Pricing Catastrophe Reinsurance with Reinstatement Provisions Using a Catastrophe Model

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Abstract

In recent years catastrophe reinsurers' use of catastrophe models has been increasing until currently virtually all of the catastrophe reinsurers in the world use a catastrophe model to aid them in their pricing and portfolio management decisions.

This paper explicitly models various types of reinstatement provisions, including reinstatements that are limited by the number of occurrences and by the aggregate losses; and reinstatement premiums based on the size of loss and by the time elapsed to the first occurrence. The paper also investigates the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered.

This is an expansion of the methods developed in papers by Leroy J. Simon and Bjorn Sundt, which were written before the widespread use of catastrophe models.

The catastrophe model used for this paper is the Insurance / Investment Risk Assessment System (IRAS) produced by Risk Management Solutions, Inc.

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Introduction

In recent years catastrophe reinsurers' use of catastrophe models has been increasing until currently virtually all of the catastrophe reinsurers in the world use a catastrophe model to aid them in their pricing and portfolio management decisions.

Leroy Simon's 1972 paper [1] on catastrophe reinsurance investigated the relationships between various provisions of catastrophe reinsurance treaties to ensure consistency in pricing between contracts. In his paper he assumes that each loss causes a total loss to the layer of reinsurance. Bjorn Sundt expanded on this theme in his paper in 1991 [2], focusing on reinstatements based on aggregate losses. This paper applies the methods outlined in these previous works to the output of a catastrophe model to calculate a fair premium for a catastrophe treaty when reinstatement premium is considered.

The paper develops the fair premium for catastrophe reinsurance with various types of reinstatement provisions, including reinstatements that are limited by the number of occurrences and by the aggregate losses; and reinstatement premiums based on the size of loss and by the time elapsed to the first occurrence. The paper also investigates the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered.

The catastrophe model used for this paper is the Insurance / Investment Risk Assessment System (IRAS) produced by Risk Management Solutions, Inc.

As background, we start with some descriptions of reinstatement provisions and how they are applied. We then describe an event loss table, the output of the catastrophe model that gives us all of the information that we need to perform the calculations. Next we turn our attention to the calculation of the fair premium of catastrophe treaties with various types of reinstatement provisions. First we discuss reinstatement provisions that limit the number of occurrences, then reinstatement provisions that limit the aggregate losses. Finally we investigate reinstatement premiums that are pro rata as to time.

Reinstatement Provisions

A common feature of many catastrophe reinsurance contracts is a reinstatement provision. A reinstatement provision puts a limit on either the number of occurrences or the aggregate losses that will be paid under the contract. For example, if a contract has a provision for one reinstatement based on the number of occurrences, then the reinsurer will be responsible for at most two occurrences (original occurrence plus one reinstatement). If the contract has a provision for one reinstatement based on aggregate losses, and the limit is $1 million, then the reinsurer will be responsible for at most $2 million in aggregate, regardless of the number of occurrences.

The reinstatements may be free or paid. If the reinstatements are free, then all of the premium is paid up front. For paid reinstatements, a portion of the premium is paid following the occurrence of an event. For example, if a contract has a provision for one paid reinstatement, then after the first event the cedant will pay some premium to the reinsurer to reinstate the coverage for a second occurrence. This additional premium is called the reinstatement premium. The reinstatement premium may vary based on the amount of reinstatement (pro rata to full limit) or the time remaining in the contract (pro rata to full time). In this paper we will limit the discussion to reinstatement premium that is pro rata to full limit, and is either 100% to time or pro rata to full time.

Event Loss Table

An "event" as we use it in this paper is a scenario taken from the set of all possible outcomes. For example, event *e* might be an earthquake of magnitude 7.3 on the San Andreas fault centered two miles off the coast of San Francisco; and event *h* might be a category 3 hurricane making landfall in Dade county Florida with a specific track, central pressure, etc. The final product from an IRAS analysis is a table of events with their expected losses and annual occurrence rates. The set of events in the Event Loss Table (ELT) represents the full range of possible outcomes that can occur to a portfolio.

Suppose that we have a catastrophe treaty of *LMT* excess *ATT*, where *LMT* is the limit of the treaty, and *ATT* is the attachment point of the treaty. Denote the gross loss for the j^{th} event as $GLOSSj$ and the expected loss to the catastrophe treaty as Lj. We have

$$L_j = \int_{ATT}^{ATT+LMT}(GLOSS_j - ATT)f_j(GLOSS_j)dGLOSS_j + LMT[1 - F_j(ATT + LMT)] \quad (1)$$

where

$f_j(GLOSS_j)$ = probability density of the gross loss given that event j has occurred
$F_j(ATT + LMT)$ = cumulative probability that the gross loss ≤ ATT + LMT, given that event j has occurred

In the ELT shown below in Table 1, λ_j is the annual rate of occurrence for event j, and L_j is the expected loss to the catastrophe treaty for event j, calculated from equation (1).

Table 1 Event-Loss Table (ELT)

Event	Rate	Expected Loss
1	λ_1	L_1
2	λ_2	L_2
:	:	:
j	λ_j	L_j
:	:	:

J	λ_J	L_J

We assume here that each event is an independent random variable, each with a Poisson frequency distribution[1]. We assume that the occurrence of one event will have no effect on the rate or the expected loss of any other event. We look at these multi-events (the occurrence of one or more events) as a compound Poisson process[2] with a total rate equal to:

$$\lambda = \sum_j \lambda_j \qquad (2)$$

Hence, the probability of exactly n occurrences in a year for this process is given by

$$p(n) = \frac{\lambda^n e^{-\lambda}}{n!} \qquad (3)$$

The average annual loss (AAL) for event j is given by the expected frequency times the expected severity, which, given our Poisson frequency assumption, is $\lambda_j L_j$. Because we assume that each event is an independent random variable, the total AAL is the sum of the AAL's for all events:

$$AAL = \sum_j \lambda_j L_j \qquad (4)$$

This represents the pure premium of a treaty with unlimited free reinstatements.

With all of this as background, we now turn our attention to the calculation of the fair premium of catastrophe treaties with various types of reinstatement provisions. First we discuss reinstatement provisions that limit the number of occurrences, then reinstatement provisions that limit the aggregate losses. For each of these cases we assume that the reinstatement premiums are pro rata as to limit and 100% as to time. Finally we investigate reinstatement premiums that are pro rata as to time, where we calculate the expected arriving time for the occurrence of an event.

Reinstatements Limited by the Number of Occurrences

The reinstatement premium will be paid whenever an event occurs with losses to the catastrophe treaty and the reinstatements are not already used up. The amount of reinstatement premium (P_{reinst}) is

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[1] Other frequency distributions, such as Negative Binomial, may be appropriate for some perils or regions. The use of these distributions is beyond the scope of this paper.

[2] For more information on Poisson processes, see references [4], [5], and [6]

$$P_{reinst} = R \cdot c \cdot L \tag{5}$$

where

R = Premium rate paid up front (rate on line)
c = fraction of reinstatement premium rate versus up-front premium rate
L = loss to the catastrophe treaty, which is a random variable.

It can be seen that the reinstatement premium formula (5) is pro rata as to limit by noting that R, the rate on line, equals the premium (P) divided by the limit (LMT):

$$P_{reinst} = \frac{P}{LMT} \cdot c \cdot L = P \cdot c \cdot \frac{L}{LMT} \tag{6}$$

We calculate the fair up-front premium rate (ignoring expense and risk load charges) by setting the expected premium collections equal to the expected loss payments.

First we calculate the expected loss to the catastrophe treaty as the expected severity times the expected frequency.

The expected loss given an event has occurred (expected severity) is given by

$$S(L) = \frac{\sum_j \lambda_j L_j}{\lambda} \tag{7}$$

To calculate the expected frequency, we make use of the limited expected value function[3]. The expected number of occurrences limited to k occurrences is given by

$$\begin{aligned} E(n;k) &= \sum_{n=1}^{\infty} \min(n,k) \cdot p(n) \\ &= \sum_{n=1}^{k-1} n \cdot p(n) + k \cdot \left(1 - F(k-1)\right) \end{aligned} \tag{8}$$

where

$p(n)$ = the probability that exactly n events will occur, as calculated by equation (3)
$F(k\text{-}1)$ = the cumulative probability that k-1 or fewer events will occur.

Let *nor* be the number of occurrence reinstatements allowed. The total number of occurrences covered by the contract is *nor*+1 (one original occurrence + *nor* additional occurrences). We define $E_O(L;nor + 1)$ to be the expected loss limited to *nor*+1 occurrences, which is the expected severity times the expected frequency:

$$E_O(L;nor+1) = S(L) \cdot E(n;nor+1) \tag{9}$$

[3] For more information on the limited expected value function, see Hogg and Klugman[3]

The subscript O in $E_O(L;nor + 1)$ stands for occurrence, to differentiate this from the case where the reinstatements are limited by the aggregate losses, which we will discuss later.

The expected premium collected is equal to the up front premium plus any reinstatement premiums collected.

$$E(P) = R \cdot LMT + R \cdot c \cdot E_O(L;nor) \tag{10}$$

where

R = rate on line for the contract

$E_O(L;nor)$ = expected loss limited to *nor* occurrences (no reinstatement premium is collected following the *nor*+1[th] occurrence).

Setting the expected premium equal to the expected losses, we get

$$R \cdot LMT + R \cdot c \cdot E_O(L;nor) = E_O(L;nor + 1) \tag{11}$$

Solving for R, we get the fair up front rate on line:

$$R = \frac{E_O(L;nor+1)}{\left(LMT + c \cdot E_O(L;nor)\right)} \tag{12}$$

For example, assume that we have a simple event loss table (ELT) with ATT = \$2 million and LMT = \$2 million as shown in Table 2:

Table 2 Sample ELT

Event	Annual Rate	Gross Loss	Cat. Loss[4]
1	0.1	5 million	2 million
2	0.2	3 million	1 million

For this case, the expected severity for the catastrophe treaty is

$$S(L) = \frac{0.1 \cdot 2 + 0.2 \cdot 1}{0.1 + 0.2} = 1.333 \text{ million}$$

The expected losses and premium rates with various numbers of reinstatements are given in Table 3 for c = 1.0:

Table 3 Expected Losses and Fair Premium Rates when c = 1

Number of Reinst.	Expected Loss (in \$million)	Rate on Line

[4] For simplicity, the losses to the catastrophe treaty in this table are calculated assuming that the gross losses are constant. The actual output from the computer model calculates the catastrophe losses using equation (1).

0	0.34558	0.17279
1	0.39482	0.16833
2	0.39962	0.16687
3	0.39998	0.16668
∞	0.40000	0.16667

In this example, as the number of reinstatements increases, the up-front premium decreases, because the expected additional reinstatement premiums outweigh the higher expected losses. There can be situations where this is not the case, and the up-front premium increases as the number of reinstatements increases. This can happen, for example, when the expected severity is very low relative to the limit.

If $c = 0$ (free reinstatements), then equation (12) reduces to

$$R = \frac{E_O(L; nor + 1)}{LMT} \tag{13}$$

and the fair up-front premium rates are shown in Table 4:

Table 4 Expected Losses and Fair Premium Rates when $c = 0$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.34558	0.17279
1	0.39482	0.19741
2	0.39962	0.19981
3	0.39998	0.19999
∞	0.40000	0.20000

In this case, as the number of reinstatements increases, the up-front premium also increases, since the losses would be higher (because losses for more occurrences are paid), but there are no additional reinstatement premiums (because $c = 0$).

It is not uncommon to set the pure premium to the average annual loss from equation (4), which is $0.4 million in this example. If the rate on line is based on this pure premium, then it is equivalent to collecting up-front premium with unlimited free reinstatements, as shown in the last row of Table 4.

Reinstatements Limited by Aggregate Losses

When the reinstatements are limited by the number of occurrences, there can be some situations in which the buyer of the reinsurance will have a difficult decision to make. For example, suppose that one event has occurred with a very small loss to the catastrophe treaty. If the insurer makes a claim, it will use up one reinstatement for a small recovery. If it doesn't make a claim, then perhaps no other events follow, and it loses a chance of recovery. To avoid this dilemma, it is common practice to limit the reinstatement by aggregate losses rather than by number of occurrences.

Here the number of reinstatements refers not to the number of occurrences, but to the number of limits. Thus, a contract with *nlr* reinstatements will pay at most *nlr*+1 times *LMT*, regardless of the number of occurrences.

We calculate the fair up-front premium rate by again setting the expected premium collections equal to the expected loss payments.

To calculate the expected losses, we must first calculate the aggregate loss distribution. In this compound Poisson process, the probability of exactly n occurrences is given in equation (3). Given n events have occurred, the aggregate loss is calculated in equation (14):

$$A = L_1 + L_2 + \cdots + L_n \tag{14}$$

The distribution of A can be obtained by Panjer's recursive approach [4] and [5], by the use of Fourier transforms as described by Heckman & Meyers [6], or by a simulation approach. Let $f(A)$ and $F(A)$ be the probability density function and cumulative probability distribution of the aggregate losses obtained by one of these approaches. Note that this distribution is for the aggregate losses, not separated into the frequency and severity pieces as we did for the reinstatements based on the number of occurrences.

For a continuous aggregate loss distribution, the limited expected value of A limited to A_m is:

$$\begin{aligned} E(A;A_m) &= \int_0^{\infty} \min(A, A_m) f(A) dA \\ &= \int_0^{A_m} A f(A) dA + A_m \left(1 - F(A_m)\right) \end{aligned} \tag{15-C}$$

For a discrete aggregate loss distribution, the limited expected value of A limited to A_m is:

$$\begin{aligned} E(A;A_m) &= \sum_{i=1}^{\infty} \min(A_i, A_m) \cdot f(A_i) \\ &= \sum_{i=1}^{m-1} A_i \cdot f(A_i) + A_m \cdot \left(1 - F(A_{m-1})\right) \end{aligned} \tag{15-D}$$

where the A_i's are sorted in ascending order.

Because a contract with nlr reinstatements will pay at most $nlr+1$ times LMT, the expected loss for a treaty with nlr reinstatements is then the limited expected value of the aggregate losses limited to $(nlr+1)\cdot LMT$. We define the expected loss for the treaty as $E_A(L;nlr+1)$:

$$E_A(L;nlr+1) = E(A;(nlr+1)\cdot LMT) \tag{16}$$

The expected reinstatement premium is proportional to the aggregate losses capped at the treaty limit. If nlr reinstatements are allowed, then the expected reinstatement premium is proportional to the aggregate loss capped at nlr limits. Adding the up-front premium, we get the total expected premium:

$$E(P) = R\cdot LMT + R\cdot c\cdot E_A(L;nlr) \tag{17}$$

Setting the expected premium equal to the expected loss, we get:

$$R\cdot LMT + R\cdot c\cdot E_A(L;nlr) = E_A(L;nlr+1) \tag{18}$$

Solving for R, we get the fair up-front premium rate with nlr reinstatements:

$$R = \frac{E_A(L;nlr+1)}{\left(LMT + c\cdot E_A(L;nlr)\right)} \tag{19}$$

For an example, we used the same event loss table as for the occurrence-limited example (Table 2), and calculated the aggregate loss distribution using Panjer's approach[5] (see Appendix A for the calculations). The probability distribution is shown in Table 5:

Table 5 Aggregate Loss Distribution

Aggregate loss A (in \$million)	Probability $f(A)$	Cumulative $F(A)$
0	0.7408182	0.7408182
1	0.1481636	0.8889818
2	0.0888982	0.9778800
3	0.0158041	0.9936841
4	0.0052351	0.9989192
5	0.0008416	0.9997608
6	0.0002026	0.9999634
7	0.0000298	0.9999932
8	0.0000058	0.9999990

[5] Here we make a simplifying assumption that the losses to the catastrophe treaty, given that an event has occurred, are constant. The actual output from the computer model shows not only the expected loss, but the coefficient of variation of the losses, from which a distribution can be assumed.

9	0.0000008	0.9999998
10	0.0000001	0.9999999

For $c = 1.0$, we have results as shown in Table 6:

Table 6 Expected Losses and Fair Premium Rates when $c = 1$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.37020	0.18510
1	0.39864	0.16819
2	0.39996	0.16674
3	0.40000	0.16667
∞	0.40000	0.16667

Comparing Table 6 with Table 3, notice that the expected loss for occurrence-based with no reinstatement is lower than the expected loss for aggregate-based with no reinstatement. This is because for aggregate-based, more than one occurrence will be paid if the aggregate loss of the first occurrence is less than the limit. For example, if a contract has a provision for nr reinstatements, then the occurrence-based reinstatements provide $nr+1$ occurrences which have loss values less than or equal to the limit; the aggregate-based reinstatements provide $nr+1$ limits of coverage for as many occurrences as needed (at least $nr+1$) to reach the aggregate limit. Also note that for one or more reinstatements, the aggregate-based rate on line is less than the occurrence-based rate on line. This again is because the expected additional reinstatement premiums outweigh the higher additional losses.

And for $c = 0$, we have results as shown in Table 7:

Table 7 Expected Losses and Fair Premium Rates when $c = 0$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.37020	0.18510
1	0.39864	0.19932
2	0.39996	0.19998
3	0.40000	0.20000
∞	0.40000	0.20000

There is a significant difference between the fair premium rate for no reinstatement and the fair premium rate based on AAL, which is equivalent to unlimited free reinstatements. In the above examples, the up-front premium rates are 0.1729 and 0.1851 for occurrence-based and aggregate-based, respectively, versus 0.2 based on the AAL. The difference increases with the increase of the total occurrence rate λ, particularly for occurrence-based contracts. Table 8 shows the impact of the total occurrence rate on the premium rates, keeping the severity distribution unchanged.

Table 8 Impact of Total Occurrence Rate on Premium

λ	Occurrence-Based*	Aggregate-Based*	Unlimited Free Reinstatements
0.03	0.0197	0.0199	0.02
0.3	0.1729	0.1851	0.20
3.0	0.6335	0.9004	2.00
3000	0.6667	1.0000	2000

* No reinstatements

The limiting case of the premium rate for an occurrence-based contract with no reinstatements as $\lambda \to \infty$ is the expected severity divided by the limit, since it is a certainty that an event will occur, and when it does occur the expected loss is equal to the expected severity. The limiting case of the premium rate for an aggregate-based contract with no reinstatements as $\lambda \to \infty$ is unity, since it is a certainty that the full aggregate limit will be paid.

Reinstatement Premiums Pro Rata for Time

Often, the reinstatement premium is proportional to the remaining time in the reinsurance contract after an occurrence. Given a loss, the reinstatement premium would be

$$P_{reinst} = R \cdot c \cdot L \cdot (1-t) \tag{20}$$

And the total collected reinstatement premium for a contract limited by *nor* number of occurrences[6] is

$$TotP_{reinst} = \sum_{i=1}^{\min(n,nor)} R \cdot c \cdot L_i \cdot (1-t_i) \tag{21}$$

where t is the time of the loss in years (assuming a one-year contract) and n is the number of occurrences in the year. The time remaining in the contract is $1 - t$. For example, if a loss occurs on October 1st of an annual contract with an effective date of January 1st, then $t = 0.75$, and the time remaining is 0.25.

The expected value of the total collected reinstatement premium is

$$E[TotP_{reinst}] = E\left[\sum_{i=1}^{\min(n,nor)} R \cdot c \cdot L_i \cdot (1-t_i)\right] \tag{22}$$

$$= R \cdot c \cdot E\left[\sum_{i=1}^{\min(n,nor)} L_i \cdot (1-t_i)\right] \tag{23}$$

[6] Reinstatements limited by the aggregate losses are left for further study.

Since the L_i's are independent of the t_i's,

$$= R \cdot c \cdot \sum_{i=1}^{\min(n,nor)} E(L_i) \cdot E(1-t_i) \tag{24}$$

Since the L_i's are independent of each other, $E(L_i)$ equals the expected severity:

$$= R \cdot c \cdot \sum_{i=1}^{\min(n,nor)} S(L) \cdot E(1-t_i) \tag{25}$$

$$= R \cdot c \cdot S(L) \cdot \sum_{i=1}^{\min(n,nor)} E(1-t_i) \tag{26}$$

Since $E_O(L;nor) = S(L) \cdot E(n;nor)$,

$$= R \cdot c \cdot E_o(L;nor) \cdot \frac{\sum_{i=1}^{nor} RT_i}{E(n;nor)} \tag{27}$$

where RT_i is the expected time remaining after the i^{th} occurrence.

Adding the up-front premium, we get the total expected premium collections:

$$E(P) = R \cdot c \cdot E_o(L;nor) \cdot \frac{\sum_{i=1}^{nor} RT_i}{E(n;nor)} \tag{28}$$

To calculate the fair premium amount, we set the expected premium collections from equation (28) equal to the expected losses from equation (9). Letting $\theta_k = \frac{\sum_{i=1}^{k} RT_i}{E(n;k)}$, we get

$$R \cdot LMT + R \cdot c \cdot E_o(L;nor) \cdot \theta_{nor} = E_o(L;nor+1) \tag{29}$$

Solving for R, we get the fair up front rate on line:

$$R = \frac{E_O(L;nor+1)}{\left(LMT + c \cdot E_O(L;nor) \cdot \theta_{nor}\right)} \tag{30}$$

We calculate the expected remaining time RT_k by integrating the distribution of the arriving time. Given the assumption of a Poisson process, the distribution of the arriving time for the k^{th} occurrence is given by a Gamma distribution, as shown in equation (31):

$$f_k(t) = \frac{\lambda(\lambda t)^{(k-1)} e^{-\lambda t}}{(k-1)!} \tag{31}$$

The expected time remaining after the k^{th} occurrence is

$$RT_k = \int_0^1 (1-t) f_k(t) dt \tag{32}$$

For k = 1, this reduces to equation (33). See Appendix B for the derivation.

$$RT_1 = \frac{\lambda + e^{-\lambda} - 1}{\lambda} \tag{33}$$

Table 9 shows the expected remaining time after the first occurrence for various λ values.

Table 9 Expected Remaining Times

λ	RT_1	θ_1
0.003	0.0015	0.5002
0.03	0.0149	0.5025
0.3	0.1361	0.5250
3.0	0.6833	0.7191
30	0.9667	0.9667
3000	0.9997	0.9997

The limiting case of θ_1 as $\lambda \to 0$ is 0.5, and the limiting case of θ_1 as $\lambda \to \infty$ is unity.

The expected losses and premium rates with various numbers of reinstatements are given in Table 10 for $c = 1.0$, using the event loss table from Table 2:

Table 10 Expected Losses and Fair Premium Rates When $c = 1$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.34558	0.17279
1	0.39482	0.18090
2	0.39962	0.18176
3	0.39998	0.18180
∞	0.40000	0.18182

It is interesting to observe that the summation of the remaining time for a one year period, $\sum_{k=1}^{nr} RT_k$, converges to $\lambda / 2$ when nr approaches infinity (see Appendix C for the proof). Since $E(n;\infty) = \lambda$, θ_k converges to 0.5. Hence, the fair premium converges to

$$R = \frac{E_O(L;\infty)}{(LMT + c \cdot E_O(L;\infty) \cdot \theta_k)} = \frac{0.4}{(2.0 + 1 \cdot 0.4 \cdot 0.5)} = 0.18182$$

Comparing Table 10 with Table 3, the up-front premium when considering the remaining time is higher because the cost of a reinstatement after an occurrence is lower.

It should be noted that although earthquakes occur uniformly throughout the year, hurricanes and tornadoes are seasonal. Particularly, along the Atlantic coast, most hurricane landfalls are in September or October. Thus, the above derivation would need to be modified to account for this seasonality. The consideration of seasonality is beyond the scope of this paper.

Summary

This paper has shown how to use the output from a catastrophe model to calculate the fair premium of catastrophe treaties with reinstatement provisions. The basis for the analysis is the catastrophe model's event loss table, which contains all of the information needed to make the calculations.

The paper also investigated the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered. Some of the findings:

- Basing the up-front premium on the average annual loss to a treaty, disregarding reinstatements, is equivalent to assuming that there are unlimited free reinstatements. If, on the other hand, reinstatements are limited and paid, then the up-front premium will be lower because fewer losses will be covered (because the reinstatements are limited) and some of the premium will be paid after an event has occurred (because the reinstatements are paid).

- Unless the expected severity is very small relative to the limit, the more paid reinstatements allowed the lower the up-front premium will be. This is because the additional reinstatement premiums expected to be collected will outweigh the additional expected losses.

- Reinstatement provisions based on aggregate losses will have higher expected losses than those based on the number of occurrences. In general, if the number of reinstatements is one or more, the up-front premiums will be less for aggregate-based reinstatements than for occurrence-based reinstatements. This again is because the additional expected reinstatement premiums will outweigh the higher expected losses.

- If the reinstatement premium is proportional to the remaining time in the reinsurance contract after an occurrence, then the up-front premium should be higher because less reinstatement premiums will be collected.

In this paper we did not consider expenses or risk loads, which are areas for further study. Other areas that deserve further study are reinstatement provisions that are limited by aggregate losses and have reinstatement premiums pro rata for time; and the effect of seasonality on the expected reinstatement premiums.

Acknowledgment

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Appendix A

Calculation of the aggregate loss distribution

We use the recursive method as described in "The Aggregate Claims Distribution and Stop Loss Reinsurance" by Harry H. Panjer. Mr. Panjer uses for his examples fixed benefit life insurance claims. Here we make the translation that an event that causes loss to the catastrophe treaty is one claim.

Using Mr. Panjer's notation, our event loss table (Table 2) is show below as Table A1:

Table A1 Event Loss Table

j	Loss Amount $= jU$	Rate $= \theta_j$	$j\theta_j = E_j$
1	\$1,000,000	0.2	0.2
2	\$2,000,000	0.1	0.2
Total		0.3	0.4

U is the greatest common divisor of the loss amounts for the claims, in this case \$1,000,000. Then j is the loss amount divided by U.

Note that the sum of the E_j's is the average annual loss.

Let P_i represent the probability that the aggregate loss will be exactly iU, and n be the number of events in the event loss table. Mr. Panjer derives the recursive formula for P_i:

$$P_i = \frac{1}{i} \sum_{\substack{j=1 \\ E_j \neq 0}}^{\min(i,n)} E_j P_{i-j} \tag{A1}$$

where

$$P_0 = \exp\left(-\sum_{\substack{j=1 \\ E_j \neq 0}}^{n} \theta_j \right) \tag{A2}$$

Applying these formulas to the values in our event loss table, we get:

$P_0 = \exp(-0.3) = 0.7408$
$P_1 = 0.2 * 0.7408 = 0.1482$
$P_2 = (1/2) * (0.2 * 0.1482 + 0.2 * 0.7408) = 0.0889$
$P_3 = (1/3) * (0.2 * 0.0889 + 0.2 * 0.1482) = 0.0158$
$P_4 = (1/4) * (0.2 * 0.0158 + 0.2 * 0.0889) = 0.0052$
etc.

These are the probabilities $f(A)$ shown in Table 5.

Appendix B

Expected time remaining after the first occurrence

$$RT_1 = \int_0^1 (1-t) f_1(t)\,dt \tag{B1}$$

$$= \int_0^1 f_1(t)\,dt - \int_0^1 t f_1(t)\,dt \tag{B2}$$

$$= \int_0^1 \lambda e^{-\lambda t}\,dt - \int_0^1 t\lambda e^{-\lambda t}\,dt \tag{B3}$$

$$= \lambda\left[\frac{-e^{-\lambda t}}{\lambda}\right]_0^1 - \lambda\left[\frac{e^{-\lambda t}}{\lambda^2}(-\lambda t - 1)\right]_0^1 \tag{B4}$$

$$= \lambda\left(\frac{-e^{-\lambda}}{\lambda} + \frac{1}{\lambda}\right) + \lambda\left(\frac{e^{-\lambda}}{\lambda^2}(\lambda + 1) - \frac{1}{\lambda^2}\right) \tag{B5}$$

$$= \frac{\lambda + e^{-\lambda} - 1}{\lambda} \tag{B6}$$

Appendix C

Proof of the convergence of the summation of remaining times

Assuming the contract period T is one year, we have

$$\sum_{k=1}^{nr} RT_k = \int_0^1 \left\{ \sum_{k=1}^{\infty} (1-t) f_{T_k}(t) \right\} dt \tag{C1}$$

$$= \int_0^1 \left\{ \sum_{k=1}^{\infty} (1-t) \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} \right\} dt \tag{C2}$$

Set $k' = k-1$, and we have

$$\sum_{k=1}^{nr} RT_k = \int_0^1 \lambda(1-t) \left\{ \sum_{k'=0}^{\infty} \frac{(\lambda t)^{k'} e^{-\lambda t}}{k'!} \right\} dt \tag{C3}$$

$$= \int_0^1 \lambda(1-t) dt = 0.5\lambda \tag{C4}$$