UNDERWRITING BEST SHARES OF EXCESS LAYERS

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ABSTRACT

The application of utility theory to insurance is incomplete without the inclusion of the structure of risk sharing; to limit the use of utility to accept or reject decisions misses its power to explain the real world of excess insurance and reinsurance. The fine subdivision of risk which is routinely achieved by insurance institutions can be explained on this basis. Practical methods of evaluating and pricing excess layers are given. This model appears to be in broad agreement with the behavior of experienced underwriters and thus can form the decision basis of a reinsurer's assumption of risk.

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INTRODUCTION

The inherent nature of the world of underwriting involves the selection of shares of all the risks available for participation. It appears that the objective is to maximize the utility of the portfolio of all retained risks. In this paper we show that utility and risk sharing go together, that the best share structure is ideal for representing reinsurance, and that risk retention is limited by choice. In other words both vertical (proportional) and layered (sharing) can improve the risk characteristics of the risk retained.

In insurance applications, utility never indicates a direct limit to the number of risks an insurer ought to accept. In reaction to this realization, Richard G. Woll [1982] wrote "The implication of exponential utility flies directly in the face of the historical and intuitive notion that there is always some limit to the amount of business one is willing to write with a given amount of capital". He fails to recognize that the problem would be still worse with a utility function whose local absolute risk aversion function declines with wealth. We choose to use the only form of utility function which provides additivity of certainty equivalents of independents risks.

There is a literature on best share in insurance; Borch [1968] and several others have sought Pareto optimal (jointly optimal) share solutions where the objective function is based on utility. Indeed, the idea of best share has a broader literature than insurance. However both the insurance and the other applications are all oriented toward Pareto optimal solutions. For example, see Raviv [1992] and Howard Raiffa's book "Decision Analysis: Introductory Lectures on Choices under Uncertainty", [1968]. Ming Yeh [1985], also provides many references on Pareto Optimality.

The problem with joint optimality for the buyer and seller is that it is not a realistic description of a market situation in which there are many parties. Pareto optimality suggests that there are only two parties to the potential treaty; the two parties must either reach agreement on the terms of the treaty or there will be no treaty. In reality, the market for reinsurance is well developed. Furthermore, the reinsurer have the institution of retrocession available to them so that if one found a good risk which is too large, then he would be able to retrocede out part of it. Consequently, in such a realistic market, a party who wants to share risk need only decide how much of it to retain.

The condition of Pareto optimality often makes the analysis impractical because it requires the knowledge of the party's risk aversion level. The determination of the best share to retain of a risk is easier without the constraint of Pareto optimality and is a more useful tool for underwriters. Cozzolino [1974] gives a formulation of the risk sharing problem for general concave utility functions where conditions for existence and uniqueness of a best share are shown. Dionne [1990] also gives an analysis of the share and how it depends upon the risk characteristics.

An optimum share must exist in the interior or at the boundaries. Cozzolino [1974] proves that the optimum cannot be at zero when the expected profit is positive. This means that some share, however small, will exist in spite of the degree of riskiness. See Samuelson in [1957].

Searching for the optimum share is straightforward because it is a single variable search. Without the additivity property of exponential utility, the result for one policy would depend upon the answers for all other policies. Leaving a difficult joint optimization problem to solve with the assumption of exponential utility and independent risks, all that is needed is the to evaluate the function.

Another characteristic of the optimum share, seen in Gupta and Cozzolino [1974], is that the certainty equivalent generally has a shallow peak at the optimum share. This suggests that the best share may be insensitive to variation within a few points of the optimum. Human judgment may therefore be quite good at finding good solutions in many customary situations. In both reinsurance and in oil exploration situations the decisions are made within the "experienced operator" type of decision making; no formal analysis is performed.

Best share is a useful model of reinsurance, whether or not it exactly represents the actual forms of reinsurance. It can also be formulated with a constraint to limit the total capacity underwritten. James M. Stone [1983], stated that risk is not reduced by "vertical" (proportional) sharing of risk. He suggests that "exposure ratio" is so important a characterization of risk that vertical sharing, which does not change the exposure ratio of one risk, must not be very important. Only the "excess of loss" form of sharing, also known as layering produces coverages with different exposure ratios. On a utility basis, proportional sharing possesses all of the benefits of reinsurance.

Recalling Woll's [1982] objection that utility ought to limit the number of risks accepted by an insurer, although intuitively appealing, the existence of unlimited numbers of good risks, is not available. This is an aspect of "parameter risk" which explains the existence of limited portfolios. What is clearly provided by the combination of risk sharing and utility is a reduction in the amount of risk retained net of reinsurance.

OPTIMUM SHARES OF EXCESS LAYERS

Under CAPM, the suggestion is often made that all corporations, including insurance corporations, ought to be neutral toward risk, see Mayers and Smith [1982,1987]. This idea, usually associated with the theory of finance, further suggests that expected value pricing ought to be the effective basis of insurance pricing. It is possible that the elaborate sharing mechanisms practiced in the real world of insurance are so effective that the resulting premium structure is close to the expected value limit. See Mayers and Smith [1982]. Clearly insurance companies do not behave as though they are risk neutral but their efforts to subdivide risk, as evidence by the structure of the industry, may result in the naive appearance that they are.

Adopting our proposed model may actually enable the industry to increase its efficiency in achieving optimal levels of retained risk. The opportunity stream seen by a reinsurer usually includes many layers of risk on an excess basis. In the Lloyds market, the underwriter is invited to participate by signing up for a share, of one or more layers. Many of the "slips" one observes contain shares in the magnitude of one or two percent. Thus many insurance institutions illustrate the need for a methodology for finding the best share.

The same model is useful as the basis of pricing and as a basis for pricing and underwriting, depending only upon whether you assume that the price is exogenously determined or is the subject of determination. The actuary determines the appropriate price and risk-load of a given risk. For example, if a 5% load is recommended for primary coverage, what is an "appropriate" size of risk load for an excess layer of $1 million excess of $5 million? Underwriting differs in the respect that underwriting decisions are usually about whether to accept the proposed risk at a given price. The share is often up to the underwriter, as in the case of Lloyds, but the price is not his main decision variable.

ILLUSTRATION EXAMPLE

The probability framework used here is the piecewise constant Pareto for severity and the Poisson distribution for frequency. Choices of these distributions reflect current practice, as reported by Bear and Nemlick [1990], and Bear, Englander, and Hess [1992]. The value system used is Risk Adjusted Cost, (RAC), in Cozzolino, [1978], based upon utility theory and the complete family of exponential utility functions for the method of risk measurement. We take a perspective of large commercial lines rather than either personal lines or small commercial accounts. On the insurer side, this represents the perspective of the reinsurer or excess carrier. In this context we seek to evaluate the best share to underwrite of an excess layer.

Table 1 gives the example probability distribution of loss severity for each of the size layers and the Poisson frequencies for each layer. It is useful to define some notation for the layers and their aggregate loss amounts. Let the ith layer start at loss size L_{i-1}, and end at loss size L_i. Let a_i be the sum of all losses whose size is in that layer. It is the aggregate loss for layer i.

TABLE 1

PROBABILITY DISTRIBUTION OF SEVERITY

LAYER	(1) LOSS LAYER FROM LOW END	(2) DEFINITIONS TO HIGH END	(3) FREQUENCY IN LAYER	(4) EXPECTED LOSS IN LAYER
1	$0	$500	18.528	$4,632
2	$500	$1,000	10.703	$8,027
3	$1,000	$5,000	12.364	$37,092
4	$5,000	$10,000	4.301	$32,258
5	$10,000	$50,000	4.970	$149,100
6	$50,000	$100,000	2.495	$187,125
7	$100,000	$500,000	2.878	$863,400
8	$500,000	$1,000,000	1.001	$750,750
9	$1,000,000	$2,000,000	0.760	$1,140,000
		TOTALS:	58.000	$3,172,384

The set of aggregate losses, a_i, for each layer, i, are independent random variables when the frequency within each layer has a Poisson distribution. The aggregate excess loss for layer i, is e_i. This is not a set of independent random variables, but the a's help to compute both the variance and Risk Adjusted Cost of the e_i's. This "piecewise constant Pareto distribution" is chosen to be useful because people naturally think of the layer's properties as constant within the layer. A fortuitous analytical property of this distribution *is the additivity of layer variances so that the variance of the aggregate loss for all layers is the sum of the layer variances. This additivity is true also for the contributions to Risk* Adjusted Cost.

For the piecewise constant Pareto we set the height of the density function for every layer as that which the best fitting Pareto has at the point which is the center of the layer. The mean aggregate loss for the ith layer, between lower point L_{i-1} and upper point, L_i, is EL as given by equation (1), where F_i is the mean frequency, in layer i, multiplied times the mean size of a rectangular distribution.

$$EL\ (L_{i-1},L_i) = F_i\{L_{i-1}+L_i\}/2 \qquad (1)$$

When discussing policy limits, it is useful to know the mean aggregate loss when the layer has zero width because the lower and upper end points are identical. Then the mean aggregate loss is the severity times the mean frequency.

The RAC formula for the piecewise Pareto with Poisson frequency is given by Cozzolino [1978], and is shown by equation (2). The symbol r stands for the (local absolute) risk aversion level, as defined by Pratt [1964]. For operational ease, we will later replace r by its reciprocal, S which will stand for "Surplus", sometimes called "risk tolerance" in the literature of exponential utility. When the loss interval is degenerate, having zero width,

$$RAC\ (L_{i-1},L_i) = (F_i/r)\ [\ \frac{\{\exp(r\ L_i)-\exp(r\ L_{i-1})\}}{r\ (L_i-L_{i-1})} - 1\ ] \qquad (2)$$

The Risk Adjusted Cost of that layer is given by equation (3).

$$RAC\ (L_{i-1},L_i) = \{F_i/r\}\ \{\exp(r\ L_i) - 1\} \qquad (3)$$

Expected value rates for excess layers are often computed in practice by subtraction of rates for the corresponding "ground-up" layers. But subtraction is not correct when risk load is present. Miccolis [1977] gives a correction formula for when the risk load is based upon variance. Cozzolino [1989] gives the similar correction formula based upon exponential utility, as shown in equation (4).

$$RAC(z,y) = \frac{RAC(x,y) - RAC(x,z)}{e^{r(z-x)}} \qquad (4)$$

When x is zero, equation (4) gives the risk loaded premium for the layer from z to y, in terms of the rate from zero to y less the rate for the layer from zero to x. The denominator represents a correction relative to pure subtraction. This formula will be used to compute the Risk Adjusted Cost (premium) for an excess layer.

RISK SHARING

The task is to compute the best share of an excess layer and to examine its properties. Once you have the risk adjusted cost formula for a layer, you can easily compute the RAC for any share of the same layer. We assume that you get equal shares of both the loss and of the premium.

Miccolis [1977] has shown that there is a positive correlation between the aggregate loss from any two layers of the same risk.

The correlation has some important implications:

1) Reinsurer need to be distinct from primary carriers so that the risk is spread among many independent companies.

2) Risk retention from a insured's perspective is logically in the form of ground-up retentions.

3) We will compute the risk-loaded premium for all ground-up retentions. From the insurer's view, it is necessary to decide which of one or more excess layers to accept. Because of the dependence between any two layers, evaluation of accepting two layers is more complicated than the separate evaluations of the two layers.

4) Due to positive correlation between any pair of layers the excess carrier might prefer to insure just one layer rather than any combination of layers. All that can be stated in general is that all combinations or available layers must be evaluated.

5) For the case where the insurer is considering layers of two separate (unique) risks, the presence of parameter risk results in a positive correlation. Thus the value of insuring any pair of layers is less than the value obtained when computing layer values separately, because of the positive correlation. The greater the separation between the two layers, the lower the expected correlation.

THE BEST SHARE

The best share cannot be determined from the RAC alone because that just measures the cost of risk. The profit of underwriting the risk is the premium for the share underwritten less the RAC of the share underwritten. This is the objective to be maximized. The RAV (Risk Adjusted Value) of share "a" is given by (5):

$$RAV(a) = Pa - RAC(a), \qquad (5)$$

where P is the Premium for retaining the whole risk.

Illustrative market premiums for all the ground-up policies are given in Table 2. They are 150% of expected loss; the risk-load is 33% of premium. Since different layers have different risk characteristics, this may be unrealistic pricing.

TABLE 2

MARKET PREMIUMS

LAYER	POLICY LIMIT	PREMIUM
1	$500	$36,552
2	$1,000	$62,142
3	$5,000	$197,664
4	$10,000	$304,574
5	$50,000	$881,714
6	$100,000	$1,323,201
7	$500,000	$3,243,201
8	$1,000,000	$4,188,522
9	$2,000,000	$4,758,570

Since the premiums were based on the expected loss for each ground-up policy, their calculations were preceded by the calculations of expected losses, whose results are shown in columns 5 through 9 of Table 3.

TABLE 3

EXPECTED LOSSES FOR GROUND-UP COMBINATIONS OF LOSS LAYERS

LAYER	(5) EXPECTED LOSS IN LAYER	(6) FREQUENCY ABOVE THE LAYER	(7) EXP LOSS BELOW & INCL LAYER	(8) EXP LOSS AT THIS DEDUCTIBLE	(9) EXP LOSS FOR THIS DEDUCTIBLE
1	$4,632	39.4720	$4,632	$19,736	$24,368
2	$8,027	28.7690	$12,659	$28,769	$41,428
3	$37,092	16.4050	$49,751	$82,025	$131,776
4	$32,258	12.1040	$82,009	$121,040	$203,049
5	$149,100	7.1340	$231,109	$356,700	$587,809
6	$187,125	4.6390	$418,234	$463,900	$882,134
7	$863,400	1.7610	$1,281,634	$880,500	$2,162,134
8	$750,750	0.7600	$2,032,384	$760,000	$2,792,384
9	$1,140,000	0.0000	$3,172,384	$0	$3,172,384
Total	$3,172,384				

The formulas, in terms of spreadsheet columns, are as follows:

Column (5) is the aggregate expected loss in the layer found by multiplying the midpoint loss size by the frequency in the layer.

Column (6) is the total frequency less the sum of layer frequencies in the rows up to the current row.

Column (7) is the cumulative sum of layer expected losses up to the current row.

Column (8) is the product of (7) multiplied by the top size of the current layer. This represents losses that penetrate the current layer and so the policy pays the policy limit.

Column (9) is the sum of (7) and (8); the sum of losses below the policy limit and those losses which penetrate the policy limit and the insured receives exactly the policy limit.

This type of analysis was first published by Allen and Duvall in [1971].

TABLE 4

SPREADSHEET FOR RAC CALCULATION

LAYER	(1) LOSS LAYER FROM LOW END	(2) DEFINITIONS TO HIGH END	(3) LOSS LAYER MIDPOINT	(4) FREQUENCY IN LAYER	(5) RAC LOSS IN LAYER
1	$0	$500	$250	18.528	4,633
2	$500	$1,000	$750	10.703	8,030
3	$1,000	$5,000	$3,000	12.364	37,156
4	$5,000	$10,000	$7,500	4.301	32,383
5	$10,000	$50,000	$30,000	4.970	151,700
6	$50,000	$100,000	$75,000	2.495	194,601
7	$100,000	$500,000	$300,000	2.878	1,032,845
8	$500,000	$1,000,000	$750,000	1.001	1,140,260
9	$1,000,000	$2,000,000	$1,500,000	0.760	2,789,788
			Totals	58.000	5,391,397

The calculation of the Risk Adjusted Cost uses equations (3), and (4), shown earlier. It is also organized into a spreadsheet. The first five columns are like those of Table 1 except the last column on the top part is RAC instead of expected loss. The last three columns of the new Table 4 are like those of Table 3 except that they are about RAC rather than expected loss.

TABLE 4 (CONTINUED)

LAYER	(6) FREQUENCY ABOVE THE LAYER	(7) RAC BELOW & INCL LAYER	(8) RAC AT THIS DEDUCTIBLE	(9) RAC FOR THIS DEDUCTIBLE
1	39.4720	4,633	19,741	24,374
2	28.7690	12,663	28,783	41,447
3	16.4050	49,819	82,230	132,050
4	12.1040	82,202	121,647	203,850
5	7.1340	233,903	365,768	599,671
6	4.6390	428,504	487,888	916,392
7	1.7610	1,461,349	1,142,398	2,603,747
8	0.7600	2,601,609	1,305,894	3,907,503
9	0.0000	5,391,397	0	5,391,397

The Risk Adjusted Cost values in Table 4 were computed for a risk aversion level which is the reciprocal of one million dollars.

The exponential utility implied by this statement can be described herein as the one representing a surplus (mentioned earlier) of one million dollars. The use, in insurance applications, of the reciprocal of the risk aversion level was popularized by Van Slyke [1985] who relates the idea to that of capacity.

The underwriter's decision here is how high to set the limit of the "primary policy", given that the use of reinsurance will reduce the risk retained. Clearly the risk aversion of the insurers explains the structure that some write primary business while others write excess business.

The best shares are computed for each of the nine ground-up policies and shown in Table 5 below. The best shares were found by simple searching in the share domain for the share giving the largest Risk Adjusted Value. Often the best share is 100% which is simply found by determining that the share 99% has a lesser risk adjusted value than does 100%. The concavity of the RAV as a function of share, as proven in [3], allows this conclusion. Table 5 is based upon a surplus value of $2 million. We see in the results that only the full risk (coverage to $2 million) is risky enough that less than 100% is preferred. If these surplus values are the same as those we are accustomed to for real property/casualty insurers, the results seem to suggest that this company is more like $20 to $50 million in terms of the surplus we are used to dealing with. As we gain experience with the use of utility theory, these judgments will become easier.

Another aspect of the results in Table 5 is that our insurer would prefer to insure a primary layer to limits of $1 million rather than any other of the limits considered, since this choice gives the largest RAV in the column. The fact that this insurer also prefers to retain 100% of that policy is just further evidence that its risk is not considered exceptionally large by this insurer.

TABLE 5

BEST SHARES AND THEIR RAV'S FOR GROUND-UP POLICIES

LAYER	POLICY LIMIT	EXPECTED LOSS @ 100%	BEST SHARE	RAV OF BEST SHARE
1	$500	$24,368	100.00%	$12,181
2	$1,000	$41,428	100.00%	$21,705
3	$5,000	$131,776	100.00%	$65,751
4	$10,000	$203,049	100.00%	$102,125
5	$50,000	$587,809	100.00%	$288,020
6	$100,000	$882,134	100.00%	$424,197
7	$500,000	$2,162,134	100.00%	$876,884
8	$1,000,000	$2,792,384	100.00%	$915,002
9	$2,000,000	$3,172,384	87.50%	$761,894

EVALUATION OF EXCESS LAYERS

An excess layer is any layer that attaches at a loss size greater than zero, whereas any layer that attaches at zero loss size is a so-called "ground-up", or "primary" layer. The excess layer has an aggregate loss with at least some positive correlation with every other layer of the same risk. Miccolis [1977] showed how to compute these correlations and how to compute the risk reduction gained from splitting one layer into two. Whereas his work was limited to variance as a measure of risk, Cozzolino, [1989] extended it to exponential utility as the risk measure. The application of this concept for evaluating layer pricing utilizes equation (4).

Thus the RAC's for the ground-up coverages will now be directly useful in computing the RAC's for the excess layers. Subtraction is correct for the expected monetary value of the excess layers while corrected subtraction is used for the RAC values.

Table 6 shows the RAC calculations of excess layers. With this spreadsheet, we can compute the RAV as a function of the share retained and then search for the best share for every layer separately. First, we need to know the premiums for each layer. They are given in Table 7 above. Again the premiums have been determined as 150% of the expected loss for that layer.

TABLE 6

EVALUATION OF EXCESS LAYERS FOR SURPLUS OF $1 MILLION

LAYER	LOSS LAYER FROM LOW END	DEFINITIONS TO HIGH END	FREQUENCY IN LAYER
1	$0	$500	18.528
2	$500	$1,000	10.703
3	$1,000	$5,000	12.364
4	$5,000	$10,000	4.301
5	$10,000	$50,000	4.970
6	$50,000	$100,000	2.495
7	$100,000	$500,000	2.878
8	$500,000	$1,000,000	1.001
9	$1,000,000	$2,000,000	0.760
		Total	58.000

TABLE 6 (CONTINUED)

EVALUATION OF EXCESS LAYERS FOR SURPLUS OF $1 MILLION

GRND UP TO THIS LIMIT	RAC GRND UP TO THIS LIMIT	EXPECT LOSS FOR THIS EXCESS LAYR	RAC FOR THIS EXCESS LAYR	% RISK LOAD
$24,368	$24,371	$24,368	$24,371	0.01%
$41,428	$41,437	$17,060	$17,062	0.01%
$131,776	$131,913	$90,348	$90,430	0.09%
$203,049	$203,449	$71,273	$71,357	0.12%
$587,809	$593,694	$384,760	$388,299	0.92%
$882,134	$899,004	$294,325	$297,772	1.17%
$2,162,134	$2,366,317	$1,280,000	$1,395,752	9.04%
$2,792,384	$3,270,520	$630,250	$704,194	11.73%
$3,172,384	$4,015,928	$380,000	$452,113	18.98%

For these eight excess layers, and the first also, we have determined the best share for an insurer whose surplus is $1,000,000. The answers are in Table 8.

TABLE 7

PREMIUM FOR EXCESS LAYERS (*)

LAYER	POLICY LAYER	COMMON NAME	PREMIUM
1	0 TO $500	$500 EX $0	$36,552
2	500 TO $1000	$500 EX$500	$25,590
3	1 TO 5 K	$4K EX $1K	$135,522
4	5 TO 10 K	$5K EX $5K	$106,909
5	10 TO 50 K	$40K EX $10K	$577,140
6	50 TO 100 K	$50K EX $50K	$441,488
7	100 TO 500K	$.4M EX $.1M	$1,920,000
8	$.5M TO $1M	$.5M EX $.5M	$945,375
9	$1M TO $2M	$1M EX $1M	$570,000

(*) K = Thousand and M = Million

TABLE 8

RISK ADJUSTED VALUES OF EXCESS LAYERS (*)

LAYER	PREMIUM FOR THIS SHARE EXCS LAYER	RAV OF THIS SHARE EXCS LAYER	BEST SHARE FOR THIS EXCS LAYER	RAV OF BEST SHARE EXCS LAYER
1	$36,552	$12,178	100.000%	$12,178
2	$25,590	$8,526	100.000%	$8,526
3	$135,522	$45,010	100.000%	$45,010
4	$106,909	$35,467	100.000%	$35,467
5	$577,140	$185,257	100.000%	$188,257
6	$441,488	$140,213	100.000%	$143,213
7	$1,920,000	$393,218	100.000%	$393,218
8	$945,375	$154,607	92.400%	$155,855
9	$570,000	$24,106	59.000%	$60,655

(*) SHARE = 100.000% and SURPLUS = $1,000,000

If the positive correlation between any two layers prevents the underwriter from taking two or more layers, then the layer with the best value should be selected by the underwriter. This is Layer 7. The top two layers are the only ones whose best shares are less than 100%. This may result from their high risk or from their inadequate price. In order to convey a better understanding of the best share idea, the Risk Adjusted Values of the layers are computed for several shares retained so that a graph can be made. Table 9 gives the results.

TABLE 9

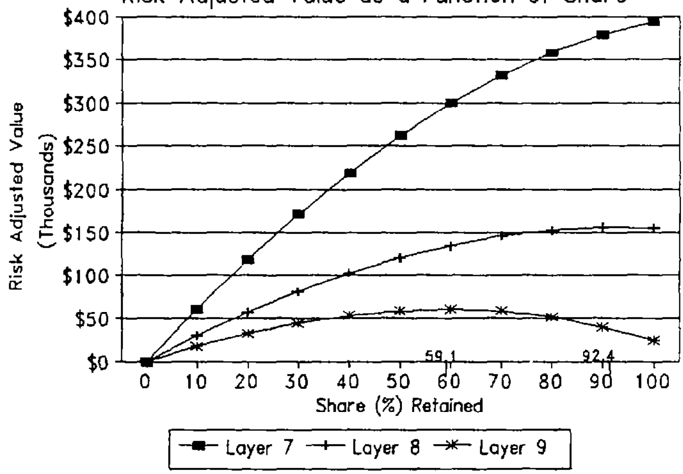
RISK ADJUSTED VALUES OF SHARES

SHARE	LAYER 7	LAYER 8	LAYER 9
0.00%	$0	$0	$0
10.00%	$61,797	$30,124	$17,701
20.00%	$119,079	$57,384	$32,670
30.00%	$171,679	$81,645	$44,691
40.00%	$219,421	$102,763	$53,533
50.00%	$262,124	$120,590	$58,944
59.10%	$296,447	$133,821	$60,655
60.00%	$299,601	$134,970	$60,650
70.00%	$331,656	$145,738	$58,354
80.00%	$358,085	$152,722	$51,736
90.00%	$378,679	$155,741	$40,446
92.40%	$382,729	$155,855	$37,000
100.00%	$393,218	$154,607	$24,106

Figure 1 shows the RAV's of these top three layers (see Table 9) as functions of the share variable in order to show the two main shapes. Layer 7 has an RAV which increases everywhere within the zero to 100% range, offering no interior optimum. Layer 8 attains its optimum at 92.4% while layer 9 has its maximum at 59.1% for our insurer. Therefore, the best share is the largest possible one which is 100%. The curves for layer 8 and layer 9 peak within the same interval. In other words, layers 8 and 9 have the best shares.

Figure 1

Risk Adjusted Value as a Function of Share

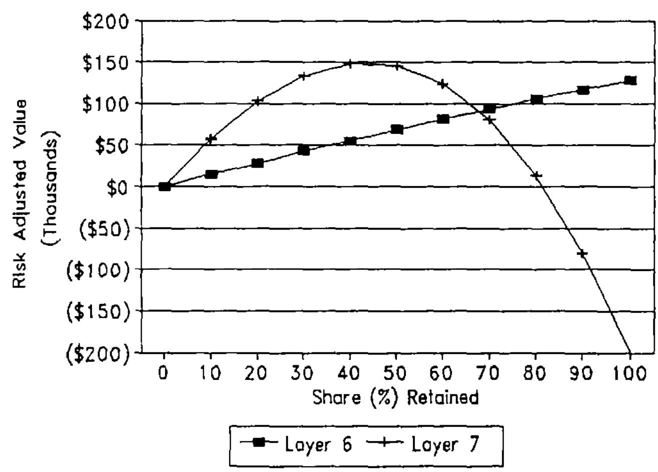


Recall that the set of possible primary layers were evaluated at a surplus level or $2 million and the set of possible excess layers at a surplus level of $1 million. That only means that the results for the two groups of possible policies are not directly comparable. But if comparability is desired, the analysis can be repeated at many different surplus levels and presented in terms of risk profile curves. This gives one graph for each layer which shows the Risk Adjusted Value at all possible surplus levels. Some broad comparisons are possible. This technique was presented by Cozzolino [1978].

The analysis was repeated at a surplus level of $370,370 and the values were compared to the values of primary layers 6 and 7 at the first surplus level used. The shares from zero to 100% were evaluated. These results are displayed in Figure 2. Layer 7, having more risk than layer 6, has a best share in the interior, while layer 6 has a best share the boundary of 100%. The decision making implications are evident.

Figure 2

The Best Share to Retain



In order to evaluate the retention of two non-adjacent layers of one risk, it is necessary to evaluate the Risk Adjusted Value of the probability distribution found in Table 10. This illustrates that the probability distribution for any possible pair is easy to write. This simply sets to zero the frequency in the layers not covered and the spreadsheet formulas are unchanged.

TABLE 10

NON-ADJACENT LAYER EVALUATION ILLUSTRATION

LAYER	(1) LOSS LAYER DEFINITIONS: FROM LOW END	(2) TO HIGH END	(3) FREQUENCY IN LAYER	(4) EXPECTED LOSS IN LAYER
1	$0	$500	0.000	$0
2	$500	$1,000	10.703	$8,027
3	$1,000	$5,000	0.000	$0
4	$5,000	$10,000	4.301	$32,258
5	$10,000	$50,000	0.000	$0
		Totals	15.004	$40,285

CONCLUSIONS

The application of utility theory to insurance is incomplete without including the structure of risk sharing; to limit the use of utility to accept or reject decisions misses its power to explain the real world of excess insurance and reinsurance. The fine subdivision of risk which is routinely achieved by insurance institutions can be explained on this basis. This paper shows practical methods of evaluating and pricing excess layers. We find that there are many cases where a share between zero and 100% of an excess layer is preferred. In other situations, 100% is the best share. The size of the best share is a function of the insurer's risk aversion level and of the risk characteristics of the layer. We find, as predicted by Samuelson in 1957, that there exists a positive best share if the expected profit from underwriting a layer is positive. Our model appears to be in broad agreement with the behavior of experienced underwriters, and thus can form the decision basis for a reinsurer's assumption of risk.

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