Independent Claim Report Lags and Bias in Forecasts Using Age-To-Age Factor Methodology

by Stewart H. Gleason, Ph.D., ACAS

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I. Introduction

In his 1985 Proceedings paper "A Simulation Test of Prediction Errors of Loss Reserve Estimation Techniques", J. Stanard [1] pointed out an apparent bias in forecasts of ultimate claims when commonly used reserving methods were applied to simulated data. The approach was to specify a stochastic model of claims emergence and use it to generate data to be used as input to various reserving methods. One of the methods selected was the familiar age-to-age factor method and it was found to produce overstated forecasts of ultimate claims in certain cases.

Stanard's simulation model assumes that the report lag of each claim is independent. This hypothesis has been put forth in other work, particularly that of E. Weissner [2], [3]. The work presented here will show analytically that when report lags are assumed to be independent, the age-to-age factor method is biased.

This will be shown in two special cases of claim count development. First, it will be assumed that the ultimate number of claims for an accident period has a Poisson distribution. In this case, the assumption of independent report lags implies the independence of the total number of claims reported in any two periods. This is a special case of what will here be called the assumption of independent increments. A general argument may then be given to show that the age-to-age factor methodology gives biased results when the underlying process is known to have independent development increments.

The situation where the ultimate number of claims has a negative binomial distribution is also addressed and is in fact the model specified by Stanard. In this case, assuming that report lags are independent does not imply that increments are independent and a somewhat different argument is required.

The arguments presented here will make use of Jensen's Inequality. Stanard notes in Appendix A of his paper that the observed bias is likely due to the fact that the expected value of the quotient of two random variables is not necessarily equal to the quotient of their expected values, i.e.

$$\frac{E[X]}{E[Y]} \neq E\left[\frac{X}{Y}\right].$$

Jensen's Inequality may be used to show that, when the right conditions are specified,

$$E\left[\frac{X}{Y}\right] > \frac{E[X]}{E[Y]}$$

These quotients will arise in what follows as the usual claims development or age-to-age factors. In addition, it will be demonstrated that weighted average forecasts exhibit a smaller bias than straight average estimates.

II. Preliminaries

Notation and Assumptions

For simplicity, claims activity segmented into n consecutive, non-overlapping time periods of equal length will be considered. $X_{i,j}$ will denote the number of incidents occurring in period i which are reported as claims in period $i+j-1$ (or with lag $j-1$). The incremental development triangle at the end of the nth period is displayed as:

Number of Accident Period i Claims Reported With Lag j-1

				Lag + 1			
Accident Period	1	2	...	n-i+1	...	n-1	n
1	X_{11}	X_{12}	...	$X_{1,n-i+1}$	...	$X_{1,n-1}$	$X_{1,n}$
2	X_{21}	X_{22}	...	$X_{2,n-i+1}$	...	$X_{2,n-1}$	
$\vdots$	$\vdots$				$\iddots$		
i	$X_{i,1}$			$X_{i,n-i+1}$			
$\vdots$	$\vdots$		$\iddots$				
n-1	$X_{n-1,1}$	$X_{n-1,2}$					
n	$X_{n,1}$						

This data is more commonly summarized as a cumulative development triangle

Number of Accident Period i Claims Reported With Lag $\leq j$-1

				Lag + 1			
Accident Period	1	2	...	n-i+1	...	n-1	n
1	S_{11}	S_{12}	...	$S_{1,n-i+1}$	...	$S_{1,n-1}$	$S_{1,n}$
2	S_{21}	S_{22}	...	$S_{2,n-i+1}$	...	$S_{2,n-1}$	
$\vdots$	$\vdots$				$\iddots$		
i	$S_{i,1}$			$S_{i,n-i+1}$			
$\vdots$	$\vdots$		$\iddots$				
n-1	$S_{n-1,1}$	$S_{n-1,2}$					
n	$S_{n,1}$						

where

$$S_{i,j} = \sum_{k=1}^{j} X_{i,k}.$$

The assumptions will be stated in terms of the $X_{i,j}$.

The basic problem for data given in this format is to deduce the number of incidents occurring in each accident period from the number reported through period n and from the pattern, consistent from period to period, in which they are reported. It is sufficient for what is intended here to consider only the problem of forecasting the next reporting increment.

There are two assumptions which will be imposed on the claims process. First, one assumes that the increments at the same age of development for different accident periods are independent, identically distributed and nonnegative random variables:

(A) For each j, the $X_{i,j}$ are independent, identically distributed and nonnegative.

One also assumes to begin with that, for a given accident period i, the development increments are independent of what has taken place up to that point in time:

(B) For each i, $X_{i,j}$ is independent of $X_{i,k}$ for $k<j$.

These independence assumptions are sufficient to demonstrate bias in the age-to-age factor estimates. Later, it will be shown condition (B) is satisfied if the ultimate claim count distribution is Poisson.

Jensen's Inequality

Proved here is a special case of a key analytical tool to be used in the demonstration. Readers familiar with the *Actuarial Mathematics* text [4] will recall a version of this for functions of one real variable. References for the multivariate statement used here may be found in [5] and [6].

Jensen's Inequality Let f be a function defined on a set $A \subseteq \mathbf{R}^n$ which takes only positive real values. Let μ be a probability measure on A with

$$E[f(\mathbf{X})] \equiv \int_A f(x_1,\ldots,x_n)\, d\mu(x_1,\ldots,x_n)$$

finite and non-zero. Provided f is not constant on every set of non-zero probability,

$$E\left[{}^{1}\!/\!_{f(\mathbf{X})}\right] > {}^{1}\!/\!_{E[f(\mathbf{X})]}.$$

Proof. Let $\gamma = E[f(\mathbf{X})]$. Consider the tangent line to the curve $s = 1/t$ at $t = \gamma$ which has the equation

$$s = -\frac{1}{\gamma^2}t + \frac{2}{\gamma}.$$

This line is always below the graph of $s = 1/t$ and so

$$\frac{1}{t} \geq -\frac{1}{\gamma^2}t + \frac{2}{\gamma}.$$

The range of f is such that, for each $\mathbf{x}$ in A,

$$\frac{1}{f(\mathbf{x})} \geq -\frac{1}{\gamma^2}f(\mathbf{x}) + \frac{2}{\gamma}.$$

Integrating each side with respect to μ gives

$$\int_A \frac{1}{f(x_1,\ldots,x_n)} d\mu(x_1,\ldots,x_n) \geq -\frac{1}{\gamma^2}\int_A f(x_1,\ldots,x_n)d\mu(x_1,\ldots,x_n) + \frac{2}{\gamma} = \frac{1}{\gamma}.$$

If equality held in this expression, then it would be the case that

$$\frac{1}{f(\mathbf{x})} = -\frac{1}{\gamma^2}f(\mathbf{x}) + \frac{2}{\gamma} \text{ or } f(\mathbf{x}) = \gamma$$

for all $\mathbf{x}$ except in a set having probability zero. This situation was ruled out and the result is now clear.

III. The Basic Argument

Using the familiar weighted average forecast, the age-to-age methodology predicts the next cumulative value as

$$\hat{S}_{i,n-i+2} = \frac{\sum_{k=1}^{i-1} S_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1}$$

provided that $\sum_{k=1}^{i-1} S_{k,n-i+1}$ and $S_{i,n-i+1}$ are non-zero. It is easier to work with the implied forecast of the change:

$$\begin{aligned}
\hat{X}_{i,n-i+2} &\equiv \hat{S}_{i,n-i+2} - S_{i,n-i+1} \\
&= \left(\frac{\sum_{k=1}^{i-1} S_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} - 1 \right) \cdot S_{i,n-i+1} \\
&= \frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1}
\end{aligned}$$

Theorem 1. Given the independence conditions (A) & (B) stated above, the expected value of the weighted average forecast $\hat{X}_{i,n-i+2}$ is always greater than the expected value of the actual change. That is,

$$E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right] > E\left[X_{i,n-i+2} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right]$$
$$\equiv E\left[X_{\cdot,n-i+2}\right]$$

Proof. To see this, one observes that

$$\begin{aligned}
&E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right] \\
&= E\left[S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right] \\
&\quad \cdot E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right] \\
&= E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right] \cdot E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]
\end{aligned}$$

due to the independence of accident periods. Because of the independence of increments, it is also true that

$$\begin{aligned}
&E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right] \cdot E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] \\
&= E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right] \cdot (i-1) \cdot E\left[X_{k,n-i+2}\right] \\
&\quad \cdot E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]
\end{aligned}$$

Using Jensen's Inequality, with $f(\mathbf{x}) = x_1 + \cdots + x_{i-1}$, one deduces that

$$E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] > \frac{1}{E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]}$$

This may be strengthened slightly by noting that

$$E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] \leq E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i-1\right]$$
$$= (i-1) \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$

Combing these steps produces

$$E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0\right]$$
$$> E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right] \cdot E\left[X_{k,n-i+2}\right] / E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$
$$\equiv E\left[X_{i,n-i+2}\right]$$

This completes the proof.

Readers will observe at this point that the result is really only a statement of fact regarding ratios of independent variables and does not make much use of the underlying process. This should not be surprising since *the age-to-age factor methodology doesn't either.* Intuition provides a guide in the construction of forecasts in a natural way by relying on the identical distributions by "lag". The conclusion to be drawn here is not that the age-to-age factor method is biased absolutely but rather that it is not compatible with a claims process assumed to have independent increments.

IV. Independent Increments From Independent Claims Lags: The Poisson Case

It will now be shown that condition (B) holds when the report lags are independent and when the distribution of ultimate accident period claims is Poisson with mean λ. This in turn relies on the observation that, in this case, the number of claims reported with lag j-1 is also Poisson with mean $p_j \lambda$, where p_j is the probability that a claim from accident period i is reported in period $i+j-1$.

Proposition 1. When the distribution of ultimate claims is Poisson with mean λ and the report lags are independent, the number of claims reported with lag j-1 is also Poisson with mean $p_j \lambda$.

Proof. Let N be the ultimate number of claims and N_j be the number reported with lag j-1, then

$$
\begin{aligned}
\Pr\{N_j = J\} &= \sum_{n=0}^{\infty} \Pr\{N_j = J | N = n\} \cdot \Pr\{N = n\} \\
&= \sum_{n=J}^{\infty} \frac{n!}{J!(n-J)!} p_j^J (1-p_j)^{n-J} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \frac{(\lambda p_j)^J}{J!} \sum_{n=0}^{\infty} \frac{[\lambda(1-p_j)]^n}{n!} \\
&= e^{-\lambda p_j} \frac{(\lambda p_j)^J}{J!} .
\end{aligned}
$$

Showing that N_j and N_k are independent is accomplished by a similar calculation.

Proposition 2. When the distribution of ultimate claims is Poisson with mean λ and the report lags are independent, the number of claims reported with lag j-1 and with lag k-1 are independent.

Proof. It need only be shown that

$$
\Pr\{N_j = J, N_k = K\} = \Pr\{N_j = J\} \cdot \Pr\{N_k = K\}.
$$

To this end, one proceeds as before and sees that

$$
\begin{aligned}
\Pr\{N_j = J, N_k = K\} &= \sum_{n=0}^{\infty} \Pr\{N_j = J, N_k = K | N = n\} \cdot \Pr\{N = n\} \\
&= \sum_{n=J+K}^{\infty} \frac{n!}{J!K!(n-J-K)!} p_j^J p_k^K (1-p_j-p_k)^{n-J-K} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \frac{(\lambda p_j)^J}{J!} \frac{(\lambda p_k)^K}{K!} \sum_{n=0}^{\infty} \frac{[\lambda(1-p_j-p_k)]^n}{n!} \\
&= e^{-\lambda p_j} \frac{(\lambda p_j)^J}{J!} \cdot e^{-\lambda p_k} \frac{(\lambda p_k)^K}{K!} \\
&= \Pr\{N_j = J\} \cdot \Pr\{N_k = K\} .
\end{aligned}
$$

One now knows that the Poisson case satisfies condition (B) and the hypothesis of Theorem 2. One obtains as an implication

Theorem 2. When the distribution of ultimate claims is Poisson and the report lags of individual claims are independent, the weighted average forecast $\hat{X}_{i,n-i+2}$ is biased.

V. *The Negative Binomial Case*

It will be shown presently that bias is present in Stanard's "Claim Counts Only" scenario as well. In his paper, claim count triangles are generated by drawing a number of claims from (a normal approximation to) a negative binomial distribution and then drawing for each claim a report period. The latter is determined by drawing a value from the convolution of a uniform time-to-accident distribution and an exponential report lag distribution. The exact form of the report lag distribution is not important here.

Proposition 1 has a counterpart when ultimate claims have a negative binomial distribution. The form of the negative binomial distribution that will be used for a random variable M is given by

$$\Pr\{M = m\} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\frac{e^{-\lambda}\lambda^{m}}{m!}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda$$
$$\equiv \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)\cdot m!}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{m}.$$

Proposition 3. When the distribution of ultimate claims is negative binomial with parameters α and β and the report lags are independent, the number of claims reported with lag j-1 is also negative binomial with parameters α and $\beta_j = \dfrac{\beta}{p_j}$.

Proof. Let N be the ultimate number of claims and N_j be the number reported with lag j-1, then

$$\Pr\{N_j = J\} = \sum_{n=0}^{\infty}\Pr\{N_j = J|N = n\}\cdot\Pr\{N = n\}$$
$$= \sum_{n=J}^{\infty}\frac{n!}{J!(n-J)!}p_j^J(1-p_j)^{n-J}\cdot\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\frac{e^{-\lambda}\lambda^{n}}{n!}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda$$
$$= \frac{\beta^{\alpha}p_j^J}{\Gamma(\alpha)J!}\int_0^{\infty}\left\{\sum_{n=0}^{\infty}\frac{\left[(1-p_j)\lambda\right]^n}{n!}\right\}\lambda^{\alpha+J-1}e^{-(\beta+1)\lambda}d\lambda$$
$$= \frac{\beta^{\alpha}p_j^J}{\Gamma(\alpha)J!}\int_0^{\infty}\lambda^{\alpha+J-1}e^{-(\beta+p_j)\lambda}d\lambda$$
$$= \frac{\Gamma(\alpha+J)}{\Gamma(\alpha)\cdot J!}\left(\frac{\beta/p_j}{\beta/p_j+1}\right)^{\alpha}\left(\frac{1}{\beta/p_j+1}\right)^{J}.$$

Unfortunately, Proposition 2 has no analogue as the increments are not independent as demonstrated in the following

Proposition 4a. When the distribution of ultimate claims is negative binomial variate M with parameters α and β, then

$$\Pr\{N_j = J|N_k = K\} = \frac{\Gamma(\alpha+J+K)}{\Gamma(\alpha+K)\cdot J!}\left(\frac{\beta+p_k}{\beta+p_j+p_k}\right)^{K+\alpha}\left(\frac{p_j}{\beta+p_j+p_k}\right)^J$$

Proof. The following calculation suffices.

$$\Pr\{N_j = J|N_k = K\} = \Pr\{N_j = J, N_k = K\}/\Pr\{N_k = K\}$$

$$= \frac{\sum_{n=J+K}^{\infty}\frac{n!}{J!K!(n-J-K)!}p_j^J p_k^K(1-p_j-p_k)^{n-J-K}\cdot\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^n}{\sum_{n=K}^{\infty}\frac{n!}{K!(n-K)!}p_k^K(1-p_k)^{n-K}\cdot\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^n}$$

$$= \frac{p_j^J}{J!}\left(\frac{1}{\beta+1}\right)^J\frac{\sum_{n=0}^{\infty}\frac{(1-p_j-p_k)^n}{n!}\Gamma(\alpha+n+J+K)\left(\frac{1}{\beta+1}\right)^n}{\sum_{n=0}^{\infty}\frac{(1-p_k)^n}{n!}\cdot\Gamma(\alpha+n+K)\left(\frac{1}{\beta+1}\right)^n}$$

$$= \frac{\Gamma(\alpha+J+K)}{\Gamma(\alpha+K)\cdot J!}\left(\frac{\beta+p_k}{\beta+p_j+p_k}\right)^{K+\alpha}\left(\frac{p_j}{\beta+p_j+p_k}\right)^J$$

Of this the following is direct consequence.

Proposition 4b. When the distribution of ultimate claims is negative binomial variate M with parameters α and β, then

$$E\left[N_j|N_k\right] = \frac{(N_k+\alpha)p_j}{(\beta+p_k)}$$

Two very similar expressions for the conditional probability and expectation are also required for the case where N_k is not given but is known to be non-zero.

Proposition 5a. When the distribution of ultimate claims is negative binomial variate M with parameters α and β, then

$$\Pr\{N_j = J|N_k > 0\} = \frac{\Gamma(\alpha+J)}{\Gamma(\alpha)\cdot J!}p_j^J\left\{\frac{1}{(\beta+p_j)^{\alpha+J}} - \frac{1}{(\beta+p_j+p_k)^{\alpha+J}}\right\}\Big/\left\{\frac{1}{\beta^{\alpha}} - \frac{1}{(\beta+p_k)^{\alpha}}\right\}$$

Proof. Proceeding in a now familiar fashion, one sees that

$$\Pr\{N_j = J|N_k > 0\} = \sum_{K=1}^{\infty}\Pr\{N_j = J, N_k = K\}/\Pr\{N_k > 0\}$$

$$= \frac{\sum_{K=1}^{\infty}\sum_{n=J+K}^{\infty}\frac{n!}{J!K!(n-J-K)!}p_j^J p_k^K(1-p_j-p_k)^{n-J-K}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty\frac{e^{-\lambda}\lambda^n}{n!}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda}{\left\{1-\left(\frac{\beta}{\beta+p_k}\right)^\alpha\right\}}$$

$$= \frac{\frac{\beta^\alpha p_j^J}{\Gamma(\alpha)J!}\int_0^\infty\left\{\sum_{K=1}^{\infty}\frac{(\lambda p_k)^K}{K!}\sum_{n=0}^{\infty}\frac{[(1-p_j-p_k)\lambda]^n}{n!}\right\}\lambda^{\alpha+J-1}e^{-(\beta+1)\lambda}d\lambda}{\left\{1-\left(\frac{\beta}{\beta+p_k}\right)^\alpha\right\}}$$

$$= \frac{\frac{\beta^\alpha p_j^J}{\Gamma(\alpha)J!}\left\{\int_0^\infty\lambda^{\alpha+J-1}e^{-(\beta+p_j)\lambda}d\lambda-\int_0^\infty\lambda^{\alpha+J-1}e^{-(\beta+p_j+p_k)\lambda}d\lambda\right\}}{\left\{1-\left(\frac{\beta}{\beta+p_k}\right)^\alpha\right\}}$$

$$= \frac{\Gamma(\alpha+J)}{\Gamma(\alpha)\cdot J!}p_j^J\left\{\frac{1}{(\beta+p_j)^{\alpha+J}}-\frac{1}{(\beta+p_j+p_k)^{\alpha+J}}\right\}\Big/\left\{\frac{1}{\beta^\alpha}-\frac{1}{(\beta+p_k)^\alpha}\right\}$$

This leads immediately to

Proposition 5b. When the distribution of ultimate claims is negative binomial variate M with parameters α and β, then

$$E[N_j|N_k > 0] = \frac{\alpha p_j}{\beta(\beta+p_k)}\left\{\frac{(\beta+p_k)^{\alpha+1}-\beta^{\alpha+1}}{(\beta+p_k)^\alpha-\beta^\alpha}\right\}$$

Proof. It is straightforward to sum the expression from Proposition 5a over J:

$$E\left[N_j \middle| N_k > 0\right] = \sum_{J=1}^{\infty} \frac{\Gamma(\alpha+J)}{\Gamma(\alpha)\cdot(J-1)!} p_j^{\,J} \left\{ \frac{1}{(\beta+p_j)^{\alpha+J}} - \frac{1}{(\beta+p_j+p_k)^{\alpha+J}} \right\} \Bigg/ \left\{ \frac{1}{\beta^{\alpha}} - \frac{1}{(\beta+p_k)^{\alpha}} \right\}$$

$$= \alpha p_j \frac{\left\{ \begin{array}{l} \dfrac{1}{\beta^{\alpha+1}} \sum_{J=1}^{\infty} \dfrac{\Gamma(\alpha+1+J-1)}{\Gamma(\alpha+1)\cdot(J-1)!} \left(\dfrac{\beta}{\beta+p_j}\right)^{\alpha+1} \left(\dfrac{p_j}{\beta+p_j}\right)^{J-1} \\ -\dfrac{1}{(\beta+p_k)^{\alpha+1}} \sum_{J=0}^{\infty} \dfrac{\Gamma(\alpha+1+J-1)}{\Gamma(\alpha+1)\cdot(J-1)!} \left(\dfrac{\beta+p_k}{\beta+p_j+p_k}\right)^{\alpha+1} \left(\dfrac{p_j}{\beta+p_j+p_k}\right)^{J-1} \end{array} \right\}}{\left\{ \dfrac{1}{\beta^{\alpha}} - \dfrac{1}{(\beta+p_k)^{\alpha}} \right\}}$$

$$= \frac{\alpha p_j}{\beta(\beta+p_k)} \left\{ \frac{(\beta+p_k)^{\alpha+1} - \beta^{\alpha+1}}{(\beta+p_k)^{\alpha} - \beta^{\alpha}} \right\}$$

The final task may now be addressed.

Theorem 3. When the distribution of ultimate claims is negative binomial, the weighted average forecast $\hat{X}_{i,n-i+2}$ is biased. That is,

$$\Omega \equiv E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0 \right] > E\left[X_{i,n-i+2} \middle| S_{i,n-i+1} > 0\right]$$

Proof. First, due to independence between accident periods, one may write

$$\Omega \equiv E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0, S_{i,n-i+1} > 0 \right]$$

$$= E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right] \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$

In the proof of Theorem 1, it was possible to separate the expectation operator containing the quotient. As has been shown, however, independence of increments does not hold here and some other mechanism must be employed. To this end, one fixes the $S_{k,n-i+1}$ and computes the expectation in successive steps.

$$\Omega = E\left[E\left[\sum_{k=1}^{i-1} X_{k,n-i+2} \middle| S_{k,n-i+1}, k=1,\ldots,i-1\right] \middle/ \sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$
$$= E\left[\sum_{k=1}^{i-1} E\left[X_{k,n-i+2} \middle| S_{k,n-i+1}\right] \middle/ \sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$

Proposition 4b may now be applied to show that

$$\Omega = \frac{p_{n-i+2}}{\beta + p_*} \cdot E\left[\sum_{k=1}^{i-1} \left(S_{k,n-i+1} + \alpha\right) \middle/ \sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right] \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$
$$= \frac{p_{n-i+2}}{\beta + p_*} \cdot \left(1 + (i-1)\alpha E\left[1 \middle/ \sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]\right) \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$

where $p_* = p_1 + \cdots + p_{n-i+1}$. It is at this stage when Jensen's Inequality may again be utilized and one observes that as in the case of independent increments that

$$\Omega > \frac{p_{n-i+2}}{\beta + p_*} \cdot \left(1 + (i-1)\alpha \middle/ E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \middle| \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]\right) \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$
$$\geq \frac{p_{n-i+2}}{\beta + p_*} \cdot \left(1 + \alpha \middle/ E\left[S_{\bullet,n-i+1} \middle| S_{\bullet,n-i+1} > 0\right]\right) \cdot E\left[S_{i,n-i+1} \middle| S_{i,n-i+1} > 0\right]$$

One makes use of Proposition 3 to see that

$$E\left[S_{\bullet,n-i+1} \middle| S_{\bullet,n-i+1} > 0\right] = \frac{\alpha p_*}{\beta\left(1 - \left(\dfrac{\beta}{\beta + p_*}\right)^{\alpha}\right)}$$

which may be substituted into the previous expression. Doing so produces

$$\Omega > \frac{\alpha}{\beta} \frac{p_{n-i+2}}{\beta + p_*} \cdot \left(\frac{p_* + \beta\left(1 - \left(\dfrac{\beta}{\beta + p_*}\right)^{\alpha}\right)}{\left(1 - \left(\dfrac{\beta}{\beta + p_*}\right)^{\alpha}\right)}\right)$$
$$= \frac{\alpha}{\beta} \frac{p_{n-i+2}}{\beta + p_*} \cdot \left(\frac{\left(\beta + p_*\right)^{\alpha+1} - \beta^{\alpha+1}}{\left(\beta + p_*\right)^{\alpha} - \beta^{\alpha}}\right)$$

Proposition 5b identifies the final expression as being precisely $E\left[X_{i,n-i+2} \middle| S_{i,n-i+1} > 0\right]$. Therefore,

$$\Omega > E\left[X_{i,n-i+2} \middle| S_{i,n-i+1} > 0\right].$$

VI. Straight Average Factors

In this section, it will be demonstrated that the "straight" average estimator

$$\overline{X}_{i,n-i+2} \equiv \frac{1}{(i-1)} \sum_{k=1}^{i-1} \frac{X_{k,n-i+2}}{S_{k,n-i+1}} \cdot S_{i,n-i+1}$$

cannot reduce or eliminate the bias seen in the weighted average estimator. For brevity, attention is restricted to the case of independent increments.

Theorem 4. When both are defined, the expected value of the unweighted average prediction is greater than the expected value of the weighted average prediction. That is,

$$E\left[\frac{1}{i-1} \sum_{k=1}^{i-1} \frac{X_{k,n-i+2}}{S_{k,n-i+1}} \cdot S_{i,n-i+1} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i\right] \geq E\left[\left(\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}}\right) \cdot S_{i,n-i+1} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i\right]$$

Proof. Again making use of the independence and symmetry between the periods, showing this is equivalent to proving that

$$\frac{1}{i-1} \sum_{k=1}^{i-1} E\left[\frac{X_{k,n-i+2}}{S_{k,n-i+1}} \middle| S_{k,n-i+1} > 0\right] \geq \sum_{k=1}^{i-1} E\left[\frac{X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i-1\right]$$

or

$$E\left[\sum_{k=1}^{i-1} \frac{1}{S_{k,n-i+1}} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i-1\right] \geq (i-1)^2 \cdot E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i-1\right]$$

After rewriting the left hand side and arranging terms to one side, this becomes

$$E\left[\frac{\sum_{k=1}^{i-1} \prod_{l=1, l\neq k}^{i-1} S_{l,n-i+1}}{\prod_{k=1}^{i-1} S_{k,n-i+1}} - \frac{(i-1)^2}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \middle| S_{k,n-i+1} > 0, k = 1,\ldots,i-1\right] \geq 0 .$$

Thus it is the goal to determine that the quantity inside the brackets is non-negative. After cross multiplication, the resulting numerator is

$$\sum_{k=1}^{i-1}\left(\sum_{m=1}^{i-1} S_{m,n-i+1}\right)\prod_{l=1,l\neq k}^{i-1} S_{l,n-i+1} - (i-1)^2\prod_{k=1}^{i-1} S_{k,n-i+1}$$
$$= \sum_{k=1}^{i-1}\sum_{m=1,m\neq k}^{i-1}\left(S_{m,n-i+1}\prod_{l=1,l\neq k}^{i-1} S_{l,n-i+1} - \prod_{k=1}^{i-1} S_{k,n-i+1}\right)$$
$$= \sum_{k=1}^{i-1}\sum_{m=1,m\neq k}^{i-1}\left(\prod_{l=1,l\neq k,m}^{i-1} S_{l,n-i+1}\right)\left(S_{m,n-i+1}{}^2 - S_{k,n-i+1}S_{m,n-i+1}\right)$$

The inner sum may be broken into two steps first summing from m=1 to k-1 and then from k+1 to i-1. For the latter, one interchanges the order of summation and interchange the roles of k and m to find that the numerator may be written as

$$\sum_{k=2}^{i-1}\sum_{m=1}^{k-1}\left(\prod_{l=1,l\neq k,m}^{i-1} S_{l,n-i+1}\right)\left(S_{m,n-i+1} - S_{k,n-i+1}\right)^2$$

which is clearly non-negative.

VII. Conclusion

It is not the purpose of this paper to advocate one set of assumptions regarding the independence of report lags over another. Indeed, if one believes that expected development increments are directly proportional to the accumulated total claims at a given point in time, then one might conclude that methods based on independent increment assumptions produce understated results.

It is, however, apparent that Stanard's simulation test of the development method produced the correct observation. If one believes that individual report lags are independent, then the loss development methods will produce overstated results. One thing that the analytical work presented here does not show is the magnitude of the bias. Stanard's work produced measures of that in specific cases. The key point is that there is a fundamental incompatibility between loss development techniques and methods relying on independent report lags.

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