Portfolio Optimization and the Capital Asset Pricing Model: A Matrix Approach

by Leigh J. Halliwell

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Abstract

Actuaries are acquainted with the basic ideas of Modern Portfolio Theory and the Capital Asset Pricing Model (CAPM). Briefly, portfolios are formed by weighting risky assets with varying means, variances, and covariances. Each portfolio can be plotted in the X-Y plane by its total return, with the standard deviation as the x-coordinate and the mean as the y-coordinate. It is plausibly asserted that the resulting subspace of returns has an envelope, which is called the efficient frontier. The efficient frontier contains the returns which offer the greatest mean for a given standard deviation, or the least standard deviation for a given mean, and therefore would correspond to portfolios chosen by perfectly informed and rational investors. However, when a riskless asset is introduced, represented by $R_f = (0, \mu_f)$*, one point on the efficient frontier becomes preferable to the others, the point at which a line through* R_f *becomes tangent to the efficient frontier. Since this point is optimal, it will be chosen by all informed and rational investors, which is to say that it will correspond to the portfolio of an efficient market. This market point,* R_m*, is the point* (σ_m, μ_m)*; and the CAPM equation for the* i^{th} *asset is readily derived:* $\mu_i = \mu_f + \beta_i(\mu_m - \mu_f)$*, where* $\beta_i = \frac{\operatorname{cov}(R_i, R_m)}{\operatorname{var}(R_m)}$*. This article shows how the aforementioned argument can be made rigorous through fairly simple matrix algebra, which will foster a deeper understanding of and appreciation for the theory. Moreover, the article offers an easy method for determining the optimal, or market, portfolio. Finally, there will be a few remarks as to why CAPM theory may falter under empirical testing.*

I. Portfolio Optimization and the CAPM in Theory

Consider a universe of n risky assets. The return of the i^{th} asset, denoted R_i, is a random variable with $E(R_i) = \mu_i$ and $Var(R_i) = \sigma_i^2$, for i = 1, 2, ... , n. Let $Cov(R_i, R_j) = \sigma_{ij}$, which implies that $\sigma_{ii} = \sigma_i^2$. Now, instead of regarding the R_i's individually as random scalars, consider the (n x 1) column vector whose elements are the R_i's. Let us call this random vector **R**, using bold type for vectors and matrices; and let us represent it by writing a typical element within matrix brackets. So $\mathbf{R} = [\ R_{i1}\ ]$, or just $[\ R_i\ ]$. [1]

Define the expectation of a matrix as the matrix of the expectations, or $E(\mathbf{X}) = [\ E(X_{ij})\ ]$. Therefore, $E(\mathbf{R}) = [\ E(R_i)\ ] = [\ \mu_i\ ] = \mathbf{M}$. Also, if **X** and **Y** are two column vectors, define $Cov(\mathbf{X}, \mathbf{Y}) = E(\ (\mathbf{X}-E(\mathbf{X}))\ (\mathbf{Y}-E(\mathbf{Y}))'\ ) = [\ E\{\ (X_i - E(X_i))\ (Y_j - E(Y_j))\ \}\ ]$, where the prime (') is the operator for matrix transposition. If **X** is (n x 1) and **Y** is (m x 1), then their covariance is an (n x m) matrix. So $Var(\mathbf{R}) = [\ E(\ (R_i - \mu_i)\ (R_j - \mu_j)\ )\ ] = [\ \sigma_{ij}\ ] = \Sigma$. Obviously, variances of column vectors are symmetric matrices. We will write $\mathbf{R} \sim [\mathbf{M}, \Sigma]$ as shorthand for saying that **R** is distributed with mean **M** and variance Σ.

If **A** is a non-stochastic matrix conformable with **X**, so that $\mathbf{Y} = \mathbf{AX}$ is defined, then $E(\mathbf{Y}) = [\ E(Y_{ij})\ ] = [\ E(\ \Sigma\ a_{ik}\ X_{kj}\ )\ ] = [\ \Sigma\ a_{ik}\ E(X_{kj})\ ] = \mathbf{A}\ E(\mathbf{X})$. Similarly, if **XA** is defined, then $E(\mathbf{XA}) = E(\mathbf{X})\ \mathbf{A}$. Therefore, given the meaning of Cov(**X**, **Y**) above, if **AX** and **BY** are defined, then

$$\begin{aligned} Cov(\mathbf{AX}, \mathbf{BY}) &= E(\ (\mathbf{AX}-E(\mathbf{AX}))\ (\mathbf{BY}-E(\mathbf{BY}))'\ ) \\ &= E(\ \mathbf{A}(\mathbf{X}-E(\mathbf{X}))\ (\mathbf{B}(\mathbf{Y}-E(\mathbf{Y})))'\ ) \\ &= E(\ \mathbf{A}(\mathbf{X}-E(\mathbf{X}))\ (\mathbf{Y}-E(\mathbf{Y}))'\ \mathbf{B}') \\ &= \mathbf{A}\ E(\ (\mathbf{X}-E(\mathbf{X}))\ (\mathbf{Y}-E(\mathbf{Y}))'\ )\ \mathbf{B}' \\ &= \mathbf{A}\ Cov(\mathbf{X}, \mathbf{Y})\ \mathbf{B}'. \end{aligned}$$

Therefore, if a non-stochastic matrix Ω' is conformable with **R**, then $\Omega'\mathbf{R}$ has mean $\Omega'\mathbf{M}$ and variance $\Omega' Var(\mathbf{R})(\Omega')' = \Omega'\Sigma\Omega$, or $\Omega'\mathbf{R} \sim [\Omega'\mathbf{M}, \Omega'\Sigma\Omega]$.

The goal of portfolio optimization is to find an (n x 1) vector $\mathbf{\Omega^*}$, given $\mathbf{R} \sim [\mathbf{M}, \Sigma]$, such that $\mathbf{\Omega^{*\prime}R}$ offers the greatest ratio of expected return in excess of the risk-free return μ_0 to its standard deviation. Let $\mathbf{R_0} \sim [\Sigma_0 = \mathbf{0}, \mathbf{M_0}]$ denote the risk-free return, which is a trivial (1 x 1) random vector. $\Sigma_0 = \mathbf{0} = [0]$ and $\mathbf{M_0} = [\mu_0]$ are (1 x 1) matrices. To be precise, a (1 x 1) matrix is not the same as a scalar, since a scalar can multiply any matrix, whereas a (1 x 1) matrix can only premultiply a (1 x n) matrix or postmultiply an (n x 1) matrix.

Let $\mathbf{J}$ be the (n x 1) vector all of whose elements are ones. Then $\mathbf{R} - \mathbf{JR_0} = [\, R_i - \mu_0 \,]$, which represents the return in excess of the risk-free return. The optimization problem is thus to maximize $(E(\mathbf{\Omega'R}) - \mathbf{\Omega'JR_0})\,(\mathrm{Var}(\mathbf{\Omega'R}))^{-1/2}$, or equivalently, $E(\mathbf{\Omega'}(\mathbf{R} - \mathbf{JR_0}))$ $\mathrm{Var}(\mathbf{\Omega'}(\mathbf{R} - \mathbf{JR_0}))^{-1/2}$, for some $\mathbf{\Omega} = \mathbf{\Omega^*}$. To simplify further calculations, we may relativize μ_0 as 0, which is in effect to substitute $\mathbf{R} + \mathbf{JR_0}$ for $\mathbf{R}$. This will not affect the maximization, and later we can convert our results back into absolute form by substituting $\mathbf{R} - \mathbf{JR_0}$ for $\mathbf{R}$.

So, in relative form, we wish to maximize $E(\mathbf{\Omega'R})\,(\mathrm{Var}(\mathbf{\Omega'R}))^{-1/2}$. Now $\mathrm{Var}(\mathbf{\Omega'R}) = \mathbf{\Omega'\Sigma\Omega}$ is a (1 x 1) matrix, whose only element must be nonnegative since it represents the variance of a scalar random variable. In matrix theory, Σ is said to be nonnegative definite. A symmetric matrix Σ such that $\mathbf{\Omega'\Sigma\Omega} > [0]$ for any non-zero column vector $\mathbf{\Omega}$ is said to be positive definite. We make the assumption that Σ is positive definite; otherwise, our universe of risky assets would not be risky in some combination. Texts in elementary matrix theory show the proof that if Σ is positive definite, then Σ^{-1} exists and is also positive definite. The other assumption which we will make is that $(\Sigma^{-1}\mathbf{M})'\,\mathbf{J}$ is non-zero, which implies that $\mathbf{M}$ is non-zero. The purpose of the second assumption will become apparent below.

Therefore, for all non-zero $\mathbf{\Omega}$, Var($\mathbf{\Omega}$'R) > [0], and (Var($\mathbf{\Omega}$'R))$^{-1/2}$ exists. We will make one more modification by seeking optimize the square: E($\mathbf{\Omega}$'R)^2 (Var($\mathbf{\Omega}$'R))$^{-1}$. One might think that this would lead to the worst $\mathbf{\Omega}^*$ if E($\mathbf{\Omega}^*$'R) < [0]; however, it will turn out that in such a case the optimal investment will be negative, or a disinvestment. Hence, the goal is to maximize some function of $\mathbf{\Omega}$, $\Phi(\mathbf{\Omega})$ = E($\mathbf{\Omega}$'R)^2 (Var($\mathbf{\Omega}$'R))$^{-1}$ = ($\mathbf{\Omega}$'M)^2 ($\mathbf{\Omega}$'$\Sigma\mathbf{\Omega}$)$^{-1}$.

Although the derivation is too involved to be presented here, an optimal $\mathbf{\Omega}^*$ is Σ^{-1}M. Now Σ^{-1} must exist since Σ is positive definite. Furthermore, E($\mathbf{\Omega}^*$'R) = (Σ^{-1}M)' M = M'(Σ^{-1})'M = M'Σ^{-1}M. And Var($\mathbf{\Omega}^*$'R) = (Σ^{-1}M)'Σ(Σ^{-1}M) = (M'Σ^{-1})Σ(Σ^{-1}M) = M'Σ^{-1}M = E($\mathbf{\Omega}^*$'R). Since M'Σ^{-1}M > [0] for our non-zero M, Var($\mathbf{\Omega}^*$'R)$^{-1}$ exists. Therefore, $\Phi(\mathbf{\Omega}^*)$ = M'Σ^{-1}M. Also note that E($\mathbf{\Omega}^*$'R) > [0], irrespective of how many negative elements M contains. However, negative elements in M are likely to produce negative investment elements in $\mathbf{\Omega}^*$.

Now consider:

$$
\begin{aligned}
\Phi(\Omega) &= (\Omega'M)^2\,(\Omega'\Sigma\Omega)^{-1} \\
&= (\Omega'M)^2\,(\Omega'\Sigma\Omega)^{-1}\,(M'\Sigma^{-1}M)^{-1}\,(M'\Sigma^{-1}M) \\
&= (\Omega'M)^2\,\mathrm{Var}(\Omega'R)^{-1}\,\mathrm{Var}(\Omega^{*\prime}R)^{-1}\,\Phi(\Omega^*) \\
&= (\Omega'\Sigma\Sigma^{-1}M)^2\,\mathrm{Var}(\Omega'R)^{-1}\,\mathrm{Var}(\Omega^{*\prime}R)^{-1}\,\Phi(\Omega^*) \\
&= (\Omega'\Sigma\Omega^*)^2\,\mathrm{Var}(\Omega'R)^{-1}\,\mathrm{Var}(\Omega^{*\prime}R)^{-1}\,\Phi(\Omega^*) \\
&= \mathrm{Cov}(\Omega'R,\ \Omega^{*\prime}R)^2\,\mathrm{Var}(\Omega'R)^{-1}\,\mathrm{Var}(\Omega^{*\prime}R)^{-1}\,\Phi(\Omega^*) \\
&= \rho(\Omega'R,\ \Omega^{*\prime}R)^2\,\Phi(\Omega^*),
\end{aligned}
$$

which is less than or equal to $\Phi(\mathbf{\Omega}^*)$, since [0] <= ρ^2 <= [1]. Thus there is no investment strategy superior to $\mathbf{\Omega}^*$.

$\Sigma^{-1}\mathbf{M}$ is not the only optimal value, since $\Phi(k\Omega) = \Phi(\Omega)$ for any non-zero scalar k. This shows that the optimization is not affected by the total amount of the investment, which in matrix terms is $k\Omega^{*\prime}\mathbf{J} = k\mathbf{M}'\Sigma^{-1}\mathbf{J}$. Since a return is relative to the initial investment, we may define the optimal return $\mathbf{R}^*$ as $\Omega^{*\prime}\mathbf{R}\,(\Omega^{*\prime}\mathbf{J})^{-1}$. By our second assumption, $\Omega^{*\prime}\mathbf{J} = (\Sigma^{-1}\mathbf{M})'\,\mathbf{J}$ is nonsingular, so the inverse exists.

Before investigating the properties of $\mathbf{R}^*$ we should ask about the practicality of a singular $\Omega^{*\prime}\mathbf{J}$, i.e., what if $\mathbf{M}'\Sigma^{-1}\mathbf{J} = [0]$? If this were the case, then the optimal return would be attained by a total investment of zero (dollars, or other units of money), whether this meant that zero would be invested in every asset or that positive and negative investments would net to zero. Either way, each investor would have a net position in the market of zero, which means that the value of the whole market of risky assets would be zero. Because this is unrealistic, we may assume $\Omega^{*\prime}\mathbf{J}$ to be nonsingular.

Since $\mathbf{R}^* = \Omega^{*\prime}\mathbf{R}\,(\Omega^{*\prime}\mathbf{J})^{-1}$, $\mathbf{R}^* \sim [\mathbf{M}'\Sigma^{-1}\mathbf{M}(\Omega^{*\prime}\mathbf{J})^{-1}, \mathbf{M}'\Sigma^{-1}\mathbf{M}(\Omega^{*\prime}\mathbf{J})^{-2}]$. Notice that $\mathrm{Var}(\mathbf{R}^*) = \mathrm{E}(\mathbf{R}^*)\,(\Omega^{*\prime}\mathbf{J})^{-1}$. Also, $\mathrm{Cov}(\mathbf{R}, \mathbf{R}^*) = \mathrm{Cov}(\mathbf{R}, \Omega^{*\prime}\mathbf{R}\,(\Omega^{*\prime}\mathbf{J})^{-1}) =$
$\mathrm{Cov}(\mathbf{R}, \mathbf{R})\,\Omega^*\,(\Omega^{*\prime}\mathbf{J})^{-1} = \Sigma\,\Omega^*\,(\Omega^{*\prime}\mathbf{J})^{-1} = \Sigma\,\Sigma^{-1}\mathbf{M}\,(\Omega^{*\prime}\mathbf{J})^{-1} = \mathbf{M}\,(\Omega^{*\prime}\mathbf{J})^{-1} =$
$\mathrm{E}(\mathbf{R})\,(\Omega^{*\prime}\mathbf{J})^{-1}$.

As an (n x1) vector we may write the CAPM beta as follows:

$$\begin{aligned}
\mathbf{B} &= \mathrm{Cov}(\mathbf{R}, \mathbf{R}^*)\,(\mathrm{Var}(\mathbf{R}^*))^{-1} \\
&= \mathrm{E}(\mathbf{R})\,(\Omega^{*\prime}\mathbf{J})^{-1}\,(\mathrm{E}(\mathbf{R}^*)\,(\Omega^{*\prime}\mathbf{J})^{-1})^{-1} \\
&= \mathrm{E}(\mathbf{R})\,(\Omega^{*\prime}\mathbf{J})^{-1}\,((\Omega^{*\prime}\mathbf{J})^{-1})^{-1}\,(\mathrm{E}(\mathbf{R}^*))^{-1} \\
&= \mathrm{E}(\mathbf{R})\,\mathrm{E}(\mathbf{R}^*)^{-1}.
\end{aligned}$$

Therefore, $\mathrm{E}(\mathbf{R}) = \mathbf{B}\,\mathrm{E}(\mathbf{R}^*)$, which is the CAPM equation in relative form. As mentioned earlier, the absolute form of the equation is obtained by substituting $\mathbf{R} - \mathbf{J}R_0$ for $\mathbf{R}$. So
$\mathrm{E}(\mathbf{R} - \mathbf{J}R_0) = \mathbf{B}\,\mathrm{E}(\Omega^{*\prime}\{\mathbf{R} - \mathbf{J}R_0\}(\Omega^{*\prime}\mathbf{J})^{-1}) = \mathbf{B}\,(\mathrm{E}(\mathbf{R}^*) - \Omega^{*\prime}\mathbf{J}R_0(\Omega^{*\prime}\mathbf{J})^{-1})$

$= \mathbf{B}\,(E(\mathbf{R}^*) - \mathbf{R}_0)$. Therefore, $E(\mathbf{R}) = \mathbf{J}\mathbf{R}_0 + \mathbf{B}\,(E(\mathbf{R}^*) - \mathbf{R}_0)$.

Although $\mathbf{R}^*$ has been called the optimal return, it is also represents the *market* return. The argument for this is theoretical: namely, that if every investor is fully informed and rational, then every investor will combine assets proportionately to $\mathbf{\Omega}^*$. This means that the whole market itself of risky assets will allocate value according to $\mathbf{\Omega}^*$, and will have the return characteristics of $\mathbf{R}^*$. Reasons why this may not happen in practice will be presented later.

In concluding this section, let us derive the familiar theorem that the beta of a portfolio is the weighted average of the betas of the portfolio's assets. Letting $\mathbf{\Omega}$ be the asset allocation, the portfolio's beta is $\mathrm{Cov}(\mathbf{\Omega}'\mathbf{R}, \mathbf{R}^*)(\mathrm{Var}(\mathbf{R}^*))^{-1} = \mathbf{\Omega}'\mathrm{Cov}(\mathbf{R}, \mathbf{R}^*)\,(\mathrm{Var}(\mathbf{R}^*))^{-1}$ $= \mathbf{\Omega}'\mathbf{B}$.

II. An Illustration of the Theory

If the author's argument has not been clear enough, perhaps an example will be of help. Consider the simple case of a two-asset universe. Suppose asset A to be priced so as to have an expected return of 0.08, or 8 percent. We regard return as a dimensionless number: $\dfrac{x_1}{x_0} - 1$, where x_0 and x_1 represent initial and terminal wealth respectively. *Rate* of return is return per time, and has units of $(\text{time})^{-1}$. It makes no difference to the example whether we deal with returns or with rates of return; however, actuaries should ensure the dimensional consistency of their formulae. Suppose that the variance of asset A's return is 0.10. Next, let asset B have an expected return of 0.02 and a variance of

0.04. And let the covariance of A and B be -0.06. Finally, suppose the risk-free return to be 0.04.

The returns in excess of the risk-free return are 0.04 and -0.02 respectively. One might wonder why asset B with its substandard return would exist in the market. The answer lies in the negative covariance. Asset B has value not in itself, but in its tendency to cancel out the variance of asset A. Using notation from above, we write:

$$\mathbf{R} \sim \left[\mathrm{M} = \begin{bmatrix} 0.04 \\ -0.02 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.10 & -0.06 \\ -0.06 & 0.04 \end{bmatrix} \right].$$

M is expressed in relative form; Σ is a positive definite matrix. The numbers were chosen so that the example would not be cluttered with fractions or repeating decimals:

$$\Sigma^{-1} = \begin{bmatrix} 100 & 150 \\ 150 & 250 \end{bmatrix}, \text{ and}$$

$$\Omega^{*} = \Sigma^{-1}\mathrm{M} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \propto \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Therefore, the market, in order to obtain the optimal return, will allocate value among assets A and B in equal proportions. Hence, the optimal return is:

$$\begin{aligned} \mathbf{R}^{*} &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \mathbf{R} \\ &\sim \left[\begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \mathrm{M}, \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \Sigma \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \right] \\ &\sim \left[[0.01], [0.005] \right]. \end{aligned}$$

$$\begin{aligned}
\text{Also, } \operatorname{cov}(\mathbf{R},\mathbf{R}^*) &= \operatorname{cov}(\mathbf{R},[0.5 \quad 0.5]\mathbf{R}) \\
&= \operatorname{cov}(\mathbf{R},\mathbf{R})\begin{bmatrix}0.5\\0.5\end{bmatrix} \\
&= \Sigma\begin{bmatrix}0.5\\0.5\end{bmatrix} \\
&= \begin{bmatrix}0.02\\-0.01\end{bmatrix}.
\end{aligned}$$

Therefore, $\mathbf{B} = \operatorname{cov}(\mathbf{R}, \mathbf{R}^*)\operatorname{var}(\mathbf{R}^*)^{-1} = \begin{bmatrix}0.02\\-0.01\end{bmatrix}[0.005]^{-1} = \begin{bmatrix}4.0\\-2.0\end{bmatrix}$. So, the CAPM equation in relative form is true: $E(\mathbf{R}) = \begin{bmatrix}0.04\\-0.02\end{bmatrix} = \begin{bmatrix}4.0\\-2.0\end{bmatrix}[0.01] = \mathbf{B}\,E(\mathbf{R}^*)$, as well as the equation in absolute form:

$$\begin{aligned}
E(\mathbf{R}) &= \begin{bmatrix}0.08\\0.02\end{bmatrix} \\
&= \begin{bmatrix}0.04\\0.04\end{bmatrix} + \begin{bmatrix}4.0\\-2.0\end{bmatrix}([0.05]-[0.04]) \\
&= \begin{bmatrix}1\\1\end{bmatrix}[0.04] + \begin{bmatrix}4.0\\-2.0\end{bmatrix}([0.05]-[0.04]) \\
&= \mathbf{JR}_0 + \mathbf{B}(E(\mathbf{R}^*)-\mathbf{R}_0).
\end{aligned}$$

Also, note that the market-weighted beta is $[0.5 \quad 0.5]\begin{bmatrix}4.0\\-2.0\end{bmatrix} = [1.0]$, as expected.[2]

The econometric material in the CAS part 10 exam induced the author to study matrix theory from an econometric perspective.[3] This effort has repaid me with a generous dividend, and I hope that many readers will have their appetites whetted to undertake similar studies. In my first draft of this article I had not seen the matrix application, and was tediously proving just the two-asset case of the CAPM by considering the efficient frontier as a parametric equation in one parameter.[4] I am convinced that matrix theory is a powerful tool in its own right, rather than just a convenient shorthand, and that in statistics the econometricians are far ahead of us actuaries precisely because of their matrix approach to this subject.

III. Portfolio Optimization and the CAPM in Practice

Throughout the article we have been speaking of a perfectly informed and rational investor. However, we know that no two investors have the same beliefs about the future, and no two have the same utilities. For example, a socially conscious investor who refuses to purchase tobacco stocks, or South African gold stocks, is undoubtedly shaving from the optimal return. However, the loss is compensated by his perceived loyalty to virtue. No two investors are alike; and perhaps the perfectly informed and rational investor is a far-away ideal.

Furthermore, we cannot obtain the needed M and Σ matrices. Indeed, the first problem is to define what belongs to the universe of assets. In the standard applications of the CAPM "the market" is proxied by the S&P 500 index. Granted that the S&P 500 makes up about two-thirds of the market value of US. stocks, what about the stocks of the rest of the world? And what about the other risky assets of the world, which is just about everything except US. treasury securities? What about real estate? And perhaps commodities, such as wheat, oil, and gold, should be included -- perhaps even collectibles, such as rare coins and art. In other words, although we speak glibly of "the market," no one really knows its extent. Anything that can traded, perhaps even insurance loss portfolios, might be part of the market. So probably our proxies are rather bad ones, and partly responsible for the mixed results of CAPM tests.

And even with a limited universe of say 500 stocks, there remains a problem in estimating 500 betas and one equity risk premium (r_m-r_f). The problem is well known to actuaries as the dilemma between stability and responsiveness, viz., that by the time you have enough observations to perform a good estimation, the underlying parameters have more or less drifted. So the CAPM might be perfectly corroborated, if only we knew the current parameters, rather than the outdated ones. Perhaps "the market" has some great collective intuition, which transcends the knowledge of individual investors. The logical positivist would balk at such a statement, which is more or less the capitalist's credo. However, the notion that there really is an "invisible hand" in human affairs which directs toward the greatest good is somewhat reasonable, even if difficult to verify -- as difficult to verify as the CAPM itself.

The CAPM is of one piece with the efficient market hypothesis. It is of no help in the selection of stocks or of any other asset. In fact, it dictates that every investor's portfolio be a microcosm of the whole market. If the market really were the S&P 500, for example, then the CAPM would have everyone invested in a mutual fund indexed to the S&P 500, which is called passive investing. Herein lies a parting conundrum: although passive investing should be optimal, the market needs to be winnowed and sifted by active investors endeavoring to outperform it.

Notes

1 It is presumed that the reader has some familiarity with matrix algebra. Therefore, some of the steps in the derivations may involve the application of multiple matrix theorems. Some of the basic properties of matricies are stated here, and may be of help if the reader is puzzled by a derivation:

A. Matrix multiplication is associative: A(BC)=(AB)C.
B. Matrix multiplication is not commutative; however, (1 x 1) matrices commute.
C. Matrix multiplication is distributive: A(B+C)=AB+AC.
D. Transposition of a product behaves thus: (AB)' = B'A'.
E. Similarly, with matrix inversion, $(AB)^{-1}=B^{-1}A^{-1}$, if A and B are nonsingular.
F. By definition, A is symmetric if and only if A'=A.
G. Every (1 x 1) matrix is symmetric.
H. If A is nonsingular, then $(A^{-1})^{-1} = A$. Also, $(A^{-1})'=(A')^{-1}$.

2 For those who wonder if the example might be contrived in that the optimal combination of assets was 50/50, we modify the example by changing the risk-free return from 0.04 to 0.03. The reader can verify:

$$\mathbf{R} \sim \left[M = \begin{bmatrix} 0.05 \\ -0.01 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.10 & -0.06 \\ -0.06 & 0.04 \end{bmatrix} \right].$$

$$\Omega^* = \Sigma^{-1} M = \begin{bmatrix} 3.5 \\ 5.0 \end{bmatrix} \propto \begin{bmatrix} 7/17 \\ 10/17 \end{bmatrix}$$

$$\begin{aligned} \mathbf{R}^* &= [7/17 \quad 10/17]\mathbf{R} \\ &\sim [[1/68],[1/578]]. \end{aligned}$$

$$\begin{aligned} \operatorname{cov}(\mathbf{R},\mathbf{R}^*) &= \operatorname{cov}(\mathbf{R},[7/17 \quad 10/17]\mathbf{R}) \\ &= \operatorname{cov}(\mathbf{R},\mathbf{R}) \begin{bmatrix} 7/17 \\ 10/17 \end{bmatrix} \\ &= \Sigma \begin{bmatrix} 7/17 \\ 10/17 \end{bmatrix} \\ &= \begin{bmatrix} 1/170 \\ -1/850 \end{bmatrix}. \end{aligned}$$

So $\mathbf{B} = \text{cov}(\mathbf{R}, \mathbf{R}^*)\,\text{var}(\mathbf{R}^*)^{-1} = \begin{bmatrix} 1/170 \\ -1/850 \end{bmatrix}\left[1/578\right]^{-1} = \begin{bmatrix} 17/5 \\ -17/25 \end{bmatrix}$. The CAPM equation

in relative form checks: $\text{E}(\mathbf{R}) = \begin{bmatrix} 0.05 \\ -0.01 \end{bmatrix} = \begin{bmatrix} 17/5 \\ -17/25 \end{bmatrix}\left[1/68\right] = \mathbf{B}\,\text{E}(\mathbf{R}^*)$. Also, the

market-weighted beta is $\begin{bmatrix} 7/17 & 10/17 \end{bmatrix}\begin{bmatrix} 17/5 \\ -17/25 \end{bmatrix} = \left[1.0\right]$.

3 For those interested in studying econometrics, the author recommends Introduction to the Theory and Practice of Econometrics, 2nd edition, by G. G. Judge, R. C. Hill, *et al.* (New York: John Wiley & Sons, 1988). The seventy-five page appendix on matrix theory alone makes the book worth reading.

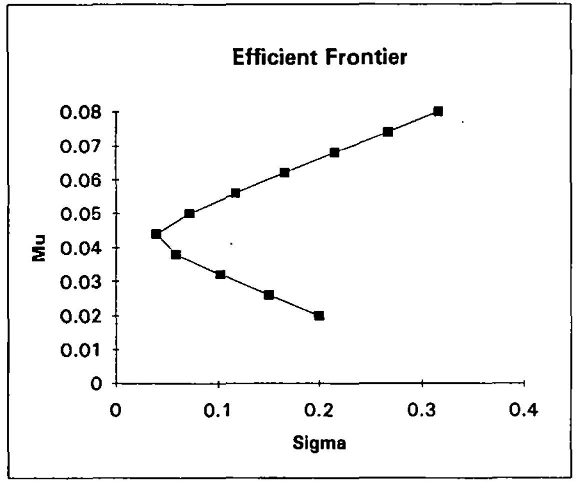
4 See the following Appendix for a speadsheet of the two-asset example.

APPENDIX

CAPM Illustration showing Optimal Mix at 50/50

	Mu	Sigma^2	Cov(A,B)
Asset A	0.08	0.1	-0.06
Asset B	0.02	0.04	
Risk-free	0.04		

Wgt(A)	Wgt(B)	Sigma	Mu	(Mu-0.04)/ Sigma
0%	100%	0.2	0.02	-0.1
10%	90%	0.150333	0.026	-0.09313
20%	80%	0.10198	0.032	-0.07845
30%	70%	0.05831	0.038	-0.0343
40%	60%	0.04	0.044	0.1
50%	50%	0.070711	0.05	0.141421
60%	40%	0.116619	0.056	0.137199
70%	30%	0.165529	0.062	0.132907
80%	20%	0.215407	0.068	0.129987
90%	10%	0.265707	0.074	0.127961
100%	0%	0.316228	0.08	0.126491



Mu = Wgt(A)*Mu(A) + Wgt(B)*Mu(B)

Sigma = SQRT{Wgt(A)^2*Sigma(A)^2 + 2*Wgt(A)*Wgt(B)*Cov(A, B) + Wgt(B)^2*Sigma(B)^2}