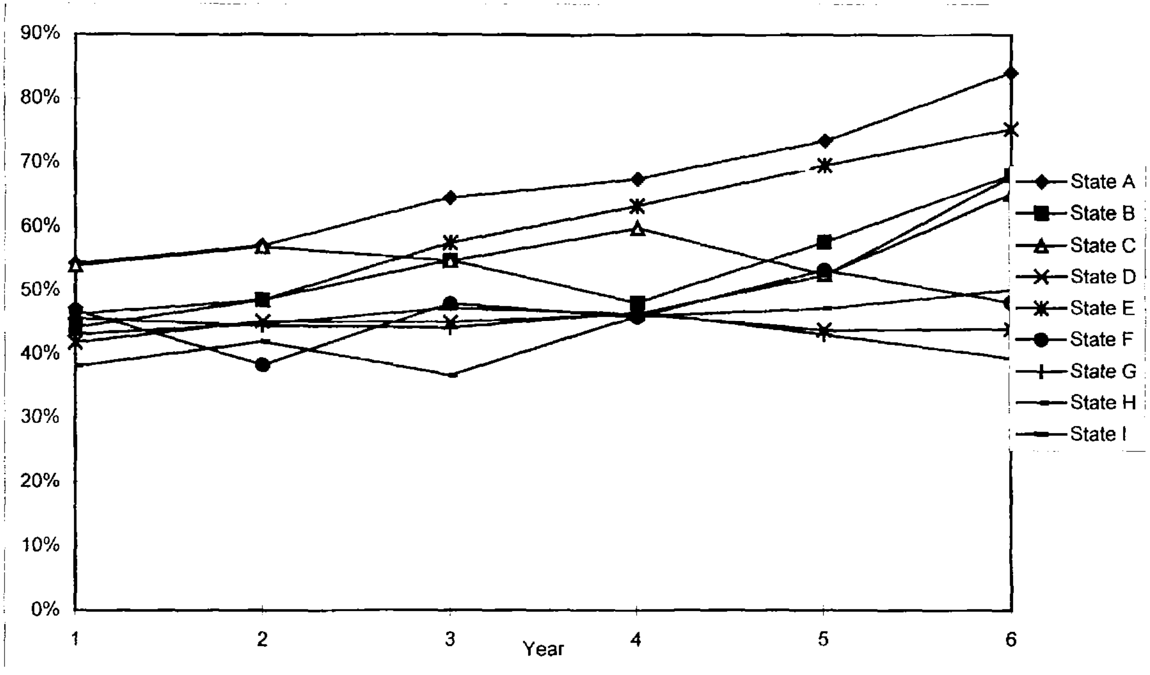


Exhibit 16

Example 7 Random-Effects Trend Model (Cont'd)

Solution of Fixed Effects

X'y	X'X																	
4.01	6	21																
15.05	21	91																
3.22			6	21														
11.96			21	91														
3.43					6	21												
12.26					21	91												
2.66							6	21										
9.36							21	91										
3.61									6	21								
13.70									21	91								
2.81											6	21						
10.08											21	91						
2.64													6	21				
9.07													21	91				
2.61															6	21		
9.55															21	91		
3.02																	6	21
11.30																	21	91

	$\hat{\beta}$	$(X'X)^{-1}$																	
A	46.8%	0.8667	-0.2000																
	5.7%	-0.2000	0.0571																
B	39.6%			0.8667	-0.2000														
	4.0%			-0.2000	0.0571														
C	52.4%					0.8667	-0.2000												
	1.4%					-0.2000	0.0571												
D	43.6%							0.8667	-0.2000										
	0.2%							-0.2000	0.0571										
E	38.7%									0.8667	-0.2000								
	6.1%									-0.2000	0.0571								
F	41.9%											0.8667	-0.2000						
	1.4%											-0.2000	0.0571						
G	47.2%													0.8667	-0.2000				
	-0.9%													-0.2000	0.0571				
H	34.9%															0.8667	-0.2000		
	2.4%															-0.2000	0.0571		
I	35.7%																	0.8667	-0.2000
	4.2%																	-0.2000	0.0571

		$Var[\hat{\beta}]$																	
t	54	0.0011	-0.0003	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
k	18	-0.0003	0.0001	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
df	36	0	0	0.0011	-0.0003	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\hat{\sigma}^2$	0.001	0	0	-0.0003	0.0001	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0.0011	-0.0003	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	-0.0003	0.0001	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0.0011	-0.0003	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	-0.0003	0.0001	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0.0011	-0.0003	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	-0.0003	0.0001	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0.0011	-0.0003	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	-0.0003	0.0001	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0.0011	-0.0003	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	-0.0003	0.0001	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.0011	-0.0003	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.0003	0.0001	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.0011	-0.0003
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.0003	0.0001

Exhibit 17

Example 7: Random-Effects Trend Model (Cont'd)

Estimation of the Variance of the Random Effects

	$\hat{\beta}$	$\hat{\beta}-\hat{\beta}_0$	$(\hat{\beta}-\hat{\beta}_0)(\hat{\beta}-\hat{\beta}_0)'$		Var[v+e]		– Var[e]		= Var[v]	
A	46.8%	4.5%	0.0020	0.0013	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	5.7%	3.0%	0.0013	0.0009	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
B	39.6%	-2.7%	0.0007	-0.0003	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	4.0%	1.3%	-0.0003	0.0002	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
C	52.4%	10.1%	0.0102	-0.0014	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	1.4%	-1.3%	-0.0014	0.0002	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
D	43.6%	1.3%	0.0002	-0.0003	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	0.2%	-2.5%	-0.0003	0.0006	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
E	38.7%	-3.6%	0.0013	-0.0012	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	6.1%	3.4%	-0.0012	0.0011	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
F	41.9%	-0.4%	0.0000	0.0001	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	1.4%	-1.3%	0.0001	0.0002	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
G	47.2%	4.9%	0.0024	-0.0018	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	-0.9%	-3.7%	-0.0018	0.0013	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
H	34.9%	-7.4%	0.0055	0.0002	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	2.4%	-0.3%	0.0002	0.0000	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005
I	35.7%	-6.6%	0.0043	-0.0010	0.0033	-0.0005	0.0011	-0.0003	0.0022	-0.0003
	4.2%	1.4%	-0.0010	0.0002	-0.0005	0.0006	-0.0003	0.0001	-0.0003	0.0005

	$\hat{\beta}_0$
Mean	42.3%
	2.7%

Var[v+e]

0.0033	-0.0005
-0.0005	0.0006

V = Var[v]

0.0022	-0.0003
-0.0003	0.0005

$\left(\Phi_i = X_i V X_i' + \hat{\sigma}^2 I_6\right)$

Φ for each group

0.0034	0.0024	0.0026	0.0029	0.0031	0.0033
0.0024	0.0044	0.0039	0.0046	0.0054	0.0062
0.0026	0.0039	0.0064	0.0064	0.0077	0.0090
0.0029	0.0046	0.0064	0.0095	0.0100	0.0118
0.0031	0.0054	0.0077	0.0100	0.0136	0.0147
0.0033	0.0062	0.0090	0.0118	0.0147	0.0188

Φ^{-1} for each group

509.28	-204.33	-134.11	-63.89	6.34	76.56
-204.33	618.15	-127.04	-88.39	-49.75	-11.10
-134.11	-127.04	663.87	-112.90	-105.83	-98.76
-63.89	-88.39	-112.90	646.43	-161.91	-186.42
6.34	-49.75	-105.83	-161.91	565.84	-274.08
76.56	-11.10	-98.76	-186.42	-274.08	422.10

Exhibit 18

Example 7: Random-Effects Trend Model (Cont'd)

Estimation of the General Parameter β_0

State	Year	y_1	X_1		$y_1 - X_1\hat{\beta}$	Φ_{11}^{-1}						
A	1	54.3%	1	1	9.2%	509.28	-204.33	-134.11	-63.89	6.34	76.56	0
A	2	57.2%	1	2	9.4%	-204.33	618.15	-127.04	-88.39	-49.75	-11.10	0
A	3	64.6%	1	3	14.1%	-134.11	-127.04	663.87	-112.90	-105.83	-98.76	0
A	4	67.6%	1	4	14.3%	-63.89	-88.39	-112.90	646.43	-161.91	-186.42	0
A	5	73.5%	1	5	17.5%	6.34	-49.75	-105.83	-161.91	565.84	-274.08	0
A	6	84.1%	1	6	25.4%	76.56	-11.10	-98.76	-186.42	-274.08	422.10	0
B	1	44.2%	1	1	-0.8%	0	0	0	0	0	0	509.28
B	2	48.6%	1	2	0.8%	0	0	0	0	0	0	-204.33
B	3	54.8%	1	3	4.3%	0	0	0	0	0	0	-134.11
B	4	48.2%	1	4	-5.0%	0	0	0	0	0	0	-63.89
B	5	57.7%	1	5	1.8%	0	0	0	0	0	0	6.34
B	6	68.2%	1	6	9.5%	0	0	0	0	0	0	76.56
C	1	53.9%	1	1	8.9%	0	0	0	0	0	0	0
C	2	57.0%	1	2	9.2%	0	0	0	0	0	0	0
C	3	54.8%	1	3	4.2%	0	0	0	0	0	0	0
C	4	59.9%	1	4	6.7%	0	0	0	0	0	0	0
C	5	52.7%	1	5	-3.3%	0	0	0	0	0	0	0
C	6	65.2%	1	6	6.5%	0	0	0	0	0	0	0
D	1	41.8%	1	1	-3.2%	0	0	0	0	0	0	0
D	2	45.2%	1	2	-2.6%	0	0	0	0	0	0	0
D	3	45.1%	1	3	-5.4%	0	0	0	0	0	0	0
D	4	46.4%	1	4	-6.8%	0	0	0	0	0	0	0
D	5	43.9%	1	5	-12.1%	0	0	0	0	0	0	0
D	6	44.0%	1	6	-14.7%	0	0	0	0	0	0	0
E	1	46.3%	1	1	1.3%	0	0	0	0	0	0	0
E	2	48.6%	1	2	0.8%	0	0	0	0	0	0	0
E	3	57.6%	1	3	7.1%	0	0	0	0	0	0	0
E	4	63.3%	1	4	10.1%	0	0	0	0	0	0	0
E	5	69.6%	1	5	13.7%	0	0	0	0	0	0	0
E	6	75.4%	1	6	16.7%	0	0	0	0	0	0	0
F	1	46.9%	1	1	1.8%	0	0	0	0	0	0	0
F	2	38.4%	1	2	-9.3%	0	0	0	0	0	0	0
F	3	48.1%	1	3	-2.5%	0	0	0	0	0	0	0
F	4	46.0%	1	4	-7.3%	0	0	0	0	0	0	0
F	5	53.4%	1	5	-2.6%	0	0	0	0	0	0	0
F	6	48.2%	1	6	-10.5%	0	0	0	0	0	0	0
G	1	45.7%	1	1	0.6%	0	0	0	0	0	0	0
G	2	44.5%	1	2	-3.3%	0	0	0	0	0	0	0
G	3	44.2%	1	3	-6.3%	0	0	0	0	0	0	0
G	4	46.7%	1	4	-6.5%	0	0	0	0	0	0	0
G	5	43.2%	1	5	-12.8%	0	0	0	0	0	0	0
G	6	39.5%	1	6	-19.2%	0	0	0	0	0	0	0
H	1	38.2%	1	1	-6.9%	0	0	0	0	0	0	0
H	2	42.0%	1	2	-5.7%	0	0	0	0	0	0	0
H	3	36.8%	1	3	-13.7%	0	0	0	0	0	0	0
H	4	46.1%	1	4	-7.2%	0	0	0	0	0	0	0
H	5	47.3%	1	5	-8.7%	0	0	0	0	0	0	0
H	6	50.2%	1	6	-8.5%	0	0	0	0	0	0	0
I	1	43.1%	1	1	-2.0%	0	0	0	0	0	0	0
I	2	44.8%	1	2	-2.9%	0	0	0	0	0	0	0
I	3	47.3%	1	3	-3.2%	0	0	0	0	0	0	0
I	4	46.4%	1	4	-6.8%	0	0	0	0	0	0	0
I	5	52.5%	1	5	-3.5%	0	0	0	0	0	0	0
I	6	67.9%	1	6	9.2%	0	0	0	0	0	0	0

$X_1'\Phi_{11}^{-1}y_1$	$X_1'\Phi_{11}^{-1}X_1$		df
1429.4	3190	2925	52
1724.4	2925	17823	

$X_1'\Phi_{11}^{-1}$						
189.853	137.541	85.228	32.915	-19.398	-71.711	189.853
-66.221	-18.064	30.092	78.248	126.404	174.560	-66.221

$\hat{\beta}_0$	$(X_1'\Phi_{11}^{-1}X_1)^{-1}$		$\hat{\sigma}^2$
42.3%	0.0004	-6E-05	1.000
2.7%	-6E-05	7E-05	

Exhibit 19

Example 7: Random-Effects Trend Model (Cont'd)

Estimation of the Group Parameters

Estimations	
State	y_2
A	-
	-
B	-
	-
C	-
	-
D	-
	-
E	-
	-
F	-
	-
G	-
	-
H	-
	-
I	-
	-

X_2

1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1
1	0
0	1

Φ_{21}

0.0019	0.0016	0.0013	0.0011	0.0008	0.0005	0
0.0002	0.0008	0.0013	0.0018	0.0023	0.0028	0
0	0	0	0	0	0	0.0019
0	0	0	0	0	0	0.0002
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$$\hat{y}_2 = X_2\hat{\beta}_0 + \Phi_{21}\Phi_{11}^{-1}\left(y_1 - X_1\hat{\beta}_0\right)$$

A	45.8%
	5.9%
B	40.6%
	3.8%
C	49.1%
	2.1%
D	42.8%
	0.5%
E	40.3%
	5.7%
F	41.8%
	1.5%
G	45.2%
	-0.4%
H	37.1%
	2.0%
I	38.0%
	3.6%

$\Phi_{21}\Phi_{11}^{-1}$

0.4399	0.31	0.1801	0.0502	-0.0796	-0.2095	0
-0.0896	-0.0493	-0.009	0.0313	0.0715	0.1118	0
0	0	0	0	0	0	0.4399
0	0	0	0	0	0	-0.0896
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Exhibit 20

Example 7: Random-Effects Trend Model (Cont'd)

Credibility-Weighted Estimates

X_i

1	1
1	2
1	3
1	4
1	5
1	6

$T_i^{-1} = (X_i V X_i' + \Sigma_i)^{-1}$

509.28	-204.33	-134.11	-63.89	6.34	76.56
-204.33	618.15	-127.04	-88.39	-49.75	-11.10
-134.11	-127.04	663.87	-112.90	-105.83	-98.76
-63.89	-88.39	-112.90	646.43	-161.91	-186.42
6.34	-49.75	-105.83	-161.91	565.84	-274.08
76.56	-11.10	-98.76	-186.42	-274.08	422.10

$X_i' T_i^{-1} X_i$

354.43	325.02
325.02	1980.29

$Z_i = V X_i' T_i^{-1} X_i$

0.691	0.146
0.067	0.939

State (*i*)	$I_2 - Z_i$		Prior	Z_i		Empirical	Posterior
A	0.309	-0.146	42.3%	0.691	0.146	46.8%	45.8%
	-0.067	0.061	2.7%	0.067	0.939	5.7%	5.9%
B			42.3%			39.6%	40.6%
			2.7%			4.0%	3.8%
C			42.3%			52.4%	49.1%
			2.7%			1.4%	2.1%
D			42.3%			43.6%	42.8%
			2.7%			0.2%	0.5%
E			42.3%			38.7%	40.3%
			2.7%			6.1%	5.7%
F			42.3%			41.9%	41.8%
			2.7%			1.4%	1.5%
G			42.3%			47.2%	45.2%
			2.7%			-0.9%	-0.4%
H			42.3%			34.9%	37.1%
			2.7%			2.4%	2.0%
I			42.3%			35.7%	38.0%
			2.7%			4.2%	3.6%
Unweighted Mean							42.3%
							2.7%

Appendix A

Groups of Statistical Models

The basis of credibility is a grand statistical model which is a group of statistical submodels. Suppose that we have n linear models of the form $\mathbf{y}_i = X_i\beta_i + \mathbf{e}_i$, where $\mathrm{Var}[\mathbf{e}_i] = \Sigma_i$, for $i = 1, \ldots, n$. As to the dimensions of the matrices, $\mathbf{y}_i$ and $\mathbf{e}_i$ are $(t_i\times 1)$, X_i is $(t_i\times k_i)$, β_i is $(k_i\times 1)$, and Σ_i is $(t_i\times t_i)$. We assume that each Σ_i is non-singular and that each X_i is of full column rank, i.e., $\mathrm{rank}(X_i) = k_i$. These assumptions ensure that each $X_i'\Sigma_i^{-1}X_i$ is non-singular. The best linear unbiased estimator [7:Appendix C] of each β_i is $\hat{\beta}_i = \left(X_i'\Sigma_i^{-1}X_i\right)^{-1} X_i'\Sigma_i^{-1}\mathbf{y}_i$, and $\mathrm{Var}[\hat{\beta}_i] = \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}$.

Let $t = t_1 + \ldots + t_n$, and $k = k_1 + \ldots + k_n$. The first model of models is as follows:

$$\mathbf{y}_{(t\times 1)} = X_{(t\times k)}\beta_{(k\times 1)} + \mathbf{e}_{(t\times 1)}$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix},$$

$$\text{where } \mathrm{Var}[\mathbf{e}] = \mathrm{Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix} = \Sigma_{(t\times t)}.$$

The best linear unbiased estimator of β is:

$$
\begin{aligned}
\hat{\beta} &= \left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathbf{y} \\
&= \left(\begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix}' \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}^{-1} \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix}' \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} \\
&= \left(\begin{bmatrix} X_1' & & \\ & \ddots & \\ & & X_n' \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & & \\ & \ddots & \\ & & \Sigma_n^{-1} \end{bmatrix} \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1' & & \\ & \ddots & \\ & & X_n' \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & & \\ & \ddots & \\ & & \Sigma_n^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} \\
&= \left(\begin{bmatrix} X_1'\Sigma_1^{-1}X_1 & & \\ & \ddots & \\ & & X_n'\Sigma_n^{-1}X_n \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1'\Sigma_1^{-1}\mathbf{y}_1 \\ \vdots \\ X_n'\Sigma_n^{-1}\mathbf{y}_n \end{bmatrix} \\
&= \begin{bmatrix} \left(X_1'\Sigma_1^{-1}X_1\right)^{-1} & & \\ & \ddots & \\ & & \left(X_n'\Sigma_n^{-1}X_n\right)^{-1} \end{bmatrix} \begin{bmatrix} X_1'\Sigma_1^{-1}\mathbf{y}_1 \\ \vdots \\ X_n'\Sigma_n^{-1}\mathbf{y}_n \end{bmatrix} \\
&= \begin{bmatrix} \left(X_1'\Sigma_1^{-1}X_1\right)^{-1}X_1'\Sigma_1^{-1}\mathbf{y}_1 \\ \vdots \\ \left(X_n'\Sigma_n^{-1}X_n\right)^{-1}X_n'\Sigma_n^{-1}\mathbf{y}_n \end{bmatrix} \\
&= \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_n \end{bmatrix}
\end{aligned}
$$

Also, $\operatorname{Var}\left[\hat{\beta}\right] = \left(X'\Sigma^{-1}X\right)^{-1} = \begin{bmatrix} \left(X_1'\Sigma_1^{-1}X_1\right)^{-1} & & \\ & \ddots & \\ & & \left(X_n'\Sigma_n^{-1}X_n\right)^{-1} \end{bmatrix} = \begin{bmatrix} \operatorname{Var}\left[\hat{\beta}_1\right] & & \\ & \ddots & \\ & & \operatorname{Var}\left[\hat{\beta}_n\right] \end{bmatrix}$. In this grand model the submodels appear together, but they are unrelated.

But this leads us to a second model of models, the one that forms the basis of credibility. Instead of n models and n betas, let there be n models and one beta. In this model $k = k_1 = \ldots = k_n$:

$$\mathbf{y}_{(t\times 1)} = X_{(t\times k)}\beta_{(k\times 1)} + \mathbf{e}_{(t\times 1)}$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\beta + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix},$$

$$\text{where } \mathrm{Var}[\mathbf{e}] = \mathrm{Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix} = \Sigma_{(t\times t)}.$$

The estimator of β is:

$$\hat{\beta} = \left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathbf{y}$$

$$= \left(\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}' \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right)^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}' \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} X_1' & \cdots & X_n' \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & & \\ & \ddots & \\ & & \Sigma_n^{-1} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right)^{-1} \begin{bmatrix} X_1' & \cdots & X_n' \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & & \\ & \ddots & \\ & & \Sigma_n^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

$$= \left(X_1'\Sigma_1^{-1}X_1 + \ldots + X_n'\Sigma_n^{-1}X_n\right)^{-1}\left(X_1'\Sigma_1^{-1}\mathbf{y}_1 + \ldots + X_n'\Sigma_n^{-1}\mathbf{y}_n\right)$$

$$= \left(X_1'\Sigma_1^{-1}X_1 + \ldots + X_n'\Sigma_n^{-1}X_n\right)^{-1}\left(X_1'\Sigma_1^{-1}X_1\left(X_1'\Sigma_1^{-1}X_1\right)^{-1}X_1'\Sigma_1^{-1}\mathbf{y}_1 + \ldots + X_n'\Sigma_n^{-1}X_n\left(X_n'\Sigma_n^{-1}X_n\right)^{-1}X_n'\Sigma_n^{-1}\mathbf{y}_n\right)$$

$$= \left(\mathrm{Var}^{-1}\left[\hat{\beta}_1\right] + \ldots + \mathrm{Var}^{-1}\left[\hat{\beta}_n\right]\right)^{-1}\left(\mathrm{Var}^{-1}\left[\hat{\beta}_1\right]\hat{\beta}_1 + \ldots + \mathrm{Var}^{-1}\left[\hat{\beta}_n\right]\hat{\beta}_n\right)$$

$$= \left(\mathrm{Var}\left[\hat{\beta}\right]\right)\left(\mathrm{Var}^{-1}\left[\hat{\beta}_1\right]\hat{\beta}_1 + \ldots + \mathrm{Var}^{-1}\left[\hat{\beta}_n\right]\hat{\beta}_n\right)$$

The estimator of this grand model is a matrix-weighted average of the estimators of the submodels. The weights themselves, which are $(k\times k)$ matrices, are the inverses of the variances of the estimators. This is a k-dimensional form of the well-known rule that non-covarying estimates of the same parameter are best averaged according to weights inversely proportional to their variances. Judge [8:287] notes that a matrix-weighted average of two vectors need not be collinear with the two vectors, unlike a scalar-weighted average, which must be collinear.

A third model of models looks like the first, but has a general variance matrix:

$$\mathbf{y}_{(t\times 1)} = \mathrm{X}_{(t\times k)}\beta_{(k\times 1)} + \mathbf{e}_{(t\times 1)}$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} \mathrm{X}_1 & & \\ & \ddots & \\ & & \mathrm{X}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix},$$

$$\text{where } \mathrm{Var}[\mathbf{e}] = \mathrm{Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{bmatrix} = \Sigma_{(t\times t)}$$

Because we are accustomed to regarding the variance matrix as block diagonal, as in the first grand model, the submodels are *seemingly* unrelated. Models of seemingly unrelated models are discussed in [7:Appendix H] and [8:444-466].

The remainder of this appendix will be devoted to proving that the estimator of the second grand model is better than the estimators of its submodels. Both the proof itself and the precise meaning of 'better' require a discussion of non-negative definite and positive definite matrices. In an earlier paper [7:Appendix A] the author discussed such matrices, and developed many basic theorems concerning them (cf. also [1:459-461] and [8:960ff]). This discussion dovetails with that of the earlier paper, and anything simply asserted here will be found proven there.

Let A and B be square matrices of the same dimension, say $(n\times n)$, and let x be an $(n\times 1)$ vector. The (1×1) matrices x'Ax and x'Bx are called quadratic forms in x (Judge [8:959]). Let '~' stand for one of the five following comparison relations among the real numbers:

'$<$', '$\le$', '$=$', '$\ge$', and '$>$'. What might '$A \sim B$' mean? In the case of equality, we know that '$A = B$' means that corresponding elements are A and B are equal (elementwise equality). So it would be natural to define '$A \sim B$' as elementwise '$\sim$', as is already the case with '$=$'.

But there is another very useful definition: $A \sim B$ if and only if for every non-zero x, $\{x'Ax\}_{11} \sim \{x'Bx\}_{11}$. (Of course, a zero x will result in equality.) The operator $\{\bullet\}_{ij}$ yields the ij^{th} element of the matrix inside the brackets, which is a scalar result. Being (1×1) matrices, $x'Ax$ and $x'Bx$ have only one element; thus, $\{\bullet\}_{11}$ makes quadratic forms comparable on a scalar basis. According to this definition, $A \sim B$ depends on the *matrices* A and B, rather than on the *elements* of A and B. But the matrices must be reduced to the *definite* level of (1×1) quadratic forms in order to invite comparison. If '$\sim$' in the first sense is elementwise comparison, we might say that '$\sim$' in the second sense is definite comparison, perhaps distinguishing it with dots '$.\sim.$'. Therefore, $A \;.\!\sim\!.\; B$ if and only if for every non-zero x, $\{x'Ax\}_{11} \sim \{x'Bx\}_{11}$.

Let C be an $(n \times n)$ matrix. Obviously, if for all non-zero x, $\{x'Ax\}_{11} \sim \{x'Bx\}_{11}$, and for all non-zero x, $\{x'Bx\}_{11} \sim \{x'Cx\}_{11}$, then for all non-zero x, $\{x'Ax\}_{11} \sim \{x'Cx\}_{11}$. So the five definite comparisons are transitive. Also, adding or subtracting the same amount from both sides of a scalar comparison does not affect the comparison. Hence,

$$
\begin{aligned}
\mathrm{A}\ .\!\sim\!.\ \mathrm{B} &\Leftrightarrow \forall \mathrm{x} \neq 0,\ \mathrm{x'Ax}\ .\!\sim\!.\ \mathrm{x'Bx} \\
&\Leftrightarrow \forall \mathrm{x} \neq 0,\ (\mathrm{x'Ax} - \mathrm{x'Bx})\ .\!\sim\!.\ (\mathrm{x'Bx} - \mathrm{x'Bx}) \\
&\Leftrightarrow \forall \mathrm{x} \neq 0,\ \mathrm{x'(A - B)x}\ .\!\sim\!.\ \mathrm{x'(B - B)x} \\
&\Leftrightarrow \forall \mathrm{x} \neq 0,\ \mathrm{x'(A - B)x}\ .\!\sim\!.\ \mathrm{x'}0_{(n\times n)}\mathrm{x} \\
&\Leftrightarrow (\mathrm{A} - \mathrm{B})\ .\!\sim\!.\ 0
\end{aligned}
$$

So A compares definitely with B as (A – B) compares definitely with the zero matrix. Similarly, multiplying or dividing both sides of a scalar comparison by a positive scalar does not affect the comparison; so if $k > 0$, then kA .~. kB.

As for inequalities, if A .≤. [.≥.] B, then B .≥. [.≤.] A. And if A .≤. B and B .≤. A, then A .=. B. So far, definite comparisons behave like scalar comparisons. But the scalar comparison '$a \leq b$' is equivalent to '($a < b$) or ($a = b$)'. It is different with the definite comparison: 'A .≤. B' means 'for all x, $\{\mathrm{x'Ax}\}_{11} \leq \{\mathrm{x'Bx}\}_{11}$'. It is possible that for some values of x the relation is '<' and for other values it is '='. Thus 'A .≤. B' is not equivalent to '(A .<. B) or (A .=. B)'. One must be cautious in handling the compound comparisons '.≤.' and '.≥.'; for instance, it is tempting but fallacious to argue that if A .≤. B and not (A .=. B), then A .<. B. In a similar vein, according to the law of trichotomy, for any two scalars a and b, ($a < b$) or ($a = b$) or ($a > b$). But it is *not* true that for any two ($n\times n$) matrices A and B, (A .<. B) or (A .=. B) or (A .>. B).

As for equalities, since every (1×1) matrix is symmetric, x'Ax = (x'Ax)' = x'A'x. So, for all non-zero x, $\{\mathrm{x'Ax}\}_{11} = \{\mathrm{x'A'x}\}_{11}$, implying that A .=. A' and that (A – A') .=. 0. Moreover, if A .=. 0, then A' = –A (skew symmetry). For if A .=. 0, then for all non-zero x, $\{\mathrm{x'Ax}\}_{11} =$

0. But $\{x'Ax\}_{11} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$. If one of the xs (say x_k) equals one and the rest are zero, then $\{x'Ax\}_{11} = a_{kk}x_k^2 = a_{kk} = 0$. And if two of the xs (say x_k and x_l) equal one and the rest are zero, then $\{x'Ax\}_{11} = a_{kk}x_k^2 + a_{kl}x_kx_l + a_{lk}x_lx_k + a_{ll}x_l^2 = a_{kk} + a_{kl} + a_{lk} + a_{ll} = a_{kl} + a_{lk} = 0$. For all k and l, $a_{lk} = - a_{kl}$, which makes A skew symmetric. Conversely, if A is skew symmetric then A .=. (A + A)/2 .=. (A – (–A))/2 .=. (A – A')/2 .=. 0/2 .=. 0. Therefore, A .=. 0 if and only if A is skew symmetric. Moreover, if A is symmetric and A .=. 0, then A is both symmetric (A' = A) and skew symmetric (A' = –A), which implies that A = –A = 0. Finally, if A and B are symmetric and A .=. B, then A – B is both symmetric and .=. 0. Hence, A – B = 0; so A = B.

A matrix A is *non-negative definite* [*positive definite*] if and only if A is symmetric and A .≥. [.>.] 0. Obviously, if A is positive definite then it is non-negative definite, but not necessarily *vice versa*. It is a theorem that if A is a non-negative definite matrix, then A is positive definite if and only if A^{-1} exists (or A is non-singular). Another theorem is that A is non-negative definite if and only if there exists a square matrix W, such that A = WW'. Such a W is sometimes called a square root matrix of A. If A is positive definite, then it is non-singular and every square root matrix of it must be non-singular. In such circumstances, $A^{-1} = (WW')^{-1} = (W')^{-1}(W)^{-1} = (W^{-1})'(W^{-1})$, which is non-negative definite. But since A^{-1} is non-singular, it must also be positive definite. Therefore, if A is positive definite, then so too is A^{-1}.

If $\mathbf{x}$ is an $(n\times 1)$ random vector with $\text{Var}[\mathbf{x}] = \Sigma$, and A is an $(m\times n)$ non-stochastic matrix, then $A\mathbf{x}$ is an $(m\times 1)$ random vector with $\text{Var}[A\mathbf{x}] = A\Sigma A'$. If A is $(1\times n)$, then $A\mathbf{x}$ is a (1×1) random vector, whose element must be non-negative. Hence, a variance matrix, which must be symmetric, must also be $.\geq.\ 0$; otherwise some non-zero linear combination of the elements of the random vector would imply a scalar random variable with a negative variance. In other words, every variance matrix is non-negative definite.

We would like to compare two $(n\times n)$ variance matrices $\text{Var}[\mathbf{x}_1] = \Sigma_1$ and $\text{Var}[\mathbf{x}_2] = \Sigma_2$. If $\Sigma_1 .<. [.\leq.]\ \Sigma_2$, then the variance of every non-zero linear combination of $\mathbf{x}_1$ is less than [less than or equal to] the variance of the same linear combination of $\mathbf{x}_2$. If Σ_1 and Σ_2 are the variance matrices of two estimates of an unknown parameter and $\Sigma_1 .<. \Sigma_2$, then Σ_1 is the better estimate. If $\Sigma_1 .\leq. \Sigma_2$, then Σ_1 may not be better; however, it is at least as good. But if, in addition, $\Sigma_1 \neq \Sigma_2$, then not $(\Sigma_1 .=. \Sigma_2)$ and Σ_1 is again better.

So, turning back to the second grand model, we will prove that $\forall i, \text{Var}[\hat{\beta}] .<. \text{Var}[\hat{\beta}_i]$, which is equivalent to $\text{Var}[\hat{\beta}_i] - \text{Var}[\hat{\beta}] .>. 0$. This too is equivalent to the statement that $\text{Var}[\hat{\beta}_i] - \text{Var}[\hat{\beta}]$ is positive definite.

As above, $\text{Var}\left[\hat{\beta}\right] = \left(\text{Var}^{-1}\left[\hat{\beta}_1\right] + \ldots + \text{Var}^{-1}\left[\hat{\beta}_n\right]\right)^{-1}$, or $\text{Var}^{-1}\left[\hat{\beta}\right] = \text{Var}^{-1}\left[\hat{\beta}_1\right] + \ldots + \text{Var}^{-1}\left[\hat{\beta}_n\right]$. Being a variance matrix, each $\text{Var}^{-1}\left[\hat{\beta}_i\right]$ is non-negative definite. And being non-singular, each is positive definite. Therefore, $\text{Var}^{-1}\left[\hat{\beta}\right] - \text{Var}^{-1}\left[\hat{\beta}_i\right]$ is positive definite. So there exists a non-singular ($k \times k$) matrix W such that $\text{Var}^{-1}[\hat{\beta}] = \text{Var}^{-1}[\hat{\beta}_t] + WW' = \text{Var}^{-1}[\hat{\beta}_t] + WI_kW'$, or

$\text{Var}[\hat{\beta}] = \left(\text{Var}^{-1}[\hat{\beta}_t] + WI_kW'\right)^{-1}$.

Now it is a theorem that if $D^{-1} + CA^{-1}B$ exists and is non-singular, then $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$. As the first part of the proof:

$$
\begin{aligned}
(A + BDC)\left(A^{-1} - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}\right) &= AA^{-1} - AA^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&\quad + BDCA^{-1} - BDCA^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&= I - B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&\quad + BDCA^{-1} - BDCA^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&= I - BDD^{-1}\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&\quad + BDCA^{-1} - BDCA^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&= I + BDCA^{-1} \\
&\quad - BD\left(D^{-1} + CA^{-1}B\right)\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1} \\
&= I + BDCA^{-1} - BDCA^{-1} \\
&= I
\end{aligned}
$$

Reversing the order of the multiplication is the second and final part of the proof:

$$
\begin{aligned}
\left(A^{-1} - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}\right)(A + BDC) &= A^{-1}A + A^{-1}BDC \\
&\quad - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}A - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}BDC \\
&= I + A^{-1}BDC \\
&\quad - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}C - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}BDC \\
&= I + A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}\left(D^{-1} + CA^{-1}B\right)DC \\
&\quad - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}C - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}BDC \\
&= I + A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}\left(D^{-1}\right)DC - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}C \\
&= I + A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}C - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}C \\
&= I
\end{aligned}
$$

Therefore, using this theorem:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}] &= \left(\mathrm{Var}^{-1}[\hat{\beta}_i] + \mathrm{W I}_k \mathrm{W}'\right)^{-1} \\
&= \left(\mathrm{Var}^{-1}[\hat{\beta}_i]\right)^{-1} - \left(\mathrm{Var}^{-1}[\hat{\beta}_i]\right)^{-1} \mathrm{W}\left(\mathrm{I}_k^{-1} + \mathrm{W}'\left(\mathrm{Var}^{-1}[\hat{\beta}_i]\right)^{-1}\mathrm{W}\right)^{-1} \mathrm{W}'\left(\mathrm{Var}^{-1}[\hat{\beta}_i]\right)^{-1} \\
&= \mathrm{Var}[\hat{\beta}_i] - \mathrm{Var}[\hat{\beta}_i]\mathrm{W}\left(\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)^{-1}\mathrm{W}'\mathrm{Var}[\hat{\beta}_i] \\
&= \mathrm{Var}[\hat{\beta}_i] - \left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)\left(\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)^{-1}\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)' \\
&.<.\ \mathrm{Var}[\hat{\beta}_i]
\end{aligned}$$

Some explanation is in order: $\mathrm{Var}[\hat{\beta}_i]$ is positive definite and thus factorable as, say, UU'. So $\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W} = \mathrm{I}_k + \mathrm{W}'\mathrm{UU}'\mathrm{W} = \mathrm{I}_k + \mathrm{W}'\mathrm{U}(\mathrm{W}'\mathrm{U})'$. The identity matrix is positive definite, and to it is added a non-negative definite matrix. Hence, $\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W}$ is positive definite. It follows that $\left(\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)^{-1}$ is positive definite and factorable as, say, VV', where V is non-singular. Thus, $\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)\left(\mathrm{I}_k + \mathrm{W}'\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)^{-1}\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)'$ is factorable as $\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)\mathrm{VV}'\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)' = \left(\mathrm{Var}[\hat{\beta}_i]\mathrm{WV}\right)\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{WV}\right)'$. This is a non-negative definite matrix. But inasmuch as the square root matrix consists of the product of three non-singular matrices, the root itself is non-singular, and so too is the root times its transpose. Therefore, $\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)\mathrm{VV}'\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{W}\right)' = \left(\mathrm{Var}[\hat{\beta}_i]\mathrm{WV}\right)\left(\mathrm{Var}[\hat{\beta}_i]\mathrm{WV}\right)'$ is positive definite, and so $\mathrm{Var}[\hat{\beta}] .<. \mathrm{Var}[\hat{\beta}_i]$.

So the grand model is better than every submodel. But an even more powerful statement can be made. Consider a partial grand model, consisting of some, but not all, of the submodels. Let $\tilde{\beta}$ be the estimator of the partial model. Then

$\mathrm{Var}^{-1}\left[\hat{\beta}\right] = \mathrm{Var}^{-1}\left[\tilde{\beta}\right] + \ldots + \sum_i \mathrm{Var}^{-1}\left[\hat{\beta}_i\right]$, where the subscript i is ranges over the submodels left out of the partial grand model. Then, by similar reasoning, $\mathrm{Var}^{-1}\left[\hat{\beta}\right] . < . \mathrm{Var}^{-1}\left[\tilde{\beta}\right]$. This goes to show that the more submodels, the better the estimate.

$\mathrm{Var}^{-1}\left[\hat{\beta}\right] = \mathrm{Var}^{-1}\left[\hat{\beta}_1\right] + \ldots + \mathrm{Var}^{-1}\left[\hat{\beta}_n\right]$ is called a harmonic sum. It is a k-dimensional equation. But there is an interesting 1-dimensional analogue in electricity, which may help the reader to understand the meaning of the statement 'the more submodels, the better the estimate'. A group of n resistors in parallel, whose resistances are $r_1, \ldots, r_n$, has an overall resistance R such that $\frac{1}{R} = \frac{1}{r_1} + \cdots + \frac{1}{r_n}$ (a harmonic sum). Every extra resistor added in parallel allows a little more current to flow through group, which in effect reduces the overall resistance. If the extra resistor is of high resistance (almost an insulator), then the reduction is small; if it is of low resistance (almost a short), then the reduction is great. The variance of an additional submodel is like the resistance of an additional resistor: when the variance is high, the extra group provides little additional information, so the reduction of variance of the estimate of the grand model is small (but a reduction nonetheless). When the variance is low, the extra group provides much additional information, with a great reduction of overall variance. Of course, the assumption implicit throughout is that each submodel is an appropriate model; otherwise, information could be created *ex nihilo*.

The case of a grand model in which some $X_i'\Sigma_i^{-1}X_i$ may be singular deserves a discussion. We will consider a model with only two submodels, in which $\mathrm{Var}[\hat{\beta}_1] = \left(X_1'\Sigma_1^{-1}X_1\right)^{-1}$

exists, but $X_2'\Sigma_2^{-1}X_2$ may be singular. In this case, the second submodel, though informative, may not be sufficiently informative for a unique estimate of β. For example, if β were (2×1), the second submodel might be a non-sample judgment that the first element of β has a mean of 1 and a variance of 2:

$$\mathbf{y}_2 = X_2\beta + \mathbf{e}_2$$
$$[1] = [1 \quad 0]\beta + \mathbf{e}_2,$$

where $\text{Var}[\mathbf{e}_2] = \Sigma_2 = [2]$. In this example, β cannot be uniquely estimated because

$$X_2'\Sigma_2^{-1}X_2 = \begin{bmatrix}1\\0\end{bmatrix}[2]^{-1}[1 \quad 0] = \begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix},$$ which is singular.

$X_2'\Sigma_2^{-1}X_2$ is non-negative definite; therefore, for all non-zero u, $\{u'(X_2'\Sigma_2^{-1}X_2)u\}_{11} \geq 0$. We will define a set Z, possibly empty, of all non-zero u such that $\{u'(X_2'\Sigma_2^{-1}X_2)u\}_{11} = 0$. But $u'(X_2'\Sigma_2^{-1}X_2)u = (X_2u)'\Sigma_2^{-1}(X_2u)$. Since Σ_2^{-1} is positive definite, $\{(X_2u)'\Sigma_2^{-1}(X_2u)\}_{11} = 0$ if and only if $X_2u = 0$. Therefore, $u \in Z$ if and only if u is non-zero and $X_2u = 0$. Recall that X_2 is $(t_2 \times k)$. At the beginning of the appendix it was assumed that $\text{rank}(X_2) = k$, but now we will relax this assumption. Let $\text{rank}(X_2) = j \leq k$. Then the set of all u such that $X_2u = 0$ is a $(k - j)$-dimensional linear subspace of k-space. Z is this subspace less the zero vector (so if $\text{rank}(X_2) = j = k$, then Z is empty and $X_2'\Sigma_2^{-1}X_2$ is positive definite).

Therefore, in the grand model:

$$
\begin{aligned}
\text{Var}[\hat{\beta}] &= \left(\text{Var}^{-1}[\hat{\beta}_1] + X_2'\Sigma_2^{-1}X_2\right)^{-1} \\
&= \left(\text{Var}^{-1}[\hat{\beta}_1]\right)^{-1} - \left(\text{Var}^{-1}[\hat{\beta}_1]\right)^{-1} X_2'\left((\Sigma_2^{-1})^{-1} + X_2\left(\text{Var}^{-1}\hat{\beta}_1\right)^{-1} X_2'\right)^{-1} X_2\left(\text{Var}^{-1}[\hat{\beta}_1]\right)^{-1} \\
&= \text{Var}[\hat{\beta}_1] - \text{Var}[\hat{\beta}_1]X_2'\left(\Sigma_2 + X_2\text{Var}[\hat{\beta}_1]X_2'\right)^{-1} X_2\text{Var}[\hat{\beta}_1] \\
&= \text{Var}[\hat{\beta}_1] - \left(X_2\text{Var}[\hat{\beta}_1]\right)'\left(\Sigma_2 + X_2\text{Var}[\hat{\beta}_1]X_2'\right)^{-1}\left(X_2\text{Var}[\hat{\beta}_1]\right) \\
&.\leq.\ \text{Var}[\hat{\beta}_1]
\end{aligned}
$$

The inequality follows from the fact that $\left(X_2\text{Var}[\hat{\beta}_1]\right)'\left(\Sigma_2 + X_2\text{Var}[\hat{\beta}_1]X_2'\right)^{-1}\left(X_2\text{Var}[\hat{\beta}_1]\right)$ is non-negative definite. But strict inequality, which represents an efficiency gain, depends on $\left\{u'\left(X_2\text{Var}[\hat{\beta}_1]\right)'\left(\Sigma_2 + X_2\text{Var}[\hat{\beta}_1]X_2'\right)^{-1}\left(X_2\text{Var}[\hat{\beta}_1]\right)u\right\}_{11} > 0$. Since $\left(\Sigma_2 + X_2\text{Var}[\hat{\beta}_1]X_2'\right)^{-1}$ is positive definite, strict inequality is thwarted only when $X_2\text{Var}[\hat{\beta}_1]u = 0$. And $X_2\text{Var}[\hat{\beta}_1]u = 0$ if and only if $\text{Var}[\hat{\beta}_1]u \in Z$. Since $\text{Var}[\hat{\beta}_1]$ is non-singular, there exists a $(k-j)$-dimensional subspace of k-space, Z^*, formed by premultiplying each member of Z by $\text{Var}^{-1}[\hat{\beta}_1]$. When rank($X_2$) = $j \leq k$, $\text{Var}[\hat{\beta}].<.\text{Var}[\hat{\beta}_1]$ except in the $(k-j)$-dimensional subspace Z^*, within which $\text{Var}[\hat{\beta}].=.\text{Var}[\hat{\beta}_1]$ (so $\text{Var}[\hat{\beta}] = \text{Var}[\hat{\beta}_1]$).

Appendix B

A Bayesian Interpretation of Prior Information

Consider the model $\mathbf{y}_{(t\times 1)} = X_{(t\times k)}\beta_{(k\times 1)} + \mathbf{e}_{(t\times 1)}$. What makes this model Bayesian is that β is stochastic. Let us assume that β is multivariate normal with mean β_0 and variance V, i.e., $\beta \sim N(\beta_0, V)$. We will assume also that $\mathbf{e} \sim N(0, \Sigma)$, and that $\mathbf{e}$ is independent of β. Being variance matrices, Σ and V must be non-negative definite. But we will further assume that both matrices are positive definite, which implies that their inverses exist and that their determinants are positive.

The probability density function of $\beta \sim N(\beta_0, V)$ is [8:49f.]:

$$f_\beta(\beta) = (2\pi)^{-\frac{k}{2}}|V|^{-\frac{1}{2}}e^{-\frac{1}{2}(\beta-\beta_0)'V^{-1}(\beta-\beta_0)} \propto e^{-\frac{1}{2}(\beta-\beta_0)'V^{-1}(\beta-\beta_0)}$$

As for the random vector $\mathbf{y}$ given that $\beta = \beta$, or $\mathbf{y}|\beta = \beta$:

$$\begin{aligned}\mathbf{y}|\beta = \beta &= (X\beta + \mathbf{e})|\beta = \beta \\ &= X\beta + (\mathbf{e}|\beta = \beta) \\ &= X\beta + \mathbf{e}\end{aligned}$$

The last equation follows from $\mathbf{e}$'s being independent of β. Hence, $\mathbf{y}|\beta = \beta \sim N(X\beta, \Sigma)$; so its probability density function is:

$$f_{\mathbf{y}|\beta=\beta}(y) = (2\pi)^{-\frac{t}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(y-X\beta)'\Sigma^{-1}(y-X\beta)} \propto e^{-\frac{1}{2}(y-X\beta)'\Sigma^{-1}(y-X\beta)}$$

Therefore, according to Bayes' theorem, the probability density function of $\beta|\mathbf{y} = y$ is:

$$f_{\beta|y=y}(\beta) = \frac{f_{y|\beta=\beta}(y) f_\beta(\beta)}{f_y(y)} \propto f_{y|\beta=\beta}(y) f_\beta(\beta)$$

$$\propto \left(e^{-\frac{1}{2}(y-X\beta)'\Sigma^{-1}(y-X\beta)} \right)\left(e^{-\frac{1}{2}(\beta-\beta_0)'V^{-1}(\beta-\beta_0)} \right)$$

$$\propto e^{-\frac{1}{2}\left((y-X\beta)'\Sigma^{-1}(y-X\beta)+(\beta-\beta_0)'V^{-1}(\beta-\beta_0)\right)}$$

We will now expand the exponent of this density function:

$$\begin{aligned}(y-X\beta)'\Sigma^{-1}(y-X\beta)+(\beta-\beta_0)'V^{-1}(\beta-\beta_0) &= y'\Sigma^{-1}y-(X\beta)'\Sigma^{-1}y-y'\Sigma^{-1}(X\beta)+(X\beta)'\Sigma^{-1}(X\beta)\\ &\quad +\beta'V^{-1}\beta-\beta_0'V^{-1}\beta-\beta'V^{-1}\beta_0+\beta_0'V^{-1}\beta_0\\ &= y'\Sigma^{-1}y-\beta'X'\Sigma^{-1}y-y'\Sigma^{-1}X\beta+\beta'X'\Sigma^{-1}X\beta\\ &\quad +\beta'V^{-1}\beta-\beta_0'V^{-1}\beta-\beta'V^{-1}\beta_0+\beta_0'V^{-1}\beta_0\\ &= -\beta'X'\Sigma^{-1}y-y'\Sigma^{-1}X\beta+\beta'X'\Sigma^{-1}X\beta\\ &\quad +\beta'V^{-1}\beta-\beta_0'V^{-1}\beta-\beta'V^{-1}\beta_0+c\end{aligned}$$

The two terms of the expansion which did not involve β were absorbed into the term c, which will be a catch-all for all terms not involving β.

Next we will perform a multivariate "completion of the square" with respect to β. To do this we must recognize that since V is positive definite, V^{-1} exists and is positive definite. It is similar with Σ, so $X'\Sigma^{-1}X$ exists and is non-negative definite. This implies that $X'\Sigma^{-1}X + V^{-1}$ is positive definite. Therefore, there exists a nonsingular $(k \times k)$ matrix W such that $W'W = X'\Sigma^{-1}X + V^{-1}$. So we continue:

$$
\begin{aligned}
(y - X\beta)'\Sigma^{-1}(y - X\beta) + (\beta - \beta_0)'V^{-1}(\beta - \beta_0) &= -\beta'X'\Sigma^{-1}y - y'\Sigma^{-1}X\beta + \beta'X'\Sigma^{-1}X\beta \\
&\quad + \beta'V^{-1}\beta - \beta_0'V^{-1}\beta - \beta'V^{-1}\beta_0 + c \\
&= \beta'(X'\Sigma^{-1}X + V^{-1})\beta - \beta'(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - (y'\Sigma^{-1}X + \beta_0'V^{-1})\beta + c \\
&= \beta'(X'\Sigma^{-1}X + V^{-1})\beta - \beta'(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - (X'\Sigma^{-1}y + V^{-1}\beta_0)'\beta + c \\
&= \beta'W'W\beta - \beta'W'(W')^{-1}(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - (X'\Sigma^{-1}y + V^{-1}\beta_0)'W^{-1}W\beta + c \\
&= (W\beta)'W\beta - (W\beta)'(W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - (X'\Sigma^{-1}y + V^{-1}\beta_0)'W^{-1}(W\beta) + c \\
&= (W\beta)'W\beta - (W\beta)'(W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - \left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'(W\beta) + c \\
&= (W\beta)'W\beta - (W\beta)'(W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0) \\
&\quad - \left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'(W\beta) \\
&\quad + \left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) \\
&\quad - \left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c \\
&= \left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) \\
&\quad - \left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left((W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c \\
&= \left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c
\end{aligned}
$$

In the last equation a term not involving β has been absorbed into the catch-all term c. Now we can simplify:

$$
\begin{aligned}
(y - X\beta)'\Sigma^{-1}(y - X\beta) + (\beta - \beta_0)'V^{-1}(\beta - \beta_0) &= \left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left(W\beta - (W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c \\
&= \left(W\beta - WW^{-1}(W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left(W\beta - WW^{-1}(W^{-1})'(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c \\
&= \left(W\beta - W(W'W)^{-1}(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'\left(W\beta - W(W'W)^{-1}(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c \\
&= \left(\beta - (W'W)^{-1}(X'\Sigma^{-1}y + V^{-1}\beta_0)\right)'W'W\left(\beta - (W'W)^{-1}(X'\Sigma^{-1}y + V^{-1}\beta_0)\right) + c
\end{aligned}
$$

Therefore, the probability density function of $\beta|y = y$ is:

$$f_{\beta|y=y}(\beta) \propto e^{-\frac{1}{2}\left((y-X\beta)'\Sigma^{-1}(y-X\beta)+(\beta-\beta_0)'V^{-1}(\beta-\beta_0)\right)}$$

$$\propto e^{-\frac{1}{2}\left(\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)'W'W\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)+c\right)}$$

$$\propto e^{-\frac{1}{2}\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)'W'W\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)}e^{-\frac{1}{2}c}$$

$$\propto e^{-\frac{1}{2}\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)'W'W\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)}$$

$$\propto e^{-\frac{1}{2}\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)'\left((W'W)^{-1}\right)^{-1}\left(\beta-(W'W)^{-1}(X'\Sigma^{-1}y+V^{-1}\beta_0)\right)}$$

The term having c in the exponent was absorbed into the proportionality, since c does not depend on β. This function is proportional to the probability density function of a normal random vector whose mean is $(W'W)^{-1}\left(X'\Sigma^{-1}y + V^{-1}\beta_0\right)$ and whose variance is $(W'W)^{-1}$; therefore, $\beta|\mathbf{y} = y \sim N\left((W'W)^{-1}\left(X'\Sigma^{-1}y + V^{-1}\beta_0\right),\ (W'W)^{-1}\right)$, or:

$$\beta|\mathbf{y} = y \sim N\left(\left(X'\Sigma^{-1}X + V^{-1}\right)^{-1}\left(X'\Sigma^{-1}y + V^{-1}\beta_0\right),\ \left(X'\Sigma^{-1}X + V^{-1}\right)^{-1}\right)$$

This is the same result as that obtained from the mixed linear statistical model:

$$\begin{bmatrix} \mathbf{y} \\ \beta_0 \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix}\beta + \begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix}, \text{ where } \mathrm{Var}\begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \Sigma & \\ & V \end{bmatrix},$$

which mixes the sample information $\mathbf{y} = X\beta + \mathbf{e}$ with the non-sample information $\beta_0 = \beta + \mathbf{v}$. Because the results are the same, Judge says that estimating the β of such a model, i.e., mixed estimation, is a "quasi-Bayesian approach" [8:877].

It may seem when R is not an identity matrix that the mixed model $\begin{bmatrix}\mathbf{y}\\ \mathbf{r}\end{bmatrix}=\begin{bmatrix}X\\ R\end{bmatrix}\beta+\begin{bmatrix}\mathbf{e}\\ \mathbf{v}\end{bmatrix}$, where $\mathrm{Var}\begin{bmatrix}\mathbf{e}\\ \mathbf{v}\end{bmatrix}=\begin{bmatrix}\Sigma & \\ & V\end{bmatrix}$, has no Bayesian interpretation. However, if R is of full row rank (which is not a restrictive condition), there exists an S such that $\begin{bmatrix}R\\ S\end{bmatrix}=Q_{(k\times k)}$ is non-singular. Add to the non-sample information thus:

$$\begin{bmatrix}\mathbf{y}\\ \mathbf{r}\\ \mathbf{s}\end{bmatrix}=\begin{bmatrix}X\\ R\\ S\end{bmatrix}\beta+\begin{bmatrix}\mathbf{e}\\ \mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}, \text{ where } \mathrm{Var}\begin{bmatrix}\mathbf{e}\\ \mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}\Sigma & & \\ & V_1 & \\ & & V_2\end{bmatrix}$$

Letting $\gamma = Q\beta$, a one-to-one transformation because $\beta = Q^{-1}\gamma$, we can transform:

$$\begin{bmatrix}\mathbf{y}\\ \begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\end{bmatrix}=\begin{bmatrix}X\\ \begin{bmatrix}R\\ S\end{bmatrix}\end{bmatrix}\beta+\begin{bmatrix}\mathbf{e}\\ \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}X\\ Q\end{bmatrix}\beta+\begin{bmatrix}\mathbf{e}\\ \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}XQ^{-1}\\ I_k\end{bmatrix}Q\beta+\begin{bmatrix}\mathbf{e}\\ \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}XQ^{-1}\\ I_k\end{bmatrix}\gamma+\begin{bmatrix}\mathbf{e}\\ \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\end{bmatrix}, \text{ where } \mathrm{Var}\begin{bmatrix}\mathbf{e}\\ \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\end{bmatrix}=\begin{bmatrix}\Sigma & \\ & \begin{bmatrix}V_1 & \\ & V_2\end{bmatrix}\end{bmatrix}$$

The transformed model does admit of a Bayesian interpretation. Both the mixed estimator and the Bayesian estimator are the same:

$$\hat{\gamma} = \left((XQ^{-1})' \Sigma^{-1} (XQ^{-1}) + \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^{-1} \right)^{-1} \left((XQ^{-1})' \Sigma^{-1} \mathbf{y} + \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= \left(Q'^{-1} X' \Sigma^{-1} X Q^{-1} + \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \right)^{-1} \left(Q'^{-1} X' \Sigma^{-1} \mathbf{y} + \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= \left(Q'^{-1} X' \Sigma^{-1} X Q^{-1} + Q'^{-1} Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} Q Q^{-1} \right)^{-1} \left(Q'^{-1} X' \Sigma^{-1} \mathbf{y} + \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= Q \left(X' \Sigma^{-1} X + Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} Q \right)^{-1} Q' \left(Q'^{-1} X' \Sigma^{-1} \mathbf{y} + \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= Q \left(X' \Sigma^{-1} X + Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} Q \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

Therefore:

$$\hat{\beta} = Q^{-1} \hat{\gamma}$$

$$= \left(X' \Sigma^{-1} X + Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} Q \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + Q' \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= \left(X' \Sigma^{-1} X + \begin{bmatrix} R' & S' \end{bmatrix} \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + \begin{bmatrix} R' & S' \end{bmatrix} \begin{bmatrix} V_1^{-1} & \\ & V_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \right)$$

$$= \left(X' \Sigma^{-1} X + R' V_1^{-1} R + S' V_2^{-1} S \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + R' V_1^{-1} \mathbf{r} + S' V_2^{-1} \mathbf{s} \right)$$

However, this model has extraneous non-sample information. But if V_2, the variance of the extraneous non-sample information, is allowed to approach infinity, this extraneous information will have no effect. Hence:

$$\lim_{V_2 \to \infty} \hat{\beta} = \lim_{V_2^{-1} \to 0} \hat{\beta}$$

$$= \lim_{V_2^{-1} \to 0} \left(X' \Sigma^{-1} X + R' V_1^{-1} R + S' V_2^{-1} S \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + R' V_1^{-1} \mathbf{r} + S' V_2^{-1} \mathbf{s} \right)$$

$$= \left(X' \Sigma^{-1} X + R' V_1^{-1} R \right)^{-1} \left(X' \Sigma^{-1} \mathbf{y} + R' V_1^{-1} \mathbf{r} \right)$$

Thus, in general, a Bayesian formulation, suitably transformed and taken to a limit, can be made equivalent to the mixed model.

Appendix C

The Limiting Behavior of a Stochastic Constraint

The model $\begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} X \\ R \end{bmatrix} \beta + \begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix}$, where $\text{Var}\begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}$, contains the stochastic constraint $\mathbf{r} = R\beta + \mathbf{v}$. The constraint loosens as $\text{Var}[\mathbf{v}] = V$ increases, and tightens as it decreases. In the limit, as V approaches 0, the constraint is non-stochastic, or absolute. The problem of estimating β in the model $\mathbf{y} = X\beta + \mathbf{e}$ subject to the non-stochastic constraint that $R\beta = r$ has been solved by many authors, e.g., [1:20-23], [6:35-42], and [8:235-240]. In this appendix we will demonstrate that that the same solution obtains from a stochastically constrained model as the variance of the constraint approaches zero.

Consider the model $\begin{bmatrix} \mathbf{y}_{(t\times 1)} \\ \mathbf{r}_{(j\times 1)} \end{bmatrix} = \begin{bmatrix} X_{(t\times k)} \\ R_{(j\times k)} \end{bmatrix} \beta_{(k\times 1)} + \begin{bmatrix} \mathbf{e}_{(t\times 1)} \\ \mathbf{v}_{(j\times 1)} \end{bmatrix}$, where $\text{Var}\begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \Sigma_{(t\times t)} & \\ & V_{(k\times k)} \end{bmatrix}$.

We will assume that both Σ and V are positive definite, so that their inverses exist. Also, assume that R is of full row rank, i.e., rank(R) = j. This means that the j constraints on β contain no redundancy. We will also assume that the $(k\times k)$ matrix $X'\Sigma^{-1}X$ has an inverse. Normally this is guaranteed by assuming that X is of full column rank. From these assumptions it follows that the $(j\times j)$ matrix $R(X'\Sigma^{-1}X)^{-1}R'$ has an inverse, which inverse we will call $H = (R(X'\Sigma^{-1}X)^{-1}R')^{-1}$.

The best linear unbiased estimator of β, sometimes in this context called the mixed estimator ([1:25] and [8:877]), is:

$$\hat{\beta} = \left(\begin{bmatrix} X \\ R \end{bmatrix}' \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}^{-1} \begin{bmatrix} X \\ R \end{bmatrix} \right)^{-1} \begin{bmatrix} X \\ R \end{bmatrix}' \begin{bmatrix} \Sigma & \\ & V \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix}$$

$$= \left(\begin{bmatrix} X' & R' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & \\ & V^{-1} \end{bmatrix} \begin{bmatrix} X \\ R \end{bmatrix} \right)^{-1} \begin{bmatrix} X' & R' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & \\ & V^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{r} \end{bmatrix}$$

$$= \left(X'\Sigma^{-1}X + R'V^{-1}R \right)^{-1} \left(X'\Sigma^{-1}\mathbf{y} + R'V^{-1}\mathbf{r} \right)$$

The expectation of the estimator is β (hence unbiased), and the variance thereof is $\left(X'\Sigma^{-1}X + R'V^{-1}R \right)^{-1}$. Therefore, $\hat{\beta} = \mathrm{Var}\left[\hat{\beta}\right]\left(X'\Sigma^{-1}\mathbf{y} + R'V^{-1}\mathbf{r} \right)$. Evaluating this expression as V approaches 0 is complicated due to the fact that as V approaches 0, V^{-1} approaches infinity. Thus, $\hat{\beta} = \mathrm{Var}\left[\hat{\beta}\right]\left(X'\Sigma^{-1}\mathbf{y} + R'V^{-1}\mathbf{r} \right) \rightarrow (\infty)^{-1}\infty$, an indeterminate form. The trick is to transform the expression so as to remove V^{-1}.

In Appendix A we proved that $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$, provided that the inverses exist. We can apply this theorem to the variance of the estimator:

$$\mathrm{Var}\left[\hat{\beta}\right] = \left(X'\Sigma^{-1}X + R'V^{-1}R \right)^{-1}$$

$$= \left(X'\Sigma^{-1}X \right)^{-1} - \left(X'\Sigma^{-1}X \right)^{-1} R' \left(\left(V^{-1} \right)^{-1} + R\left(X'\Sigma^{-1}X \right)^{-1} R' \right)^{-1} R \left(X'\Sigma^{-1}X \right)^{-1}$$

$$= \left(X'\Sigma^{-1}X \right)^{-1} - \left(X'\Sigma^{-1}X \right)^{-1} R' \left(V + R\left(X'\Sigma^{-1}X \right)^{-1} R' \right)^{-1} R \left(X'\Sigma^{-1}X \right)^{-1}$$

Because of the assumptions, all the inverses exist; in particular, $V + R(X'\Sigma^{-1}X)^{-1}R'$ is the sum of positive definite V and non-negative definite $R(X'\Sigma^{-1}X)^{-1}R'$. Therefore, it is positive definite, and hence non-singular. This expression has no V^{-1}, so:

$$\lim_{V \to 0} \mathrm{Var}\left[\hat{\beta}\right] = \left(X'\Sigma^{-1}X \right)^{-1} - \left(X'\Sigma^{-1}X \right)^{-1} R' \left(R\left(X'\Sigma^{-1}X \right)^{-1} R' \right)^{-1} R \left(X'\Sigma^{-1}X \right)^{-1}$$

$$= \left(X'\Sigma^{-1}X \right)^{-1} - \left(X'\Sigma^{-1}X \right)^{-1} R'HR \left(X'\Sigma^{-1}X \right)^{-1}$$

Also worth noting is:

$$\begin{aligned}
\lim_{V\to 0} R\,\mathrm{Var}\left[\hat{\beta}\right] &= R \lim_{V\to 0} \mathrm{Var}\left[\hat{\beta}\right] \\
&= R\left(\left(X'\Sigma^{-1}X\right)^{-1} - \left(X'\Sigma^{-1}X\right)^{-1} R'\left(R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1} R\left(X'\Sigma^{-1}X\right)^{-1}\right) \\
&= R\left(X'\Sigma^{-1}X\right)^{-1} - R\left(X'\Sigma^{-1}X\right)^{-1} R'HR\left(X'\Sigma^{-1}X\right)^{-1} \\
&= R\left(X'\Sigma^{-1}X\right)^{-1} - H^{-1}HR\left(X'\Sigma^{-1}X\right)^{-1} \\
&= R\left(X'\Sigma^{-1}X\right)^{-1} - R\left(X'\Sigma^{-1}X\right)^{-1} \\
&= 0
\end{aligned}$$

Now we are ready to remove the remaining V^{-1} from the estimator:

$$\begin{aligned}
\hat{\beta} &= \mathrm{Var}\left[\hat{\beta}\right]\left(X'\Sigma^{-1}\mathbf{y} + R'V^{-1}\mathbf{r}\right) \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \mathrm{Var}\left[\hat{\beta}\right]R'V^{-1}\mathbf{r} \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \left(\left(X'\Sigma^{-1}X\right)^{-1} - \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1} R\left(X'\Sigma^{-1}X\right)^{-1}\right)R'V^{-1}\mathbf{r} \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \left(X'\Sigma^{-1}X\right)^{-1} R'V^{-1}\mathbf{r} \\
&\quad - \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1} R\left(X'\Sigma^{-1}X\right)^{-1} R'V^{-1}\mathbf{r} \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1}\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)V^{-1}\mathbf{r} \\
&\quad - \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1} R\left(X'\Sigma^{-1}X\right)^{-1} R'V^{-1}\mathbf{r} \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1} VV^{-1}\mathbf{r} \\
&= \mathrm{Var}\left[\hat{\beta}\right]X'\Sigma^{-1}\mathbf{y} + \left(X'\Sigma^{-1}X\right)^{-1} R'\left(V + R\left(X'\Sigma^{-1}X\right)^{-1} R'\right)^{-1}\mathbf{r}
\end{aligned}$$

Therefore:

$$
\begin{aligned}
\lim_{V\to 0}\hat{\beta} &= \lim_{V\to 0}\left(\operatorname{Var}[\hat{\beta}]X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'\left(V + R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\right)\\
&= \lim_{V\to 0}\left(\operatorname{Var}[\hat{\beta}]X'\Sigma^{-1}y\right) + \lim_{V\to 0}(X'\Sigma^{-1}X)^{-1}R'\left(V + R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\\
&= \left(\lim_{V\to 0}\operatorname{Var}[\hat{\beta}]\right)X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'\left(R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\\
&= \left(\lim_{V\to 0}\operatorname{Var}[\hat{\beta}]\right)X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'Hr\\
&= \left((X'\Sigma^{-1}X)^{-1} - (X'\Sigma^{-1}X)^{-1}R'HR(X'\Sigma^{-1}X)^{-1}\right)X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'Hr\\
&= \left(I_k - (X'\Sigma^{-1}X)^{-1}R'HR\right)(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'Hr
\end{aligned}
$$

In the limit the non-stochastic constraint is satisfied:

$$
\begin{aligned}
\lim_{V\to 0}R\hat{\beta} &= \lim_{V\to 0}\left(R\left(\operatorname{Var}[\hat{\beta}]X'\Sigma^{-1}y + (X'\Sigma^{-1}X)^{-1}R'\left(V + R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\right)\right)\\
&= \lim_{V\to 0}\left(R\operatorname{Var}[\hat{\beta}]X'\Sigma^{-1}y\right) + \lim_{V\to 0}R(X'\Sigma^{-1}X)^{-1}R'\left(V + R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\\
&= \left(\lim_{V\to 0}R\operatorname{Var}[\hat{\beta}]\right)X'\Sigma^{-1}y + R(X'\Sigma^{-1}X)^{-1}R'\left(R(X'\Sigma^{-1}X)^{-1}R'\right)^{-1}r\\
&= (0)X'\Sigma^{-1}y + I_j r\\
&= r
\end{aligned}
$$

In an earlier paper [6:35f.] the author derived the formula for the non-stochastically constrained estimator $\beta^* = \left(I_k - (X'X)^{-1}R'HR\right)(X'X)^{-1}X'y + (X'X)^{-1}R'Hr$. We see that the formulas are identical except for the presence of Σ^{-1} in the middle of X'X and X'y. (Remember too that H contains an X'X.) But the earlier paper simplistically assumed Var[**e**] = Σ to be some scalar multiple of an identity matrix, i.e., $\sigma^2 I$, [6:35]. The general model can be reduced to the simpler model by a transformation [8:329f.]: If Var[**e**] = Σ = $\sigma^2\Phi$, where Φ is positive definite, then Φ^{-1} is also positive definite and there exists a non-singular W such that Φ^{-1} = W'W (cf. Appendix A). Transform the general model by premultiplying it by W (a one-to-one transformation): $W\mathbf{y} = WX\beta + W\mathbf{e}$, where $\text{Var}[W\mathbf{e}] = W\text{Var}[\mathbf{e}]W' = W\sigma^2\Phi W' = \sigma^2 W(W'W)^{-1}W' = \sigma^2 WW^{-1}(W')^{-1}W' = \sigma^2 I_t$. The transformed model $(W\mathbf{y}) = (WX)\beta + (W\mathbf{e})$ has the scalar-identity variance of the simpler model, so the term corresponding to $X'X$ is $(WX)'(WX) = X'W'WX = X'\Phi^{-1}X$. Similarly, the term corresponding to $X'\mathbf{y}$ is $(WX)'(W\mathbf{y}) = X'\Phi^{-1}\mathbf{y}$. The formula for β^* is so constructed as to be invariant to the scale of Φ; hence, Φ can be replaced by Σ with the result:

$$\beta^* = \left(I_k - \left(X'\Sigma^{-1}X\right)^{-1}R'HR\right)\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathbf{y} + \left(X'\Sigma^{-1}X\right)^{-1}R'Hr,$$

where $H = (R(X'\Sigma^{-1}X)^{-1}R')^{-1}$. Therefore, we have demonstrated that the non-stochastically constrained model is a limiting case of the stochastically constrained model.

Amemiya [1:25f.] performs a similar demonstration, but with the simplistic assumption that $\text{Var}\begin{bmatrix}\mathbf{e}\\ \mathbf{v}\end{bmatrix} = \sigma^2\begin{bmatrix} I_t & \\ & \frac{1}{\lambda^2}I_l\end{bmatrix}$. (His notation is different, but this is in effect his reasoning.) The limiting case results from letting λ^2 approach infinity. Our demonstration is more powerful, since it allows $\text{Var}[\mathbf{v}]$ to approach zero in any manner, not just as a shrinking scalar multiple of an identity matrix.

Appendix D

Estimating the Mean and the Variance of a Multivariate Random Sample

The variance of the error term of a linear statistical model is usually assumed to be known to within a proportionality constant, i.e., $\mathrm{Var}[\mathbf{e}] \propto \Phi$. But in the case of a multivariate random sample the entire variance can be estimated. We start with n $(k\times 1)$ random vectors $\mathbf{y}_1, \ldots, \mathbf{y}_n$, which are randomly sampled from a population of unknown mean and variance, μ and Σ. According to the definition of variance ([7:Appendix A] and [8:43]), $\Sigma = \mathrm{Var}[\mathbf{y}_i] = \mathrm{E}\left[(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)'\right]$.

The mean and the variance will be estimated from the linear model:

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}_{(nk\times 1)} = \begin{bmatrix} \mathrm{I}_k \\ \vdots \\ \mathrm{I}_k \end{bmatrix}_{(nk\times k)} \mu_{(k\times 1)} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}_{(nk\times 1)}, \text{ where } \mathrm{Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma & & \\ & \ddots & \\ & & \Sigma \end{bmatrix}_{(nk\times nk)}$$

The variance matrix is block diagonal in Σ, because random sampling implies independent, identically distributed trials. The best linear unbiased estimator of μ happens not to depend on the unknown Σ:

$$\hat{\mu} = \left(\begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix}' \begin{bmatrix} \Sigma & & \\ & \ddots & \\ & & \Sigma \end{bmatrix}^{-1} \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix} \right)^{-1} \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix}' \begin{bmatrix} \Sigma & & \\ & \ddots & \\ & & \Sigma \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & & \\ & \ddots & \\ & & \Sigma^{-1} \end{bmatrix} \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix} \right)^{-1} \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & & \\ & \ddots & \\ & & \Sigma^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

$$= \left(\Sigma^{-1} + \ldots + \Sigma^{-1} \right)^{-1} \left(\Sigma^{-1}\mathbf{y}_1 + \ldots + \Sigma^{-1}\mathbf{y}_n \right)$$

$$= \left(n\Sigma^{-1} \right)^{-1} \left(\Sigma^{-1}\mathbf{y}_1 + \ldots + \Sigma^{-1}\mathbf{y}_n \right)$$

$$= \frac{1}{n} \Sigma \left(\Sigma^{-1}\mathbf{y}_1 + \ldots + \Sigma^{-1}\mathbf{y}_n \right)$$

$$= \frac{1}{n} \left(\mathbf{y}_1 + \ldots + \mathbf{y}_n \right)$$

(That the true Σ might be singular does not impugn the validity of the estimator.) Since the estimator is unbiased, $E[\hat{\mu}] = \mu$. The variance is:

$$\mathrm{Var}[\hat{\mu}] = E\left[(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \right] = \left(\begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix}' \begin{bmatrix} \Sigma & & \\ & \ddots & \\ & & \Sigma \end{bmatrix}^{-1} \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix} \right)^{-1} = \frac{1}{n} \Sigma$$

For future reference it is noted here that $\sum_1^n (\mathbf{y}_i - \hat{\mu}) = 0$ and $\sum_1^n (\mathbf{y}_i - \mu) = \sum_1^n (\hat{\mu} - \mu)$.

Now consider the function $\Psi(\nu) = \sum_1^n (\mathbf{y}_i - \nu)(\mathbf{y}_i - \nu)'$. This function can be minimized:

$$
\begin{aligned}
\Psi(\nu) &= \sum_{1}^{n}(\mathbf{y}_i - \nu)(\mathbf{y}_i - \nu)' \\
&= \sum_{1}^{n}\big((\mathbf{y}_i - \hat{\mu}) - (\nu - \hat{\mu})\big)\big((\mathbf{y}_i - \hat{\mu}) - (\nu - \hat{\mu})\big) \\
&= \sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})(\mathbf{y}_i - \hat{\mu})' - \sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})(\nu - \hat{\mu})' - \sum_{1}^{n}(\nu - \hat{\mu})(\mathbf{y}_i - \hat{\mu})' + \sum_{1}^{n}(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&= \Psi(\hat{\mu}) - \sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})(\nu - \hat{\mu})' - \sum_{1}^{n}(\nu - \hat{\mu})(\mathbf{y}_i - \hat{\mu})' + \sum_{1}^{n}(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&= \Psi(\hat{\mu}) - \left(\sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})\right)(\nu - \hat{\mu})' - (\nu - \hat{\mu})\left(\sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})'\right) + \sum_{1}^{n}(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&= \Psi(\hat{\mu}) - (0)(\nu - \hat{\mu})' - (\nu - \hat{\mu})(0) + \sum_{1}^{n}(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&= \Psi(\hat{\mu}) + \sum_{1}^{n}(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&= \Psi(\hat{\mu}) + n(\nu - \hat{\mu})(\nu - \hat{\mu})' \\
&\,.\!\geq\!.\, \Psi(\hat{\mu})
\end{aligned}
$$

The matrix inequality (cf. Appendix A) holds because $n(\nu - \hat{\mu})(\nu - \hat{\mu})'$ is non-negative definite, with equality obtaining if and only if $\nu = \hat{\mu}$. Due to the existence and uniqueness of this minimum, we could have defined $\hat{\mu}$ as the minimizing argument of Ψ, rather than as the best linear unbiased estimator of the model above.

The minimum of Ψ is:

$$
\begin{aligned}
\Psi(\hat{\mu}) &= \sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})(\mathbf{y}_i - \hat{\mu})' \\
&= \sum_{1}^{n}((\mathbf{y}_i - \mu) - (\hat{\mu} - \mu))((\mathbf{y}_i - \mu) - (\hat{\mu} - \mu))' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - \sum_{1}^{n}(\mathbf{y}_i - \mu)(\hat{\mu} - \mu)' - \sum_{1}^{n}(\hat{\mu} - \mu)(\mathbf{y}_i - \mu)' + \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - \left(\sum_{1}^{n}(\mathbf{y}_i - \mu)\right)(\hat{\mu} - \mu)' - (\hat{\mu} - \mu)\left(\sum_{1}^{n}(\mathbf{y}_i - \mu)'\right) + \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - \left(\sum_{1}^{n}(\hat{\mu} - \mu)\right)(\hat{\mu} - \mu)' - (\hat{\mu} - \mu)\left(\sum_{1}^{n}(\hat{\mu} - \mu)'\right) + \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' - \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' + \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - \sum_{1}^{n}(\hat{\mu} - \mu)(\hat{\mu} - \mu)' \\
&= \sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - n(\hat{\mu} - \mu)(\hat{\mu} - \mu)'
\end{aligned}
$$

But the importance of this minimum lies in its expected value:

$$
\begin{aligned}
\mathrm{E}\left[\Psi(\hat{\mu})\right] &= \mathrm{E}\left[\sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)' - n(\hat{\mu} - \mu)(\hat{\mu} - \mu)'\right] \\
&= \mathrm{E}\left[\sum_{1}^{n}(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)'\right] - \mathrm{E}\left[n(\hat{\mu} - \mu)(\hat{\mu} - \mu)'\right] \\
&= \sum_{1}^{n}\mathrm{E}\left[(\mathbf{y}_i - \mu)(\mathbf{y}_i - \mu)'\right] - n\mathrm{E}\left[(\hat{\mu} - \mu)(\hat{\mu} - \mu)'\right] \\
&= \sum_{1}^{n}\mathrm{Var}[\mathbf{y}_i] - n\mathrm{Var}[\hat{\mu}] \\
&= \sum_{1}^{n}\Sigma - n\left(\frac{1}{n}\Sigma\right) \\
&= n\Sigma - \Sigma \\
&= (n-1)\Sigma
\end{aligned}
$$

Therefore, $\hat{\Sigma} = \frac{1}{n-1}\Psi(\hat{\mu}) = \frac{1}{n-1}\sum_{1}^{n}(\mathbf{y}_i - \hat{\mu})(\mathbf{y}_i - \hat{\mu})'$ is an unbiased estimator of Σ.

Appendix E

Credibility and the Random-Effects Model

Appendix A introduced groups of statistical models. The first model consisted of n linear models of the form $\mathbf{y}_i = X_i\beta_i + \mathbf{e}_i$, where $\mathrm{Var}[\mathbf{e}_i] = \Sigma_i$, for i = 1, ..., n. $\mathbf{y}_i$ and $\mathbf{e}_i$ were $(t_i{\times}1)$, X_i was $(t_i{\times}k_i)$, β_i is $(k_i{\times}1)$, and Σ_i was $(t_i{\times}t_i)$. Each Σ_i was non-singular, and each X_i was of full column rank, i.e., $\mathrm{rank}(X_i) = k_i$, which ensured that each $(X_i'\Sigma_i^{-1}X_i)^{-1}$ existed. These specifications will be adopted here, but with the additional specification that all the k_is are equal: $k = k_1 = \ldots = k_n$. The model then appears as:

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}$$

$$\text{where } \mathrm{Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}$$

As shown in Appendix A, the best linear unbiased estimator of β is:

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \\ \hat{\beta}_n \end{bmatrix} = \begin{bmatrix} (X_1'\Sigma_1^{-1}X_1)^{-1}X_1'\Sigma_1^{-1}\mathbf{y}_1 \\ \vdots \\ (X_n'\Sigma_n^{-1}X_n)^{-1}X_n'\Sigma_n^{-1}\mathbf{y}_n \end{bmatrix} = \begin{bmatrix} \mathrm{Var}[\hat{\beta}_1]X_1'\Sigma_1^{-1}\mathbf{y}_1 \\ \vdots \\ \mathrm{Var}[\hat{\beta}_n]X_n'\Sigma_n^{-1}\mathbf{y}_n \end{bmatrix}$$

This is called a fixed-effects model because every submodel is given its own β_i.

The second model of models was like the first, but with the constraint that all the β_is be equal: $\beta_0 = \beta_1 = \ldots = \beta_n$. The model then appears as:

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \beta_0 + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}$$

$$\text{where Var}\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_n \end{bmatrix}$$

Again, as shown in Appendix A, the best linear unbiased estimator of β_0 is:

$$\begin{aligned} \hat{\beta}_0 &= \left(X_1'\Sigma_1^{-1}X_1 + \ldots + X_n'\Sigma_n^{-1}X_n\right)^{-1}\left(X_1'\Sigma_1^{-1}\mathbf{y}_1 + \ldots + X_n'\Sigma_n^{-1}\mathbf{y}_n\right) \\ &= \left(\text{Var}^{-1}\left[\hat{\beta}_1\right] + \ldots + \text{Var}^{-1}\left[\hat{\beta}_n\right]\right)^{-1}\left(\text{Var}^{-1}\left[\hat{\beta}_1\right]\hat{\beta}_1 + \ldots + \text{Var}^{-1}\left[\hat{\beta}_n\right]\hat{\beta}_n\right) \\ &= \left(\text{Var}\left[\hat{\beta}_0\right]\right)\left(\text{Var}^{-1}\left[\hat{\beta}_1\right]\hat{\beta}_1 + \ldots + \text{Var}^{-1}\left[\hat{\beta}_n\right]\hat{\beta}_n\right) \end{aligned}$$

This too is a fixed-effects model, but with only one fixed effect.

Now an attractive basis of a credibility model is the belief that the parameters (here γ_is) constitute a random sample from a distribution of mean γ_0 and variance V. So $\gamma_i = \gamma_0 + \mathbf{v}_i$, where $E[\mathbf{v}_i] = 0$, $\text{Var}[\mathbf{v}_i] = V$, and the $\mathbf{v}_i$s do not covary either with each other or with the $\mathbf{e}_i$s. This transforms the fixed-effects model into the random-effects model (with estimations):

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \\ \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \\ I_k & & \\ & \ddots & \\ & & I_k \end{bmatrix} \begin{bmatrix} \gamma_0 + \mathbf{v}_1 \\ \vdots \\ \gamma_0 + \mathbf{v}_n \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} X_1 \\ \vdots \\ X_n \\ I_k \\ \vdots \\ I_k \end{bmatrix} \gamma_0 + \begin{bmatrix} X_1\mathbf{v}_1 + \mathbf{e}_1 \\ \vdots \\ X_n\mathbf{v}_n + \mathbf{e}_n \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

Instead of each submodel having its own fixed effect γ_i, there is one fixed-effect (γ_0) for the whole model and each submodel has its own random effect $\mathbf{v}_i$. Therefore, the estimations of this random-effects model have a non-zero error term, and the variance matrix is:

$$\operatorname{Var}\begin{bmatrix} X_1\mathbf{v}_1+\mathbf{e}_1 \\ \vdots \\ X_n\mathbf{v}_n+\mathbf{e}_n \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} = \left[\begin{array}{ccc:ccc} X_1VX_1'+\Sigma_1 & & & X_1V & & \\ & \ddots & & & \ddots & \\ & & X_nVX_n'+\Sigma_n & & & X_nV \\ \hdashline VX_1' & & & V & & \\ & \ddots & & & \ddots & \\ & & VX_n' & & & V \end{array}\right] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

The best linear unbiased estimations are:

$$\begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_n \end{bmatrix} = \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix} \hat{\gamma}_0 + T_{21}T_{11}^{-1}\left(\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} - \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \hat{\gamma}_0 \right),$$

$$\text{where } \hat{\gamma}_0 = \left(\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}' T_{11}^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}' T_{11}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

$$= \operatorname{Var}[\hat{\gamma}_0] \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}' T_{11}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

Let us define the n blocks of T_{11} as $T_{11i} = X_iVX_i' + \Sigma_i$. T_{11i} is $(t_i \times t_i)$ and positive definite. We will also use the shorthand expression of 'T_i' for 'T_{11i}'. Then the estimations may be written as:

$$\begin{bmatrix}\hat{\gamma}_1\\ \vdots\\ \hat{\gamma}_n\end{bmatrix}=\begin{bmatrix}I_k\\ \vdots\\ I_k\end{bmatrix}\hat{\gamma}_0+T_{21}T_{11}^{-1}\left(\begin{bmatrix}\mathbf{y}_1\\ \vdots\\ \mathbf{y}_n\end{bmatrix}-\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}\hat{\gamma}_0\right)$$

$$=\begin{bmatrix}\hat{\gamma}_0\\ \vdots\\ \hat{\gamma}_0\end{bmatrix}+\begin{bmatrix}VX_1' & & \\ & \ddots & \\ & & VX_n'\end{bmatrix}\begin{bmatrix}T_1 & & \\ & \ddots & \\ & & T_n\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{y}_1-X_1\hat{\gamma}_0\\ \vdots\\ \mathbf{y}_n-X_n\hat{\gamma}_0\end{bmatrix}$$

$$=\begin{bmatrix}\hat{\gamma}_0\\ \vdots\\ \hat{\gamma}_0\end{bmatrix}+\begin{bmatrix}VX_1' & & \\ & \ddots & \\ & & VX_n'\end{bmatrix}\begin{bmatrix}T_1^{-1} & & \\ & \ddots & \\ & & T_n^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1-X_1\hat{\gamma}_0\\ \vdots\\ \mathbf{y}_n-X_n\hat{\gamma}_0\end{bmatrix}$$

$$=\begin{bmatrix}\hat{\gamma}_0+VX_1'T_1^{-1}(\mathbf{y}_1-X_1\hat{\gamma}_0)\\ \vdots\\ \hat{\gamma}_0+VX_n'T_n^{-1}(\mathbf{y}_n-X_n\hat{\gamma}_0)\end{bmatrix},$$

$$\text{where } \hat{\gamma}_0=\left(\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}'\begin{bmatrix}T_1 & & \\ & \ddots & \\ & & T_n\end{bmatrix}^{-1}\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}\right)^{-1}\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}'\begin{bmatrix}T_1 & & \\ & \ddots & \\ & & T_n\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{y}_1\\ \vdots\\ \mathbf{y}_n\end{bmatrix}$$

$$=\left(\begin{bmatrix}X_1' & \cdots & X_n'\end{bmatrix}\begin{bmatrix}T_1^{-1} & & \\ & \ddots & \\ & & T_n^{-1}\end{bmatrix}\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}\right)^{-1}\begin{bmatrix}X_1' & \cdots & X_n'\end{bmatrix}\begin{bmatrix}T_1^{-1} & & \\ & \ddots & \\ & & T_n^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \vdots\\ \mathbf{y}_n\end{bmatrix}$$

$$=\left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}\mathbf{y}_1+\cdots+X_n'T_n^{-1}\mathbf{y}_n\right)$$

$$=\mathrm{Var}[\hat{\gamma}_0]\left(X_1'T_1^{-1}\mathbf{y}_1+\cdots+X_n'T_n^{-1}\mathbf{y}_n\right)$$

The penultimate expression for $\hat{\gamma}_0$ looks like the expression for $\hat{\beta}_0$ except that it contains terms with T_i^{-1} instead of terms with Σ_i^{-1}. But this small difference has great effects, which must be investigated. As a beginning, borrowing a theorem from Appendix A, viz., that $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$, we have:

$$T_i^{-1} = \left(\Sigma_i + X_i V X_i'\right)^{-1}$$
$$= \Sigma_i^{-1} - \Sigma_i^{-1} X_i \left(V^{-1} + X_i'\Sigma_i^{-1}X_i\right)^{-1} X_i'\Sigma_i^{-1}$$

Then $X_i'T_i^{-1}X_i = X_i'\Sigma_i^{-1}X_i - X_i'\Sigma_i^{-1}X_i\left(V^{-1} + X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i$. Moreover:

$$\begin{aligned}
X_i'T_i^{-1}\mathbf{y}_i &= X_i'\Sigma_i^{-1}\mathbf{y}_i - X_i'\Sigma_i^{-1}X_i\left(V^{-1} + X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i \\
&= X_i'\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i - X_i'\Sigma_i^{-1}X_i\left(V^{-1} + X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i \\
&= \left(X_i'\Sigma_i^{-1}X_i - X_i'\Sigma_i^{-1}X_i\left(V^{-1} + X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i\right)\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i \\
&= X_i'T_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i \\
&= X_i'T_i^{-1}X_i\hat{\beta}_i
\end{aligned}$$

And finally:

$$\begin{aligned}
\hat{\gamma}_0 &= \left(X_1'T_1^{-1}X_1 + \cdots + X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}\mathbf{y}_1 + \cdots + X_n'T_n^{-1}\mathbf{y}_n\right) \\
&= \left(X_1'T_1^{-1}X_1 + \cdots + X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1\hat{\beta}_1 + \cdots + X_n'T_n^{-1}X_n\hat{\beta}_n\right) \\
&= \mathrm{Var}[\hat{\gamma}_0]\left(X_1'T_1^{-1}X_1\hat{\beta}_1 + \cdots + X_n'T_n^{-1}X_n\hat{\beta}_n\right)
\end{aligned}$$

Therefore, the estimator of the grand parameter of this credibility model (γ_0) is like the estimator of the grand parameter of the non-credibility model (β_0) in that both are weighted averages of the estimators of the fixed-effects model (the $\hat{\beta}_i$s). The difference is that the weights of the credibility model are $X_i'T_i^{-1}X_i$, whereas those of the non-credibility model are $X_i'\Sigma_i^{-1}X_i$.

There is a danger of using the fixed-effects estimators $\hat{\beta}_i = \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i$ in this random-effects model. Whichever model is assumed:

$$\begin{aligned}\mathrm{Var}\left[\hat{\beta}_i\right] &= \mathrm{Var}\left[\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathbf{y}_i\right]\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathrm{Var}\left[\mathbf{y}_i\right]\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\end{aligned}$$

However, under the fixed-effects model:

$$\begin{aligned}\mathrm{Var}\left[\hat{\beta}_i\right] &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathrm{Var}\left[\mathbf{y}_i\right]\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\Sigma_i\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\end{aligned}$$

But under the random-effects model:

$$\begin{aligned}\mathrm{Var}\left[\hat{\beta}_i\right] &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\mathrm{Var}\left[\mathbf{y}_i\right]\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}T_i\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}(X_iVX_i'+\Sigma_i)\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_iVX'\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}+\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}\Sigma_i\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= V+\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= V+\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\end{aligned}$$

Further manipulation (again, using the theorem from Appendix A cited above) yields:

$$\begin{aligned}\mathrm{Var}\left[\hat{\beta}_i\right] &= V+\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(\left(\left(X_i'\Sigma_i^{-1}X_i\right)^{-1}+V\right)^{-1}\right)^{-1}\\ &= \left(X_i'\Sigma_i^{-1}X_i-X_i'\Sigma_i^{-1}X_i\left(V^{-1}+X_i'\Sigma_i^{-1}X_i\right)^{-1}X_i'\Sigma_i^{-1}X_i\right)^{-1}\\ &= \left(X_i'T_i^{-1}X_i\right)^{-1}\end{aligned}$$

So, the variance of this estimator under the random-effects model differs from that under the fixed-effects model either by the addition of V to the latter or by the substitution of T for Σ in the latter. With the use of the correct variance, the true formula for $\mathrm{Var}\left[\hat{\gamma}_0\right]$ results:

$$
\begin{aligned}
\mathrm{Var}[\hat{\gamma}_0] &= \mathrm{Var}\left[\left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1\hat{\beta}_1+\cdots+X_n'T_n^{-1}X_n\hat{\beta}_n\right)\right] \\
&= \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\mathrm{Var}\left[X_1'T_1^{-1}X_1\hat{\beta}_1+\cdots+X_n'T_n^{-1}X_n\hat{\beta}_n\right] \\
&\quad \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1} \\
&= \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1\mathrm{Var}\left[\hat{\beta}_1\right]X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\mathrm{Var}\left[\hat{\beta}_n\right]X_n'T_n^{-1}X_n\right) \\
&\quad \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1} \\
&= \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1\left(X_1'T_1^{-1}X_1\right)^{-1}X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\left(X_n'T_n^{-1}X_n\right)^{-1}X_n'T_n^{-1}X_n\right) \\
&\quad \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1} \\
&= \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right) \\
&\quad \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1} \\
&= \left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}
\end{aligned}
$$

Limiting cases for V of the random-effects model are important. The first limiting case, as $V \to 0$, is simple. As $V \to 0$, the $\mathbf{v}_i$s $\to$ zero vectors, and model approaches:

$$
\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \\ \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \\ I_k & & \\ & \ddots & \\ & & I_k \end{bmatrix} \begin{bmatrix} \gamma_0 + 0 \\ \vdots \\ \gamma_0 + 0 \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \\ I_k \\ \vdots \\ I_k \end{bmatrix} \gamma_0 + \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

But this is the fixed-effects model with the constraint that all the γ_is be equal: $\gamma_0 = \gamma_1 = \ldots = \gamma_n$. And as $V \to 0$, $T_i \to \Sigma_i$, and:

$$
\begin{aligned}
\lim_{V\to 0}\hat{\gamma}_0 &= \lim_{V\to 0}\left(X_1'T_1^{-1}X_1+\cdots+X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}\mathbf{y}_1+\cdots+X_n'T_n^{-1}\mathbf{y}_n\right) \\
&= \left(X_1'\Sigma_1^{-1}X_1+\cdots+X_n'\Sigma_n^{-1}X_n\right)^{-1}\left(X_1'\Sigma_1^{-1}\mathbf{y}_1+\cdots+X_n'\Sigma_n^{-1}\mathbf{y}_n\right) \\
&= \mathrm{Var}\left[\hat{\beta}_0\right]\left(X_1'\Sigma_1^{-1}\mathbf{y}_1+\cdots+X_n'\Sigma_n^{-1}\mathbf{y}_n\right) \\
&= \hat{\beta}_0
\end{aligned}
$$

In typical actuarial parlance, the submodels of this case have no credibility.

The opposite limiting case is for V to approach infinity. But the more precise meaning is that $V^{-1} \to 0$. It is best to see what happens to $\hat{\gamma}_0$ in this case. Inasmuch as $\hat{\gamma}_0 = \left(X_1'T_1^{-1}X_1 + \cdots + X_n'T_n^{-1}X_n\right)^{-1}\left(X_1'T_1^{-1}X_1\hat{\beta}_1 + \cdots + X_n'T_n^{-1}X_n\hat{\beta}_n\right)$ and $X_i'T_i^{-1}X_i \to 0$ as $V^{-1} \to 0$, the limit is the indeterminate form $0^{-1}0$. However, since $V + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1} = \left(X_i'T_i^{-1}X_i\right)^{-1}$:

$$\begin{aligned}
X_i'T_i^{-1}X_i &= \left(V + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}\right)^{-1} \\
&= \left(\left(I_k + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}V^{-1}\right)V\right)^{-1} \\
&= V^{-1}\left(I_k + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}V^{-1}\right)^{-1} \\
&= V^{-1}U_i
\end{aligned}$$

Therefore:

$$\begin{aligned}
\hat{\gamma}_0 &= \left(V^{-1}U_1 + \cdots + V^{-1}U_n\right)^{-1}\left(V^{-1}U_1\hat{\beta}_1 + \cdots + V^{-1}U_n\hat{\beta}_n\right) \\
&= \left(V^{-1}\left(U_1 + \cdots + U_n\right)\right)^{-1}\left(V^{-1}\left(U_1\hat{\beta}_1 + \cdots + U_n\hat{\beta}_n\right)\right) \\
&= \left(U_1 + \cdots + U_n\right)^{-1}VV^{-1}\left(U_1\hat{\beta}_1 + \cdots + U_n\hat{\beta}_n\right) \\
&= \left(U_1 + \cdots + U_n\right)^{-1}\left(U_1\hat{\beta}_1 + \cdots + U_n\hat{\beta}_n\right)
\end{aligned}$$

But $\lim_{V^{-1}\to 0} U_i = \lim_{V^{-1}\to 0}\left(I_k + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}V^{-1}\right)^{-1} = \lim_{V^{-1}\to 0}\left(I_k + \left(X_i'\Sigma_i^{-1}X_i\right)^{-1}0\right)^{-1} = I_k$. Hence:

$$\begin{aligned}
\lim_{V^{-1}\to 0}\hat{\gamma}_0 &= \lim_{V^{-1}\to 0}\left(U_1 + \cdots + U_n\right)^{-1}\left(U_1\hat{\beta}_1 + \cdots + U_n\hat{\beta}_n\right) \\
&= \left(I_k + \cdots + I_k\right)^{-1}\left(I_k\hat{\beta}_1 + \cdots + I_k\hat{\beta}_n\right) \\
&= \frac{1}{n}\left(\hat{\beta}_1 + \cdots + \hat{\beta}_n\right)
\end{aligned}$$

In actuarial parlance, the submodels of this case have full credibility. Therefore, it makes sense here for $\hat{\gamma}_0$ to be the simple average of the fixed-effects estimators.

We turn now to the credibility estimators, and elaborate the formula:

$$\begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_n \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_0 + VX_1'T_1^{-1}(\mathbf{y}_1 - X_1\hat{\gamma}_0) \\ \vdots \\ \hat{\gamma}_0 + VX_n'T_n^{-1}(\mathbf{y}_n - X_n\hat{\gamma}_0) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\gamma}_0 + VX_1'T_1^{-1}\mathbf{y}_1 - VX_1'T_1^{-1}X_1\hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 + VX_n'T_n^{-1}\mathbf{y}_n - VX_n'T_n^{-1}X_n\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\gamma}_0 + VX_1'T_1^{-1}X_1\hat{\beta}_1 - VX_1'T_1^{-1}X_1\hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 + VX_n'T_n^{-1}X_n\hat{\beta}_n - VX_n'T_n^{-1}X_n\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} VX_1'T_1^{-1}X_1\hat{\beta}_1 + (I_k - VX_1'T_1^{-1}X_1)\hat{\gamma}_0 \\ \vdots \\ VX_n'T_n^{-1}X_n\hat{\beta}_n + (I_k - VX_n'T_n^{-1}X_n)\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} Z_1\hat{\beta}_1 + (I_k - Z_1)\hat{\gamma}_0 \\ \vdots \\ Z_n\hat{\beta}_n + (I_k - Z_n)\hat{\gamma}_0 \end{bmatrix}$$

So the credibility estimators are matrix-weighted averages of the fixed-effects estimators and the estimator of the grand parameter. This is a k-dimensional generalization of what actuaries call credibility weighting. (See the remark in Appendix A on how matrix-weighted averages differ from scalar-weighted averages.)

In the first limiting case, $\lim_{V \to 0} Z_i = \lim_{V \to 0} VX_iT_i^{-1}X_i' = 0X_i\Sigma_i^{-1}X_i' = 0$; so, $\lim_{V \to 0} \hat{\gamma}_i = \hat{\gamma}_0$. This is to say that as the submodels lose credibility, random-effects model approaches the constrained fixed-effects model, or the one-fixed-effect model. In the opposite limiting case, $\lim_{V^{-1} \to 0} Z_i = \lim_{V^{-1} \to 0} VX_iT_i^{-1}X_i' = \lim_{V^{-1} \to 0} VV^{-1}U_i = \lim_{V^{-1} \to 0} U_i = I_k$; so, $\lim_{V^{-1} \to 0} \hat{\gamma}_i = \hat{\beta}_i$. This means that as the submodels gain credibility, the random-effects model approaches the *n*-fixed-effects model.

In the linear model $\mathbf{y} = X\beta + \mathbf{e}$, where $Var[\mathbf{e}] = \Sigma$, $\hat{\beta} = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}\mathbf{y}$. Therefore:

$$\begin{aligned} X'\Sigma^{-1}\left(\mathbf{y} - X\hat{\beta}\right) &= X'\Sigma^{-1}\mathbf{y} - X'\Sigma^{-1}X\hat{\beta} \\ &= X'\Sigma^{-1}\mathbf{y} - X'\Sigma^{-1}X\left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}\mathbf{y} \\ &= X'\Sigma^{-1}\mathbf{y} - X'\Sigma^{-1}\mathbf{y} \\ &= 0 \end{aligned}$$

We will apply this identity in the following derivation:

$$\begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_n \end{bmatrix} = \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \left(\begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix} + \begin{bmatrix} VX_1' & & \\ & \ddots & \\ & & VX_n' \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}^{-1} \begin{bmatrix} y_1 - X_1\hat{\gamma}_0 \\ \vdots \\ y_n - X_n\hat{\gamma}_0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} VX_1' & & \\ & \ddots & \\ & & VX_n' \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}^{-1} \begin{bmatrix} y_1 - X_1\hat{\gamma}_0 \\ \vdots \\ y_n - X_n\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix} + \begin{bmatrix} VX_1' & \cdots & VX_n' \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}^{-1} \begin{bmatrix} y_1 - X_1\hat{\gamma}_0 \\ \vdots \\ y_n - X_n\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix} + V\begin{bmatrix} X_1' & \cdots & X_n' \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}^{-1} \begin{bmatrix} y_1 - X_1\hat{\gamma}_0 \\ \vdots \\ y_n - X_n\hat{\gamma}_0 \end{bmatrix}$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix} + VX'T^{-1}(y - X\hat{\gamma}_0)$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix} + V(0)$$

$$= \begin{bmatrix} I_k & \cdots & I_k \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_0 \end{bmatrix}$$

This shows that the simple average of the credibility estimators equals the grand parameter. Earlier we saw that in the fixed effects model, $\hat{\beta}_0$ was a weighted average of the $\hat{\beta}_i$s, the weights being proportional to the inverses of the variances of the $\hat{\beta}_i$s. This average is aristocratic in that the better $\hat{\beta}_i$s (i.e., those with the smaller variances) receive more weight. But in the random effects model, $\hat{\gamma}_0$ is a simple average of the $\hat{\gamma}_i$s. This suggests an interpretation of credibility: credibility democratizes submodels. After a credibility adjustment, every submodel is entitled to one vote in determining the grand parameter. Of course, the weaker submodels are adjusted more vigorously.

Appendix F

A SAS® Procedure for Credibility Problems

According to Appendix E, many credibility problems can be expressed as random-effects statistical models. There is a SAS® procedure, PROC MIXED, which is very versatile with random-effects models. This procedure formulates the model as [12:575f., 634] $\mathbf{y} = X\beta + Z\gamma + \varepsilon$, where $E\begin{bmatrix}\gamma \\ \varepsilon\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$, and $\mathrm{Var}\begin{bmatrix}\gamma \\ \varepsilon\end{bmatrix} = \begin{bmatrix}G & 0 \\ 0 & R\end{bmatrix}$. $Z\gamma + \varepsilon$ is the total error term, with a mean of 0 and a variance of $V = ZGZ' + R$. We know that the best linear unbiased estimator of β is $(X'V^{-1}X)^{-1}X'V^{-1}\mathbf{y}$. To estimate γ, we would use the estimator equation $\gamma = 0\beta + \gamma$; so $\hat{\gamma} = 0\hat{\beta} + \mathrm{Cov}[\gamma,\, Z\gamma + \varepsilon]V^{-1}(\mathbf{y} - X\hat{\beta}) = GZ'V^{-1}(\mathbf{y} - X\hat{\beta})$ [12:641]. But the most powerful feature of this procedure is that the variance matrices may be specified with an unknown parameter vector, viz., $G(\theta)$ and $R(\theta)$. The procedure will estimate θ, whether by variance components or by maximum likelihood [12:588, 639f.]. This model is more general than the random-effects examples of this paper; and estimating θ is a more general problem than estimating the random-effects variance of those examples. The following code succinctly solves the problem posed by Gary Venter [13] and treated as Example 5 of this paper:

```
/** This SAS program uses PROC MIXED to solve the problem on page 433 **/
/** of Gary Venter's "Credibility," Foundations of Casualty Actuarial **/
/** Science, Casualty Actuarial Society, 1990.                        **/

data data1;
   input risk year1-year6;
   cards;
1 0.430 0.375 2.341 0.175 1.016 0.466
2 0.247 1.587 1.939 0.712 0.054 0.261
3 0.661 0.237 0.063 0.250 0.602 0.700
```

```
4 0.182 0.351 0.011 0.022 0.019 0.252
5 0.311 0.664 1.002 0.038 0.370 2.502
6 0.301 0.253 0.044 0.109 2.105 0.891
7 0.219 1.186 0.431 1.405 0.241 0.804
8 0.002 0.058 0.235 0.018 0.713 0.208
9 0.796 0.260 0.932 0.857 0.129 0.349
;

proc transpose data=data1 out=data1 (rename=(_name_=time col1=x));
   by risk;

proc mixed data=data1;
   class risk;
   model x= /p s;
   random intercept /g s subject=risk;
run;
```

Once the time is invested to learn how to use routines like PROC MIXED, many complicated problems can be solved easily and quickly. However, it is possible to go overboard and to pose problems that are so complicated that one might unknowingly misuse the software. In such cases, a wrong answer may go undetected because intuition has been overwhelmed by the complexity.

The Analysis of the Effect of Tort Reform Legislation on Expected Liability Insurance Losses

by Allan Kerin, FCAS, MAAA,
and Jason Israel, ACAS

Abstract:

This paper presents a framework for possible methodologies to evaluate the effect of tort reform legislation on expected liability insurance losses and loss adjustment expense. An analysis of the most common types of reforms and the difficulties that may be encountered when evaluating their effects is presented. The direct(non-behavioral) effect on General Liability losses of a hypothetical reform which caps punitive damages and non-economic compensatory losses and which eliminates joint and several liability is analyzed using methodologies developed at ISO.

Note:

Due to the highly subjective nature of many tort reforms and their often complex influence on potential litigants' behavior, it is extremely difficult to predict their impact. In the past, many actuaries have taken the view that the best way to approach tort reform is to let the effect of highly subjective reforms be reflected in the loss experience. This is a valid approach since the real impact of the reform will be reflected in the actual experience. ISO has been studying this issue and is in the process of trying to develop a methodology which will permit the reflection of the effect of highly subjective reforms upon losses earlier and with greater precision. In this paper we provide an overview of common types of tort reforms and a discussion of the difficulties encountered when evaluating the impact of these reforms. We also discuss a methodology that is being evaluated at ISO to reflect the direct(non-behavioral) effect on General Liability losses of a hypothetical reform, which caps punitive damages and non-economic compensatory damages and eliminates joint and several liability. These analyses produce only preliminary estimates for only certain types of reforms. We caution against overestimating either the precision of the results presented here or the broadness of their application.

I) BACKGROUND

During the last several years a number of individuals and groups have expressed concern about rising liability insurance costs and about the possibly detrimental effect of high levels of litigation on our national economic efficiency and on the rate of technological innovation in some industries (e.g., pharmaceuticals, aviation). They have proposed statutory changes in the tort system intended to reduce or stabilize litigation expenses, especially for businesses and government agencies. These proposed statutes, which are intended to modify existing statutes and existing case law, have commonly been referred to as tort reforms. Although few such reforms have been enacted at the federal level a number of states have enacted tort reforms. Some reforms have been applicable to certain types of cases, such as, Medical Malpractice and Employer Liability, while others have affected a wide range of cases.

Many of these reforms, to the extent that they are effective, will affect insurance liability losses. The actuarial question of how to prospectively estimate the effect of these reforms on expected losses (including loss adjustment expense) is, therefore, one of increasing importance. For reasons that will be discussed in greater detail below the effect of most reforms can only be estimated by making a number of judgmental modeling assumptions. In some cases data based analyses are not possible at all and polling of attorneys and other experts might produce the best estimates.

Work on this subject performed in several actuarial areas in ISO during the last year has helped form a framework for the analysis of the effect of several different types of reforms. In this paper we will discuss the issues encountered when analyzing tort reforms. We will also provide an example of an analysis of the direct impact of several reforms on General Liability losses.

II) TYPES OF TORT REFORMS

Most tort reform provisions that have been enacted in the past several years can be characterized as falling in one of the following categories:

1) Limiting the amount of specific type(s) of damages that can be paid to a claimant in total or by a specific tortfeasor. Such as:
    a) Monetary caps on damages or on specific kinds of damages (e.g., punitive damages, non-economic compensatory damages).
    b) Changes to comparative negligence statutes and/or case law.
    c) Changes to joint and several liability statutes and/or case law.

2) Restricting the conditions under which specific type(s) of damages can be paid. Such as:
    a) Changing definitions of types/degrees of negligence.
    b) Changing type/degree of negligence (e.g., gross negligence, intentional acts) required to award specific types of damages (e.g., punitive damages).
    c) Changing contributory negligence statutes and/or case law.
    d) Changing statutes of limitation and/or repose.

3) Modifying the rules of evidence. Such as:
    a) Changing standards of proof.
    b) Changing types of evidence that may be considered in determining fault or evaluating damages (e.g., information on available collateral sources of recovery).

4) Other changes to legal procedures intended to change potential litigants behavior. Such as:
    a) Revised limits on contingency fee percentages
    b) Making the losing side in a civil trial responsible for the legal expenses of the winner
    c) Encouraging or mandating mediation or arbitration

Of these four major categories of tort reforms limitations on the amount of damages is the area that is most readily analyzed. Statistically reported insurance data can be used to calculate claim size distributions for all indemnity losses combined. For some lines of insurance there is a limited amount of closed claim data that can be used to support assumptions about the distribution of these losses by type of award (economic, general, punitive, etc.), by number of tortfeasors and by degree of contributory negligence. The primary generally available multi-state source that we have found for this type of information is the biennial NAIC Closed Claim Survey for Commercial General Liability. To the extent that additional closed claim data sources are not available for other lines of insurance the effect of tort reforms on these lines must be evaluated indirectly by making judgmental adjustments to the results obtained for General Liability.

Even for lines of insurance where closed claim data is available to evaluate specific reforms, two major conceptual and practical limitations exist. First, in many cases detailed information is only available for claims that go to trial and are resolved by a verdict. This is a small minority of the actual claims that enter the system since most claims are resolved by negotiated settlement at an earlier stage in litigation or after the initial verdict while appeals are pending. Therefore, assumptions about the relationship between the size and composition of awards directed by verdicts and the size and composition of negotiated settlements must be of major importance in any tort reform analysis. Second, any static analysis of this relationship between awards and settlements made under existing conditions must be further adjusted to reflect behavioral changes on the part of claimants, defendants and attorneys resulting from the changes in the risk/benefit scenarios that they face as a result of the reforms. (By risk/benefit scenario we mean the set of possible favorable and unfavorable outcomes faced by each potential participant in the liability claim process and the probability associated with each outcome.)

III) BEHAVIORAL CHANGES: AN EXAMPLE OF THE LIMITATIONS ON PRECISION OF TORT REFORM ESTIMATES

Even when the direct effects of a reform can be accurately estimated using closed claim and statistically reported data, indirect "behavioral" effects of that reform , which may be of far greater magnitude, may be subject to a far less precise degree of analysis. A very clear example of this situation can be seen in any monetary cap on punitive damages.

As noted above, most cases do not go to trial. Most are resolved by negotiated settlements rather than by verdicts. Punitive damages are only awarded in cases that are resolved by a verdict. We can assume as a working hypothesis that cases that are resolved by settlements have an average implicit provision for punitive damages, that is a specific function of the average punitive damage award that is included in verdicts for similar cases. Of course, the choice of this function may rely largely on informed judgment.

Even if the implicit provision for punitive damages in cases that are resolved by settlements can be accurately estimated under pre-reform conditions, a potentially more significant factor will be even more difficult to estimate. This is the behavioral effect that might result from imposing monetary caps on punitive damages. This effect will be manifest in at least three aspects of the process. The first is the propensity of potential claimants to pursue claims. The second is the propensity of claimants and defendants to go to trial rather than to negotiate. The third is comprised of the possible changes that may occur in the functional relationship between verdict size and composition for cases that go to trial and negotiated settlement amounts for similar cases that do not go to trial.

In short, even when the change in expected losses resulting from a reform can be estimated analytically from data under the assumption that participants' behavior will not change, the actual change in expected losses may be highly dependent on behavioral changes induced by changes in the risk/benefit scenarios faced by the participants. The effects of these behavioral changes may be estimable only through an analysis that includes a number of important judgmentally chosen assumptions.

The significance of behavioral changes is often stressed by proponents of specific reforms, including those advocating monetary caps on punitive damages. Defenders of the status quo have pointed out that only a very small number of claims go to trial and that only a minority of those claims result in awards of punitive damages. One argument that opponents of restrictions on punitive damages make is that the overall effect of punitive damages is grossly exaggerated and that punitive damages do not adversely affect economic efficiency but rather serve to deter the most egregious forms of conduct at a relatively small cost to the entire liability system[1]. Proponents of additional limitations on punitive damages respond that the possibility of large punitive damage awards, especially for cases where potential compensatory damages are relatively low, significantly affects the risk/benefit scenarios faced by plaintiffs, defendants and attorneys. They maintain that punitive damages greatly enhance the bargaining position of plaintiffs resulting in a greater propensity by potential claimants to make claims and a greater willingness by defendants to settle claims rather than risk potentially ruinous punitive damages that could result if they insisted on going to trial.[2]

Some proponents of punitive damage caps and other tort reforms claim that these behavioral effects are the truly significant factors that must be considered when evaluating the possible monetary effects of tort reforms. To the extent that this is true, we as actuaries, are faced with the difficulty of having to rely on the least quantifiable and verifiable aspect of our analyses to measure what may be among the quantitatively most significant factors.

---

1 Robin, Topping, "Around the Island Crime & Courts Law and Order: Contract Still Open on Litigation Reform", Newsday , 24 May 1994.

2 Steven Hayward, "The Role of Punitive Damages in Civil Litigation: New Evidence from Lawsuit Filings", (San Francisco: Pacific Research Institute for Public Policy),8.

IV) ALLOCATED LOSS ADJUSTMENT EXPENSE(ALAE)

Many reforms may affect expected ALAE in different ways than expected indemnity losses. Both the direct and behavioral effects of each tort reform on expected ALAE must be considered. Consider an example where the major result of an enacted tort reform (e.g., loser pays winner's expenses) is a behavioral change resulting in fewer frivolous claims being made. If most of these claims were formerly successfully contested the effect on expected ALAE may be proportionally much greater than the effect on expected indemnity losses. However, if most of the claims were formerly settled by the defendant to avoid court costs the effect of the reform on ALAE may be proportionally much smaller than the effect on expected indemnity losses. The relationship between indemnity losses and ALAE must be modeled throughout every stage of a thorough tort reform analysis.

V) OUTLINE OF PROCEDURE FOR ANALYZING THE EFFECT OF TORT REFORM LEGISLATION ON EXPECTED LIABILITY LOSSES AND ALAE

The analysis of the effect of tort reforms on liability losses can be divided into the following seven steps.

1) Analyze the content of the tort reform legislation.

2) Evaluate the possible interactions among the various reforms that were concurrently enacted.

3) Evaluate data sources available to aid in the analysis of each reform and develop the best strategy for analyzing each reform as well as for measuring the effect of any interactions found in Step 2.

4) Perform the analyses designed in Step 3 and test results for reasonableness and consistency. If possible compare with the results of past reforms in the same or different jurisdiction.

5) Evaluate the effect of behavioral changes that may result from changes in the risk/benefit scenarios faced by potential claimants, defendants and attorneys. Modify the analyses performed in Step 4 to reflect this analysis.

6) Evaluate the probability of specific provisions of the reform being overturned or modified under judicial review of the relevant appellate courts. If necessary modify short term pricing decisions to reflect these contingencies.

7) Evaluate any mitigating factors that might temper the effects of the above analysis, such as changes in tactics by plaintiffs' attorneys to circumvent the impact of the reforms.

A discussion of each of these seven steps follows.

STEP 1: ANALYZE THE CONTENT OF THE TORT REFORM LEGISLATION

This is a significant and often a difficult task. The present changes in the statutes have to be analyzed and any earlier changes that might affect prior loss experience used in the tort reform analysis must be identified. A legislative and judicial history extending several years into the past is often needed. It is often necessary to consult with attorneys that are knowledgeable in this area. This may add considerably to the expense of the analysis.

Tracking the changes in the language of all of the relevant statutes may be an arduous and expensive job. However, this is often far easier than interpreting the interactions between the changes in the statutes and case law and judicial practice. In this area local legal experience may be especially valuable. This is a key part of the analysis both retrospectively (in interpreting the history of past reforms, as well as the current legal environment in the jurisdiction) and prospectively in evaluating how the statutory provisions of the current reforms will be interpreted by trial and appellate courts. In some cases reform statutes may be, knowingly or not, largely cosmetic in that they may just codify the existing case law.

It is optimal when analyzing significant tort reform legislation to have an effective working relationship between actuaries and attorneys. In-house attorneys who are experts in insurance law may provide a great deal of guidance. Consultation with local attorneys may also be desirable in order to accurately analyze the history of procedures in civil trials in the relevant jurisdiction.

These issues are compounded when multi-state data is used in analyses of tort reform statutes. The relevant aspects of the legal environment in each state whose data is included in the analysis should be evaluated throughout the experience period of the study.

STEP 2: EVALUATE THE POSSIBLE INTERACTIONS AMONG THE VARIOUS REFORMS THAT WERE CONCURRENTLY ENACTED

There are a number of possible different interactions. These should be carefully analyzed by the actuary, where necessary in consultation with an attorney. Comparative negligence provisions are closely related to joint and several liability provisions. Monetary caps on specific types of damages may often interact with other reforms that affect those damages. Numerous other interactions are possible.

STEP 3: EVALUATE DATA SOURCES AVAILABLE TO AID IN THE ANALYSIS OF EACH REFORM AND PLAN THE BEST STRATEGY FOR ANALYZING EACH REFORM AS WELL AS FOR MEASURING THE EFFECT OF THE INTERACTIONS FOUND IN STEP 2.

As noted in the earlier sections of this paper, data may exist that can be incorporated into the analysis of some reforms, such as monetary caps on damages. However, other reforms may only be subject to a non-data based analysis. Informed assumptions, expert opinions of knowledgeable parties (e.g., local attorneys and claims adjusters) and analogies to reforms with more readily quantifiable effects are among the strategies that may have to be employed for these reforms. Comparison with changes in loss levels in other jurisdictions after similar reforms may be possible. However, in these cases it may be difficult to control for other factors affecting loss levels.

If data from a longer time period than originally expected and or from additional states is included in the analysis then the legal histories produced in Step 1 will have to be extended.

STEP 4: PERFORM THE ANALYSES DESIGNED IN STEP 3 AND TEST THE RESULTS FOR REASONABLENESS AND CONSISTENCY.

Reasonableness can be examined by analyzing the effects of past reforms in the same or different jurisdictions when such information is available. In some cases comparisons may be made with loss levels in states that have legal systems that are similar to the post reform system in the state being studied. Of course, controlling for other factors may be difficult when making historical analogies or making direct comparisons with other jurisdictions. Hopefully, as more reforms are evaluated actuaries will benefit from the experience gained and will be better able to analyze the reasonableness of results.

STEP 5: EVALUATE THE EFFECT OF BEHAVIORAL CHANGES THAT MAY RESULT FROM CHANGES IN THE RISK-BENEFIT SCENARIOS FACED BY POTENTIAL CLAIMANTS, DEFENDANTS AND ATTORNEYS. MODIFY THE ANALYSES PERFORMED IN STEP 4 TO REFLECT THIS ANALYSIS.

This is one of the most difficult and important aspects of tort reform analysis. (A discussion of possible behavioral changes related to monetary caps on punitive damage awards can be found in Section III of this paper.) Almost any reform can be expected to have some behavioral effect. An effective reform will change the probabilities of recovery and/or the expense of pursuing a legal claim for at least some potential claimants. These changes can influence the decisions of prospective claimants, defendants and attorneys on whether or not to pursue specific claims, defenses and negotiations. In fact, many proponents of tort reform stress the importance of behavioral changes. In their opinion the current tort system encourages frivolous litigation which is detrimental to efficiency and serves as a disincentive to technological innovation. A stated goal of many proponents of tort reform is to make it more risky and on average less profitable to pursue frivolous claims and thereby to deter legal action through behavioral change.

STEP 6: EVALUATE THE PROBABILITY OF SPECIFIC PROVISIONS OF THE REFORM BEING OVERTURNED OR MODIFIED UNDER JUDICIAL REVIEW OF THE APPELLATE COURTS. IF NECESSARY MODIFY SHORT TERM PRICING DECISIONS TO REFLECT THESE CONTINGENCIES.

Tort reform legislation is often challenged in the courts. Frequently these challenges are at least partially successful. Even when challenges are not successful, they may significantly delay the full impact of the reforms. For example, consider the extensive tort reform statute that was enacted in Illinois during 1995. Illinois courts overturned major provisions of this act in decisions that were issued in February, May and September of 1996. The February ruling struck a section of the statute that gave defendants greater access to the medical records of plaintiffs in many cases [3]. The May ruling found the act's $500,000 cap on pain and suffering awards to be unconstitutional[4]. The September decision struck down provisions dealing with suits concerning unsafe products [5]. The ultimate fate of these and other provisions will probably depend on subsequent decisions by higher appellate courts.

When a reform is passed that seems to have a significant probability of being successfully challenged in the courts a delayed implementation of revisions to insurance prices might be appropriate. Alternatively, a loss cost or premium discount might be adjusted to reflect the likelihood that the tort reform provisions might be rescinded or significantly modified. It may be possible to estimate the probability of various outcomes to court challenges and the percent of the total expected savings that would be associated with each outcome. An average expected saving that reflects the probability of successful challenges could then be calculated and used in place of the full savings estimated under the assumption that the entire reform is upheld. This strategy adds an additional layer of complexity to the analysis. Additionally, it may not be favorably viewed by regulators. In using this strategy a more complex set of assumptions are substituted for the simpler assumption that the provisions of the act will not be significantly modified by judicial action. In either case, the effect of the enacted tort reform should be reevaluated after all significant court challenges are resolved.

STEP 7: EVALUATE ANY MITIGATING FACTORS THAT MIGHT TEMPER THE EFFECTS PREDICTED BY THE ABOVE ANALYSIS, SUCH AS CHANGES IN TACTICS BY PLAINTIFF'S ATTORNEYS TO CIRCUMVENT THE IMPACT OF THE REFORMS.

After enactment of any tort reform provision, plaintiff's attorneys will re-evaluate their legal strategies. In some cases there may be alternate legal strategies that prove effective in at least

---

[3] Andrew Fegelman, "Judge Overturns Tort Reform on Medical Record Access.", The Chicago Tribune, 28 February 1996.

[4] Andrew Fegelman & Rick Pearson, "State Cap on Jury Awards Removed: Judge Rules Law Unconstitutional.", The Chicago Tribune, 23 May 1996

[5] Andrew Fegelman, "Another Tort Change Knocked Down: Product Liability Provision Ruled Unconstitutional.", The Chicago Tribune, 18 September 1996

partially mitigating the effect of the enacted reforms. For example, the recent restriction on Federal suits for securities fraud has been followed by an increased number of these cases being brought in the state courts. Changes in the jurisdiction, the legal grounds for a claim, types of damages or the choice of defendants may at times help the claimant to partially or totally avoid the impact of enacted reforms on expected compensation[8].

The rules of the civil justice system are comprised of an intricately entwined mixture of statute and case law, in some cases including principles of common law that go back to colonial times. Even when laws are not successfully challenged in an appellate court the details of their actual implementation may not be completely determined until a number of cases have been tried. It is possible that a court charged with interpreting newly enacted tort reforms will interpret them narrowly in order to preserve rights that existed under former law.

Juries' attitudes may also mitigate the effect of some tort reforms. In cases where there is a great deal of sympathy for the claimant and/or a sense of repugnance at the conduct of the defendant, the jurors' sense of justice may result in decisions that at least partially offset the practical effect of the enacted reforms. For example, limitations on or abolition of punitive damages may cause juries in some cases to award larger amounts in compensatory damages than they would have formerly.

Evaluating these factors is extremely difficult. The legal philosophy of the appellate judges in the state, as well as popular attitudes toward a number of issues can have a decisive effect on how judges and juries shape the post-tort reform system and on the resulting degree of effectiveness of the enacted reforms.

---

[8] Beckett, Paul, "Reform Rings Hollow for Firms Worried About Class Action Law Suits", Wall Street Journal, 4 April 1997.

VI) ALTERNATE ANALYTICAL STRATEGIES

We have considered two ways in which to analyze individual tort reform provisions (i.e., in which to perform Steps 3 and 4 of the analysis outlined above). The first such strategy is to use any available data to model the loss generation process before and after the enactment of the reforms and to calculate an effect of the reform as a function of the ratio of the post reform losses to the pre-reform losses. When adequate data for such an analysis does not exist expert opinion, historical analogies and logical arguments are relied upon. Our work to date has centered on developing applications of this strategy to price specific reforms..

An alternative strategy, that we have not yet attempted, would fit a least squares model for loss cost levels to multi-state data where the various states included represent a broad range of legal environments. The fitted values of the regression coefficients for categorical variables identifying different types of civil law procedures could theoretically serve as the basis for estimates of the differences in loss levels that would occur under specific alternative civil law provisions. Other factors that could also influence differences in loss levels among states would also be included in order to remove their effect on state specific loss levels from the analysis. The categorical variables would be evaluated on a state-by-state and year-by-year basis in order to identify differences in levels of factors that occur over time within specific states, as well as among states. The difficulties of performing such an analysis include:

1) Needing to perform an accurate statutory and case law history for each state included in the analysis throughout the experience period used.

2) The number of different provisions that could be directly modeled would be restricted by the current and historical variation in provisions among states, although some degree of extrapolation might be valid.

3) The difficulty of identifying and controlling for all major extraneous factors, such as sociological, political and economic differences among states.

VII) EFFECT ON INCREASED LIMITS COVERAGE

Many tort reforms impact different size claims differently. This is most obvious for monetary caps, which generally will have a minimal effect on small claims. Larger claims are more likely to involve punitive damages and joint and several liability and are therefore more likely to be affected by reforms in these areas. In many cases the most accurate reflection of changes in expected losses due to tort reforms would be to revise both base loss costs or rates and increased limits tables.

Revising increased limits tables to reflect the effect of tort reforms on expected losses raises several practical and theoretical questions. For credibility reasons increased limits tables are often calculated on a countrywide or multi-state (e.g., all tort states for Personal Auto BI) basis. Revising increased limits tables to reflect individual state tort reforms could result in an explosion in the number of such tables. Individual state increased limits tables may in many cases depend on sparse claim size detail data and require new credibility procedures. The additional cost of computing, updating and applying a significantly increased number of tables must be considered and weighed against the possible increase in accuracy attainable, in light of credibility limitations.

An alternative to individual state increased limits tables is grouping states by tort system. ISO already does this to a limited extent for Personal Auto Bodily Injury Liability by grouping states into a tort state group and five groups of No-Fault states. Refining this system for Personal Auto (breaking up the tort state group) and extending it to other lines is theoretically possible.

Grouping by tort system is certainly preferable to ad-hoc adjustments to countrywide or multi-state tables to reflect tort reforms enacted in individual states. Such ad-hoc adjustments can lead to severe inconsistencies. For example, State A might still have a more "plaintiff friendly" tort system even after the enactment of tort reforms than State B does in the absence of any such reforms. If modifications are made to give State A a less steep increased table and State B remains on the countrywide table an obvious inequity would result. In summary, the current countrywide and multi-state increased tables are not pre-reform tables. Instead they are tables that reflect the averages of losses by claim and/or occurrence size incurred under a wide range of legal and other variables among and often within states. Treating these tables as a pre-reform base that can be adjusted incrementally, without tempering, to reflect recently enacted tort reforms can result in significant inaccuracies.

Evaluating increased limits that vary by state group to reflect differences in legal systems among the states is, therefore, an area that deserves further research. Such tables may be more accurate both in a static legal environment and as a way of dealing with tort reforms whose proportional effect differs by loss size. However, the following factors must be considered:

1) Grouping states by legal system for the purpose of calculating increased limits factors requires a thorough state-by-state analysis of each state's tort system including any changes that have occurred during the experience period used for increased limits reviews. Even a thorough review of current and past statutes may not be sufficient due to the importance of case law and judicial procedures in determining the frequency and disposition of claims.

2) Many other factors which may affect loss size distributions significantly differ among (and within) states besides the relevant components of the legal system. Some examples include, types of industry, conditions of roads, level of traffic and safety enforcement, levels of past pollution, income distribution, unemployment levels, political and social attitudes that may be reflected in decisions by juries, judges and other participants in the tort process, etc.

VIII) EXAMPLE OF THE EVALUATION OF THE EFFECT OF THREE TORT REFORM PROVISIONS ON GENERAL LIABILITY LOSSES

Up to this point we have discussed in considerable detail the difficulties that are faced when evaluating the effect of tort reforms on expected losses. Now we will, more optimistically, present an example of an analysis of the direct (non-behavioral) effect of several of the more readily evaluated reforms on expected General Liability losses.

Given the considerations discussed above we have limited the scope of this analysis in the following ways.

1) We have analyzed only the direct, non-behavioral, effects of the reforms.

2) We have restricted our attention to reforms that are representative of the first category of reforms described in Section I: "Limiting the amount of specific type(s) of damages that can be paid to a claimant in total or by a specific tortfeasor."

3) We have analyzed the effect of the modeled reforms only for premises and operations General Liability(GL) claims.

4) We have restricted our analysis to indemnity losses.

The three reform types that we have analyzed are:

1) Cap on Non-Economic Damage Awards

2) Cap on Punitive Damage Awards

3) Repeal of Joint & Several Liability.

This analysis produces rough estimates for only certain types of reforms. We caution against over-estimating either the precision of the results, or the broadness of its applicability.

METHODOLOGY

We use simulation to model the effect of these reforms. This allows us more flexibility than a purely analytic method in integrating data from different sources from which the probability distributions of a number of variables are estimated using a variety of discrete and continuous functions.

The ISO occurrence indemnity size distributions provide the framework for our simulation. (Since, a high percentage of these occurrences have a single claimant, we used this occurrence distribution as a proxy for a General Liability premises and operations claim size distribution.) For simplicity, we use the truncated Pareto approximation, rather than the full mixed Pareto model which is used in ISO's General Liability increased limits reviews. Although experience has shown us that the truncated Pareto distribution is a reasonable model for liability occurrence and claim size distributions and that it fits the ISO General Liability occurrence size data well, we recommend evaluating alternate distributions when other data sources are used. We can invert the truncated Pareto, using formulas shown in Exhibit 1. This inverted function is used to generate the sizes of our simulated claims. (A similar analysis can be done if a distribution other than the truncated Pareto is used to model occurrence or claim size.) The ISO data was also used to estimate the percent of total losses that are attributable to bodily injury (BI) rather than property damage (PD) by loss size interval.

For all other information we turned to the 1991, 1993 and 1995 NAIC closed claim surveys. Using them we can make assumptions and estimates about our simulated claims.

Unfortunately, the NAIC surveys only include sizable bodily injury settlements and verdicts. For property damage, we have no such resource.

As noted above, a fundamental problem is that most GL claims are settled by negotiation and do not result in a verdict. However, a breakdown of damages by type (punitive, non-economic, economic) is only available for the small portion of claims that are resolved by verdict. If a reform caps a portion of an award, we must determine what indirect impact it will have on the settlements. While it seems reasonable that a settlement reflects an expected average verdict for that claim, we know that settlements tend to be smaller than verdicts. Is this a reflection of the possibility of a $0 verdict (which would not get into our average), or is it a different body of claims? Here we assume reforms impact settlements of $X the same as verdicts of $X.
We have estimated the following quantities using the NAIC closed claim data:

1) Ratio of average claim size for claims with a punitive damage component to average claim size for all claims.

2) Ratio of average claim size for multiple defendant claims impacted by joint and several liability to average claim size for all multiple defendant claims.

3) Probability of a claim involving multiple defendants.

4) Probability of a multiple defendant claim being impacted by joint and several liability.

5) Probability distribution of non-economic loss amount as a percent of total compensatory amount.

The population of claims available in the NAIC surveys is relatively sparse and for certain important categories of claims it is extremely small. Information is only available for 19 verdicts that itemized punitive damages in the combined data from the 1991, 1993 and 1995 surveys. This limited data source does not support detailed modeling of many of the relationships among the different variables being studied. Many assumptions about these relationships and hence many aspects of the structure of the model that we have developed to evaluate the effect of these reforms are based largely on judgment. We hope that additional data sources will become available that will support further testing and refinement of these assumptions.

Using data from itemized verdicts, we made the following assumptions:

1) Large total awards are more likely to have a punitive damage component.

2) If there is a punitive component, the portion of the total indemnity that it comprises is uniformly distributed from 0% to 100%.

3) For General Liability, the ratio of non-economic to economic damages is independent of award size.

4) Large awards with multiple defendants are more likely to involve joint and several liability.

In addition based on judgment we have assumed:

1) The probability of a claim being BI varies by size of loss.

2) PD claims have no significant chance of involving punitive or non-economic awards.

3) The likelihood that multiple defendants are involved and the number of defendants is independent of the modeled defendant's pre-reform award amount.

4) The claim size distribution of claims with a punitive damage component represents a scalar expansion of the claim size distribution for all claims (i. e., If c is the ratio of the average claim size for claims including a punitive damage component to the average claim size for all claims and \$X is the value of the nth percentile (for any real number n, $0<n<100$) of the claim size distribution for all claims then \$cX(c times \$X) is the nth percentile of the claim size distribution for claims including a punitive damage component.)

5) The claim size distribution of multiple defendant claims impacted by joint and several liability represents a scalar expansion of the claim size distribution for all multiple defendant occurrences

We could simulate each probabilistic characteristic of each simulated claim. Instead we choose to only simulate claim size from the inverted truncated Pareto distribution. For each simulated claim, we model each possible combination of values of the other variables and weigh all of the resulting combinations by weights derived from the empirical probability distributions estimated from the closed claim study data. This event tree structure embedded in the simulation reduces the risk of significant bias resulting from a very large claim having an extreme value of one or more of the variables other than claim size.

For each simulated claim, 168 scenarios representing possible combinations of values of the other modeled variables are weighed together to produce the estimated loss before and after each combination of the reforms being analyzed. The variables that are represented by these 168 scenarios are:

1) BI vs. PD

2) Single vs. Multiple Defendant

3) Impacted by Joint and Several Liability vs. not Impacted

4) Percent of award comprised by a punitive damages (the mode of this distribution is 0%)

5) Percent of compensatory damages that are non-economic.

Each possible combination of reform provisions are applied to the simulated claims. For this analysis we model a reform comprised of the following components:

1) Cap on Non-Economic Damage Awards -$250,000 per plaintiff.

2) Cap on Punitive Awards -Greater of $100,000 or 3 x Economic per plaintiff

3) Repeal of Joint & Several Liability - Total

Exhibit 2 shows the impact on one simulated claim.

Finally, we apply policy limits to the simulated claim, both before and after the reform. A reform that limits large losses may have little effect if the policy limits are often exceeded both before and after the application of the reform.

We generated a large number of claims under the 168 scenarios. For each combination we calculated the average indemnity impact on the above reform package at several policy limits. Exhibit 3 summarizes the results of this analysis.

EFFICIENCY OF SIMULATION

Differences in the provisions of the reforms and in characteristics of the population of insurance policies being considered will affect the convergence rate of the simulation. Evaluating the effect on policies with higher limits of liability will often require more iterations since more variation is present further out in the tail of the claim size distribution.

Our early simulations required at least hundred thousand occurrences to produce convergence for the relative impact. Millions of simulations were necessary for severity convergence, requiring over a week on a personal computer.

We improved the efficiency of our simulation using two related techniques, re-weighting and stratified sampling. Re-weighting entailed generating more occurrences of larger size, but giving them proportionally less weight. This is accomplished by modifying the function which assigns a Pareto distribution value to each randomly generated uniform distribution value. A compensating weighting function is applied to avoid the introduction of bias in the resulting claim size distribution.

Stratified sampling involves fixing the number of simulations within intervals. We cycled our generation of uniform random values within 500 equal probability intervals. This ensures adequate coverage of every part of the distribution.

ADDITIONAL AREAS FOR RESEARCH

There are a number of areas that require further research. We must develop methodologies to evaluate additional types of reforms. We need to develop methodologies to estimate the impact of the behavioral effects of reforms. We need to incorporate a consideration of the likelihood and potential impact of repeal or reinterpretation. Certainly, efforts to develop or find new or existing data sources should be pursued.

IX) CONCLUSION

Actuaries are often called upon to evaluate the effect of law changes on expected insurance losses. The imposition, modification and on occasion elimination of automobile No-Fault systems in a number of states; changes in uninsured/under-insured motorist statutes; and mandated changes in Workers' Compensation benefit levels are common examples of such situations. The changes which are now referred to as tort reforms are often less well defined in their scope and impact than the above examples. They also, often, affect all lines of insurance rather than specific lines and coverages. They may often have minimal effect. At times they may only represent the codification of existing case law. At other times their effect may be significant, but only indirectly manifested, through behavioral changes that may or may not have been intended by the drafters and proponents of the legislation. The accurate analysis of tort reforms may be difficult and costly. The limits on accuracy may be significant even when talent and expense are not limiting factors.

However, in many cases waiting may not be an acceptable initial pricing strategy. First, insurers may be required by statute and/or regulation to reflect the effect of reforms immediately or by a specified data. Second, due to the slow development of some liability claims it may take a number of years for the full effect of changes to enter into the data. Third, some changes may have significant effects and the potential error resulting from delaying reflection of the change may be greater than the potential error resulting from analyses based on limited or imperfect data.

There may be political and regulatory pressure to reflect changes, even if their effect is at first questionable. Trade groups for a number of industries as well as Think Tanks, political groups and elected officials have made substantial, perhaps sometimes overstated, claims about the cost savings and other benefits that might result from the reforms that they support. When such reforms are enacted (even if they have been weakened significantly by amendment) elected officials and the public expect significant savings to be realized quickly. Actuaries must evaluate these changes as accurately as possible using the limited information that is available.

We hope that his paper contributes to the continuing evolution of more accurate methods of analyzing the effect of tort reforms and other changes in the legal environment on expected insurance losses.

VIII) EXHIBITS

1 - Truncated Pareto Formulas

2 - Impact of Sample Reform on one simulated claim.

3 - Average Impact of Sample Reform by Policy Limit.

REFERENCES

[1] Beckett, Paul. "Reform Rings Hollow for Firms Worried About Class-Action Suits." Wall Street Journal, 4 April 1997.

[2] Claim Evaluation Project. Los Angeles: Hamilton, Rabinovitz and Alschuler, Inc., 1987.

[3] Claim File Data Analysis: Technical Analysis of Study Results. New York: ISO Data, Inc., 1988.

[4] Claim File Data Analysis: Overview. New York: ISO Data Inc., 1988.

[5] Closed Claim Survey for Commercial General Liability: Survey Results, 1993. New York: ISO Data Inc., 1993.

[6] Closed Claim Survey for Commercial General Liability: Statistical Compilations, 1991. New York: ISO Data Inc., 1991.

[7] Closed Claim Survey for Commercial General Liability: Survey Results, 1991. New York: ISO Data Inc., 1991.

[8] Closed Claim Survey for Commercial General Liability: Statistical Compilations, 1993. New York: ISO Data Inc., 1994.

[9] Closed Claim Survey for Commercial General Liability: Survey Results, 1993. New York: ISO Data Inc., 1994.

[10] Closed Claim Survey for Commercial General Liability: Statistical Compilations, 1995. New York: ISO Data Inc., 1996.

[11] Closed Claim Survey for Commercial General Liability: Survey Results, 1995. New York: ISO Data Inc., 1996.

[12] Fegelman, Andrew. "Another Tort Change Knocked Down Product Liability Provision Ruled Unconstitutional." Chicago Tribune, 18 September 1996.

[13] Fegelman, Andrew. "Judge Overturns Tort Reform on Medical Record Access Plaintiff Privacy Rights Upheld." Chicago Tribune, 28 February 1996.

[14] Fegelman, Andrew and Rick Pearson. "State Cap on Jury Awards Removed Judge Rules Law Unconstitutional." Chicago Tribune, 23 May 1996.

[15] Insurance Services Office, "The Impact of Tort Reforms on Liability Insurance." Commercial Casualty Actuarial Panel Meeting Agenda(ISO Circular AS-CA-97-13), 10 April 1997.

[16] Marquardt, Dick. A Study of the Effect of Tort Reform on Insurance Rates and Availability and Its Impact on the Civil Justice System. Olympia, Washington: Washington Insurance Department, 1991

[17] Property and Casualty Insurance Industry Data and the Case for Tort Reform, Documenting the Cost of Expanding Tort Liability Doctrines Beyond Traditional Common Law Boundaries. Alliance of American Insurers, American Insurance Association, National Association of Independent Insurers, 1986.

[18] Raybill, James F. "The current legislative agenda: Ohio and federal bills of concern to CPAS." The Ohio CPA Journal 54 (June 1995): 3-14.

[19] Reno, Robert. "Reno at Large GOP Betrays Constituents by Helping the Swindlers." Newsday, 28 June 1995.

[20] Starkman, Dean. "Securities Class-Action Suits Seem Immune to Effects of a New Law." Wall StreetJournal, 12 November 1996.

[21] Topping, Robin. "Around the Island Crime and Courts Law and Order." Newsday, 24 May 1997.

[22] "1997 Tort Reform Rate Rollback Information." July 25, 1997, by the Texas Department of Insurance, [on-line]; available from http:// www.tdi.state.tx.us; Internet; accessed August 11,1997.

Truncated Pareto Formulas

Definitions

B,Q	= Pareto parameters
T	= truncation point
P	= probability that an occurrence is less than T
S	= average size of an occurrence less than T
ABT,BBT	= exponential fit parameters (from P,S and T)

Formulas for Truncated Pareto:

(1) Probability Density Function:

$$h(X) = \begin{cases} e^{ABT \cdot X + BBT} & \text{for } 0 \le X \le T \quad \text{Exponential fit} \\ \dfrac{Q(1-P)(T+B)^Q}{(X+B)^{Q+1}} & \text{for } T < X \quad \text{Pareto distribution above T} \end{cases}$$

(2) Cumulative Distribution Function:

$$H(X) = \begin{cases} \dfrac{(e^{ABT \cdot X} - 1)e^{BBT}}{ABT} & \text{for } 0 \le X \le T \\ 1 - (1-P)\left(\dfrac{T+B}{X+B}\right)^Q & \text{for } T < X \end{cases}$$

(3) Average Loss Size when Losses are limited to Policy Limit K
(Limited Average Severity):

$$LAS = E\langle \min(X,K) \rangle = \begin{cases} K - \dfrac{(e^{ABT \cdot K} - 1 - ABT \cdot K)e^{BBT}}{ABT^2} & \text{for } 0 \le K \le T \\ PS + \left(\dfrac{1-P}{Q-1}\right)\left[(B+QT) - (B+K)\left(\dfrac{B+T}{B+K}\right)^Q\right] & \text{for } (T < K) \text{ and } (Q \ne 1) \end{cases}$$

Inversion of Truncated Pareto Formulas

Starting with the cumulative distribution function.

$$H(X)=\begin{cases}\dfrac{\left(e^{ABT\cdot X}-1\right)e^{BBT}}{ABT} & \text{for } 0\le X\le T.\\[2ex] 1-(1-P)\left(\dfrac{T+B}{X+B}\right)^{Q} & \text{for } T<X\end{cases}$$

Solving for X in terms of H gives us:

$$X(H)=\begin{cases}\dfrac{\ln\left(1+ABT\cdot H\cdot e^{-BBT}\right)}{ABT} & \text{for } 0\le H\le P.\\[2ex] (B+T)\left(\dfrac{1-P}{1-H}\right)^{1/Q}-B & \text{for } P<H\le 1.\end{cases}$$

By generating uniform random values for H (from 0 to 1), X(H) gives us simulated indemnity values for our truncated Pareto distribution.

Notation and Parameters

Occurrence Indemnity Size Model - Truncated Pareto Distribution

B	33,947.174	Pareto Scalar
Q	1.300	Pareto Thinness of Tail
P	0.869	Probability of an Occurrence being less than T.
S	2,925.631	Average (indemnity) size of an Occurrence less than T.
T	10,000	Truncation point of Model
ABT	-0.0002797	1st Parameter below truncation point (from P,S,T)
BBT	-8.2591837	2nd Parameter below truncation point (from P,S,T)
	From Prem-Ops Table 2.	

Non-Economic Damages Model: Cycle %NE through empirical quantiles:

0.0%	18.2%	38.6%	56.5%	70.0%	82.7%	93.4%	100.0%

BIwt	0.87	Avg.Weight of BI, in layer above $100,000.
	0.60	Avg.Weight of BI, in layer below $100,000.
Pun_sz	2.0	= AvgSev(occurrences with punitive)/AvgSev(All occurrence)
Pun_a	5.0%	Overall Probability of a BI occurrence having a punitive component.
Mult	0.40	Chance of a claim involving Multiple defendants
JS_sz	1.20	Relative Size of J&S claims. (From Closed claim study: $280k / $231k)
JS_a	15.0%	Overall J&S Prob, given a BI occurrence with multiple defendants.
Xc	250,000	Parameter in Estimates of Total Size of All-defendant award and J&S impact.
Js_Sm	0.60	Impact of (Elimination of) J&S on claims smaller than Xc.
Js_Lg	0.30	Marginal Impact of (Elimination of) J&S on claims larger than Xc.

I. Simulate One Occurrence

1 H = Random Variable underlying simulated indemnity size.
We could generate this from a uniform distribution from 0 to 1.
For our study we used stratified sampling:
generating one from 0 to 0.002, the next from 0.002 to 0.004, etc.
We also used reweighting to generate more large values of H, but giving each less weight
= 0.989988

2 X = Indemnity. Uncensored, Pre-reform.
We invert the CDF representing the Occurrence Size Distribution.
For this study we used a Truncated Pareto Approximation (See Exhibit 1)
X= Ln(1+abt H Exp(-bbt))/abt OR (B+T)[(1-P)/(1-H)]^(1/Q) -B.
= 283,640

3 PunProb= probability that a BI occurrence of size X involves punitive damages.
= p(pun|x)
= p(pun) * [f(x|pun) / f(x)]
Assuming that the pdf of f(x|pun) represents a scalar expansion of f(x)
(that the distribution of punitives is the same, except for a constant multiplier):
f(x|pun) = f(x / Pun_sz) / Pun_sz

= (Pun_a/Pun_sz) * [f(x/Pun_sz) / f(x)]
= (Pun_a/Pun_sz) * [Q(1-P)(T+B)Q([X/Pun_sz]+B)-(Q+1) / Q(1-P)(T+B)Q(X+B)-(Q+1)]
= (Pun_a/Pun_sz) * [([X/Pun_sz]+B)-(Q+1) / (X+B)-(Q+1)]
= (Pun_a/Pun_sz) * [(X+B) / ([X/Pun_sz]+B)] Q+1
= (0.05 / 2.00) * 3.90
= 0.09747

4 JSProb= Probability that an occurrence of size X is impacted by Joint & Several Liability,
given that it has multiple defendants.
Using the same assumptions as in PunProb.
= p(JS) * [f(x|JS) / f(x)]
= (JS_a/JS_sz) * [f(x/JS_sz) / f(x)]
= (JS_a/JS_sz) * [(X+B) / ([X/JS_sz] +B)]$^{Q-1}$
= (0.15 / 1.20) * 1.45
= 0.18109

5 Simulate various scenarios underlying this occurrence.

Single Defendant, Multiple Defendants w/o Joint Liability, Multiple Defendants with J&S.

Each of the those 3 are broken into 7 possiblities:
If BI, assume six possibilities, with varying punitive components:
One with No Punitive (Punitive = 0% of Award)
Five with Varying Punitives of 10%, 30%, 50%, 70% or 90%.

If PD, assume only one possibililty, punitive of 0%.

Each of the above 21 are then calculated with 8 values of NE%.
If BI, percentage of non-punitive (compensatory) damages given by non-economic damages

For PD, we currently assume the entire damages are economic, so NE% has no effect.

The Following 8 values of NE% are used with equal weight:

0.0%	18.2%	38.6%	56.5%	70.0%	82.7%	93.4%	100.0%

This makes 168 (=3x7x8) distinct scenarios:

Here we display the calculations for four (of the 168) scenarios:

A) BI, Single Defendant, No Punitive
If the insured is the only defendant, then Joint and Several cannot apply.

B) BI, Single Defendant, 90% Punitive
As "A" above, but the **same size** loss now consists mostly of punitive.

C) BI, Multiple Defendants, but without Joint & Several, 50% Punitive
Now the insured's loss is part of a larger verdict. The verdict is half-punitive.

D) BI, Multiple Defendants, Joint & Several invoked, 50% Punitive
Similar to "C", but part of the insured's loss was from other defendants.
Due to reform (repeal) of the J&S doctrine, this extra amount is now a savings.

Each uses **56.5%** for NE%.

Scenario:	**A**	**B**	**C**	**D**	
Defendants	Single	Single	Multiple	Multiple	
J&S Applies	n/a	n/a	No	Yes	
Punitive Damages	0%	90%	50%	50%	
Weight (BI vs PD)	0.8700	0.8700	0.8700	0.8700	BIwt or 1-BIwt
Weight (of # of Def)	0.6000	0.6000	0.4000	0.4000	Mult or 1-Mult.
Weight (of J&S)	1.0000	1.0000	0.8189	0.1811	1, JS_Prob or 1-JS_Prob.
Weight (of Punitive)	0.9025	0.0195	0.0195	0.0195	=PunProb/5 or 1-PunProb
Weight (of NE%)	0.1250	0.1250	0.1250	0.1250	= 1/ (# of NE quantiles)
6 Scenario Weight**	0.0589	0.0013	0.0007	0.0002	Product of weights

* For cases with punitive. Otherwise Weight = 1-PunProb

** The weights for the twenty-one scenarios with this NE% add up to .125.
The weights for all 168 scenarios add up to 1.000.

7 Verdict Size=	283,640	283,640	542,051	542,051

Total award (verdict or settlement) to plaintiff from ALL defendants.
If Single Defendant = X
If Multi-Defendant = X * 2 (if X< Xc)
Xc*2 + (X-Xc)*1.25 (if X> Xc)

8 Xjs =	283,640	283,640	283,640	160,092

Loss: J&S reduced — If J&S impacted this occurrence, how large would it have been without it?
If Multi-Defendant = X * JS_Sm (if X < Xc)
Xc*JS_Sm + (X-Xc)* JS_Lg (if X > Xc)

9 PunOld (after J&S)=	-	255,276	141,820	80,046

Punitive calculated as our scenario %, of the post-Joint & Several loss.
= Xjs * Pun%

10 NeOld(after J&S)=	160,257	16,026	80,128	45,226

Non-Economic = Xjs * (1-Pun%) * NE%

11 EcoOld(after J&S)=	123,384	12,338	61,692	34,820

Economic = Xjs - NeOld- PunOld

Scenario:	A	B	C	D	
12 Cap on Non-Eco					
0*Economic					Variable Cap
Min. Cap		250,000			Minimum Cap
Max Cap					Maximum Cap
Net Cap per plaintiff	250,000	250,000	250,000	250,000	
Cap by defendant	250,000	250,000	130,818	130,818	= Net Cap * (X/Verdict)
13 Cap on Punitive					
3*Economic	370,151	37,015	353,688	353,688	Variable Cap
Min Cap		100,000			Minimum Cap
Max Cap					Maximum Cap
Net Cap per plaintiff	370,151	100,000	353,688	353,688	
Cap by defendant	370,151	100,000	185,075	185,075	Entire punitive cap
14 Capped Punitive	-	100,000	141,820	80,046	
15 Capped Non-Eco	160,257	16,026	80,128	45,226	
16 Xref =	283,640	128,364	283,640	160,092	

Post Reform Loss: =EcoOld + Capped Punitive + Capped Non-Eco

17 Calculate the **Limited** Loss, and calculate the average weighted across all 168 scenarios
(We cannot just calculate the Limited Average, we need the average of the Limited)

We calculate these Average Limited Losses for Five Sample Policy Limits:
$100,00 500,000 1,000,000 10,000,000 unlimited

We should calculate these values reflecting various combinations of reforms.
Elimination of Joint&Several.
J&S + Cap on Non-Economic Damages
All three (J&S, Non-Eco and Punitive) reforms.

	A	B	C	D	Weighted Avg of: Displayed 21	All 168
Post-Reform						
Min(Xref,100k)=	100,000	100,000	100,000	100,000	99,893	99,608
Min(Xref,500k)=	283,640	128,364	283,640	160,092	261,946	249,422
Min(Xref, 1 M)=	283,640	128,364	283,640	160,092	261,946	249,422
Pre-Reform						
Min(X , 100k)=	100,000	100,000	100,000	100,000	100,000	100,000
Min(X , 500k)=	283,640	283,640	283,640	283,640	283,640	283,640
Min(X , 1 M)=	283,640	283,640	283,640	283,640	283,640	283,640

Note that all 168 scenarios will be identical under pre-reform conditions.

II. Repeat simulation until results converge.
For each of limited loss in step one, calculate the Mean value.

100,000 random simulations is sufficient to generate stable % changes at each relevant limit.
But more are needed for Limited Average Severities stable in absolute dollars.

We used reweighting and stratified sampling to improve our efficiency.

The Analysis of the Effect of Tort Reform Legislation on Expected Liability Insurance Losses

Impact on One Simulated Occurrence

H Pareto	X(H) Old Indem.	Which Non-Eco	NE/NP	P(Pun>0 | x) punprob	J&s Prob	Total Losses before caps Limited to: $100,000	$1,000,000	Unlimited
0.9900	**283,640**	4	56.50%	0.0974694	0.181091	**100,000**	**283,640**	**283,640**
			BIProb	87.0%				

	Entire Verdict	Ratio of Several to Joint		Total Losses after J&S Reform Limited to: $100,000	$1,000,000	Unlimited
If Single Defendant	283,640	if Joint applies:	0.5644	**100,000**	**274,691**	**274,691**
If Multi Defendants	542,051	if not:	1.0000			

	Punitive	Scenario Weight	Losses before Cap (but after J&S): Economic	Non-Eco	Punitive	Losses after Caps: Non-Eco	Punitive	Total Losses after caps Limited to: $100,000	$1,000,000	Unlimited
BI	0.0%	0.05889	123,384	160,257	-	160,257	-	100,000	283,640	283,640
Single	10.0%	0.00127	111,045	144,231	28,364	144,231	28,364	100,000	283,640	283,640
Defend	30.0%	0.00127	86,369	112,180	85,092	112,180	85,092	100,000	283,640	283,640
	50.0%	0.00127	61,692	80,128	141,820	80,128	141,820	100,000	283,640	283,640
	70.0%	0.00127	37,015	48,077	198,548	48,077	111,045	100,000	196,137	196,137
	90.0%	0.00127	12,338	16,026	255,276	16,026	100,000	100,000	128,364	128,364
BI	0.0%	0.03215	123,384	160,257	-	130,818	-	100,000	254,202	254,202
Multi-	10.0%	0.00069	111,045	144,231	28,364	130,818	28,364	100,000	270,228	270,228
Defend	30.0%	0.00069	86,369	112,180	85,092	112,180	85,092	100,000	283,640	283,640
No J&S	50.0%	0.00069	61,692	80,128	141,820	80,128	141,820	100,000	283,640	283,640
	70.0%	0.00069	37,015	48,077	198,548	48,077	111,045	100,000	196,137	196,137
	90.0%	0.00069	12,338	16,026	255,276	16,026	52,327	80,691	80,691	80,691
BI	0.0%	0.00711	**69,640**	**90,452**	**-**	73,836	-	100,000	143,476	143,476
Multi-	10.0%	0.00015	**62,676**	**81,407**	**16,009**	73,836	16,009	100,000	152,522	152,522
Defend	30.0%	0.00015	**48,748**	**63,316**	**48,028**	63,316	48,028	100,000	160,092	160,092
	50.0%	0.00015	**34,820**	**45,226**	**80,046**	45,226	80,046	100,000	160,092	160,092
J&S	70.0%	0.00015	**20,892**	**27,136**	**112,064**	27,136	62,676	100,000	110,704	110,704
	90.0%	0.00015	**48,748**	**63,316**	**48,028**	63,316	48,028	100,000	160,092	160,092
PD										
Single	0.0%	0.00975	283,640					100,000	283,640	283,640
No J&S	0.0%	0.00532	283,640					100,000	283,640	283,640
J&S	0.0%	0.00118	**160,092**					100,000	160,092	160,092
Weighted Total		0.12500	**16,831**	**16,064**	**1,441**	14,989	923	12,487	32,743	32,743
(Normalized)			**134,652**	**128,511**	**11,529**	119,910	7,384	**99,893**	**261,946**	**261,946**

Numbers in shaded regions include reduction for impact of J&S.

Exhibit 3

The Analysis of the Effect of Tort Reform Legislation on Expected Liability Insurance Losses
Overall Effect on Average Severity

Unlimited Indemnity*, by Component

Component	Reforms: None	J&S Only	J&S and Non-Eco	All
Economic	$ 12,037	$ 11,591	$ 11,591	$ 11,591
Non-Economic	9,956	9,562	4,901	4,901
Punitive	885	840	840	487
Total	22,878	21,993	17,331	16,978

Limited Average Indemnity Severity*

Policy Limit	Reforms: None	J&S Only	J&S and Non-Eco	All
$100,000	$ 9,304	$ 9,157	$ 9,131	$ 9,126
$500,000	13,967	13,656	12,958	12,859
$1,000,000	15,599	15,199	13,889	13,740
$10,000,000	19,281	18,645	16,553	16,246
Unlimited	22,878	21,993	17,331	16,978

% Change in Limited Average Severity*

Policy Limit	Reforms: None	J&S Only	J&S and Non-Eco	All
$100,000	n/a	-1.6%	-1.9%	-1.9%
$500,000	n/a	-2.2%	-7.2%	-7.9%
$1,000,000	n/a	-2.6%	-11.0%	-11.9%
$10,000,000	n/a	-3.3%	-14.1%	-15.7%
Unlimited	n/a	-3.9%	-24.2%	-25.8%

* Result of 160,000 simulated Premises and Operations occurrences.

** Simulated Reforms:
Complete abolishment of Joint & Several Liability.
Unconditional cap of $250,000 on Non-Economic awards.
Cap on Punitive awards of greater of $100,000 or 3xeconomic.

The Concentration Charge: Reflecting Catastrophe Exposure Accumulation in Rates

by Donald F. Mango, FCAS

The Concentration Charge:
Reflecting Catastrophe Exposure Accumulation in Rates

Donald Mango, F.C.A.S.
Crum & Forster Insurance

Abstract

Diversification of exposure concentration means geographical balancing amongst capacity providers -- insurers, reinsurers, or capital market participants. But how to diversify those exposures is still unsettled. Efforts to this point have focused on balancing the exposures which have already been written by insurers -- via catastrophe reinsurance (regular or securitized), several proposed catastrophe indices, even direct exposure exchanges. This paper proposes an alternative approach: exposure balancing at the point of sale using an *insurance pricing structure* which reflects the insurer's exposure level or "*portfolio state*" -- what can be called *portfolio state dependent pricing.* Instead of one set of filed loss costs and loss cost multipliers, insurers would quote a manual rate which included a surcharge which reflects their exposure level in the area where the potential insured is located. If all carriers were required to quote on a similar basis, had similar loss costs and multipliers, a potential insured's desire to be charged the lowest premium would lead them to choose the carrier who was *least exposed in their area.*

Biography

Mr. Mango is with Zurich Centre ReSource in New York City. Prior to that he was with Crum & Forster Insurance in Morristown, New Jersey, where he was responsible for Catastrophe Management, Ceded Reinsurance, and Umbrella pricing and reserving. He holds a B.S. degree in Mechanical Engineering from Rice University.

The Concentration Charge: Reflecting Catastrophe Exposure Accumulation in Rates[1]

Donald Mango, F.C.A.S.
Crum & Forster Insurance

Section 1: Introduction

This paper will present a method for reflecting exposure concentration in property catastrophe rates via a "*concentration charge*" -- an additional charge on top of the manual rate which varies based on the insurer's exposure level in the area where the potential insured is located.

On first glance one might well ask why have a concentration charge? In a perfectly functioning economy, with plentiful reinsurance and capital market capacity, insurers would be able to diversify away exposure concentration problems. Since the market does not reward diversifiable risks, it would appear a "charge" or return for exposure concentration risks could be an arbitrage opportunity. Insurers would collect the additional money for their concentration problems, then diversify those problems away, presumably for less cost than they collected in concentration charges. Competitive markets would not allow such an arbitrage engine to exist for long. So why continue this paper?

Because the situation is not as simple as that. Diversification of exposure concentration means geographical balancing amongst capacity providers -- insurers, reinsurers, or capital market participants. But how to diversify those exposures is still unsettled. Efforts to this point have focused on balancing the exposures which have already been written by insurers -- via catastrophe reinsurance (regular or securitized), several proposed catastrophe indices, even direct exposure exchanges.

This paper proposes an alternative approach: exposure balancing at the point of sale using an *insurance pricing structure* which reflects the insurer's exposure level or "*portfolio state*" -- what can be called *portfolio state dependent pricing.* Instead of one set of filed loss costs and loss cost multipliers, insurers would quote a manual rate which includes a surcharge reflecting their exposure level in the area where the potential insured is located. If all carriers were required to quote on a similar basis, had similar loss costs and multipliers, a potential insured's desire to be charged the lowest premium[2] would lead them to choose the carrier who was *least exposed in their area.*

[1] *I would like to thank Gary Blumsohn, Matt Mosher, Clive Keatinge and Paul Kneuer for their (in)voluntary efforts providing needed peer review and feedback. I would also like to thank the anonymous reviewers on the CAS Ratemaking Committee for their helpful comments.*

[2] Ignoring for discussion purposes issues such as insurer security levels, services and/or other coverages provided, personal relationships,....

This is an important distinction: the concentration charge proposed here is not a reward for bearing a risk which can be diversified away, it is a means to let the market forces at the point of sale do the diversifying.

This approach is a departure from the current ratemaking paradigm, and significant issues stand in the way of implementation. There is no place in the current filed loss cost/LCM paradigm for PSD pricing. Adoption would require fundamental changes to the concepts underlying insurance pricing. PSD pricing is also computationally intensive and complex. Personal lines carriers with hundreds of thousands or millions of policyholders may feel the additional costs outweigh any marginal benefits. However, as will be discussed below, these are not insurmountable problems.

Perhaps the biggest concern though is unfair discrimination. Under PSD pricing potential insureds could be quoted different rates based on the month, week, or day they come in. Such apparently arbitrary pricing does not seem appropriate for an economic necessity such as insurance.

However, PSD pricing need not appear arbitrary. The public could be made aware of the concentration charge's intended purpose. It could be broken out and quoted separately from the "regular" premium. Policyholders would have a strong incentive to shop around and get several quotes. They may even feel empowered rather than powerless in tight insurance markets such as Florida. They become an active participant in improving the insurance market rather than a passive recipient of what may seem arbitrary capacity decisions by carriers.

The remainder of this paper is organized as follows. Section 2 develops the needed surplus distribution, derived from the modeled loss distribution and available funds for payment of catastrophe losses. Section 3 introduces the concept of surplus tiers, which are ranges of percent of total policyholders surplus. In Section 4 we look at the costs of exposure accumulation and the concentration charge, an annual "payback" charge which takes the form of an expense load to be applied to the new account's loss cost. In Section 5 we combine all these concepts into an approach for pricing new business. We conclude in Section 6 with a discussion of PSD pricing in relation to the provisions in the CAS "Statement of Principles Regarding Property and Casualty Insurance Ratemaking" [1].

Section 2: Needed Surplus Distribution

What is the relationship between surplus and the payment of catastrophe losses? The company has some collected funds on hand with which to pay catastrophe losses. It may be a planned or budgeted annual cat loss load, or the sum of collected loss cost provisions for catastrophe coverage (e.g. the wind load portion of the Basic Group 2 loss cost for Commercial Multi Peril). These funds will be referred to as the *catastrophe fund* (CF).

For events whose losses are less than or equal to the CF, no surplus is needed. However, surplus will be needed to cover losses in excess of the CF. This *needed surplus* is equivalent to the catastrophe loss net of a deductible equal to the CF amount. Each modeled event loss will require a different surplus amount. This means given a CF amount and a modeled loss distribution, one can develop a *needed surplus distribution*.

Using a modeled occurrence size of loss distribution[3] with event identifiers **i**, the event probabilities p_i and modeled loss amounts L_i[4], the needed surplus distribution by event NS_i is:

$$NS_i = \text{Max}\,[\,L_i - CF,\ 0\,] \qquad [2.1]$$

where L_i *= modeled loss for event i*

It will prove more convenient going forward to express NS_i as a percentage of PHS:

$$NS_i = \frac{\text{Max}\,[\,L_i - CF,\ 0\,]}{PHS} \qquad [2.2]$$

The needed surplus distribution tells us what percentage of the available surplus will be depleted by each modeled event. But different amounts of depletion can have qualitatively different impacts upon a company's ability to continue functioning post-event. To better discuss the different amounts of depletion we introduce the concept of *surplus tiers*.

Section 3: Surplus Tiers

An insurer of reasonable size should be able to withstand an event-based depletion of say **-10%**[5] of available surplus without significant disturbance to ongoing operations. This amount might be considered the limit of "acceptable variation": there will be no regulatory intervention, ratings downgrades, or loss of market position or viability.

Between **-10%** and **-20%**, the company may begin to attract the attention of regulators and rating agencies. Between **-20%** and **-30%**, regulatory bodies may step in to oversee operations and protect the interests of other policyholders; guaranty funds may be put on alert; ratings downgrades are almost certain, and with them comes possibly irreparable damage to market position and viability. Between **-30%** and **-50%**, the company would almost certainly fall under direct regulatory control. Beyond **-50%**, the company is in all likelihood headed for major reorganization, runoff, or even insolvency.

3 Annual aggregate loss distributions could also be used.

4 See Appendix A for a discussion of possible modifications to modeled losses which a company may want to consider before calculating the needed surplus distribution.

5 I have selected these breakpoints arbitrarily for discussion purposes.

These highlighted "ranges" demarcate what I call *surplus tiers*:

> Surplus tiers are ranges of surplus bounded by selected percentiles within which the ongoing operating status of the company is considered "constant."
>
> Movement from one tier to the next reflects a qualitative change in the ongoing operating status of the company.

We will be using this sample set of surplus tiers throughout the remainder of the paper:

Table 1 - Sample Surplus Tiers

Surplus Tier	Percentile Range	Impact
1	0-10%	None - Acceptable Variation
2	10-20%	Regulatory and Rating Watch
3	20-30%	Regulatory Oversight, Ratings Downgrade
4	30-50%	Regulatory Intervention
5	50-100%	Reorganization, Runoff or Insolvency

(*Note the convention that "higher" numbered tiers of surplus represent deeper shocks and more severe impairment to the company.*)

This means that each modeled event has both a needed surplus amount $\mathbf{NS}_i$ and a *corresponding surplus tier*. Events can even be referred to by their tiers -- a very severe event might be "Tier 4." These tier references are both company specific and portfolio state dependent. They will change as the exposure levels, collected premiums, and surplus of the company change.

Now that we have a framework for relating exposure levels and surplus via tiers, we turn our attention to the development of an appropriate concentration charge.

Section 4: The Concentration Charge

Should the concentration charge just be another form of risk load? If the answer is yes, then an application of one of the well known risk load methods -- from Kreps [2] or Meyers [3], for example -- would suffice. Both methods would give larger charges for adding a risk to a more exposed area, which makes intuitive sense.

However, these methods would generate a concentration charge for the addition of a risk to *any* geographic area, even those with Tier 1 exposure. This expands the concentration charge's definition beyond its intended focus: reflection of exposure accumulation beyond critical amounts. Also, these methods while being theoretically sound may not be acceptable to a regulator as support for additional surcharges. The issue of additional (marginal) surplus and an appropriate return thereon have yet to be satisfactorily resolved in the public forum.

This may be a purely semantic distinction, but I intended for the concentration charge to serve as more of an *economic indicator* than as a reward for bearing risk. I had hoped this approach could be filed and used to develop portfolio state dependent rates for catastrophe coverage. I believe this requires a concentration charge which is economically sound yet understandable and acceptable to both regulators and the public.

In that light, I propose a formula for the concentration charge which focuses on the reparation of impairment by requiring depleted surplus to be *replenished* in order for the company to continue operating as a viable going concern. The time frame for replenishment would depend on the tier: higher tiers would need to be replenished more quickly than lower. Tier 2 surplus need not be replaced within one year, but maybe over five years. Depletion to the Tier 4 level may mean regulatory supervision, so a two year turnaround may be mandated just to restore viability.

Each tier will be assigned a replenishment period. Since each event has a tier associated with it, it too will have a replenishment period. That means an incremental dollar of loss to that event exposes a dollar of surplus which must be replenished within the appropriate time period. To accomplish this replenishment, that loss dollar would need to carry an accompanying annual surplus replenishment load (as a percent of that dollar of loss) equal to the inverse of the replenishment period (in years). This expense load shall be referred to as the *concentration charge* (CC):

Table 2 - Sample Surplus Tiers and Concentration Charges

Surplus Tier	Percentile Range	Replenishment Period	Concentration Charge (CC)
1	0-10%	-	-
2	10-20%	5 Years	1/5 = 20%
3	20-30%	3 Years	1/3 = 33%
4	30-50%	2 Years	1/2 = 50%
5	50-100%	1 Year	1/1 = 100%

Summary

Before proceeding it may be helpful to review the new components to the approach:

- The needed surplus distribution by modeled event, expressed as a percentage of total surplus, associates a surplus tier with each event.
- Surplus tiers are percentile ranges of surplus within which the company's operational status is considered constant, but between which material changes in operational status are assumed to occur.
- Each tier has a different replenishment period associated with it, based on the severity of the predicament.
- The inverse of the replenishment period yields a surplus replenishment load called the concentration charge which is applied to any additional loss dollars added to that event by a new account.

These new components will now be combined into a pricing approach for a new account.

Section 5: Pricing A New Account

The first step in pricing a new account is creation of its own occurrence size-of-loss distribution, consisting of loss amounts $\mathbf{n}_i$ by event. The concentration charge dollars by event ($\mathbf{CC\$}_i$) for the new account then equals

$$CC\$_i = [\ CC_i * n_i\ ] \qquad [5.1]$$

where CC_i *= concentration charge for event i*

These dollars represent the replenishment costs of the additional loss to each event. For Tier 1 events, this charge is 0. For Tier 5 events, it is according to our example equal to an additional 100% of modeled loss for the new account -- a 100% surcharge to pay for exposure concentration.

The expected concentration charge dollars over all events (**CC$**) equals

$$CC\$ = \sum_i [\ CC\$_i * p_i\ ] \qquad [5.2]$$

where $\sum_i$ *= sum over all events*

The **concentration charge** (CC) -- the expense provision to be applied to the catastrophe loss cost -- is calculated as follows:

$$CC = CC\$ \ / \ \sum_i [\ n_i * p_i\ ] \qquad [5.3]$$

where $\sum_i [\ n_i * p_i\ ]$ *= modeled expected loss for new account*

This assumes that the ratio of

$$\frac{\text{expected concentration charge dollars}}{\text{modeled expected loss}}$$

is a suitable proxy for the required concentration charge to be applied to the filed *catastrophe*[6] loss cost.

Example: Homeowners
One might deem this detailed approach to be "continuous" PSD pricing. Computational and regulatory restrictions for a line like homeowners might call for more of a "discrete" or approximate method. An example would be *territorial loss cost multipliers*.

Begin by calculating the concentration charge for a sample policy added to each of the company's territories (could be bureau defined, county, zipcode,...). This concentration charge would be a loss-based expense to be included with the company's other expenses in developing loss cost multipliers. For example, say a company had two territories, Y and Z. Territory Z is more heavily exposed than Territory Y. Their expense loads and loss cost multipliers would be:

Table 3 - Example of Homeowners Territorial LCM's

	Expense Item	Terr. Y	Terr. Z
(1)	Premium-Based Expense Load	31%	31%
(2)	Concentration Charge	15%	30%
(3)	Loss Cost Multiplier = [1.00 + (2)] / [1.00 - (1)]	1.667	1.884

(*Note: the formula in (3) assumes the concentration charge is included as part of premium for determination of taxes, commission and other variable expense provisions. It could easily be modified to accommodate different treatments -- e.g. surcharge.*)

Territorial LCM's do represent a compromise position between PSI and PSD pricing. They would still be on file with the insurance department. An insured would be quoted the same manual rate independent of portfolio state for the period the LCM's are in effect. However, they do represent a step forward in their explicit recognition in the loss cost multiplier of the cost of exposure accumulation.

[6] Clearly the introduction of separate catastrophe loss costs and multipliers represents yet another regulatory hurdle to be overcome before this approach can be implemented. However, many cat-prone states are pushing companies to provide a cat/non-cat breakout of their "indivisible" package premiums (HO or CMP). See Walters and Morin [4] for more on separate cat rates.

Example: Large Commercial Account
Companies may wish to use the "continuous" approach when pricing a larger commercial account. The addition of a large account will likely have a substantial impact on the portfolio state, so it may be worth the extra effort to get the more exact answer from the continuous method over the approximate territorial method. Also, the locations may be so geographically dispersed that the territorial LCM method cannot be effectively applied.

Table 3 shows highlights of an example[7] showing the difference in concentration charge for adding a new account to two portfolios, LOW and HIGH. To reflect the differences in exposures, I set LOW's modeled losses equal to 50% of HIGH's by event.

Table 4 - Example of Adding a New Large Account

	Item	Identifier	LOW	HIGH
(1)	Expected Loss	$\sum_i [n_i * p_i]$	$151.78	$151.78
(2)	Expected Concentration Charge Dollars	CC$	$9.73	$33.38
(3)	Concentration Charge = (2) / (1)	CC	6.41%	21.99%

Holding all else constant, the difference between the LOW and HIGH concentration charges is due to the lower tiers exposure (see Columns (7) and (15) on Table 5).

Section 6: Portfolio State Dependent Pricing and the CAS Ratemaking Principles

Before giving PSD cat pricing further consideration, we might ask how it compares to the recommendations of the CAS "Statement of Principles Regarding Property and Casualty Insurance Ratemaking" [3].

It is important that proper actuarial procedures be employed to derive rates that protect the insurance system's financial soundness and promote equity and availability for insurance consumers.

PSD pricing produces rates which directly reflect threats to a company's *financial soundness* due to exposure accumulation. PSD pricing is *equitable* among

7 A full version of the example is included at the end of the paper in Table 5.

policyholders covered under different lines of business and/or different states, the collectibility of whose insurance is threatened by exposure accumulation. In counterpoint to the discriminatory charge against PSD pricing, one could argue that portfolio state *independent* pricing represents an implicit subsidy among property cat policyholders in high exposure areas and policyholders in other states and/or lines of business and/or companies. Excessive exposure accumulation also threatens the *availability* of insurance. If the exposure balancing promise of PSD pricing were fulfilled, it may actually lead to more availability.

Principle 1: A rate is an estimate of the expected value of future costs.
Principle 2: A rate provides for all costs associated with the transfer of risk.
Principle 3: A rate provides for the costs associated with an individual risk transfer.

PSD pricing is based on the view that the cost of adding a new cat policy depends not only on the characteristics of the policyholder (*transfer of risk*) but also on the state of the portfolio at the time it is written (*individual risk transfer*).

Principle 4: A rate is reasonable and not excessive, inadequate, or unfairly discriminatory if it is [based on Principles 1-3].

A PSD pricing process can be as objective and fair as a PSI process if it is systematic, based on sound economic principles, objectively applied, auditable, and not subject to distortion or fraud. It is not by definition unfairly discriminatory, instead reflecting the consumption and availability of a limited resource -- underwriting capacity as represented by surplus.

[It] is desirable to encourage experimentation and innovation in ratemaking.

That is the intent of this paper.

Section 7: Conclusion

The outlined approach provides a connection between

- current portfolio exposure levels,
- modeled losses,
- the resulting exposure of surplus,
- the costs of that surplus exposure, and
- required pricing for a new account based on the current portfolio state.

It reflects exposure accumulation in the rates, but requires a ratemaking paradigm shift to portfolio state dependent pricing. There are unresolved regulatory and social issues of fairness and order dependency which clearly must be addressed for this approach to ever be implemented. Still, it is meant to be a forward-looking paper, providing a conceptual framework for discussion and advancement of the science.

References

[1] Casualty Actuarial Society, "Statement of Principles Regarding Property and Casualty Insurance Ratemaking," as adopted May, 1988.

[2] Kreps, Rodney, "Reinsurer Risk Loads from Marginal Surplus Requirements," *PCAS* LXXVII, 1990, pp. 196-203.

[3] Meyers, Glenn, "Managing the Catastrophe Risk," *Incorporating Risk Factors in Dynamic Financial Analysis*, Casualty Actuarial Society Discussion Paper Program, 1995, pp. 111-150.

[4 Walters, Michael, and Morin, Francois, "Catastrophe Ratemaking Revisited (Use of Computer Models to Estimate Loss Costs)," *CAS Forum*, Winter 1996, pp.347-382.

Appendix A
Possible Adjustments to Modeled Losses and Surplus

The needed surplus distribution should reflect all payments related to a large catastrophe net of all budgeted funds. There are several cost components which a company may want to consider in addition to the modeled loss amounts produced by their catastrophe models:

1. Reinsurance recoveries (including non-recoverables and Catastrophe reinsurance reinstatement premium);
2. Model adjustments -- demand surge, fire following earthquake;
3. Non-voluntary and guaranty fund assessments;
4. Bond losses due to forced liquidation.

(1) <u>Reinsurance Recoveries</u>
Needed surplus will be reduced by the recoveries from reinsurance programs, particularly catastrophe treaties. These recoveries and those from per risk treaties as well as facultative can be built directly into many catastrophe models to give accurate net loss numbers.

However, care should be taken to reflect reasonable non-recoverable provisions. It may not be realistic to expect full recovery in a $50B industry event for example. Also, cat treaty recoveries should be net of any reinstatement premium.

(2) <u>Model Adjustments</u>
Demand surge (the localized inflation of materials and labor after an event) and fire following earthquake are just two examples of adjustments to modeled results which may warrant reflection, depending on a company's conservatism and faith in the modeled results.

(3) <u>Non-voluntary and Guaranty Fund Assessments</u>
Both of these represent costs which will vary with industry event size and company participation. The assessments could be substantial and should not be ignored. Non-voluntary pools in cat-prone states have gone from insurers of last resort to first choice providers for the difficult to insure. Insurers and the public need to know the non-voluntary facilities' exposure levels.

(4) <u>Bond Losses Due To Forced Liquidation</u>
This item differs from the others in that instead of increasing losses it would act to decrease asset value and thus surplus. The P-C insurance industry could flood the capital markets in the aftermath of a large catastrophe in their demand for cash. This create a self-feeding downward pricing spiral, causing material losses to asset value.

TABLE 5: Concentration Charge Example - Large Account

(1) Event i	(2) P(Event) p(i)	(3) HIGH Portfolio Loss H(i) ($000's)	(4) Collected Cat Loss Cost Amt CLC	(5) Needed Surplus By Event NS(i)	(6) Needed Surplus as % of PHS PHS = $500MM	(7) Surplus Tier Exposed	(8) Conc Charge CC(i)	(9) New Account Loss n(i) ($000's)	(10) Conc Charge Dollars CC$(i)
1	0.76%	200,000	10,000	190,000	38.0%	4	50.0%	1,811	905
2	0.75%	175,000	10,000	165,000	33.0%	4	50.0%	3,270	1,635
3	0.40%	150,000	10,000	140,000	28.0%	3	33.3%	2,236	745
4	0.52%	125,000	10,000	115,000	23.0%	3	33.3%	277	92
5	0.32%	100,000	10,000	90,000	18.0%	2	20.0%	2,128	426
6	0.73%	85,000	10,000	75,000	15.0%	2	20.0%	3,268	654
7	0.80%	70,000	10,000	60,000	12.0%	2	20.0%	2,900	580
8	0.65%	55,000	10,000	45,000	9.0%	1	0.0%	2,170	0
9	0.31%	40,000	10,000	30,000	6.0%	1	0.0%	2,447	0
10	0.73%	25,000	10,000	15,000	3.0%	1	0.0%	819	0
11	0.40%	15,000	10,000	5,000	1.0%	1	0.0%	1,186	0
12	0.34%	12,500	10,000	2,500	0.5%	1	0.0%	4,948	0
Else	93.29%	0	10,000	0	0.0%	1	0.0%	0	0
							Expected Value :	151.78	33.38
							Concentration Charge :	21.99%	

(11) LOW Portfolio Loss L(i) ($000's)	(12) Collected Cat Loss Cost Amt CLC	(13) Needed Surplus By Event NS(i)	(14) Needed Surplus as % of PHS PHS = $500MM	(15) Surplus Tier Exposed	(16) Conc Charge CC(i)	(17) New Account Loss n(i) ($000's)	(18) Conc Charge Dollars CC$(i)
100,000	10,000	90,000	18.0%	2	20.0%	1,811	362
87,500	10,000	77,500	15.5%	2	20.0%	3,270	654
75,000	10,000	65,000	13.0%	2	20.0%	2,236	447
62,500	10,000	52,500	10.5%	2	20.0%	277	55
50,000	10,000	40,000	8.0%	1	0.0%	2,128	0
42,500	10,000	32,500	6.5%	1	0.0%	3,268	0
35,000	10,000	25,000	5.0%	1	0.0%	2,900	0
27,500	10,000	17,500	3.5%	1	0.0%	2,170	0
20,000	10,000	10,000	2.0%	1	0.0%	2,447	0
12,500	10,000	2,500	0.5%	1	0.0%	819	0
7,500	10,000	0	0.0%	1	0.0%	1,186	0
6,250	10,000	0	0.0%	1	0.0%	4,948	0
0	10,000	0	0.0%	1	0.0%	0	0
					Expected Value :	151.78	9.73
					Concentration Charge :	6.41%	

Notes for Table 5
Concentration Charge Example - Large Account

Column (1) = the event identifier **i**
Column (2) = the event probability $\mathbf{p}_i$
Column (3) = the current losses for the HIGH portfolio $\mathbf{H}_i$
Column (4) = the collected catastrophe loss cost amount **CF** of $10MM
Column (5) = the needed surplus by event $\mathbf{NS}_i$ for the HIGH portfolio
= the maximum of [(3) - (4)] and 0
Column (6) = the needed surplus by event as a % of PHS (= $500MM).
Column (7) = HIGH surplus tier from Table 2
Column (8) = HIGH concentration charge $\mathbf{CC}_i$ from Table 2
Column (9) = the New account loss $\mathbf{n}_i$.
Column (10) = the HIGH concentration charge dollars $\mathbf{CC\$}_i$ by event
= [(8) * (9)]
Column (11) = the current losses for the LOW portfolio $\mathbf{L}_i$
Column (12) = Column (4)
Column (13) = the needed surplus by event $\mathbf{NS}_i$ for the LOW portfolio
= the maximum of [(11) - (12)] and 0
Column (14) = the needed surplus by event as a % of PHS (= $500MM).
Column (15) = LOW surplus tier from Table 2
Column (16) = LOW concentration charge $\mathbf{CC}_i$ from Table 2
Column (17) = Column (9)
Column (18) = the LOW concentration charge dollars $\mathbf{CC\$}_i$ by event
= [(16) * (17)]

Expected losses for the New account = $\sum_i$ [(2) * (9)]

Expected **CC$** for the HIGH portfolio = $\sum_i$ [(2) * (10)]

HIGH Concentration Charge **CC** = 33.38 / 151.78 = **<u>21.99%</u>**

Expected **CC$** for the LOW portfolio = $\sum_i$ [(2) * (18)]

LOW Concentration Charge **CC** = 9.73 / 151.78 = **<u>6.41%</u>**

A Frequency Based Model for Excess Wind in Property Ratemaking

by Tim McCarthy

ABSTRACT

In some geographic areas the most significant cause of variation in total dollar losses are fortuitous, non-hurricane storms. Many of the models developed to address the issue of such excess wind losses use dollar loss data only. The traditional models may muddy the distinction between large loss procedures and excess wind models, particularly in territorial analysis. Additionally, as new models are developed which address the hurricane-type risks only, overlap between the hurricane and non-hurricane losses in the traditional procedure degrades the utility of the historical database. A frequency based model for excess wind is proposed. A frequency based model has the benefit of both providing an appropriate load for non-hurricane excess wind, and making the company's internal property data more suitable for trend analysis.

A Frequency Based Model for Excess Wind in Property Ratemaking

OVERVIEW

Increasingly, property coverages are having a portion of their catastrophic losses estimated through the use of loss simulation procedures. These modeling procedures provide the long term expected losses for major catastrophic events, like hurricanes. However, they generally make no provision for smaller wind catastrophes which can represent a more significant component of a line's annual expected catastrophic losses on an ongoing basis.

As the hurricane models become more widely accepted, a data gap can exist between the historical excess wind model, which generally considered non-hurricane events along with hurricane losses, and the hurricane only loss procedure. This paper provides a procedure to develop a catastrophe or excess wind provision for non-hurricane losses. It develops a catastrophe load based upon the non-hurricane wind loss frequency. The model as developed enables data in a property book to be used for loss trend analysis.

CURRENT PROCEDURES

There are currently a number of procedures used. Most applications are variants of a procedure described by Homan [1]. He describes a procedure which ratios wind losses to total losses excluding wind. He takes historic losses over a long period (27 years) and determines the median ratio of wind to non-wind losses over the period. If a year's wind to non-wind losses are 150% or greater than the median ratio, then the excess wind ratio for the particular year is calculated as the difference between the year's excess wind ratio and the median. The excess wind ratios are totaled and divided by the number of years (27) to produce an average excess wind factor. This average excess wind factor is used to develop the excess wind loading for the year's under review.

Many excess wind procedures are variations on Homan's procedure. Chernick [2] describes a procedure where catastrophe events are identified in the database, and a catastrophe loading is developed with the defined catastrophe losses. Fitzgerald [3] provides an example where the total losses for each calendar year are ratioed to premium.

Problems with the Current Procedures

There are a number of problems with the current procedures. Among the problems are:

1. Hurricane Losses Included in the Data
2. Mix of Different Policy Forms
3. Historical Premium Adequacy
4. Changing Definitions of Historical Catastrophes
5. Geographic Distribution Changes

6. Application to Territorial Analysis

7. Applicability of the Procedure to New Products

Hurricane Losses Included in the Data

The excess wind losses using the traditional 30 year catastrophe period include hurricane (major catastrophic wind) losses which are increasingly accounted for in rate development with modeled hurricane losses. A company, with an exposure base which is susceptible to both frequent wind / hail storms and hurricane losses, may have lost some of the value of an excess wind database if it is unable to separate hurricanes from the remainder of wind losses. While such segregation may be possible for most recent years, frequently the detail from older years no longer is available. Fitzgerald [3] notes that the ISO historical database lacks information for removing hurricanes from older years.

Mix of Different Policy Forms

Coverage changes occur over time, and the applicability of the traditional excess loss procedure to older years is unknown. For example, in Homeowners many companies had a different distributional mix of Actual Cash Value (ACV) policies and Replacement Cost Coverage (RCC) policies in older years than exist during the experience period under review. Do RCC policies produce proportionally larger or smaller losses than ACV policies, given the fundamental coverage differences?

Historical Premium Adequacy

The ISO excess procedure for Extended Coverage ratios losses to premiums. When excess loss ratios are used, problems can exist with the historical premium base. How does the adequacy of the historical premium base compare with current adequacy? That is, does a particular year appear to have excess losses solely due to the inadequacy of the premiums? Even if the historical premiums were adequate, if companies have been reducing expenses over time (including policyholder dividends), the older years' premiums are excessive at today's levels.

The Changing Definitions of Historical Catastrophes

In the procedure described by Chernick [2], catastrophes are described in the database. How are such catastrophes defined? If Property Claims Service (PCS) defined catastrophes are used, then the actuary needs to be sensitive to the long term definition changes of catastrophes. Prior to the 1980's an event was defined as a catastrophe if it produced over $1 million in insurance industry losses. Until recently a $5 million industrywide loss would be defined as a PCS catastrophe. Now, the storm must generate $25 million in losses to be defined as a catastrophe. A number of issues are raised by the use of such a standard.

1. **How does a company's distribution of risks compare to the industry's?** If it has a lesser concentration of risks than the industry, then the industrywide catastrophe may not have produced many losses for the company.

Contrariwise, a company with a much greater concentration of risks in a particular area may experience significant losses to its book, yet the storm may not qualify as an industry catastrophe.

2. **How well does a national catastrophe standard translate to state pricing?** This is a problem which is akin to the geographic issue raised above. The PCS catastrophe standard is a countrywide standard. A state on the periphery of the system generating an industry catastrophe may experience few losses. Similarly, a storm which generates relatively large losses for a particular state may not surpass the threshold for it to be defined as a countrywide industry loss.

3. **How does one redefine older catastrophes at the new total dollar level?** That is, under the PCS definition in 1993, a storm would have needed to generate losses of $5 million to qualify as a catastrophe. In 1997 the break point is $25 million. What should the level of losses have been in the 1993 storm to still qualify as a catastrophe? $25 million? Some interpolated dollar amount between the time of the last definition and the most recent definition?

Geographic Distribution Changes

The traditional method does not account for geographic distributional shifts which occur over time. Fitzgerald [3] notes that there has been a population shift to areas impacted by hurricanes over the last 30 years. Have shifts occurred to or away from areas impacted by wind, hail, and tornadoes? If so, then the historical excess loss model will not adequately reflect the prospective catastrophe risk being priced.

Application to Territorial Analysis

The traditional method advanced by Homan [1] performs territorial analysis by assuming that the excess catastrophes are distributed evenly across all territories. He does state that territorial catastrophe factors can be developed, but the specifics of such a procedure are not outlined in detail. Thus, the historical procedure does not allow for area catastrophic losses to be recognized in territorial analysis.

Applicability of the Procedure to New Products

The current procedures require the availability of many years of data since the variance is a function of a series of full years' losses. When a new product is introduced if its geographic spread or susceptibility to wind losses are different than other product lines, the applicability of the current procedures to the new product may be difficult to establish.

RECOMMENDED ALTERNATIVE

The proposed alternative is to develop a catastrophe procedure based on the wind claim frequency of particular dates of loss. Why use a frequency based model versus total dollars of loss?

While total dollars of loss produce the variation in the experience of any insurer, it is generally the variation in the underlying number of claims which generates the variation in the total dollars of loss. Catastrophe procedures, which rely upon the excess loss dollars to develop a catastrophe loading, are utilizing a surrogate for the variation in claim counts. By placing reliance upon the frequency, the surrogate is being replaced by a more accurate measure of the source of variation. If a frequency model more accurately accounts for catastrophic variation, then the accuracy of the actuarial model is enhanced.

Using a frequency based wind cause of loss procedure eliminates distortions to the catastrophe factor which can be generated by other causes of loss. That is, in many traditional methods, the wind claims are ratioed to the non-wind claims. Suppose that in a particular year the wind experience is somewhat worse than usual, but that theft and fire losses have declined considerably in the particular year. In such a year, the wind losses may be considered "excess" more by virtue of the good fire and theft experience than as a result of poor wind experience. The converse can hold, wherein all, or most, causes of loss deteriorating in a particular year can exclude that particular year's wind losses from consideration in the catastrophe factor development.

THE FREQUENCY MODEL

The proposed alternative is to consider the relative quarterly frequency of wind losses to determine the catastrophe loading. That is, summarize the wind claims and losses, by day of loss, over the experience period. Calculate the frequency of the wind losses by dividing each day's wind claim counts by the quarterly earned exposures. The 2.5% of days with the highest frequency are selected to be catastrophe days[1]. The losses associated with these claims are ratioed to the historical total losses excluding the catastrophe claims to develop a catastrophe factor.

Exhibit 1 provides an example of this procedure applied to a recently introduced product line which was introduced in 1988.[2] The underlying database contains all days with wind losses, the number of earned exposures (units insured) for the quarter, the number of claims generated on the day, and the cumulative paid losses for claims generated on the day through the most recent valuation quarter. The frequency and severity are calculated from the data on the exhibit. In the exhibit, 39 days are summarized, which represent the 2.5% of worst wind days. Over an approximate eight and a half year period (approximately 3,100 coverage days) the wind claims on these 39 days generated 14.3% of the total claims which represented 20.5% of the total loss dollars. From these data a catastrophe factor of 1. 2601 was generated.[3]

---

[1] The derivation of this 2.5% criterion is discussed in the section "Catastrophe Cutoff" beginning on page 11.

[2] Only 8 and 1/2 years of data are reflected in this exhibit. The number of years used to develop a catastrophe factor generally can and should exceed this period. This exhibit reflects the experience for a recently introduced product line. This recently introduced product line was selected for this paper:

1. to show the applicability of the procedure to recently introduced product lines; and
2. to keep the example simple by including all the data on one page.

While more years are needed to develop a reliable excess wind factor, the specific length of experience to be examined has not been determined satisfactorily. One could argue that a period of approximately 15 years is reliable given that underwriting practices, coverage and geographic distributional changes render the applicability of data older than this suspect.

[3] If one examines the exhibit closely, he / she will notice that the seventh catastrophe date (1988-09-16) has only 5 claims. Because this is a recently introduced product line one could justify excluding the first or second year of data from the determination of the catastrophe load due to the instability which could be introduced to the frequencies from the rapid exposure growth. All data are

To price with this factor, the payments and reserves associated with the catastrophe days should be excluded from the calendar or accident years in the review. The factor should be applied to the incurred losses, without excess wind, to develop the prospective losses with catastrophes. Figure 1, below, demonstrates how the procedure would be applied to indications developed using calendar year data.[4] It summarizes the application of the catastrophe procedure to calendar year 1995 and 1996 incurred losses.

Figure 1[5]

Calendar Year	- - - Incurred Losses - - - Total	Excess Wind	Total Excl Excess Wind	Excess Factor	Total Adjusted Incurred
1995	12,519,591	3,611,313	8,908,278	1.2601	11,225,321
1996	7,403,814	681,212	6,722,602	1.2601	8,471,151
Total	19,923,405	4,292,525	15,630,088	N / A	19,696,472

There are additional adjustments which need to be made to the data to properly price a product.

---

included in Exhibit 1 to emphasize the advantage of this procedure over the current procedures. That is, the catastrophes are selected not by the total losses they generate (which in the case of this particular date may seem to be ridiculously small), but based upon how many claims are generated by an event relative to the book's overall size.

4 Exhibit 2 shows how the excluded excess losses are determined for calendar years 1995 and 1996. The example shown here is for illustrative purposes, and intended to show only how the catastrophe losses are removed from the experience period losses, and how the excess loss factor is applied. Application of trend, hurricane costs, and change in IBNR issues have been ignored. A more complete example would include the hurricane cost loading. Homan [4] discusses one such procedure. Finally, while calendar year losses are shown, the procedure can be applied to accident year losses.

5 The data in the table are consistent with procedures used historically in the development of loss ratio indications. Following the section on trend, an alternative procedure using the application of the frequency based model with pure premiums is developed.

1. **Reinsurance** -- For an individual company, the excess wind losses which will be covered by an aggregate occurrence treaty should be excluded. This does not necessarily mean that losses which *were* covered by catastrophe reinsurance contracts should be excluded. If historically the company had a treaty which provided cover for losses excess $1 million, and in the prospective rate period aggregate losses excess $2 million will be covered, then losses exceeding the prospective coverage retention should be excluded from the calculation of the catastrophe loading. (This presupposes that the "cost" of such a reinsurance treaty is handled as cost of doing business.)

   Aggregate occurrence reinsurance issues complicate the analysis. Should the historical losses be trended so that aggregate occurrences be excluded? It will be necessary to have long term average coverage amounts to accomplish this.

2. **Use Multiple Days of Loss** -- Aggregate catastrophe contracts covering excess wind generally consider events generating losses which occur over a 72 hour period. Rather than selecting single days, one could aggregate the days into 3 or 4 day clusters. This would provide a better matching for the adjustment noted above.

3. **Incorporating with a Hurricane Model** -- Increasingly, expected losses from hurricanes are incorporated in pricing models. If the expected losses from hurricanes are included in the indications, then all hurricane losses should be excluded from this procedure. The pricing actuary needs to understand how the expected losses from hurricanes are estimated. If only hurricanes which make landfall are considered in the hurricane model, then hurricanes which do not make landfall, but which generate insurance losses, would need to be kept in the excess wind database used in this catastrophe model. Similarly, if the hurricane model considers only "true" hurricanes (e.g. Saffir - Simpson scale 1 or greater), then tropical storms need to be retained in the excess wind database.

Catastrophe Cutoff

How was the 2.5% of worst days cutoff criterion selected?

Initially, this value was selected arbitrarily as an acceptable cutoff point.[6] However, subsequent analysis tended to support this selection. The coefficient of variation between the frequency of wind losses, excluding catastrophes, was compared to the coefficient of variation on non-wind losses. If one assumes that once the variation in wind frequency due to catastrophes is removed, that the random variation in claims is the same between wind and non-wind losses, then the ideal percentage cutoff would

---

6 An alternative I have considered, but not employed, is to establish a cutoff frequency which is considered "catastrophic". That is, if the wind frequency for a particular day exceeds, say, 4% then that day would be considered catastrophic. Thus, the total catastrophic losses would be the sum of the losses, in this example, where the daily wind frequency exceeded 4%.

occur when the coefficient of variation was the same between the frequency for the wind losses excluding catastrophe losses and the non-wind losses.

Different cutoff percentages for various products were examined to determine the cutoff point. No ideal cutoff point has been developed. Although some such equivalence could be found at the 2.5% cutoff point, the ideal cutoff point has not been conclusively identified. The inability to develop a perfect match between these coefficients of variation probably result from a violation of the underlying assumption. That is, the randomness attributable to the non-cat wind claims and the non-wind claims are probably not the same. For example, if underwriting was concentrating on a reduction in fire losses over the experience period, then the company would have introduced a systematic influence on the random variation in fire claims while not simultaneously influencing the wind claims.

However, given the improvement in the loss trend data discussed in the next section that the 2.5% cutoff criterion generated, I believe a reasonable cutoff point has been established.

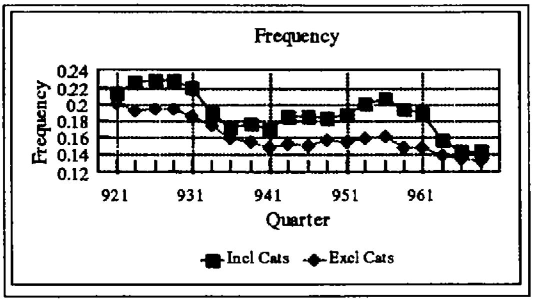
APPLICATION TO TREND

It is common for the trend used in property indications to be derived from external data. Homan [1] develops trend factors using Boeckh factors and the modified CPI. He states that these factors are surrogates for the historical and prospective changes to severity. He presents no procedure to consider changes in loss frequency.

ASB 13 [5] states that the most reliable data to be used in the development of trend is the data internal to the book of business. Historically, the use of internal data for pricing in property lines is complicated by the variance that excess wind and water introduce to the calendar year losses and claims. The frequency based catastrophe procedure eliminates much of the variance which generally makes internal data difficult to apply in the development of property loss trends.

Figure 2 below summarizes the historical calendar year frequency for the product whose catastrophe factor is developed in Exhibit 1.

Figure 2



Without analyzing any statistics associated with the chart above, it appears that the data excluding catastrophes are more stable than the data including catastrophes. Figure 3, below, summarizes the calendar year severity for the line.

Figure 3

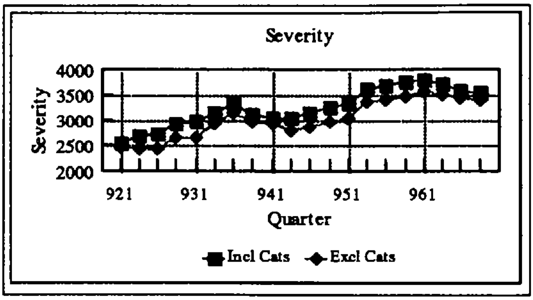


Figure 4, below, is a table which summarizes the R-squareds from linear regression performed on the data underlying the charts above.

Figure 4
R-Squareds Including and Excluding Catastrophes

	- - - Frequency - - -		- - - Severity - - -	
"Fit"	Incl Cats	Excl Cats	Incl Cats	Excl Cats
12 Point	0.2383	0.4362	0.6816	0.7804
20 Point	0.5134	0.8032	0.8399	0.8466

In each case above the quality of the fit is better using the data excluding catastrophes.

One might note that the severity has a "spike" in 1993. It should be noted that the data used in this regression include large non-catastrophic losses which are generally removed before the regression procedures are performed. They are not removed here as a complete discussion of the application of a large loss procedure is outside the scope of this paper.[7]

Analysis of the frequency excluding catastrophes may be providing insight here which is helpful in the development of equitable rates. This is a new product. Often the frequency on less mature business is greater than the frequency on mature business. The decline in frequency may be a reflection of a maturing in the book, so that developing rates which account for the lower frequency could produce lower and more equitable indications than would be developed with frequency being ignored.[8]

The more recent decline in frequency opens other areas of consideration in the development of indications. The actuary may wish to examine the source of these improvements. Has there been a shift to larger deductibles? If so, then the premium trend may need to make a provision for such a shift. Indeed, one of the advantages to using external loss trends based upon external indices which are linked to coverage amounts is that the premium trend analysis is greatly simplified. The use of internal

---

[7] Although a large loss procedure is not discussed in this paper, a general comment about the inclusion of such a procedure is in order. Large losses should be analyzed after the selection of the catastrophe days. If analyzed prior to the selection of the catastrophe days, then a large loss might be excluded twice if it is a large wind loss which occurs on a selected catastrophe date.

In the example above the 1993 large losses which would be excluded from the severity trend analysis are more than 120% greater than the 1992 and 1994 excluded losses. When the large loss procedure is employed the R-squared is increased.

[8] Because this is a new, rapidly growing product, one may want to examine the impact the exposure base is exerting on the frequency. Frequency has been calculated using earned exposures in the denominator. For this product, the exposure base may be trailing the claim counts during the rapid growth. It may be more appropriate to use an exposure base which is a weighted average of in force policies and earned exposures during the period of rapid growth. Such a weighting may provide a more accurate reflection of the frequency. If one concludes that such a weighting is needed in developing the frequency trends, then one should revisit the exposures used in determining the catastrophe days.

data in property lines will require more sophisticated analysis of the premium trend so that there is a complete matching of the trended premiums and trended losses.[9]

If the internal data provide a more accurate projection of the current and prospective loss costs, then more accurate indications will be developed. For this product, the trends that are generated by the internal data are greater than those derived using the external indices commonly employed for the line. If the internal trends truly are more accurate, then a parameter error would have been introduced to the indications. If the relationship holds over time that the internal trends are larger than the trends developed using external indices, then a systematic downward bias would exist in property indications.[10]

AN ALTERNATIVE APPLICATION OF THE FREQUENCY BASED CATASTROPHE LOAD USING PURE PREMIUMS

Figure 1 showed how the application of frequency based catastrophe load could be applied to obtain untrended calendar year losses without the hurricane catastrophe

---

9 A discussion of all the analysis needed to develop the correct premium trend is outside the scope of this paper. However, it must be emphasized that if the internal trends are to be used in the development of the indications then the actuary must be aware that distributional shifts occurring in, say, deductibles, territory, and amount of insurance are contributing to the loss trend. Since each of the items is a rating variable, premiums are also being impacted by the distributional shifts.

Ideally, an analysis of the changes in the average relativities for each of the rating variables which can be impacting the loss trends should be performed. In the absence of time or data to adequately analyze how each relativity is impacting loss trend, the average premiums at present rates can be used to develop premium trend.

10 A.M. Best [6] recently noted for the Homeowners line that "Although baseline costs (excluding catastrophe) would clearly show rate inadequacy, many regulators and even some companies are reluctant to increase rates." If companies' internal trends are generally greater than the trends developed using external data, then companies and regulators may be unaware of the full magnitude of rate deficiencies.

load. Figure 5 below provides summaries of the 1995 and 1996 calendar year incurred losses for the excess wind, non-excess wind, and non-wind causes of loss.

Figure 5

Year	Total	Excess Wind	Other Wind	Other Causes
1995	12,519,591	3,611,313	3,395,122	5,513,156
1996	7,304,814	681,212	2,859,190	3,764,412

In 1995 "Other Wind" losses (wind losses not defined to be catastrophic) were approximately $550 thousand greater than the 1996 "Other Wind" losses. The "Other Causes" losses (all losses other than those caused by wind) were approximately $1.75 million greater than the 1996 "Other Causes" losses. 1996's earned exposures were approximately 2% lower than the 1995 exposures. The catastrophe load as developed in Exhibit 1 is 26.01%. Should 1995's untrended, non-hurricane catastrophe loading be approximately $450 thousand ($1.75 million X 26.01%) greater than 1996's untrended catastrophe losses? Put differently, should increased non-wind related losses increase the level of the non-hurricane excess losses?[11] In general the answer is no. However, when one is developing indications using five years of data, the variation in the non-wind losses from year to year should offset sufficiently to limit the bias caused by this type of loading.

If one wishes to load the indications with a non-hurricane excess wind factor which is not a function of the non-excess losses, then a pure premium approach can be used.

[11] Note that the $450 thousand does not consider the non-excess wind losses. If they are considered then 1995's untrended non-hurricane excess wind losses are approximately $600 thousand greater than 1995's { ($1.75 million + $550 thousand) x 26.01% }.

The ability to develop long term severity trends with internal data enables a reasonable pure premium method to be developed. The table below outlines the pure premium approach.

Figure 6

(1) C.Y.	(2) Days of Loss	(3) Excess Pure Prem	(4) Severity Trend	(5) Trended Excess Pure Prem	(6) C.Y. Excess Pure Prem
1998	2	1.20	1.851	2.22	71.81
1989	11	175.48	1.714	300.77	77.55
1990	2	42.61	1.587	67.62	83.76
1991	2	48.88	1.469	71.80	90.48
1992	3	155.68	1.360	211.72	97.74
1993	5	81.52	1.259	102.63	105.58
1994	4	125.66	1.166	146.52	114.00
1995	7	217.98	1.080	235.42	123.07
1996	3	57.59	1.000	57.59	132.92
Average Excess Pure Premium:				132.92	

The pure premiums in column 3 are developed by taking the cumulative paid losses from the catastrophe dates within a year and dividing them by the year's earned exposures. An annual 8 percent severity trend has been developed from the internal data.[12] Column 5 contains the trended pure premiums. The average pure premium is

[12] Because of the nature of the losses a stable non-hurricane excess wind trend cannot be obtained from the excess wind data. It is assumed that the non-hurricane excess wind losses will be impacted by the same inflationary influences which impact the non-excess wind losses and the long term non-excess severity trend has been selected.

Since the wind losses are fortuitous, generally one would anticipate only applying a severity trend to the pure premiums. However, if one believes that the policy mix at the beginning of the period is sufficiently different than the policy mix at the end of the period by a rating variable which would impact historical frequency of excess wind claims (such as a shift to higher deductibles), then one could apply a frequency adjustment to the severity trend.

calculated using the average pure premiums in column 5.[13] For each calendar year, multiply the calendar year earned exposures by the overall average excess pure premium to obtain the total non-hurricane excess wind losses at current levels. To obtain the total losses at current levels the non-catastrophe experience losses (trended to current levels) are added to the non-hurricane excess losses and the hurricane expected losses.

The pure premium based frequency load is not as critical for developing the overall statewide indication as one might initially believe. The table below summarizes the differences between the total non-hurricane losses before trending to current levels.

Figure 7

C.Y.	(1) Earned Exp	(2) CY. Excess Pure Prem	(3) [(1) x (2)] Excess Using Pure Prem	(4) Excess Derived w/ Factor	(5) Total Excl Excess Wind	(6) [(3) + (5)] Pure Prem Total Losses	(7) [(4) + (5)] Factor Total Losses
1995	16,280	123.07	2,003,580	2,317,043	8,920,587	10,924,167	11,225,321
1996	15,921	132.92	2,116,219	1,748,549	6,722,602	8,838,821	8,471,983
Total			4,119,799	4,065,864		19,762,988	19,696,472

The C.Y. Excess Pure Prem in Figure 7 (column 2) is taken from column 6 of Figure 6. The data in column 4 are derived from the loss data in Figure 1. There is an approximate 1/2% difference between the untrended losses developed using the factor derived in Exhibit 1 and the pure premium method just presented. When more years of

[13] To maintain consistency with the issues discussed previously, the 1988 pure premiums are shown here and trended. If 1988 were excluded from the average pure premium calculation, the average non-hurricane excess pure premium for 1989 through 1996 is $149.26.

data are considered and the expected losses are added into the above losses trended to current levels, the percentage difference between the two methods should decline.

An advantage to the pure premium excess process just introduced is that it eliminates the leveraging effect the non-wind and non-excess wind losses generate on the excess wind factor. A disadvantage is that the average pure premium is dependent upon the selected trend factor. In the example used thus far, less than 10 years of data are used to develop the catastrophe factor. If the catastrophe factor is developed with 15 to 20 years of data, any inaccuracy of the trend factor will greatly impact the older trended pure premiums. The more inaccurate the selected trend factor is, the more inaccurate the average pure premium will be.

However, in performing analysis for other rating variables, the non-wind losses can have a greater leveraging effect on the factor application of non-hurricane excess loss load within particular cells, and the pure premium method is probably preferable.

APPLICABILITY TO OTHER PRICING ISSUES

The catastrophe procedure developed here can enhance the equitable pricing of property rating variables.

Fitzgerald [3] notes that the application of the hurricane loss models in the development of property rates has eliminated some cross subsidization across property rating territories. Historically, the hurricane losses were apportioned throughout the state, whereas the new modeling techniques enable the loading of such losses to be more accurately assigned to the proper rating territory. This frequency based model similarly enables catastrophic non-hurricane losses to be more equitably assigned to the appropriate rating territory.

In performing territorial analysis, the same catastrophe dates selected for the statewide indication are selected in determining the catastrophe loads by territory. However, catastrophe loadings are developed for areas of territories separately using the ratios for the excess wind losses versus the total losses excluding excess wind within each rating territory. The determination of these area, excess wind factors is shown in Exhibit 3. The range of factors ranges from 1.0096 in Area 1 to 1.4646 in Area 3. In developing the territorial indications, the catastrophe dates are removed from the experience period and the area excess loss factors are applied following the same process shown in Figure 1. This should generate a more equitable distribution of catastrophes to the appropriate territories and a more accurate rate. Again, a hurricane loss loading is needed for each area or territory, but is not explored here.

The historical procedure had catastrophe losses removed proportionally from the losses in each territory and the same catastrophe load was applied to each territory. This produces inaccurate indications. Consider only those territories subject to higher long term catastrophic losses. If over the experience period in the review these territories had abnormally low losses (relative to the long term historical average) then loading the average statewide catastrophe load will understate the needed rate level in these territories.

Protection class relativities can be more equitably priced with this procedure. This is particularly important if the distribution of policies by protection class varies by territory. If catastrophe losses have not been removed, and then accounted for with a catastrophe load, then the protection classes are developed with the random error from catastrophes.

If the historical catastrophe procedure has been used, then the unprotected properties' relativity is too high. Since a higher protection class indicates an increased fire risk, applying the overall catastrophe factor overstates the total losses by protection class as the average catastrophe factor is being leveraged by the higher fire losses. Similarly, the lower fire losses in protected areas understate the catastrophe losses when the average statewide factor is applied. Developing catastrophe loadings by protection class using the frequency based procedure would produce more accurate protection relativities.

Finally, the use of the frequency based procedure could facilitate the application of accident year loss data in the development of indications. When the frequency catastrophe procedure is employed, the development factors for the 15 to 27 link ratios are generally smaller with less variance.

ADVANTAGES OF PROPOSED PROCEDURE OVER CURRENT PROCEDURES

A summary of the advantages and disadvantages of the proposed method to the methods currently employed is made below.

Advantages

1. The procedure enhances the usability of internal data for loss trend analysis.
2. Hurricane losses are not considered in the database, so that the proposed procedure can be used more readily with hurricane models than the current procedures.
3. The procedure enables catastrophe analysis to be performed on new product lines.
4. The development of the loading is not a function of other causes of loss, which have the potential to distort the loading.
5. The development of the loading is not a function of premium, which has the potential to distort the loading.
6. The development of the loading is not dependent upon multi-state industry catastrophe definitions, which can distort the loading.
7. The procedure enables territorial and protection class indications to have catastrophe loadings which can be developed for each analyzed cell.
8. The above advantages develop a more equitable rate.

Disadvantages

1. The proposed procedure is more complex than those currently employed.
2. The initial development of the database may be time consuming and costly. A company may not have data which goes back very far past the years used in developing indications. Thus, it will need to build the data prospectively. Even if the data exist in an electronic archive (most probably tape), system resources will need to be utilized to retrieve the archived data.
3. It is change. It will require time to explain to people within the company and outside it. It will require changes to spreadsheets and or programs used to develop indications and filings. These issues are time consuming and can sometimes create emotional upset with individuals who have taken pride in their past work product and perceive change not as an evolutionary improvement but as an indictment of their previous work product.

CONCLUSION

Although significant problems have been identified with the current excess wind methods used in property ratemaking, in the absence of available data and alternative procedures they served the ratemaking process well. The evolution of information technology has made the application of the theory presented herein practical. Prior to the 70's much of the available claim information was highly summarized. Even when more detailed information began to be stored, obtaining summarized data in a form usable for the actuary required extensive work with the data processing department. Because of the man-hours involved in establishing an initial report process revising reports to obtain better information was difficult to schedule. Only recently with inexpensive electronic storage costs and powerful computers, which enable direct analysis by the actuary, has the proposed procedure been feasible.

There are areas which need to be explored further.

1. When a geographic distributional shift is occurring how should this be accounted for in determining statewide excess losses? Should the exposure base be adjusted to reflect the distributional shift?
2. What are the optimal number of years to which this procedure should be applied? The historic use of thirty years of data was developed to account for both excess non-hurricane wind losses and excess hurricane losses. With the advent of modeling techniques which enable expected hurricane losses to be considered separately from non-hurricane losses are thirty years of data still needed? Does the optimal number of years vary by state?

The current procedures for developing excess wind losses for property losses are undergoing a transformation. The introduction of modeled hurricane losses into the rate development procedures necessitates some degree of revision to the non-hurricane excess wind procedure. The recommended procedure compliments the incorporation of modeled hurricane losses into rate development. It also provides the added benefit of making the internal data for the product line useful for loss trend analysis.

References

[1] Homan, Mark, "Homeowners Insurance Pricing -- An Update", *Pricing Issues,* Casualty Actuarial Society *Discussion Paper Program,* 1990, Vol. II

[2] Chernick, D.R., *Private Passenger Automobile Physical Damage Ratemaking,* CAS Study Note Reading

[3] Fitzgerald, Beth; Burger, George; White, Jonathan; and Wood, Patrick; "Incorporating a Hurricane Model into Property Ratemaking";*1996 CAS Forum, Winter 1996, Including the Ratemaking Call Papers*

[4] Homan, Mark; "Reflecting Reinsurance Costs in Rate Indications for Homeowners Insurance"; *Casualty Actuarial Society Forum, Winter 1997, Including the Ratemaking Call Papers*

[5] Actuarial Standards Board of the American Academy of Actuaries, "Actuarial Standard of Practice No. 13, Trending Procedures in Property/Casualty Insurance Ratemaking"

[6] Sheffield, Martin; Ryan, Darian; Maxwell, Hope; Mosher, Matt; & Watson, Teri, "Segments' Prospects are Slipping", *Best's Review, Property and Casualty Edition,* January 1997, pp 67 - 70

Development of Excess Wind Factor
Accident Years 1988 through 1996
Evaluated as of March 31, 1997

QTR	D.O.L.	Payments	Claims	Exposure	Severity	Frequency
19922	1992-04-28	1,901,667	382	3,550	4,978.19	0.1076
19892	1989-05-04	445,312	137	1,672	3,250.45	0.0819
19892	1989-05-16	457,659	113	1,672	4,050.08	0.0676
19912	1991-04-29	562,473	167	3,427	3,368.10	0.0487
19892	1989-06-06	439,654	74	1,672	5,941.28	0.0443
19892	1989-06-07	191,922	74	1,672	2,593.54	0.0443
19883	1988-09-16	6,969	5	143	1,393.83	0.0350
19942	1994-04-25	767,533	131	3,830	5,859.03	0.0342
19952	1995-05-05	605,320	137	4,064	4,418.39	0.0337
19942	1994-04-26	565,209	125	3,830	4,521.67	0.0326
19952	1995-04-29	622,121	127	4,064	4,898.59	0.0313
19951	1995-01-18	534,098	115	3,951	4,644.33	0.0291
19952	1995-05-07	485,685	106	4,064	4,581.93	0.0261
19892	1989-05-05	114,283	42	1,672	2,721.03	0.0251
19934	1993-10-18	430,694	93	3,740	4,631.12	0.0249
19952	1995-05-28	692,207	97	4,064	7,136.16	0.0239
19902	1990-04-05	279,598	67	3,071	4,173.11	0.0218
19952	1995-06-27	312,902	87	4,064	3,596.57	0.0214
19902	1990-04-27	175,090	61	3,071	2,870.33	0.0199
19892	1989-05-13	117,219	33	1,672	3,552.08	0.0197
19931	1993-03-29	286,444	65	3,469	4,406.83	0.0187
19941	1994-03-27	246,397	69	3,712	3,570.97	0.0186
19934	1993-10-17	251,317	68	3,740	3,695.84	0.0182
19892	1989-05-15	119,477	30	1,672	3,982.55	0.0179
19942	1994-05-13	243,544	68	3,830	3,581.53	0.0178
19893	1989-07-02	138,967	40	2,266	3,474.19	0.0177
19962	1996-05-25	356,492	65	3,991	5,484.49	0.0163
19892	1989-04-29	67,829	27	1,672	2,512.20	0.0161
19892	1989-06-02	163,785	27	1,672	6,066.11	0.0161
19932	1993-05-05	295,454	57	3,560	5,183.41	0.0160
19911	1991-02-18	168,999	53	3,329	3,188.67	0.0159
19892	1989-05-01	96,471	26	1,672	3,710.40	0.0156
19931	1993-03-25	150,975	53	3,469	2,848.59	0.0153
19922	1992-04-29	229,651	54	3,550	4,252.80	0.0152
19951	1995-03-25	314,271	60	3,951	5,237.85	0.0152
19961	1996-01-17	132,113	61	4,020	2,165.79	0.0152
19884	1988-11-15	9,202	10	678	920.18	0.0147
19964	1996-10-21	228,517	57	3,936	4,009.07	0.0145
19922	1992-06-04	260,750	50	3,550	5,215.00	0.0141
"Excess"		13,468,271	3,113	NA	4,326.46	
Overall Wind		33,981,642	9,337	NA	3,639.46	
"Excess" / Overall Wind		39.63%	33.34%		118.88%	
Total All Causes		65,252,655	21,711		3,005.51	
"Excess" / Total		20.64%	14.34%		143.95%	

Excess Wind Factor = 1 + [13,468,271 / (65,252,655 - 13,468,271)] = 1.2601

Exhibit 2

Determination of Excess Wind Amounts
Calendar Years 1995 & 1996

Accident Date	--- *C.Y. Paid* ---		--- *C.Y. Ending Reserves* ---			--- *C.Y. Incurred* ---	
	1995	1996	1994	1995	1996	1995	1996
1993-05-05	10,559	0	0	0	0	10,559	0
1993-10-17	5,220	0	1,000	0	0	4,220	0
1993-10-18	2,733	2,188	2,995	3,500	0	3,238	(1,312)
1994-03-27	55,330	0	40,310	0	0	15,020	0
1994-04-25	16,495	1,080	46,615	0	0	(30,120)	1,080
1994-04-26	34,840	6,627	34,485	0	800	355	7,427
1994-05-13	12,661	0	31,270	0	0	(18,609)	0
1995-01-18	526,779	7,320	0	36,630	0	563,409	(29,310)
1995-03-25	309,636	4,635	0	24,840	0	334,476	(20,205)
1995-04-29	583,306	38,815	0	47,300	0	630,606	(8,485)
1995-05-05	581,170	24,150	0	54,080	0	635,250	(29,930)
1995-05-07	456,512	29,525	0	47,475	0	503,987	(17,950)
1995-05-28	590,584	100,736	0	45,105	1,495	635,689	57,126
1995-06-27	284,713	27,363	0	38,520	0	323,233	(11,157)
1996-01-17	0	134,298	0	0	7,140	0	141,438
1996-05-25	0	350,402	0	0	35,785	0	386,187
1996-10-21	0	174,468	0	0	31,835	0	206,303
Total	3,470,538	901,607	156,675	297,450	77,055	3,611,313	681,212

Exhibit 3

Development of Area Excess Wind Factors
Accident Years 1988 through 1996
Valued as of March 31, 1997

Total Paid Losses

	----- *Rating Areas* -----					
Year	1	2	3	4	5	Total
1988	7,294	25,251	92,205	60,309	26,072	211,131
1989	456,417	791,114	2,172,721	2,061,930	1,103,032	6,585,214
1990	191,891	642,777	2,081,750	1,242,218	716,310	4,874,946
1991	815,174	1,746,683	2,316,482	2,007,986	1,317,641	8,203,966
1992	491,388	1,336,373	3,831,662	2,373,489	1,814,160	9,847,072
1993	1,053,979	800,521	1,817,877	2,397,582	2,158,609	8,228,568
1994	989,041	871,581	2,755,530	3,121,772	1,412,900	9,150,824
1995	822,881	834,691	4,520,362	2,703,439	2,270,943	11,152,316
1996	393,363	846,549	2,101,594	1,566,133	2,090,981	6,998,620
Total	5,221,428	7,895,540	21,690,183	17,534,858	12,910,648	65,252,657

Excess Wind Paid Losses

	----- *Rating Areas* -----					
Year	1	2	3	4	5	Total
1988	3,122	2,128	9,135	779	1,007	16,171
1989	0	7,818	913,756	790,462	640,542	2,352,578
1990	5,923	51,108	378,241	16,764	2,653	454,689
1991	39,473	521,146	0	170,853	0	731,472
1992	0	0	1,726,972	404,767	260,330	2,392,069
1993	650	8,306	532,172	510,505	363,251	1,414,884
1994	0	4,203	1,131,626	638,350	48,503	1,822,682
1995	380	9,980	1,958,300	764,024	833,921	3,566,605
1996	0	11,636	229,922	108,175	367,390	717,123
Total	49,548	616,325	6,880,124	3,404,679	2,517,597	13,468,273
Excess Wind Factor	1.0096	1.0847	1.4646	1.2410	1.2422	1.2601

Actuarial Implications of Texas Tort Reform

by Michael J. Miller, FCAS, MAAA and
Jerrold W. Rapp, FCAS, MAAA

Abstract

Tort reforms are generally difficult to evaluate because historical claim loss data are rarely available in a format to facilitate analysis and because the tort reform itself may change plaintiff's behavior in a way that renders historical data unpredictive of the future. In this paper we describe our attempt to calculate the effects of tort reform in Texas using a combination of claims data, focus group information and judgment.

Background

In early 1995, a series of legislation was passed in Texas which reformed the civil justice system in ways intended to reduce the cost of litigation and the size of judgments. The key pieces of legislation pertained to punitive damages, joint and several liability, venue, frivolous lawsuits, the Deceptive Trade Practices Act and medical malpractice.

Subsequent to the passage of this series of tort reforms, the Texas Legislature passed House Bill 1988 in May 1995. H.B.1988 required the Insurance Commissioner to hold a hearing and, based on the evidence presented at the hearing, mandate by October 1, 1995 appropriate rate reductions for each line of insurance effected by the tort reforms. The rate reductions mandated by the Commissioner were to be effective for all new and renewal policies on January 1, 1996. The Commissioner was further required to annually review the rate reductions and make any appropriate changes to the reductions. If the Commissioner failed to order rate reductions by October, 1995, certain default reductions as set forth in H.B. 1988 were to be automatically applied. The default reductions are presented in Exhibit 1.

The Commissioner did hold a hearing and received evidence on the anticipated impact of the tort reforms in July, 1995. As a result of that hearing, the Commissioner ordered the rate reductions presented in Exhibit 1 be implemented effective January 1, 1996. The Commissioner has annually reviewed the ordered rate reductions and to date has made no modifications to his original order. It does now appear that the rate reductions may be increased in 1998 as a result of the 1997 hearing, but that decision was not published as of this writing.

Summary of Tort Reforms

Senate Bill 25 limited the amount of punitive damages to the greater of $200,000 or twice the amount of economic damages plus non-economic damages not to exceed $750,000. The standard for awarding punitive damages was raised from gross negligence to malice. Malice was defined to mean a specific intent to cause substantial injury or an act which involved extreme risk of which the defendant was aware, but nevertheless proceeded. This reform took effect on causes of action that occurred on or after September 1, 1995.

Senate Bill 28, effective on causes of action on or after September 1, 1995, revised the law so that a defendant would be held jointly and severally liable for damages caused by others only if the defendant was more than 50% liable. Previously a defendant could be held jointly and severally liable for damages caused by others even if the defendant was at least 11% liable. This revision meant that no more than one liable party could be held jointly and severally liable. Senate Bill 28 included an exception for toxic tort cases in which the joint and several threshold was set at 15%, rather than 50%. Previously the law provided no threshold for toxic tort cases.

Senate Bill 28 also allowed the extent of a person's liability to be included in the evidence to the jury. Previously, a person that was only 60% liable could be held 100% liable by a jury that was never told about the defendant's degree of liability. The revised law created the possibility that the jury might reduce the award to reflect the 60% liability factor. The downside to this reform was the threat from plaintiffs' lawyers that they would now sue all liable parties to an accident to ensure that 100% of the damages were recovered, thereby increasing defense costs.

Senate Bill 32 required a plaintiff to bring suit only in the county in which all, or a substantial part, of the insured event occurred, or in the county of the defendant's residence, or in the county of the defendant's principal office. Only if none of the previous conditions applied, could the suit be brought in the plaintiff's county of residence. Previously, suits could be initiated in any county in which there was some nexus to the insured event. Examples of abuse were the initiation of lawsuits against an insurer in any county in which the insurer had an agent, or the initiation of a lawsuit for a plaintiff in the proper county, but then joining of multiple plaintiffs from throughout Texas or countrywide. A person will no longer be able to join as a plaintiff unless the venue is proper as to that specific plaintiff. The intent of the new venue rules was to curb shopping for forums which had a reputation for large injury awards. This act took effect on suits brought under the Federal Employers' Liability Act and the Jones Act on January 1, 1996 and on all other suits commenced on or after September 1, 1995.

Senate Bill 31 was patterned after Federal Rule 11 and attempted to restrict frivolous pleadings and motions in a lawsuit. A frivolous lawsuit is defined as one in which there is no evidentiary support or legal basis for the pleading or is filed for an improper purpose (i.e. harassment). Suits involving frivolous pleadings and motions can and will still be litigated. There will supposedly now be an avenue to complain to the court, and the court may apply sanctions against the offending attorney or plaintiff. But Senate Bill 31 will not in itself eliminate a frivolous suit from the system. This act was effective on suits commenced on or after September 1, 1995.

The Deceptive Trade Practices Act (DTPA) generally reformed the conditions for awarding damages when a consumer is harmed by deceptive practices. The new law requires a showing that the consumer relied on the misleading or deceptive practices before damages can be awarded. There is a potential for a DTPA award against the insurer, but if there were such an award, it would not ordinarily be included in the ratemaking process, because it would not be expected to recur in the future.

House Bill 971, effective 9/1/94, was intended to discourage frivolous medical malpractice claims and reduce the number of defendants in each suit. This law provided new deadlines for filing expert reports and attempted to eliminate "junk science" by setting certain standards for expert medical testimony.

Data

The primary sources of data underlying the actuarial costing of the tort reforms were the Texas Liability Insurance Closed Claim Annual Report and a series of special data calls designed by the Texas Department of Insurance (TDI).

In accordance with its statutory mandate, the TDI annually collects information concerning closed bodily injury claims relating to the following categories of insurance: general liability, medical professional liability, other professional liability, commercial auto liability and the liability portion of commercial multi-peril. These closed claim data are regularly published in the Annual Report. Throughout this paper this report is referred to as the "regular closed claim survey."

Because the regular closed claim survey neither covered all lines of insurance nor contained all the information needed to cost all the tort reforms, it was necessary for the TDI to issue a series of special data calls. One such call, attached as Appendix A, applied to the following lines of insurance: private passenger auto, homeowners, farmowners, personal umbrella, general liability, other professional liability, commercial auto garage, commercial multi-peril and product liability. Throughout this paper, this special call for data is referred to as the "special closed claim survey."

There was also a special call for employers' liability claim data addressed to workers' compensation insurers (Appendix B) and a special call for information pertaining to deceptive trade practices and venue data (Appendix C).

In addition to the data call information, the following data were also relied upon in costing the tort reforms.

- Size of loss data for products liability prepared by the Insurance Services Office and released by the TDI.
- Annual Statement Page 14 data compiled by the TDI.
- Cause of loss data for homeowners, farmowners and tenants provided by the TDI.
- Information concerning the anticipated changes in lawsuit procedures gathered from a focus group discussion with Texas attorneys.

Costing

Punitive Damages

The actuarial impact of Senate Bill 25 was calculated by individually analyzing each claim in the TDI's regular and special closed claim surveys so as to determine the amount of punitive damages. These amounts were individually capped to the greater of $200,000 or twice the amount of economic loss plus the non-economic loss (with the non-economic loss limited to $750,000). The difference between the capped amount of punitive damages and the actual punitive damages represented the estimated savings arising from the new cap (see Line 1, Exhibit 2).

The savings due to the elimination of the gross negligence standard and the introduction of the malice standard were judged to be approximately 25% of the capped punitive damage amount. The 25% reduction factor, shown on Line 4 of Exhibit 2, was based on information gathered at the focus group discussion with Texas attorneys.

The claims data in the TDI's closed claim survey included the total claim costs, only a portion of which was below policy limits and insured. Since the purpose of the actuarial calculations was to determine the savings to the insurance system, it was necessary to divide the claims data into the primary insured portion of the claim and the excess portion. The total savings for each claim was first applied to the amount of the claim in excess of insurance and then any remaining savings was applied to the insured portion of the claim. For example, assume a $1 million claim settlement of which $800,000 was insured by the primary carrier. If the punitive damage savings was 10%, the capped claim would have been $900,000. In this case, there would have been no savings to the primary insurance system and no impact on the primary insurance rate. The claim amounts in excess of policy limits are shown on Line 6 of Exhibit 2.

The savings arising from Senate Bill 25 were calculated separately for each line of insurance. The calculations for private passenger auto and general liability are presented in Exhibit 2 as illustrations of the calculations performed for each line of insurance.

Joint and Several Liability

The impact of Senate Bill 28 was determined using two different analysis techniques. For those lines of insurance included in the regular closed claim survey, each claim in the survey affected by the joint and several rules was individually analyzed for possible savings. If the plaintiff was more than 50% at-fault, then it was assumed that none of the insured defendants would be jointly and severally responsible and the entire settlement would be saved.

The savings estimates were calculated so as to reflect the fact that some settlements will exceed insurance policy limits. In those cases, the savings to losses in excess of policy limits will have no effect on the insurance rates.

In cases involving multiple defendants, the new joint and several rules reduced the financial liability of one of the insured defendants while causing an increase in the financial liability of another insured defendant. Due to these "dissavings", a 25% reduction factor was judgmentally applied to the otherwise calculated savings (see Line 23 of Exhibit 3).

The calculations for general liability and commercial auto are presented in Exhibit 3 as illustrations of this first analysis technique.

For those lines of insurance included in the special closed claim survey, a different analysis approach was followed because of data credibility concerns. When the data in the special call were stratified by claim size, the data in each cell lacked sufficient credibility for reliable analysis. For the lines of insurance in the special call, the estimated savings provided by the respondees (see Question 4K of Appendix A) to the special call were utilized without further adjustment. These estimated savings by the respondees were on a combined basis for the joint and several, venue, deceptive trade practices and frivolous suit reforms. Examples of the calculations are shown in Exhibit 4 for the homeowners and products liability lines of insurance.

Venue

The actuarial analysis was based on claims data from the regular closed claim survey for the product liability and the environmental liability lines of insurance. Data from the TDI's special venue data call (see Appendix C) were the basis for the analysis of the general liability, commercial auto, commercial multi-peril, medical professional and other professional lines of insurance.

The first step in the analysis was to divide the claims information for each county between those events which occurred inside the county and those which occurred outside the county.

For products liability and environmental liability, the number of expected "venue" claims for each county was determined as a function of the ratio of claims from outside the county to the total number of claims for the county. If the "outside" claim ratio for the county exceeded 110% of the statewide average "outside" claim ratio, the excess number of claims for the county were considered to be "venue" claims. As an example of this calculation, consider the products liability calculation for Travis County. From Exhibit 5, page 2, one can determine that 35% (7 of 20) of the Travis County products liability claims arose from outside the county. The statewide average "outside" claim ratio was only 16% (127 of 794). The Travis County ratio exceeded 110% of the statewide average ratio by 17.4% (.35 - .16 x 1.1). The excess number of "outside" claims (.174 x 20 = 3.48 claims) is shown on page 3 of Exhibit 5. To account for losses in excess of policy limits, the number of excess claims were then judgmentally reduced by 25% to determine the number of claims for Travis County which were expected to be effected by the venue reforms. The estimated dollar savings were then calculated by multiplying the number of "venue" claims by the dollar difference between the average settlements outside the county and the statewide average settlements inside the county.

The basis for the savings estimates for general liability, commercial auto liability, commercial multi-peril and professional liability, were the responses to the special venue data call (see Question 2d, Appendix C). An example of the calculations using the special venue data call is shown for general liability in Exhibit 5, page 1.

For those lines of insurance not covered by the special venue data call, the estimated savings provided by the respondees to the special closed claim survey were utilized as presented in Exhibit 4. Again, these savings were estimated by the respondees on a combined basis for the joint and several, venue, deceptive trade practices and frivolous suit reforms.

Frivolous Pleadings and Motions

The TDI's regular closed claim survey did not include information upon which to calculate the savings from the Senate Bill 31 restrictions on frivolous pleadings and motions. No meaningful claims data were available to calculate the impact on insurance losses. Information gathered in focus group discussions suggested no significant savings on insurance losses. An estimate of 0% savings was used for the lines of insurance in the TDI's regular closed claim survey. For those lines of insurance included in the special closed claim survey, the responses to Question 4K (see Appendix A) were used as the basis for the savings estimates. Those estimates were on a combined basis for the joint and several, venue, deceptive trade practices, and frivolous suit reforms and are summarized in Exhibit 4.

Deceptive Trade Practices Act (DTPA)

Since awards under the DTPA are generally not covered by insurance, it was assumed that any savings from this reform would have negligible impact on insurance rates. Responses to the special closed claim survey tended to confirm this judgment of negligible savings to the insurance system. No explicit savings were included in the cost estimates for the DTPA reforms. But to the extent that respondees included DTPA savings when answering the special closed claim survey, some savings were included in the calculations for the lines of insurance covered by the special closed claim survey (see Exhibit 4).

Medical Negligence

House Bill 971 was intended to discourage frivolous medical malpractice claims and reduce the number of defendants in each suit. This reform was considered in the same context as the restrictions on frivolous suits in Senate Bill 31. Discussions in the focus group sessions indicated negligible savings were anticipated. No explicit measurement of savings was included in the calculations.

Out-of-State Lawsuits

Some policies issued on Texas risks give rise to claims and lawsuits outside Texas. Examples of this can be found in the auto liability and products liability coverages. These out-of-state claims are reported as Texas losses because they are associated with Texas insureds.

The Texas closed claim data upon which the savings calculations were based reportedly did not include out-of-state losses. As a result, the savings estimates derived from the data should have been applied only to that portion of the premium dollar, or to that portion of the losses, which represent in-state premiums or losses. Data published by the Insurance Research Council indicated that approximately 3% of the private passenger auto liability losses were out-of-state. Similar data were not available for commercial auto liability or products liability, but it was anticipated that the percentage was substantially greater than 3%.

One possible method of adjusting the savings estimates for the out-of-state losses would have been to judgmentally reduce the punitive damage savings by a factor of .97 for private passenger auto liability and a factor of .90 for commercial auto liability and products liability. Another reasonable approach would have been to analyze a range of estimates and select a savings from the lower end of the range.

The rate reductions which were finally ordered were generally above the range of savings indicated by the closed claim data. It was difficult to determine if any recognition. even implicit recognition, was given to the impact of out-of-state losses.

Allocated Loss Adjustment Expenses

With the possible exception of the venue reform, the various tort reforms were not expected to result in any measurable savings to allocated loss adjustment expenses (ALAE).

In the case of the joint and several reform, a greater incentive to vigorously defend a claim to avoid paying more than the insured's degree of fault was anticipated. Prior to the reforms, the precise degree of fault of each claimant was not as relevant as after the reforms.

For the punitive damage reforms, the caps on awards were anticipated to result in the injured party seeking greater amounts for economic and non-economic damages so as to increase the amount of the potential punitive damage award. This will take more effort to defend against, thus defense costs were anticipated to increase.

It is possible that there will be some ALAE savings on claims affected by the venue reforms. It was assumed the portion of ALAE savings from the venue reforms would be approximately equal to the percentage of loss savings.

With the potential for increases in ALAE arising from the joint and several and punitive damage reforms and the potential savings coming from the venue reforms, it was judged that there would be no overall savings in ALAE.

Range of Estimates

Exhibit 6 summarizes the authors' indicated savings by line of insurance and compares those results to the savings determined by the TDI actuaries.

Even though we used different procedures in some cases and the TDI included savings for behavioral changes for some lines of insurance of as much as 2.5%, our results were often very close to those of the TDI actuaries. We believe the similarity of results suggests the reasonableness of both sets of calculations.

However, there are a few areas where the differences are significant enough to be of concern. We are convinced that the TDI overestimated the savings pertaining to the venue reforms for the general liability, commercial auto, commercial multi-peril and homeowners/farmowners lines of insurance. This overestimate results primarily because the TDI assumed that Harris County was an attractive "venue" county and that the venue reforms would reduce lawsuits in that county. But the information we have from attorneys is that Harris County was not a "venue" county. Our analysis reflected information from the special venue data call and confirmed that the savings in Harris County could be expected to be considerably less than those indicated by the TDI analysis. If the TDI's calculations were adjusted to reflect the information in the special venue data call, its results would have been very similar to ours.

We are also concerned that the TDI overestimated the savings from the joint and several liability reforms for product liability (non-BI). The TDI's calculations were based on only 4 claims from the special closed claim survey where joint and several liability reform would have impacted the result. Four claims is not a sufficient base to conclude that a 16.5% savings is indicated for this line.

With these exceptions, we believe the TDI's calculations generally confirmed the reasonableness of our results and vice versa.

Commissioners Decision

In 1995 the Commissioner received testimony from several parties and ultimately ordered the rate reductions set forth in Exhibit 1. The ordered rate reductions were generally higher than the range of actuarial indications because of a subjective judgment that the tort reforms would cause a change in the claiming behavior of the plaintiffs. The authors' actuarial estimates were derived by reviewing past claims and adjusting the data for the impact of the new rules on those claims, with no explicit recognition of potential behavioral changes. The TDI actuaries developed estimates with some explicit recognition of behavioral changes, but even those estimates were overridden in the final order.

The danger of factoring "behavioral modifications" or other "unintended consequences" into the actuarial calculations is that such assumptions are necessarily arbitrary. Basing cost estimates on arbitrary assumptions can completely overshadow the actuarial cost estimates and convert objective calculations into pure guesses. On the other hand, not considering potential behavioral changes can cause the estimates to miss the mark. Whether the miss is high or low is the puzzle.

Whether or not the promised savings from the Texas tort reforms are ever realized may never be known because there are a myriad of factors affecting liability claims. It is nearly impossible to determine with certainty whether changes in claims severity or claim frequency arise from tort reform or some other phenomenon. In the recent 1997 hearing concerning the impact of these tort reforms, updated closed claim data for the year 1996 were introduced into evidence. These more recent data did not yet indicate tort reform savings different from the actuarial indications submitted in the 1995 hearing. If the 1997 hearing results in a decision to an increase the mandated rate reductions, it will be because of anticipated behavioral changes, not because of actual savings materializing in the claims data.

Exhibit 1

House Bill 1988
Rate Reductions

Coverage	Default Rate Reduction	Ordered Rate Reduction
Professional Liability		
Physicians, Other Health Care Provider	30%	3.5% to 11.5%*
Hospital	30%	3.5% to 15.0% *
Commercial Liability - Products/Completed Operations	25%	12.5%
Private Passenger Auto B.I. Liability	15%	7.5%
Commercial Auto B.I. Liability	20%	12.0%
Personal Umbrella and Excess Liability	20%	7.5%
Homeowners and Renters Liability	5%	0%
Farm/Ranch Owners Liability	5%	10%
Liability Portion of CMP	10%	12.5%
Employer's Liability Portion of Work. Comp.	10%	0%
Other Commercial Liability		
Umbrella	15%	18%
Excess Liab. For G.L., Auto, Products	15%	18%
Excess Med. Prof. - Physicians	15%	4.5% to 15% *
Excess Med. Prof. - Hospitals	15%	4.5% to 20% *
Excess Med. Prof. - Other	15%	0.5% to 17.5% *
Misc.	15%	1% to 12.5% **

* Varies by occurrence v. Claims made and timing of suits
** Varies by subline

Impact of Punitive Damages Reform

Private Passenger Automobile Bodily Injury Liability
(Including UM/UIM)

		All Claims Settlement Range				
		Less Than $20,000	$20,001- $50,000	$50,001- $100,000	Over $100,000	Wt'd Average
(1)	Total Savings Resulting From Caps on Awards	$ -	$ 4,050,000	$ -	$ 82,527	
(2)	Total Punitive Damages	4,000	4,358,383	228,977	733,873	
(3)	Remaining Punitive Damages (2) - (1)	4,000	308,383	228,977	651,346	
(4)	Estimated Savings Resulting From Elimination of Gross Negligence Standard and Adoption of Clear and Convincing Standard of Proof (3) x 0.25	1,000	77,096	57,244	162,837	
(5)	Total Estimated Savings (1) + (4)	1,000	4,127,096	57,244	245,364	
(6)	Savings Attributable to Excess of Policy Limits	-	4,111,250	23,387	48,000	
(7)	Net Savings to Insurance System (5) - (6)	1,000	15,846	33,857	197,364	
(8)	Total Primary Insurance System Paid Losses within Interval based on Survey (Special Survey, Q7a + Q7d)	1,913,379	8,689,387	21,279,076	27,594,028	
(9)	Aggregate Paid Losses in Interval for All Companies Responding to Survey	593,823,708	313,240,391	93,763,324	58,633,424	
(10)	Estimated Percentage Loss Savings In Interval based on Survey (7) / (8) Wt'd is based on aggregate paid in line (9)	0.1%	0.2%	0.2%	0.7%	**0.1%**

Source: Special Texas Closed Claim Survey

Exhibit 2
Page 2

Impact Of Punitive Damages Reform

General Liability - Non Toxic
Bodily Injury Liability

		1991	1992	1993	Total
1)	Total Savings Resulting From Caps on Awards	16,008,543	6,502,515	10,030,125	32,541,183
2)	Total Punitive Damages	30,402,613	24,841,918	23,358,465	78,602,996
3)	Remaining Punitive Damages (2) - (1)	14,394,070	18,339,403	13,328,340	46,061,813
4)	Estimated Savings Resulting From Elimination of Gross Negligence Standard and Adoption of Clear and Convincing Standard of Proof (3) x 0.25	3,598,518	4,584,851	3,332,085	11,515,453
5)	Total Estimated Savings (1) + (4)	19,607,061	11,087,366	13,362,210	44,056,636
6)	Savings Attributable to Excess of Policy Limits	10,239,367	1,960,750	6,927,000	19,127,116
7)	Net Savings to Insurance System (5) - (6)	9,367,694	9,126,616	6,435,210	24,929,520
8)	Total Primary Insurance System Paid Losses (Line 33 from Joint and Several Exhibit)	367,905,558	342,958,174	387,622,846	1,098,486,578
9)	Ratio of Insured Savings to Total Primary Insurance System Losses (7) / (8)	2.5%	2.7%	1.7%	2.3%

Source: Regular Texas Closed Claim Survey

Exhibit 3
Page 1

Impact Of Joint And Several Reform

General Liability - Non Toxic
Bodily Injury Liability

		1991	1992	1993	Total
1)	Losses Paid on Claims with Complete Settlement Information and Greater than $25,000 that were affected by Joint and Several Liability				
	a) Primary Insurer	29,874,219	29,917,003	17,905,902	77,697,124
	b) Deductible Applied to Claims	841,537	1,891,134	460,000	3,192,671
	c) *Excess Over Policy Limits*	5,069,412	3,321,104	6,850,444	15,240,960
	d) Other Insurers	35,850,853	29,724,073	39,470,189	105,045,115
	e) Total	71,636,021	64,853,314	64,686,535	201,175,870
2)	Total Savings				
	a) Primary Insured	8,835,300	6,347,199	4,457,500	19,639,999
	b) Other Insured	1,935,048	1,763,059	5,063,066	8,761,173
	c) Total	10,770,348	8,110,258	9,520,566	28,401,172
3)	Amount of Primary Insured Savings Attributable to Excess of Policy Limits	1,202,301	1,000,000	1,195,000	3,397,301
4)	Net Savings - Primary Insured (2a) - (3)	7,632,999	5,347,199	3,262,500	16,242,698
5)	Estimated Savings - Other Insured, Net of Excess Portion (2b) x [(4) / (2a)]	1,671,728	1,485,289	3,705,721	7,245,677
6)	Portion of Primary Insureds Losses Excess of Policy Limits (1c) / (1a + 1b + 1c)	0.142	0.095	0.272	0.159
7)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion [1 - ((6) x 0.75)] x (1d)	32,041,815	27,616,497	31,428,134	92,554,406
8)	Total Primary Insured Losses (1a) + (1b) + (7)	62,757,571	59,424,634	49,794,036	173,444,201
9)	Ratio of Savings to Primary Insured Losses [(4) + (5)] / (8)	14.8%	11.5%	14.0%	13.5%
10)	Loss Paid on Claims Greater than *$25,000 with Incomplete Settlement*				
	a) Primary Insurer	24,440,468	18,961,395	17,903,779	61,305,642
	b) Deductible Applied to Claims	972,256	1,547,100	214,030	2,733,386
11)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion (10a) x [(7) / (1a)]	26,213,805	17,503,334	31,424,408	73,028,537
12)	Estimated Total Net Primary Insurance System Paid Losses (10a) + (10b) + (11)	51,626,529	38,011,829	49,542,217	137,067,565
13)	Estimated Total Net Primary Insurance System Savings (12) x (9)	7,654,387	4,370,500	6,932,982	18,562,133

Source: Regular Texas Closed Claim Survey

Exhibit 3
Page 2

Impact Of Joint And Several Reform

General Liability - Non Toxic
Bodily Injury Liability

		1991	1992	1993	Total
14)	Losses Paid on Claims with Complete Settlement Information and Greater than $25,000 that were not affected by Joint and Several Liability				
	a) Primary Insurer	171,409,015	166,994,663	187,240,735	525,644,413
	b) Deductible Appied to Claims	6,004,449	7,873,253	15,839,907	29,717,609
	c) Excess Over Policy Limits	23,597,461	32,228,004	22,171,523	77,996,988
	d) Other Insurers	33,948,702	35,230,315	55,712,005	124,891,022
	e) Total	234,959,627	242,326,235	280,964,170	758,250,032
15)	Ratio of Primary Insured Excess Payments to Total Primary Insured Payments (14c) / (14a + 14b + 14c)	0.117	0.156	0.098	0.123
16)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion [1 - ((15) x 0.75)] x (14d)	30,959,674	31,118,442	51,599,214	113,355,949
17)	Total Net Primary Insurance System of Paid Losses (14a) + (14b) + (16)	208,373,138	205,986,358	254,679,856	668,717,971
18)	Estimated Total Net Primary Insurance System Savings	0	0	0	0
19)	Losses Paid on Claims with Complete Settlement Information and Between $10,000 and $25,000 that had Multiple Defendants				
	a) Primary Insurer	444,982	385,435	472,220	1,302,637
	b) Deductible Applied to Claims	15,500	23,500	5,000	44,000
	c) Excess Over Policy Limits	0	0	0	0
	d) Other Insurers	252,406	241,272	275,883	769,561
	e) Total	712,888	650,207	753,103	2,116,198
20)	Ratio of Primary Insured Excess Payments to Total Primary Insured Payments (19c) / (19a + 19b + 19c)	0.000	0.000	0.000	0.000
21)	Estimated Payments on Behalf of Other Insureds. Net of Excess Portion [1 - ((20) x 0.75)] x (19d)	252,406	241,272	275,883	769,561
22)	Total Net Primary Insurance System of Paid Losses (19a) + (19b) + (21)	712,888	650,207	753,103	2,116,198
23)	Estimated Total Net Primary Insurance System Savings (22) x (9) x 0.25	26,424	18,690	26,347	71,646

Source: Regular Texas Closed Claim Survey

Impact Of Joint And Several Reform

General Liability - Non Toxic
Bodily Injury Liability

		1991	1992	1993	Total
24)	Losses Paid on Claims with Incomplete Settlement Information and Between $10,000 and $25,000 that had Multiple Defendants				
	a) Primary Insurer	503,010	533,188	625,637	1,661,835
	b) Deductible Applied to Claims	20,000	28,599	250	48,849
25)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion (24a) x [(21) / (19a)]	285,321	333,761	365,513	981,765
26)	Estimated Total Net Primary Insurance System Paid Losses (24a) + (24b) + (25)	808,331	895,548	991,400	2,692,449
27)	Estimated Total Net Primary Insurance System Savings (26) x (9) x 0.25	29,962	25,742	34,684	91,155
28)	Losses Paid on Claims Between $10,000 and $25,000 with Single Defendant				
	a) Primary Insurer	13,343,411	14,044,104	12,557,939	39,945,454
	b) Deductible Applied to Claims	420,156	624,077	579,252	1,623,485
29)	Total Net Primary Insurance System of Paid Losses. (28a) + (28b)	13,763,567	14,668,181	13,137,191	41,568,939
30)	Estimated Total Net Primary Insurance System Savings	0	0	0	0
31)	Losses Paid on Claims $10,000 and Under				
	a) Total Paid	29,863,534	23,321,416	18,725,043	71,909,993
32)	Total Savings	0	0	0	0
33) -	Estimated Total Net Primary Insurance System Paid Losses (8) + (12) + (17) + (22) + (26) + (29) + (31)	367,905,558	342,958,174	387,622,846	1,097,517,315
34)	Estimated Total Net Primary Insurance System Savings [(4) + (5) + (13) + (18) + (23) + (27) + (30) + (32)] x 0.75	12,761,625	8,435,565	10,471,676	31,659,982
35)	Ratio of Net Primary Insurance Savings to Total Net Primary Insurance System of Paid Losses (34) / (33)	3.5%	2.5%	2.7%	2.9%

Source: Regular Texas Closed Claim Survey

Impact Of Joint And Several Reform

Commercial Automobile - Non Toxic

Bodily Injury Liability

		1991	1992	1993	Total
1)	Losses Paid on Claims with Complete Settlement Information and Greater than $25,000 that were affected by Joint and Several Liability				
	a) Primary Insurer	17,196,115	19,514,296	13,350,760	50,061,171
	b) Deductible Applied to Claims	1,500	250	683,500	685,250
	c) Excess Over Policy Limits	4,161,950	4,672,080	1,447,173	10,281,203
	d) Other Insurers	4,390,022	8,819,068	6,324,825	19,533,915
	e) Total	25,749,587	33,005,694	21,806,258	80,561,539
2)	Total Savings				
	a) Primary Insured	6,734,638	7,234,346	3,919,848	17,895,941
	b) Other Insured	354,485	476,889	1,083,281	2,155,148
	c) Total	7,089,123	7,711,235	5,003,129	20,051,089
3)	Amount of Primary Insured Savings Attributable to Excess of Policy Limits	2,040,250	2,416,670	75,000	4,546,750
4)	Net Savings - Primary Insured (2a) - (3)	4,694,388	4,817,676	3,844,848	13,349,191
5)	Estimated Savings - Other Insured, Net of Excess Portion (2b) x [(4) / (2a)]	247,094	317,582	1,062,554	1,607,598
6)	Portion of Primary Insureds Losses Excess of Policy Limits (1c) / (1a + 1b + 1c)	0.195	0.193	0.093	0.168
7)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion [1 - ((6) x 0.75)] x (1d)	3,748,469	7,541,397	5,881,401	17,065,785
8)	Total Primary Insured Losses (1a) + (1b) + (7)	20,946,084	27,055,943	19,915,661	67,812,206
9)	Ratio of Savings to Primary Insured Losses [(4) + (5)] / (8)	23.6%	19.0%	24.6%	22.1%
10)	Loss Paid on Claims Greater than $25,000 with Incomplete Settlement				
	a) Primary Insurer	6,612,509	8,064,724	10,684,862	25,362,095
	b) Deductible Applied to Claims	34,875	500,000	251,000	785,875
11)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion (10a) x [(7) / (1a)]	1,441,418	3,116,653	4,706,995	8,645,904
12)	Estimated Total Net Primary Insurance System Paid Losses (10a) + (10b) + (11)	8,088,802	11,681,377	15,642,857	34,793,874
13)	Estimated Total Net Primary Insurance System Savings (12) x (9)	1,908,265	2,217,143	3,854,544	7,674,203

Source: Regular Texas Closed Claim Survey

Impact Of Joint And Several Reform

Commercial Automobile - Non Toxic

Bodily Injury Liability

		1991	1992	1993	Total
14)	Losses Paid on Claims with Complete Settlement Information and Greater than $25,000 that were not affected by Joint and Several Liability				
	a) Primary Insurer	259,710,240	304,880,531	302,641,396	867,232,167
	b) Deductible Appied to Claims	11,909,996	9,839,276	14,750,546	36,499,818
	c) Excess Over Policy Limits	22,886,526	43,392,330	40,356,727	106,635,583
	d) Other Insurers	10,745,143	11,630,342	11,814,408	34,189,893
	e) Total	305,251,905	369,742,479	369,563,077	1,044,557,461
15)	Ratio of Primary Insured Excess Payments to Total Primary Insured Payments (14c) / (14a + 14b + 14c)	0.078	0.121	0.113	0.106
16)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion [1 - ((15) x 0.75)] x (14d)	10,118,878	10,573,408	10,814,843	31,483,557
17)	Total Net Primary Insurance System of Paid Losses (14a) + (14b) + (16)	281,739,114	325,293,215	328,206,785	935,215,542
18)	Estimated Total Net Primary Insurance System Savings	0	0	0	0
19)	Losses Paid on Claims with Complete Settlement Information and Between $10,000 and $25,000 that had Multiple Defendants				
	a) Primary Insurer	231,770	274,786	277,061	783,617
	b) Deductible Applied to Claims	0	0	12,500	12,500
	c) Excess Over Policy Limits	0	0	0	0
	d) Other Insurers	86,481	154,738	127,000	368,219
	e) Total	318,251	429,524	416,561	1,164,336
20)	Ratio of Primary Insured Excess Payments to Total Primary Insured Payments (19c) / (19a + 19b + 19c)	0.000	0.000	0.000	0.000
21)	Estimated Payments on Behalf of Other Insureds, Net of Excess Portion [1 - ((20) x 0.75)] x (19d)	86,481	154,738	127,000	368,219
22)	Total Net Primary Insurance System of Paid Losses (19a) + (19b) + (21)	318,251	429,524	416,561	1,164,336
23)	Estimated Total Net Primary Insurance System Savings (22) x (9) x 0.25	18,770	20,381	25,661	64,202

Source: Regular Texas Closed Claim Survey

Exhibit 4
Page 2

Combined Impact of Venue, DTPA, Frivolous and Joint & Several Damages Reforms

Products Liability Non-BI Claims

		All Claims Primary Insurer Settelment Range		
		Less Than $25,000	Over $25,000	Weighted Average
(1)	Amount of Reduced Payments (Special Survey Q4k)	$ 22,000	$ 44,000	
(2)	Savings Attributable to Excess	-	-	
(3)	Net Savings to Primary Insurer	22,000	44,000	
(4)	Total Primary Insurance Paid Losses within Interval based on Survey (Special Survey, Q7a)	293,173	4,929,870	
(5)	Aggregate Paid Losses in Interval for All Companies Responding to Survey	1,307,450	4,929,870	
(6)	Estimated Percentage Loss Savings	7.5%	0.9%	**2.3%**

Source: Special Texas Closed Claim Survey

Exhibit 5
Page 1

Impact of Venue Reform

General Liability
Bodily Injury Claims

	County	Estimated Venue Savings*
	Bexar	$200,000
	Harris	$665,250
	Jim Wells	$462,500
	Nuces	$40,000
	Panola	$70,000
	Rusk	$2,293,546
(1)	Total Savings For Claims in Survey	$3,731,296
(2)	Total Settlement Amounts For Outside Claims in Special Venue Survey	$54,466,493
(3)	Overall Savings For Claims in Survey (1) / (2)	6.9%
(4)	Total amount of Settlements for Outside Claims >$100,000 that were not responded to in survey	$12,123,898
(5)	Expected Savings on No Response Claims (3) x (4)	$830,563
(6)	Total Estimated Venue Savings (1) + (5)	$4,561,859
(7)	Total 1993 Settlement Amounts including amount Paid < $10,000	$413,098,311
(8)	Estimated % Venue Savings (6) / (7)	1.1%

* Per response 2d. Used lowest response if % range was given, no offset for excess
Excludes claims where injury occured out-of-state and proper venue indicated in Q2a and Q2b.

Source: Special Venue Data Call for 1993 - Claims Over $100,000

Impact of Venue Reform

Products Liability Bodily Injury Losses

Summary of Estimated Savings For Claims > $25,000

Exhibit 5
Page 2

	County Suit Filed	Claims from Inside County			Claims from Outside County			All Claims		
		Number	Total Settlement	Average Settlement	Number	Total Settlement	Average Settlement	Number	Total Settlement	Average Settlement
Total 1991 - 1993	Anderson	1	1,600,000	1,600,000	0	0	0	1	1,600,000	1,600,000
	Andrews	2	186,046	93,023	1	40,000	40,000	3	226,046	75,349
	Angelina	4	778,433	194,608	0	0	0	4	778,433	194,608
	Aransas	0	0	0	0	0	0	0	0	0
	Archer	0	0	0	0	0	0	0	0	0
	Armstrong	0	0	0	0	0	0	0	0	0
	Atascosa	0	0	0	0	0	0	0	0	0
	Austin	1	30,000	30,000	0	0	0	1	30,000	30,000
	Bailey	0	0	0	0	0	0	0	0	0
	Bandera	0	0	0	0	0	0	0	0	0
	Bastrop	0	0	0	0	0	0	0	0	0
	Travis	13	4,739,019	364,540	7	4,763,000	680,429	20	9,502,019	475,101
	Trinity	1	235,000	235,000	0	0	0	1	235,000	235,000
	Tyler	1	450,000	450,000	0	0	0	1	450,000	450,000
	Upshur	1	37,500	37,500	0	0	0	1	37,500	37,500
	Upton	0	0	0	0	0	0	0	0	0
	Uvalde	0	0	0	0	0	0	0	0	0
	Val Verde	4	750,210	187,553	0	0	0	4	750,210	187,553
	Van Zandt	0	0	0	0	0	0	0	0	0
	Victoria	2	355,658	177,829	1	30,000	30,000	3	385,658	128,553
	Walker	0	0	0	0	0	0	0	0	0
	Waller	0	0	0	0	0	0	0	0	0
	Ward	0	0	0	0	0	0	0	0	0
	Washington	0	0	0	0	0	0	0	0	0
	Webb	4	281,705	70,426	0	0	0	4	281,705	70,426
	Wharton	0	0	0	0	0	0	0	0	0
	Wheeler	2	678,464	339,232	0	0	0	2	678,464	339,232
	Wichita	2	475,000	237,500	3	1,034,263	344,754	5	1,509,263	301,853
	Wilbarger	0	0	0	0	0	0	0	0	0
	Willacy	1	70,000	70,000	0	0	0	1	70,000	70,000
	Williamson	0	0	0	0	0	0	0	0	0
	Winkler	0	0	0	0	0	0	0	0	0
	Wise	2	132,500	66,250	0	0	0	2	132,500	66,250
	Wood	0	0	0	0	0	0	0	0	0
	Yoakum	0	0	0	0	0	0	0	0	0
	Young	0	0	0	0	0	0	0	0	0
	Zapata	1	240,000	240,000	0	0	0	1	240,000	240,000
	Zavala	1	300,000	300,000	0	0	0	1	300,000	300,000
	Statewide	667	253,713,841	380,381	127	65,404,391	514,995	794	319,118,232	401,912

Source: Regualar Texas Closed Claim Survey

Impact of Venue Reform
Products Liability Bodily Injury Losses
Summary of Estimated Savings For Claims > $25,000

Total 1991 - 1993

County Suit Filed	Frequency Ratios: Ratio 1: # Outside to # Inside	Ratio 2: # Outside to # Total	Ratio 3: # Inside to Population	Ratio 4: # Outside to Population	Ratio 5: Ratio 1 to SW Ratio 1	Severity Ratios: Ratio 6: All Avg. to SW All Avg.	Ratio 7: Out Avg. to SW In Avg.	Outside # Claims in Selected Counties	Excess # Outside Claims	Excess # Reduced By 25% For Excess Loss	Estimated Savings
Anderson	0.0%	0.0%	2.1%	0.0%	0.0%	398.1%	0.0%	-	-	-	-
Andrews	50.0%	33.3%	13.9%	7.0%	262.6%	18.7%	10.5%	-	-	-	-
Angelina	0.0%	0.0%	5.7%	0.0%	0.0%	48.4%	0.0%	-	-	-	-
Aransas	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Archer	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Armstrong	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Atascosa	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Austin	0.0%	0.0%	5.0%	0.0%	0.0%	7.5%	0.0%	-	-	-	-
Bailey	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Bandera	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Bastrop	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
.	.	.	.	.	.	.	.	.	.	.	.
Travis	53.8%	35.0%	2.3%	1.2%	282.8%	118.2%	178.9%	7.00	3.48	2.61	783,375
Trinity	0.0%	0.0%	8.7%	0.0%	0.0%	58.5%	0.0%	-	-	-	-
Tyler	0.0%	0.0%	6.0%	0.0%	0.0%	112.0%	0.0%	-	-	-	-
Upshur	0.0%	0.0%	3.2%	0.0%	0.0%	9.3%	0.0%	-	-	-	-
Upton	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Uvalde	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Val Verde	0.0%	0.0%	10.3%	0.0%	0.0%	46.7%	0.0%	-	-	-	-
Van Zandt	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Victoria	50.0%	33.3%	2.7%	1.3%	262.6%	32.0%	7.9%	-	-	-	-
Walker	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Waller	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Ward	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Washington	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Webb	0.0%	0.0%	3.0%	0.0%	0.0%	17.5%	0.0%	-	-	-	-
Wharton	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Wheeler	0.0%	0.0%	34.0%	0.0%	0.0%	84.4%	0.0%	-	-	-	-
Wichita	150.0%	60.0%	1.6%	2.5%	787.8%	75.1%	90.6%	-	-	-	-
Wilbarger	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Willacy	0.0%	0.0%	5.6%	0.0%	0.0%	17.4%	0.0%	-	-	-	-
Williamson	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Winkler	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Wise	0.0%	0.0%	5.8%	0.0%	0.0%	16.5%	0.0%	-	-	-	-
Wood	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Yoakum	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Young	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
Zapata	0.0%	0.0%	10.8%	0.0%	0.0%	59.7%	0.0%	-	-	-	-
Zavala	0.0%	0.0%	8.2%	0.0%	0.0%	74.6%	0.0%	-	-	-	-
Statewide	19.0%	16.0%	3.9%	0.7%	100.0%	100.0%	135.4%	39	14	11	6,075,857
							Total Settlement Amount For Claims > $25,000				323,731,699

Source: Regular Texas Closed Claim Survey

Impact of Venue Reform

Environmental Liability Bodily Injury Losses
Summary of Estimated Savings For Claims > $25,000

Estimated Venue Savings	$49,959
Total Settlement Amount For Claims > $25,000	$74,292,911
Estimated Percentage Savings	0.1%

Source: Regular Texas Closed Claim Survey

Exhibit 6

Average Paid ALAE Per Claim
By Primary Insured's % of Fault
1991-1993 Data

Line	Primary Insured's % of Fault	Number of Claims	Average Paid ALAE
General Liability	0-10	409	43,990
	11-25	487	44,504
	26-50	1,157	31,941
	51-75	879	27,636
	76-90	667	25,868
	91-100	1,794	29,141
	Total	5,393	31,605
Commercial Auto	0-10	319	13,337
	11-25	120	24,007
	26-50	471	21,891
	51-75	539	18,559
	76-90	677	13,839
	91-100	6,233	8,877
	Total	8,359	11,024
Commercial Multi-Peril	0-10	198	49,507
	11-25	235	31,171
	26-50	723	25,102
	51-75	599	18,298
	76-90	454	15,325
	91-100	1,191	17,882
	Total	3,400	21,909
Medical Professional	0-10	232	31,616
	11-25	199	44,418
	26-50	466	42,923
	51-75	253	44,148
	76-90	155	35,230
	91-100	1,421	32,011
	Total	2,726	36,058
Other Professional	0-10	10	42,755
	11-25	16	39,187
	26-50	19	127,346
	51-75	12	35,421
	76-90	8	68,663
	91-100	125	23,520
	Total	190	38,887

Source: Regular Texas Closed Claim Survey

Exhibit 6

Average Paid ALAE Per Claim
By Primary Insured's % of Fault
1991-1993 Data

Line	Primary Insured's % of Fault	Number of Claims	Average Paid ALAE
General Liability	0-10	409	43,990
	11-25	487	44,504
	26-50	1,157	31,941
	51-75	879	27,636
	76-90	667	25,868
	91-100	1,794	29,141
	Total	5,393	31,605
Commercial Auto	0-10	319	13,337
	11-25	120	24,007
	26-50	471	21,891
	51-75	539	18,559
	76-90	677	13,839
	91-100	6,233	8,877
	Total	8,359	11,024
Commercial Multi-Peril	0-10	198	49,507
	11-25	235	31,171
	26-50	723	25,102
	51-75	599	18,298
	76-90	454	15,325
	91-100	1,191	17,882
	Total	3,400	21,909
Medical Professional	0-10	232	31,616
	11-25	199	44,418
	26-50	466	42,923
	51-75	253	44,148
	76-90	155	35,230
	91-100	1,421	32,011
	Total	2,726	36,058
Other Professional	0-10	10	42,755
	11-25	16	39,187
	26-50	19	127,346
	51-75	12	35,421
	76-90	8	68,663
	91-100	125	23,520
	Total	190	38,887

Source: Regular Texas Closed Claim Survey

TEXAS DEPARTMENT OF INSURANCE

SPECIAL CLOSED CLAIM SURVEY FORM

Company Name and Address: ______________________________

____

____

NAIC Company Code: ______________ *NAIC Group Code:* __________

Claim File Identification: ______________ *Claimant Suffix:* __________

Form Completed By: ______________ *Tel:* ______________

Form Reviewed by (Coordinator): ______________ *Tel:* ______________

____

Reserved for State Use: (Do not write in this area).

IC Co. Code:______ NAIC Group Code:______ Claim File ID:______________

te Information:

Complete the following items: MM / DD / YY

1. a. Date of injury .... ____/____/____

   b. Date suit filed (indicate N/A if not applicable) .... ____/____/____

   c. Date of settlement .... ____/____/____

   d. Date claim closed .... ____/____/____

Policy Information:

2. a. Policy Type (choose one) .... ________

   1. Private passenger auto liability (including UM/UIM coverage)
   2. Homeowners multiple peril
   3. Farmowners/Ranchowners multiple peril
   4. Personal umbrella
   5. Product liability
   6. Monoline general liability
   7. Commercial auto liability
   8. Commercial multi-peril liability (including TCPP and TBOP)
   9. Other professional liability

   b. Indicate the code for the line of business that the claim was reported on under the Annual Statement. (choose one) .... ________

   030 - Farmowners multiple peril
   040 - Homeowners multiple peril
   052 - Commercial multiple peril (liability portion)
   170 - Other liability
   180 - Product liability
   192 - Other private passenger auto liability
   194 - Other commercial auto liability

   c. What is the per person policy limit? (indicate N/A if not applicable) .... $______________

   d. What is the per occurrence policy limit? (indicate N/A if not applicable) ..... $______________

   e. What is the aggregate policy limit? (indicate N/A if not applicable) .... $______________

   f. What is the deductible/self-insured retention limit? (indicate N/A if not applicable) .... $______________

Venue Information:

3. a. Indicate the county number where the insured's principal office is located if a commercial entity, or the insured's principal place of residence if not a commercial entity. .... ________

   b. Indicate the county number where the injury was alleged to have occurred.... ________

   c. Indicate the county number of plaintiff's residence at the time of the incident.. ________

NAIC Co. Code:______ NAIC Group Code:______ Claim File ID:____________

d. Indicate the county number where suit was initially filed (indicate N/A if not applicable). . . . . ________

e. Indicate the county number where the trial was held (indicate N/A if not applicable). . . . . ________

f. If the new law affecting choice of venue had been in effect when this claim was made, would it have impacted the settlement of this claim?* . . . . [ ] Unk [ ] Y [ ] N

Civil Justice and General Information:

4. a. What stage of the legal system was a settlement reached or an award made? ________
   Choose one
   1. Alternative dispute resolution
   2. Settlement, no suit filed
   3. Suit filed, settlement reached before trial
   4. During trial, before court verdict
   5. Court verdict
   6. Settlement reached after verdict

b. Was your insured a business? . . . . [ ] Y [ ] N

c. If yes to item 4(b), indicate what type of business? (indicate N/A if not applicable)

____

d. Did this claim arise from the rendering of a professional service? . . . . [ ] Y [ ] N

e. Was the claimant a business? . . . . [ ] Unk [ ] Y [ ] N

f. Were there any defendants (tort feasors) other than your insured involved in relation to this claim? . . . . [ ] Y [ ] N

g. Have all of the other defendants (tort feasors) settled relative to this claim? . . [ ] Unk [ ] Y [ ] N

h. Did this claim allege Deceptive Trade Practices Act (DTPA) violations against your insured?. . . . . [ ] Y [ ] N

If yes to item 4(h), answer items 1 and 2:

1. Were any payments for this claim due to DTPA allegations against your insured? . . . . [ ] Y [ ] N

2. If the new law limiting DTPA actions had been in effect when this claim was made, would it have impacted the settlement of this claim?* . . . . [ ] Unk [ ] Y [ ] N

*Use your most professional opinion.

AIC Co. Code:_____ NAIC Group Code:_____ Claim File ID:____________

i. If the new law punishing parties for filing frivolous pleadings had been in effect when this claim was made, would it have impacted the settlement of this claim?* .... [ ] Unk [ ] Y [ ] N

j. If the new law limiting payments for joint and several liability claims had been in effect when this claim was made, would it have impacted the settlement of this claim?* .... [ ] Unk [ ] Y [ ] N

k. If any of the responses to items 3(f), 4(h).2, 4(i), or 4(j) were Yes, estimate the amount that your payment would have been reduced? (indicate "Unknown" if applicable)* .... $____________

Allocated Loss Adjustment Expenses:

Loss adjustment expenses must be allocated on a per claim basis. Round all amounts to dollars.

5. a. Were there any allocated loss adjustment expenses paid relating to this claim? [ ] Y [ ] N

b. Indicate the amount paid for defense counsel (either outside or in-house). .. $____________

c. Indicate the amount of all other allocated loss adjustment expense. .... $____________

d. Indicate the total allocated loss adjustment expense [sum of items 5(b) + 5(c)]. $____________

Allocation of Damages:

Damages must be allocated based on the total indemnity amount indicated in item 7(e).*

6. a. 1. Economic losses .... $____________
2. Non-economic losses .... $____________
3. Exemplary damages .... $____________
4. Interest .... $____________
5. Total .... $____________

The percentage of fault allocations do not have to agree with the percentage of the settlement paid by that party. Round percentages to whole numbers.*

b. Estimated percentage of fault assigned to:
1. Injured party .... ______%
2. Your insured .... ______%
3. Other parties .... ______%
4. Total 100%

Settlement Information:

Indicate the following dollar amounts for indemnity payments as applicable to this claim. Indicate unknown where applicable. Do not indicate unknown in item 7(e). Round all amounts to dollars.

7. *Amounts paid on behalf of your insured [items 7(a) through 7(c)]*
a. Amount paid under the policy covering this loss .... $____________
b. Amount paid by either the insured or an insurer for underlying coverage .... $____________
c. Amount paid by either the insured or an insurer for coverage exceeding your policy limits .... $____________
d. *Amounts paid on behalf of other parties* .... $____________
e. **Total Amount of Settlement** .... $____________

*Use your most professional opinion.

TEXAS DEPARTMENT OF INSURANCE

SPECIAL EMPLOYERS' LIABILITY CLAIM SURVEY FORM

Company Name and Address: ______________________________
____
____
____

NAIC Company Code: ______________ *NAIC Group Code:* __________

Claim File Identification: ______________ *Claimant Suffix:* __________

Form Completed By: ______________ *Tel:* ( ) __________

Form Reviewed by (Coordinator): ______________ *Tel:* ( ) __________

Reserved for State Use: (Do not write in this area).

NAIC Co. Code: ______ **NAIC Group Code:** ______ **Claim File ID:** ____________

ite Information:

Complete the following items: MM / DD / YY

1. a. Date of injury .... ___/___/___

b. Date suit filed (indicate N/A if not applicable) .... ___/___/___

c. Date of settlement of employers' liability claim (indicate N/A if not applicable) ___/___/___

d. Is the employers' liability claim still open? .... ❑ Y ❑ N

e. Is the corresponding workers' compensation claim still open? .... ❑ Y ❑ N

Policy Information:

2. a. What is the employers' liability policy limit? .... $____________

b. What is the deductible/self-insured retention limit? (indicate N/A if not applicable) .... $____________

Venue Information:

3. a. Indicate the county number where the insured's principal office is located. ______

b. Indicate the county number where the injury was alleged to have occurred.... ______

c. Indicate the county number of plaintiff's principal office is located if a ~~commercial entity, or the plaintiff's principal place of residence, at the time of~~ the incident.......... ______

d. Indicate the county number where suit was initially filed (indicate N/A if not applicable).......... ______

e. Indicate the county number where the trial was held (indicate N/A if not applicable).......... ______

f. If the new law affecting choice of venue had been in effect when this claim was made, would it have impacted the settlement of this claim?* .... ❑ Unk ❑ Y ❑ N

Civil Justice and General Information:

4. a. Indicate the type of business of your insured.

____

b. Was the claimant a business? .... ❑ Unk ❑ Y ❑ N

c. Were there any defendants (tort feasors) other than your insured involved in relation to this claim? .... ❑ Y ❑ N

d. Have <u>all</u> of the other defendants (tort feasors) settled relative to this claim? . ❑ Unk ❑ Y ❑ N

*Use your most professional opinion.

NAIC Co. Code:______ NAIC Group Code:______ Claim File ID:__

e. Did this claim allege Deceptive Trade Practices Act (DTPA) violations against your insured? . . . . ❑ Y [

If yes to item 4(e), answer items 1 and 2:

1. Were any payments for this claim due to DTPA allegations against your insured? . . . . ❑ Y ❑

2. If the new law limiting DTPA actions had been in effect when this claim was made, would it have impacted the settlement of this claim?* . . . . ❑ Unk ❑ Y ❑ N

f. Did the payment on this claim include exemplary (punitive) damages? . . . . . ❑ Y ❑ N

If yes to 4(f), would the new law capping exemplary damages, replacing the gross negligence standard with one of malice, and changing the required level of proof from the preponderance of the evidence to clear and convincing evidence, have impacted this settlement? . . . . ❑ Y ❑ N

g. If the new law punishing parties for filing frivolous pleadings had been in effect when this claim was made, would it have impacted the settlement of this claim?* . . . . ❑ Unk ❑ Y ❑ N

h. If the new law limiting payments for joint and several liability claims had been in effect when this claim was made, would it have impacted the settlement of this claim?* . . . . ❑ Unk ❑ Y ❑ N

i. If any of the responses to items 3(f), 4(e).2, 4(f), 4(g) or 4(h) were Yes, estimate the amount that your payment would have been reduced? (indicate "Unknown" if applicable)* . . . . $____________

j. Describe the nature of the injury ________________________

k. Did the claim involve a hold harmless agreement? . . . . ❑ Y ❑ N

l. Did the claim involve action over? . . . . ❑ Y ❑ N

*Use your most professional opinion.

Allocated Loss Adjustment Expenses:

Loss adjustment expenses must be allocated on a per claim basis. Round all amounts to dollars.

5. a. Were there any allocated loss adjustment expenses paid relating to the employers' liability portion of this claim? ❑ Y ❑ N

b. Indicate the amount paid for defense counsel (either outside or in-house). . . $____________

c. Indicate the amount of all other allocated loss adjustment expense. . . . . $____________

*Use your most professional opinion.

NAIC Co. Code:______ NAIC Group Code:______ Claim File ID:____________

location of Damages:

d. Indicate the total allocated loss adjustment expense [sum of items 5(b) + 5(c)]. $__________

amages must be allocated based on the total indemnity amount indicated in item 7(e).*

6. a. 1. Economic losses .... $__________
2. Non-economic losses .... $__________
3. Exemplary damages .... $__________
4. Interest .... $__________
5. Total .... $__________

The percentage of fault allocations do not have to agree with the percentage of the settlement paid by that party. Round percentages to whole numbers.*

b. <u>Estimated percentage of fault assigned to:</u>
1. Injured party .... ______%
2. Your insured .... ______%
3. Other parties .... ______%
4. Total 100%

Settlement Information:

Indicate the following dollar amounts for indemnity payments as applicable to this claim. Indicate unknown where applicable. Do <u>not</u> indicate unknown in item 7(e). Round all amounts to dollars.

7. *Amounts paid on behalf of your insured [items 7(a) through 7(c)]*
a. Amount paid under the policy covering this loss .... $__________
b. Amount paid by either the insured or an insurer for underlying coverage .... $__________
c. Amount paid by either the insured or an insurer for coverage exceeding your policy limits .... $__________
d. *Amounts paid on behalf of other parties* .... $__________
e. **Total Amount of Settlement** .... $__________

*Use your most professional opinion.

TEXAS DEPARTMENT OF INSURANCE

SPECIAL DTPA AND VENUE CLAIM SURVEY FORM

Company Name and Address: ______________________________

____

____

____

NAIC Company Code: ______________ *NAIC Group Code:* __________

Claim File Identification: ______________ *Claimant Suffix:* __________

Form Completed By: ______________ *Tel:* ( ) __________

Form Reviewed by (Coordinator): ______________ *Tel:* ( ) __________

Reserved for State Use: (Do not write in this area).

Complete the Following Items:

1. Did this claim allege Deceptive Trade Practices Act (DTPA) violations against your insured? ❑ Y ❑ N

   If your answer is "no", go to question 2; otherwise answer the following:

   a. Were any payments for this claim due to DTPA allegations against your insured? ❑ Y ❑ N

   b. Was the amount of settlement affected by the DTPA allegations? ❑ Y ❑ N

   c. If the response to "a" or "b" was "yes", please estimate the amount of by which it affected the cost of the claim (use your most professional opinion). ____________

   d. If the new law limiting DTPA actions had been in effect when this claim was made, would it have impacted the settlement of this claim (use your most professional opinion)? ❑ Unk ❑ Y ❑ N

2. a. Indicate county number where the insured's principal office is located. ____________

   b. Indicate county number of the plaintiff's residence at the time of the incident if plaintiff is a natural person. ____________

   c. If multiple defendants, would the plaintiff have been able to establish venue under the new law against any defendant in the county in which the original suit was actually filed? ❑ Y ❑ N

   If yes, give basis

   ____

   d. If the new law governing venue had been in effect and the county of suit of this claim would not qualify as proper venue, estimate the impact of the settlement. ____________

   e. Was the suit filed in Federal or State Court? ❑ Fed ❑ State