AN ALTERNATIVE TO THE PARALLELOGRAM METHOD

Alfred D. Commodore

Abstract

The so-called "parallelogram" method is standard in actuarial practice for illustrating loss and exposure statistics as a conceptual and calculational device. Ratemaking is a prime example. In this article we propose a similar device based on three variable calculus.

Introduction

This article is a result of conclusions drawn from the following observation. Under the parallelogram method we plot a 7/1/92 accident on a 1/1/92 policy at the point, (7/1/92, 6 mos.), in the xy-plane. Thus, for example, the statement: " As of 12/31/94, Accident Year 1991 paid dollars totaled $100 million. All policies annual. " ; is represented by the following illustration.

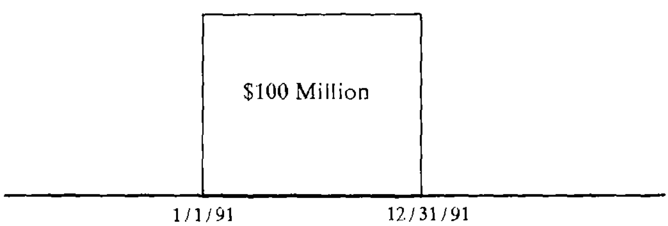


Fig. 0.1: $Paid

It is common practice to use the same picture to illustrate the statement: " Calendar Year 1991 paid dollars totaled $100 million. All policies annual. " From the plotting rule given above, however, use of Figure 0.1 would suggest that all of the calendar year payments were made on 1991 accidents. Figure 0.1 does not "fit" the statement. We propose that the rule lead, instead, to an illustration such as the following.

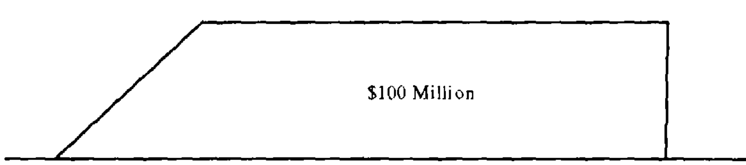


Fig. 0.2: $Paid

In Figure 0.2, $100 million is paid on accidents beginning with those on an unspecified earliest policy (i.e., the left edge of the figure) through accidents on 12/31/91. We also suggest the use of a rectangle that starts with an unspecified earliest accident date and ends with accident date 12/31/91. The idea, in general, is that the diagram should allow for allocation of the $100 million by accident year.

It is common practice to estimate the average accident date for a calendar year at the middle of the year. Figure 0.1 appears to support this conclusion. This is not the case using Figure 0.2. According to that figure, some other point appears more likely.

The basis of our conclusions (and solution) was our interpretation of the plotting scheme underlying the parallelogram method. We believe our interpretation is consistent and that the traditional illustration of calendar period can be misleading.

In this article we present an alternative to the parallelogram method. The method is probably best described as a "rectangle" method. It is simply the result of plotting a 7/1/92 accident on a 1/1/92 policy as point, (1/1/92, 6 mos.), in the xy–plane, for example. Thus, a policy year under our method is a rectangle, whereas it is drawn as a parallelogram under the parallelogram method. So what is the point of developing a new method? Our reason is that the rectangle method is simpler to work with for pictures and mathematical applications using a rectangular coordinate system.

We begin with the basic plotting scheme. We apply the method for the same purposes as the parallelogram method. We then move to three dimensions where we graphically make the distinction between accident, policy and calendar periods. With the use of 3–D we suggest that for a calendar year the average valuation date, and not the average accident date, be estimated at the middle of the year. We review the so-called "overlap" fallacy using 3–D pictures.

Mathematical applications are reserved for the appendix. The plotting method allows for ready application of basic calculus. We provide mathematical interpretations of the notions of development, trend, average date and on level factor. We model and test basic reserving methods.

Plotting Basics

To each accident on an occurrence policy we assign point (x,y) in the xy–plane, where x equals the effective date of the policy triggered and y is accident lag. Both axes are scaled using the same time unit, so that x+y equals accident date. For example, (1/1/92, 6) is the assignment of a 7/1/92 accident on a 1/1/92 policy. This is a fundamental difference from the traditional parallelogram method which places the same accident at point (7/1/92, 6). The basic definitions are as follows:

A loss is a point (x,y) representing all x+y accidents on all policies with effective date x. (1.0.a)

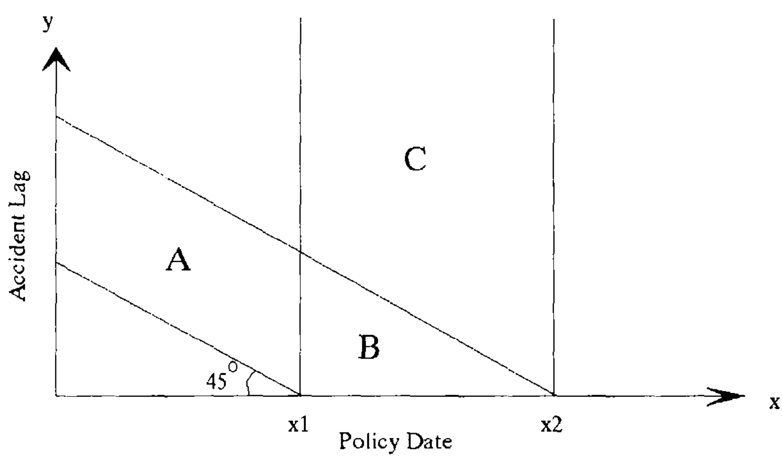
A loss region, R, is a collection of points in the plane. (1.0.b)

For example, if our data base shows 100 accidents on 7/1/92, of which 20 are on policies issued on 1/1/92, then those 20 accidents would be assigned to loss (1/1/92, 6).

Accidents on a policy with effective date x0 are assigned to the vertical line through x0. As we are working with occurrence policies, if term is at most 12 months, the line through x0 cuts off at lag 12 months.

The primary loss regions are those representing accident and policy period shown in Figure 1.0, below. For time period x1 to x2, the accident period (A+B) is the diagonal band over the interval and the policy period (B+C) is the vertical band over the interval.

In applications R is often bounded. R may not extend upward indefinitely as policy term may be at most 6 months, for example. Also, R may not extend back in time indefinitely as there is some date at which a company began writing policies. In later illustrations we may not always label the axes or the origin, (0,0). Such labels should be clear from the context.



A+B = Accident Period x1 to x2

B+C = Policy Period x1 to x2

B = Policy / Accident Period x1 to x2

Fig. 1.0 : Basic Regions[1]

[1] Accident period x1 to x2 is all (x,y) such that x+y falls between x1 and x2. Policy period x1 to x2 is all (x,y) for which x falls between x1 and x2. Finally, Policy / Accident period x1 to x2 is all (x,y) for which both x+y and x fall between x1 and x2.

Illustrations in 2-D

We identified losses with points in the plane. In this section we annotate plots of regions to represent levels of statistics associated with losses. Examples of loss statistics are dollars paid, number of claims, estimated ultimate losses, loss reserves and number salvage or subrogation recoveries.

Plotting Valuations in the xy-plane

For a single loss, a level (of a statistic) is defined as a net change in statistic over some time period. Level for a region is the sum over levels for its points. We provide instructions for illustrating valuation (i.e., determination of some statistic level) of R for four particular types of valuations for its points. The valuations are:

1) all points valued from accident to common date t ;
2) all points valued over calendar period t1 to t2 ;
3) all points valued from accident to valuation lag c ; and
4) all points valued from accident to ultimate . (2.0)

The instruction for plotting is straight forward as follows.

To illustrate level for R for the four (2.0) valuations, draw R, assign level and indicate the type of valuation.

Optional: Represent valuation of R over calendar period t1 to t2 as two diagrams, one each for valuations as of dates t1 and t2. (2.1.a)

Exhibit 2.1A contains sample plots of the four valuations. Note that we represent valuation of Accident Year 1991 over Calendar Year 1992 in two ways in the figure. The top two figures apply the two-diagram option described in (2.1.a), whereas a single diagram is used in the third figure. Note that for valuation as of date t, we include in our diagrams the diagonal line through t on the x-axis: $t = x + y$. Such lines are convenient for reference.

Policy, accident and calendar period are terms commonly used in practice. Under our plotting scheme, policy and accident period are two types of loss regions. Calendar period in this instance, however, is the second type of (2.0) valuation where R equals the entire plane. Note how we represent the calendar period information in Figure 2.1B.

All losses through 12/31/91 are represented. Given the information, we could illustrate more detailed distribution of the $35 million. For example, it may be the case that only accidents after 12/31/81 contribute to level. Figure 2.1B is our rectangle method version of Figure 0.2 from the introduction.

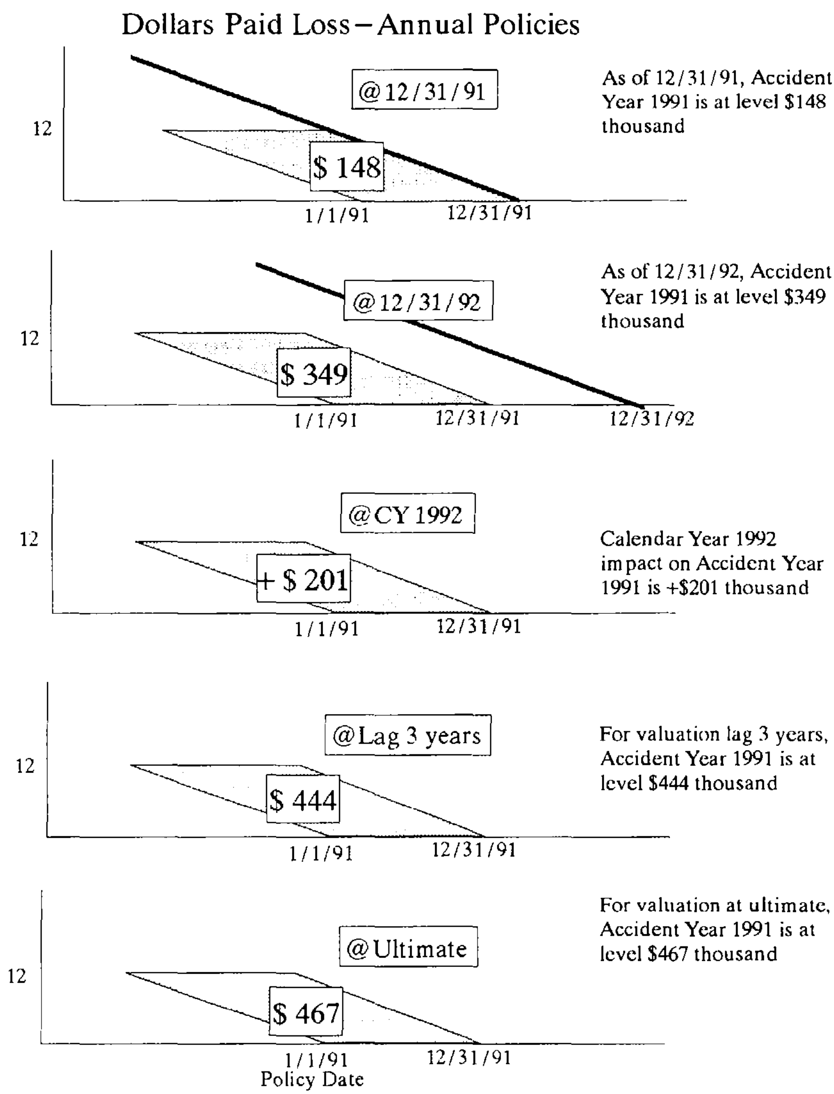


Fig. 2.1A: Sample 2-D Valuations

"Calendar Year 1991 paid dollars totaled $100 million of which $65 million was paid on 1991 accidents. All policies annual."

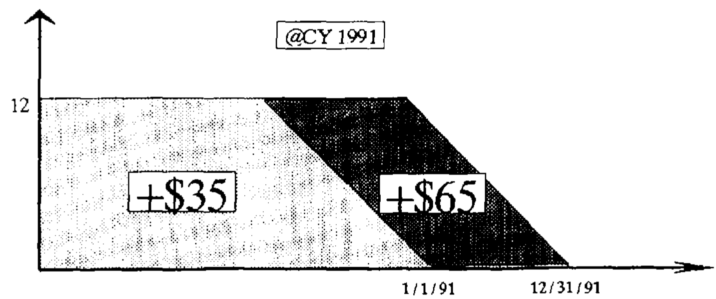


Fig. 2.1B : $Paid Losses

Average Value and Average Point of R

In addition to assigning level in a diagram, we can also calculate the average value and plot the average point. The average value is simply level divided by the area of R. The average point is intuitively where statistic level is balanced or the "center of mass " to borrow from another discipline. We assume such points, (x^*,y^*), exist and are unique. We revisit these concepts in the appendix. We add to instruction (2.1.a) as follows:

Optional: Plot average point, (x^*,y^*), and determine average value as level divided by the area of R. (2.1.b)

In Figure 2.1B, let 144 be the area (in square months) for Accident Year 1991 to which $65 million was assigned. The average value is then 0.451. An estimate of some region on the level for the accident year is the product of 0.451 and the area of the region. This is, of course, a very rough estimate.

In ratemaking, one often trends between the average accident date (x^*+y^*) or policy date (x^*) for two valuations. Points are often set using the uniformity assumption, which places the average coordinates x^* and y^* for typical accident and policy periods at the midpoints of the respective coordinate ranges. We derive this result in Example F of the appendix. It is common practice to place the average accident date for a calendar year at the middle of the year. By our scheme, however, some other point may actually be more appropriate.

Consider Figure 2.1B above. Suppose the $65 million is uniformly spread over Accident Year 1991. Thus, its average point is (1/1/91, 6). Also, let (3/1/90, 9) be the average point (possibly set through judgement) for the region of all accidents prior to 1991. The average point for R is the "weighted average" of average points for its parts. The weighting is actually done per x and y coordinate using level of statistic per part. The average point for R equal to the entire plane is therefore, (9/15/90, 7) (e.g., lag 7 = (65•6+35•9)/100), so that 4/15/91 is the average accident date.

The average point may vary with the statistic. For example, paid dollars may yield a different result from number of paid claims. Which point is more appropriate depends on how the point is to be used for some analysis or review.

Illustrations in 3-D

So far, valuations have been illustrated by assigning level to R in the xy-plane. Given R, all of our illustrations in 2-D were drawn the same. We made clear the type of (2.0) valuation drawn by adding a heading. Headings indicated the time period over which each point of a region was valued. Define z (in the same time unit as x and y) as valuation lag, so that x+y+z is a valuation date. By adding a dimension, we improve on drawings by graphically representing all variables x, y and z.

Plotting Valuations in xyz-space

In xyz-space, let (x,y,z) represent valuation of loss (x,y) at time x+y+z. We represent R in a natural way as a collection of points, (x,y,0), in the xy-plane. In illustrating valuation we plot R, however assign level to particular sets V positioned above [1] R in xyz-space. Thus: 1) the distinction between a loss region and a valuation reduces to the difference between R and V; and 2) an illustration in 2-D is the result of collapsing V onto R. Sets V are determined by (2.0) valuations as follows:

> To illustrate valuation for R as of date t, valuation lag c or at ultimate, assign level to the set V above R :
>
> i) between R and plane z=t-x-y;
> ii) between R and plane z=c; and
> iii) between R and plane z=+infinity ,
>
> respectively.
>
> To illustrate valuation for R over calendar period t1 to t2, assign level to the set V above R between planes z=t1-x-y and z=t2-x-y. (3.1.a)

[1] Sets projecting onto R in the xy-plane. These are sets of points (x,y,z) such that (x,y,0) is in R.

This instruction is analogous to (2.1.a). Plane $z = t-x-y$, for fixed t, is that through points (t,0,0), (0,t,0) and (0,0,t). Plane z = c sits above and parallel to the xy–plane. We draw the z = +infinity plane sitting above and parallel to the xy–plane by convention, consistent with the notion that valuations as of date t and lag c converge at ultimate.

Figure 3.1A is the extended version of Figure 2.1A. Region R (i.e., Accident Year 1991 with 0 and 1 corresponding to dates 1/1/90 and 1/1/91, respectively) sits at the base of the stack of valuations.

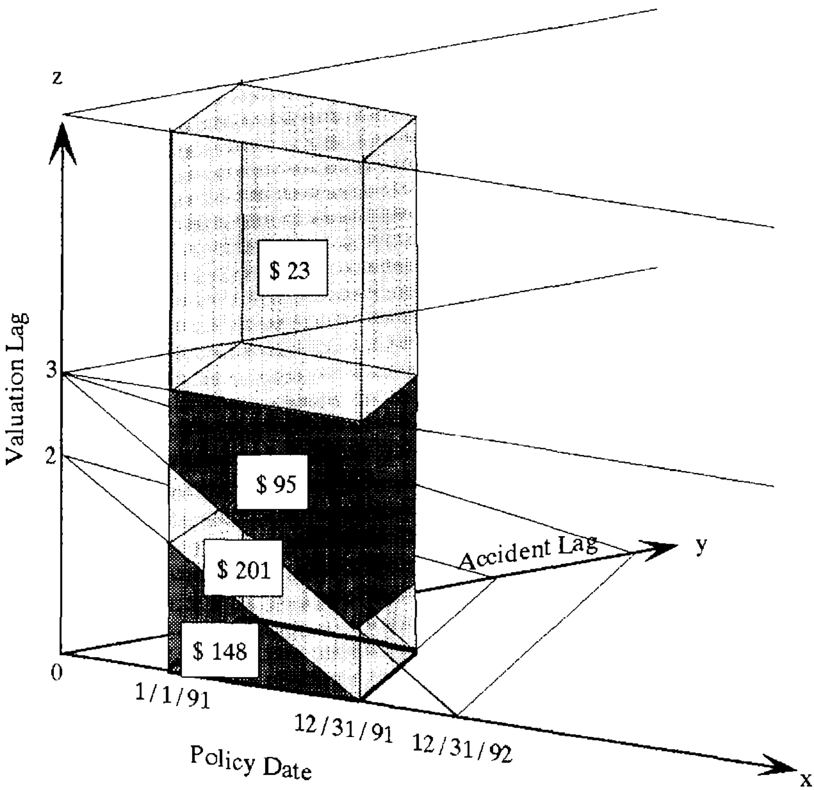


Fig. 3.1A : 3–D Version of Figure 2.1A [1]

[1] dollars in thousands

We have identified additional valuations with Figure 3.1A as follows:

- $148 thousand valuation as of 12/31/91
- $201 thousand calendar year 1992 development
- $95 thousand development from date 12/31/92 to lag 3 years
- $23 thousand development from lag 3 years to ultimate

Of course, we could have inferred these additional valuations from Figure 2.1A. The advantage here, is that each level is directly associated with a picture.

In 2-D, Figure 2.1B was the standard way we represented calendar period valuation. The standard illustration in 3-D is provided with Figure 3.1B.

"Calendar Year 1991 paid dollars totaled $100 million of which $65 million was paid on 1991 accidents. All policies annual."

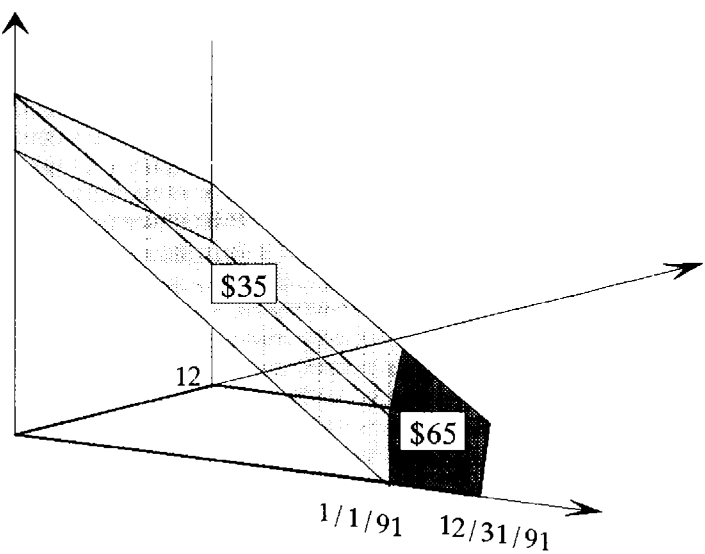


Fig. 3.1B: 3-D Version of Figure 2.1B.

As a conceptual device, we can use 3-D diagrams to show the difference between development and trending concepts applied in ratemaking. Define development as the ratio of levels for two valuations of a region R. Define trend as the ratio of levels for valuations of two distinct regions R1 and R2.

Consider Figure 3.1C below. A is level of incurred loss for experience period R1 valuated as of 12/31/92. Additional development to ultimate is B. The ultimate level for the proposed period, R2, for which new rates will be in effect is given by C. Consider the following equation.

$$C = A \cdot (A+B)/A \cdot C/(A+B).$$

The equation is certainly valid. Development is given by (A+B)/A. Trend is given by ratio C/(A+B). We can therefore arrive at C by applying two factors to A that do not "overlap". We have illustrated that there is no overlap or redundancy, but equally important, no deficiency in development and trend factors in the equation. It is conceivable, however, that a method of estimating development and trend factors may yield or allow for redundant or deficient forecasts.

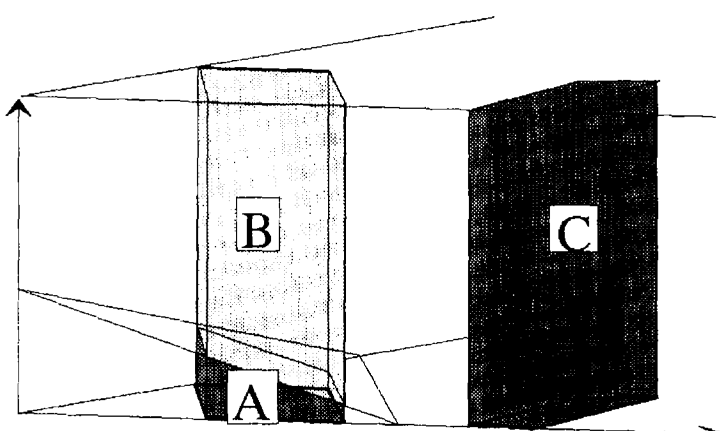


Fig. 3.1C: Statistic Development and Trend

There is another use of 3-D illustrations. In practice one adds lines to diagrams to indicate incidence of rate, statutory or other changes affecting level in a fundamental way. We can show these highlights or phases of the valuation process itself, more effectively in 3-D. Let s(x,y,z) be the "rule" for determining level for sets V. Suppose s(x,y,z) is in three distinct phases as follows.

$$s(x,y,z) = \begin{cases} s1(x,y,z) & \text{for } x+y+z <= 1/1/92 \\ s2(x,y,z) & \text{for } 1/1/92 < x+y+z \text{ and } x <= 12/31/92 \\ s3(x,y,z) & \text{for } x > 12/31/92 \end{cases}$$

It may be the case that s2 has to do with new management coming on board with a change in reserving philosophy, for example. Phase s3 covers the impact of a rate change and a law change that the company has no inkling of at the present time. It is too far off into the future. Figure 3.1D is an illustration showing how these phases of s partition xyz-space. Note that it would not be as easy to show s1 or s2 in 2-dimensions.

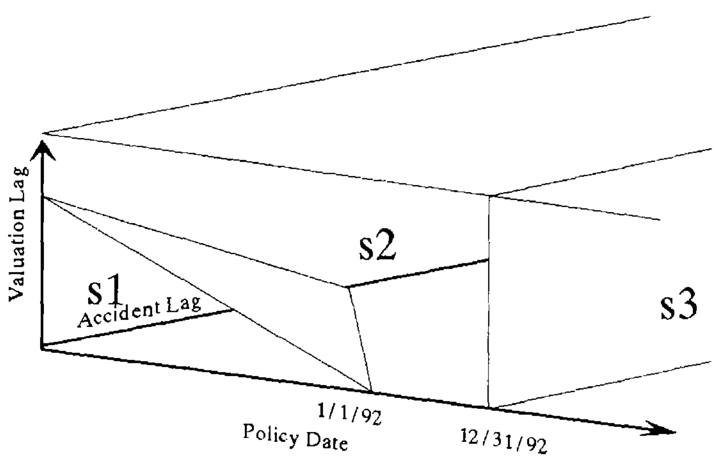


Fig. 3.1D: An s-Partition of xyz-space.

Each set V falls in one or more partitions or domains of phases of s. Denote by, V|s, the level of V given s. If V falls in s1's domain the "true" level is V|s1. Consequently, we interpret V|s2 and V|s3 as V on s2 and s3 level, respectively, provided application of s2 and s3 make sense for V. At the risk of waxing philosophical, we can only make estimates of s at any point in time. Moreover, it may be difficult to say whether those approximations do not in fact mix phases s1, s2 and s3 of s.

Illustrations in 2-D suffice if the extra detail of 3-D is not required. We derive 2-D pictures from those in 3-D by projecting or collapsing the latter onto the xy-plane. We thus lose the perspective of the z-dimension. Another useful projection is onto the xz-plane. The disadvantage in this case is that all of R and V is projected onto the x-axis. Any detail for R and V is lost.

Average Value and Average Point of V

Similar to the 2-D case, we can determine the average value and point of V. The average value is level divided by the volume of V. The average point, (x^*,y^*,z^*), is interpreted as a balance point as before. We add to instruction (3.1.a) as follows.

Optional: Plot average point (x^*,y^*,z^*) and determine average value as level divided by the volume of V. (3.1.b)

In Figure 3.1B, let 864 be the volume (in square months) for Accident Year 1991 to which $65 million is assigned. The average value is 0.075. An estimate of some set V on the level for the accident year is the product of 0.075 and the volume of V.

A 2-D diagram is the result of projecting a 3-D diagram onto R. Consequently the average point, (x^*,y^*) of (2.1.b) is the projection of (x^*,y^*,z^*) in (3.1.b), and coordinates match. As was the case for average accident and policy dates, we might use the average valuation date, $x^*+y^*+z^*$, for trending purposes. In fact, we suggest that under uniformity the average valuation date, and not the average accident date, be at the middle of the year for a calendar year valuation (see Appendix, Example F).

Consider Figure 3.1B above. Suppose the $65 million is uniformly spread over the set V for Accident Year 1991 valuated as of 12/31/91. Using methods of the appendix, we can show that its average point is (11/1/90, 6, 4). This point does not project onto the point (1/1/91, 6) estimated earlier under uniformity of level over R. Consequently, uniformity over V and R may lead to different conclusions when level is not in fact uniform over both V and R.

As in the 2-D case, we can select an average point for the set to which the $35 million is assigned, and estimate (x^*,y^*,z^*) for the entire calendar period. We take averages by coordinates as before, using levels $35 million and $65 million as weights.

Other Applications

Our plotting method assigns items or "subjects" to points in the plane by letting x equal the date of Event 1, y equal the lag in Event 2 and z equal valuation lag. This theme can be applied where the subject is losses on claims made policies, losses by accident and report date on policies, or losses on a single occurrence or claims–made policy. To this point, our subject has been losses on a book of occurrence policies.

Losses on a Book of Claims–Made Policies

Coordinates x and y represent policy date and report lag, respectively. Point (x,y) is all losses on x–dated policies reported at time x+y. In Figure 1.0 we have the interpretations (along with new labels for the axes):

A+B	Report period x1 to x2
B+C	Policy period x1 to x2
B	Policy / Report period x1 to x2

Losses on a Book of Claims by Accident and Report Date

Coordinates x and y represent accident date and report lag, respectively. Point (x,y) is all accidents on date x reported at time x+y, for all policies. In Figure 1.0 we have the interpretations (along with new labels for the axes):

A+B	Report period x1 to x2
B+C	Accident period x1 to x2
B	Accident / Report period x1 to x2
C	IBNR losses for accident period x1 to x2 as of date x2

Losses on a Single Occurrence or Claims–Made Policy

Coordinates x and y represent accident date and report lag, respectively. Point (x,y) is all losses on x–dated accidents reported at time x+y. In Figure 1.0 we have the interpretations (along with new labels for the axes):

A+B	Claims–made coverage over period x1 to x2 with retroactive active date at the origin
B+C	Occurrence policy coverage over the period x1 to x2
B	Accident / Report period coverage x1 to x2
C	Unlimited tail coverage for claims–made coverage during x1 to x2. Limit tail to d time units after period with line x+y = x2+d.

The treatment is virtually identical to that provided losses on a book of occurrence policies. We associate statistics with regions in the plane and illustrate statistic levels in 2-D and 3-D.

Conclusions

The goal of our exercise was formal construction and application of a basic plotting scheme. A similar approach can be used for the traditional parallelogram method. In fact, we made a "parallel" construction. We still did not arrive at the traditional rectangle for illustrating calendar period valuation. This was a consistency issue (at least for this author) that led to the article itself.

We feel the 2-D and 3-D illustrations are effective and consistent ways of picturing valuations. In addition to pictures, we interpreted standard concepts in the context of our plotting scheme. We interpreted the notions of average date, trend, development and on level factor.

We applied the method for the same purposes as the parallelogram method. We feel it is no more difficult to use for drawing 2-D diagrams than the traditional method. In particular, we suggest that it may be easier to use for drawing policy periods with arbitrary policy terms. The basic plotting scheme can be used in other situations. It can be used to treat exposure statistics as well.

The rectangular coordinate system is ready-made for calculus applications. We found it convenient to use densities (discussed in the appendix) g(x,y,z) and s(x,y,z) for testing and building basic reserve models, for example. More advanced mathematics regarding g and s is one area that warrents further investigation. Vector analysis in 3-D is one topic. Adding record lag, for example, we can model valuation using four variables.

References

Berquist, J. R., and Sherman, R. E. 1977. Loss reserve adequacy testing: a comprehensive, systematic approach. PCAS/64: 123-185

Cook, C. F. 1970. Trend and loss development factors. PCAS 57:1

Miller, D.L., and Davis, G.E. 1976. A refined model for premium adjustment. PCAS 55:61

McClenahan, C. L. 1975. A mathematical model for reserve analysis. PCAS/ 26 : 134-153.

Appendix

To accompany illustrations in 2-D and 3-D, we briefly introduce functions g(x,y,z) and s(x,y,z). We assume the reader familiar with basic calculus methods. Some familiarity with double and triple integrals is required.

Valuation of R

Above, we dealt with the illustration of statistic valuations. In this section we provide an overview of calculus methods used for calculating valuations. A key feature of our rectangular plotting scheme is that it is ready – made for calculus applications.

Let g(x,y,z) be density of statistic level at point (x,y) at time x+y+z. We may think of g(x,y,z) as "infinitesimal" level per area near the loss at the time of valuation.

Level for a region is the sum over levels for its points. As the reader may have guessed, we determine level for R using integration. Valuations (2.0) are determined as follows:

> Level for R valuated as of date t, over calendar period t1 to t2 or at valuation lag c, is the double integral over R of the function:
>
> i) $g(x,y,t-x-y)$;
> ii) $g(x,y,t2-x-y)-g(x,y,t1-x-y)$; and
> iii) $g(x,y,c)$,
>
> respectively.
>
> Level for R valuated at ultimate is the limit as t (resp., c) approaches infinity in i) (resp., iii)). (4.1.a)

We require g(x,y,z) equal 0 for $z<0$, with 0 and finite limit as y and z increase, respectively. These properties of g(x,y,z) reflect the fact that: 1) losses that haven't occurred have level zero; 2) after high enough accident lag level becomes insignificant; and 3) after high enough valuation lag change in level becomes insignificant.

These are preferred and not exhaustive mathematical properties for g. We assume g and R sufficiently defined so that the integrations of (4.1) are defined and finite. A simple way to satisfy 2) and 3) is to set g=0 if y is larger than policy term 6 months and set g=0 if z is larger than 10 years, for example, respectively. As an example, set

$$g(x,y,z) \;=\; 100 \bullet e^{x+y} \;-\; 100 \bullet e^{x+y-z} \quad , \qquad \text{(Eq.4.1)}$$

where x is the number of years since 1 / 1 /90 and all policies are annual term. The reader should verify that this function behaves as required (i.e., as y and z increase) and plot g(x,y,z) after fixing two of its variables.

One must be careful to check to see where in R a density in (4.1) is zero because the z-coordinate is negative. R should be partitioned before integrating, accordingly. When permitted, we derive ultimate level by letting z approach infinity in the formula for g(x,y,z), then integrate the resulting function, $f(x,y)$, over R.

The instructions for calculating average values and points are provided next, along with a formal interpretation of uniformity.

> The average value of g over R is level divided by the (geometric) area of R (i.e., the integral of 1 over R). (4.1.b)
>
> Let h(x,y) be a function. The average value of h, with respect to the valuation, is the integral of product h•g divided by the integral of g over R. (4.1.c)
>
> Let x* and y* be the average value of h=x and h=y, respectively, in (4.1.c). Then (x*,y*) is the average point of R for the valuation. (4.1.d)
>
> The uniformity assumption is the case the integrand in (4.1.a) is identically constant, ß, over R. Level equals ß times the area of R. (4.1.e)

Valuation of V

As with the 2-dimensional case, we outline how calculus methods may be used to model valuation. In this instance, however, we integrate a function describing level over set V instead of R.

Let s(x,y,z) be density of statistic level at point (x,y,z). We may think of s(x,y,z) as "infinitesimal" level per volume near the loss at the time of valuation.

Level for a region is the sum over levels for its points. We determine level for R using integration. Valuations (2.0) are determined as follows.

> Level for R valuated as of date t, over calendar period t1 to t2, at valuation lag c or at ultimate, is the triple integral of s(x,y,z) over the set V determined by the valuation in (3.1.a). (4.2.a)

This rule for calculating level for V is more straight forward than rule (4.1.a). We do not need to specify the behavior of the z variable. All such information is contained in V.

We require s(x,y,z) equal zero for z<0 and have limit 0 as both y and z increase. Losses yet to occur have level zero. Moreover, for sufficiently high accident or valuation lag, contribution to level becomes insignificant. We assume s and V sufficiently defined so that integrations in (4.2) are defined and finite.

We usually integrate s(x,y,z) over the the range for z and then over R [1]. Note that we can integrate s over paths and surfaces in 3-space in addition to sets V associated with (2.0) valuations that have been the focus of the article. For example, the rate of statistic at time t applies the integral of s(x,y,z) over the intersection of set V and the plane $z = t-x-y$. An example of s(x,y,z) is the following.

$$s(x,y,z) \;=\; 100 \cdot e^{x+y-z} \quad , \qquad \text{(Eq.4.2)}$$

where x is the number of years since 1/1/90 and all policies are annual term. The instructions for calculating average values and points are provided next, along with a formal interpretation of uniformity.

The average value of s over V is the level divided by the (geometric) volume of V (i.e., the integral of 1 over V). (4.2.b)

Let w(x,y,z) be a function. The average value of w, with respect to the valuation, is the integral of product $w \cdot s$ divided by the integral of s over V. (4.2.c)

Let x*, y* and z* be the average value of w=x, w=y and w=z, respectively, in (4.2.c). Then (x*,y*,z*) is the average point of V for the valuation. (4.2.d)

The uniformity assumption is the case s(x,y,z) is constant, ß, over V. Level equals ß times the volume of V. (4.2.e)

Applications

If we integrate $s(x,y,\mu)$ over range $0 < \mu < z$, with respect to μ, the result is a density g(x,y,z) of the type described earlier. In that case, we can use either g(x,y,z) or s(x,y,z) to determine (2.0) valuations. The results of the calculations will be the same. In particular, we would derive average points using (4.1.d) and (4.2.d) so that the average point for V would indeed project onto the average point derived for R. We do not require, however, that g be the anti-derative of some s or that s be the derivative of some g, with respect to the z-variable.

Examples follow. As our concern is mainly with (2.0) valuations, the examples apply g(x,y,z) using (4.1). In most cases an example can be restated as an application of s(x,y,z). We use the notation R | g (resp., V | s) to denote valuation of R (resp., V) using integration in (4.1) (resp., (4.2)).

[1] For valuation as of date t, over calendar period t1 to t2, at valuation lag c and at ultimate, the ranges are:
$0 < z < t-x-y$; $t1-x-y < z < t2-x-y$; $0 < z < c$; and $0 < z < +\text{infinity}$, respectively.

EXAMPLE A

We are given dollars paid loss density of (Eq.4.1)

$$g(x,y,z) \;=\; 100 \bullet e^{x+y} \;-\; 100 \bullet e^{x+y-z} \quad ,$$

where x is the number of years since 1/1/90 and all policies are annual term. We wish to determine paid losses for Accident Year 1991:

1) as of 12/31/91;
2) as of 12/31/92;
3) at common lag 3 years; and
4) at ultimate ,

as well as illustrate the four valuations in 2-D and 3-D.

We set up our solution by noting that Accident Year 1991 is the region:

$$0 < y < 1\,,\quad 1-y < x < 2-y \quad ;$$

which is also the order and limits of integration.

1) Date 12/31/91 corresponds to t=2. Using instruction (4.1.a)(i) we determine $R\,|\,g(x,y,2-x-y)$. The result is \$148 thousand[1] as follows.

$$\int_0^1 \int_{1-y}^{2-y} 100 \bullet e^{x+y} - 100 \bullet e^{2x+2y-2} \; dxdy$$

$$= \int_0^1 100 \bullet (e^{2} - e^{1}) \; dy$$

$$- \int_0^1 100 \bullet ½ \bullet e^{-2} \bullet (e^{4} - e^{2}) \; dy$$

$$= \quad 100 \bullet 4.67 \; - \; 2{,}360 \bullet e^{-2} = \; 148$$

[1] For accident period R given by $0<y<1$, $a-y<x<b-y$:

$$R\,|\,g(x,y,t-x-y) = 100 \bullet (e^{b} - e^{a}) - ½ \bullet 100 \bullet e^{-t} \bullet (e^{2b} - e^{2a})$$

$$R\,|\,g(x,y,c) = 100 \bullet (e^{b} - e^{a}) - 100 \bullet (1 - e^{-c})$$

2) Date 12/31/92 corresponds to t=3. Using (4.1.a)(i), we determine $R \mid g(x,y,3-x-y)$. The result is \$349 thousand (i.e., $467-2360 \cdot \exp(-3)$).

3) Using (4.1.a)(iii), we determine $R \mid g(x,y,3)$. The result is \$444 thousand (i.e., $467 \cdot (1-\exp(-3))$).

4) Finally, letting z approach infinity in the formula for g results in the function

$$f(x,y) = 100 \cdot e^{x+y}$$

Integration over R yields ultimate \$467 thousand.

As for illustration, we have already provided a 2-D version with Figure 2.1A and a 3-D version with Figure 3.1A. Of course, the same problem using s(x,y,z) in (Eq.4.2) yields the same results.

We use this particular density to generate "actual" data for later examples. The following results are applied.

		Accident Period a to b			
Val.	D-	1991	1992	1993	1994
Date t	value	1 2	2 3	3 4	4 5
2	1.00	148			
3	1.00	350	401		
4	1.00	424	950	1,091	
5	.800	436	1,082	2,314	2,499
5	1.00	451	1,152	2,583	2,965
5	1.20	459	1,196	2,787	3,383
6	1.00	461	1,226	3,132	7,021
At Ultimate		467	1,270	3,451	9,382

Table A : \$Paid Losses for g(x,y,z) and Regions R [1]

[1] Given: $g(x,y,z) = 100 \cdot e^{x+y} - 100 \cdot e^{x+y-Dz}$ where D>0,

and R given by: $0<y<1$, $a-y<x<b-y$; valuation as of date t is given by:

$$R \mid g(x,y,t-x-y) = 100 \cdot (e^{b} - e^{a}) - 100/(1+D) \cdot e^{-tD} \cdot (e^{2b} - e^{2a})$$

First term of right side of equation is ultimate level.

EXAMPLE B

A model helps to demonstrate ideas in an effective way. We look at how a change in payment pattern, with no change to ultimate losses, affects the outcome of a purely mechanical (i.e., "selections" are calculations) reserving model. Let g(x,y,z) be the density determining dollars paid loss as follows:

$$g(x,y,z) = \begin{cases} 100 \cdot e^{x+y} - 100 \cdot e^{x+y-z} & \text{for } x+y+z<5 \\ 100 \cdot e^{x+y} - 100 \cdot e^{x+y-Dz} & \text{otherwise,} \end{cases}$$

where x is the number of years since 1/1/90. Valuation date t=5 corresponds to date 1/1/95. All policies are annual. The ultimate density does not change with D(>0) and so D only determines the rate at which ultimate levels are reached.

Using Table A of Example A, we have generated triangles using g(x,y,z) for three choices of D in constructing Table B below. Our reserving model uses average link ratios as selected factors and repeats the last ratio to estimate the tail. Only the latest diagonal changes.

A generalization of this example is to let cumulative incurred losses be given by density i(x,y,z) where,

$$i(x,y,z) = p(x,y,z) + r(x,y,z) \quad ,$$

and p and r represent densities for cumulative payments and reserves, respectively. We can then review the effect on a mechanical reserving model when at some time, t*, there is a switch to density

$$i^*(x,y,z) = p^*(x,y,z) + r^*(x,y,z) \quad ,$$

where both p and p* converge[1] to the same (or different) ultimate function, and both r and r* converge to the zero function. For example, we might review the effect of reserve adequacy on an incurred loss model by letting p=p*. Here, we interpret a change in adequacy as a shift from r to r*. Letting r*(x,y,z) be greater than r(x,y,z) for x+y+z > t*, provides for the impact of reserve strengthening on the model.

[1] Let z approach infinity.

	Method may be deficient for slow-down in payment pattern. Case D =			*0.80*	*Method may be sufficient for no change in payment pattern. Case D =*			1.00	*Method may be redundant for speed-up in payment pattern. Case D =*			1.20
AY	12	24	36	48	12	24	36	48	12	24	36	48
1991	148	350	424	436	148	350	424	451	148	350	424	459
1992	401	950	1,082		401	950	1,152		401	950	1,196	
1993	1,091	2,314			1,091	2,583			1,091	2,787		
1994	2,499				2,965				3,383			
	2.37	1.21	1.03		2.37	1.21	1.06		2.37	1.21	1.08	
	2.37	1.14			2.37	1.21			2.37	1.26		
	2.12				2.37				2.55			
				Tail				Tail				Tail
Average	2.29	1.18	1.03	1.03	2.37	1.21	1.06	1.06	2.43	1.24	1.08	1.08
Cumu.	2.84	1.24	1.06	1.03	3.25	1.37	1.13	1.06	3.52	1.45	1.17	1.08
Est. Ultimate	7,105	2,878	1,144	448	9,646	3,549	1,305	480	11,909	4,037	1,402	497
Act. Ult.	9,382	3,451	1,270	467	9,382	3,451	1,270	467	9,382	3,451	1,270	467
Est.–Act.	($2,277)	($574)	($125)	($19)	$265	$97	$36	$13	$2,527	$585	$132	$30
Percent Diff.	–24.3%	–16.6%	–9.9%	–4.0%	2.8%	2.8%	2.8%	2.8%	26.9%	17.0%	10.4%	6.4%

Table B: $Paid Loss Projection [1]

[1] See Table A of Example A.

EXAMPLE C

In principle, g(x,y,z) can be used to determine all (2.0) valuations given our basic regions R. With Figure 3.1C in mind, we wish to estimate ultimate level for a policy period given estimated ultimate levels from past experience. Recall that for our purposes, we defined "trend" in the discussion of Figure 3.1C as the ratio of levels for two distinct loss regions.

Assume annual policy term and that we have estimated ultimates losses for Accident Years 1991–1994 (using, for example, development triangles). We wish to determine ultimate level for Policy Year 1995. Recall that $f(x,y)$ was the result of letting z in g(x,y,z) approach infinity. Thus $R \mid f$ yields ultimate level. Assume f takes the form:

$$f(x,y) \;=\; e^{A(x+y)+B} \quad .$$

For Accident Year: $0 < y < 1$, $a-y < x < b-y$; we make the approximation for $R \mid f$:

$$\begin{aligned} R \mid f &= f((a+b-1)/2, -1/2) \bullet (b-a) \\ &= \exp[\,A(a+b)\,/\,2 + B\,] \bullet (b-a) \;; \end{aligned}$$

which is the value of f at the average point of R under uniformity times the area of R. This approximation is the basis for our linear regression model shown in Table C, below.

Note that in Table C, fitted ultimate levels appear in column (4) and levels as integrations appear in column (5). We estimate Policy Year 1995 ultimate as follows:

$$\begin{aligned} \text{PY 1995 Ult.} &= (\text{AY 1994 Ult.}) \bullet \frac{(\text{PY 1995} \mid f)}{(\text{AY 1994} \mid f)} \\ &= 1{,}802 \quad . \end{aligned}$$

We could have made a similar calculation using column (4). Our preference is to use level based on integration when columns (4) and (5) are close. Note that fitting level using the model assigns the same result to two different regions that have the same average accident date and area. In particular, note the entries in columns (4) and (5) for Accident Year Ending 7/1/96 and Policy Year 1995.

$$Model^{1} : (R|f) / (b-a) = A \bullet (avg.\ acc.\ date\ of\ R) + B$$

	(1)	(2)	(3)	(3)	(4)	(5)
Accident Period	Area of Region R	Average Acc. Date	Ultimate Level(R | f)	ln [Level / (b−a)]	R | f [2] Fitted	R | f [3] Integral
AY 1991	1	1.5	1,080	6.98	1,073	1,073
AY 1992	1	2.5	1,205	7.09	1,199	1,200
AY 1993	1	3.5	1,300	7.17	1,340	1,340
AY 1994	1	4.5	1,525	7.33	1,497	1,498
AY 1995	1	5.5	–	–	1,673	1,674
AY 1996	1	6.5	–	–	1,870	1,871
PY95	1	6	–	–	1,769	1,771
AYE 7/1/96	1	6	–	–	1,769	1,770

Regression Output:

Constant		6.8114
Std Err of Y Est		0.0257
R Squared		97.9%
No. of Observations		4
Degrees of Freedom		2
X Coefficient(s)	0.1111	
Std Err of Coef.	0.0115	

[1] Values 0 and 1 correspond to dates 1/1/90 and 1/1/91, respectively.

[2] (4) = exp(0.1111 • (2) + 6.8114) • (1)

[3] The integral of f over accident period: $0 < y < 1$, $a-y < x < b-y$; is given by:

$$R|f = \exp(Aa + B) \bullet [\exp(A(b-a)) - 1] / A$$

The integral of f over policy period: $0 < y < 1$, $a < x < b$; is given by:

$$R|f = \exp(Ab + B) \bullet [\exp(A) - 1] / A^2 - \exp(Aa + B) \bullet [\exp(A) - 1] / A^2$$

Table C: Trending at Ultimate

EXAMPLE D

Given the following data, we wish to fit and project the triangle to 36 months.

Accident Year	Months Development 12	24	36
1991	148	350	424
1992	401	950	
1993	1,091		

Let 0 and 1 correspond to dates 1/1/90 and 1/1/91, respectively. Assume that all policies are effective for one year.

We assume a form for g or its integral, R | g, which is a function of the region and the type of valuation. As something of a compromise, we select the form of g but estimate R | g as the product of the integrand at a point in R times the area of R.

Let $g(x,y,z) = \exp(A\bullet(x+y+z) + B)$. We valuate over accident periods: $0<y<1$, $a-y<x<b-y$; as of date t. Integration of $g(x,y,t-x-y)$ over the region results in the function: $(b-a)\bullet\exp(At+B)$. Our first regression model is as follows.

$$\text{(I)}\quad \ln(\text{Level}/(b-a)) = A\bullet t + B \quad .$$

The integral, R | g, of $g(x,y,t-x-y)$ over accident period: $0<y<1$, $a-y<x<b-y$; is a function of t, a and b. We assume R | g takes the following form for our second regression model.

$$\text{(II)}\quad \text{Level} = Aa + Bb + Ct \quad .$$

Finally, for our third model, we assume $g(x,y,z)= \exp[Ax+By+C/(1+z)]$. We estimate the integral of $g(x,y,t-x-y) = \exp[Ax+By+C/(1+t-x-y)]$ over the accident period as the product of its value at a point times the area of the region. The point chosen is the average point, $(\{a-1+b\}/2, 1/2)$, under uniformity. The third regression model is as follows.

$$\text{(III)}\quad \ln(\text{Level}/(b-a)) = A(a-1+b)/2 + B/2 + C/(1+t-a/2-b/2) \quad .$$

The results of the regressions are shown with Table D. Model (III) gave the best results with (I) and (II) probably not feasible. A Model (I) strategy has theoretical appeal, but it may be difficult to apply regression analysis. A type-(II) approach takes the focus off of g and may be too simplistic. We lean toward Model (III) with an extra step. Once $g(x,y,z)$ has been determined, we set level based on direct integrations involving g. We took this approach in Example C.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
	Region R: $0<y<1$, $a-y<x<b-y$											
Accident	valuated as of date t				ln(Level /			1	Level	Level	Level	Level
Year	a	b	t	Level	(b−a))	(a−1+b)/2	1/2	(1+t−a/2−b/2)	Actual	Model I	Model II	Model III
1991	1	2	2	148	4.995	1.0	0.5	0.667	148	158	82	148
1991	1	2	3	350	5.857	1.0	0.5	0.400	350	350	302	328
1991	1	2	4	424	6.049	1.0	0.5	0.286	424	777	523	461
1992	2	3	3	401	5.995	2.0	0.5	0.667	401	350	580	406
1992	2	3	4	950	6.857	2.0	0.5	0.400	950	777	800	899
1993	3	4	4	1,091	6.995	3.0	0.5	0.667	1,091	777	1,077	1,115
1992	2	3	5	–	–	2.0	0.5	0.286	1,152	1,725	1,020	1,265
1993	3	4	5	–	–	3.0	0.5	0.400	2,583	1,725	1,297	2,468
1993	3	4	6	–	–	3.0	0.5	0.286	3,132	3,828	1,517	3,471

Model [1]	(I) ln(Level/(b−a)) = A•t + B	(II) Level = Aa + Bb + Ct			(III) ln(Level/(b−a)) = A(a−1+b)/2 + B/2 + C/(1+t−a/2−b/2)		
Constant	3.467		0			0	
Std Err of Y Est	0.369		153.691			0.070	
R Squared	79.5%		89.8%			99.4%	
No. of Observations	6		6			6	
Degrees of Freedom	4		3			3	
X Coefficient(s)	0.7971	912.695	−635.56	220.27	1.00955	11.9506	−2.9817
Std Err of Coef.	0.2023	303.515	287.529	97.2024	0.04404	0.20057	0.20897

Table D : Regression Analysis

[1] Formula for model determines dependent and independent columns to use in the table above. For example, Model (I) formula requires regression of column (6) on column (4). Thus, 1991 as of date t=2 is set at: 158 = exp(0.7971 • 2 + 3.467) • (2 − 1). Column (10) from Table A of Example A.

EXAMPLE E

We give an example of on level calculation for uniform and non-uniform level. Given R | g1, we define the valuation on g2 level as R | g2, with on level factor R | g2/R | g1. Policy Year 1992 valuated as of 12/31/93 is at level R | g. We wish to estimate R | 1.25•p, given

$$g(x,y,z) = \begin{cases} p(x,y,z), & \text{for } x < 10/1/92 \\ 1.25 \cdot p(x,y,z), & \text{otherwise} \end{cases} ,$$

where 1 and 2 correspond to dates 1/1/91 and 12/31/92, respectively. Assume all policies are annual. Valuation date 12/31/93 corresponds to t=3. This split for g creates two regions R1 and R2 for the policy year illustrated in Figure E.1, below. The factor to place R on 125•p level is as follows.

$$\frac{1.25 \cdot R \mid p}{R1 \mid p + 125 \cdot R2 \mid p}$$

$$= \frac{1.25}{1.00 \cdot (R1 \mid p/R \mid p) + 125 \cdot (R2 \mid p/R \mid p)} .$$

If we assume p is uniform over R, the distribution by areas yields factor:

$$\frac{1.25}{1.00 \cdot 75\% + 1.25 \cdot 25\%} = 1.176 .$$

Figure E.2 is the 3-D version Figure E.1. If we assume level R | p uniform over V=V1+V2 in Figure E.2, the distribution by volumes yields factor:

$$\frac{1.25}{1.00 \cdot 84\% + 1.25 \cdot 16\%} = 1.202 .$$

Suppose p takes the forms p=1001x+y+z and p=x+y+1001z. Levels R | p, R1 | p and R2 | p are provided in Table E.1.

Figure E.1			Integral of p(x,y,~~3-x-y~~) Over : 0<y<1 and a < x < b	
			p = 1001x+y+z	p = x+y+1001z
Region	a	to b	$(b-a) \cdot 3 + 500(b^2-a^2)$	$(b-a) \cdot 2503 - 500(b^2-a^2)$
R1	1	1.75	1,034	846
R2	1.75	2	470	157
R	1	2	1,503	1,003

Table E.1: Valuations as of 12/31/93

If p=1001x+y+z, R|g and R|125•p equal 1,622 and 1,879 units, respectively. The on level factor is therefore 1.16. The assumption that level is uniform over R yields a more accurate on level factor as p in this case is not as sensitive in the z-direction.

If p=x+y+1001z, R|g and R|125•p equal 1,042 and 1,254 units, respectively. The on level factor is therefore 1.203. The assumption that level is uniform over V yields a more accurate on level factor as p in this case is sensitive in the z-direction.

Note that for valuation at lag 3 years and over Calendar Year 1994, both uniformity assumptions would give an on level factor of 1.176.

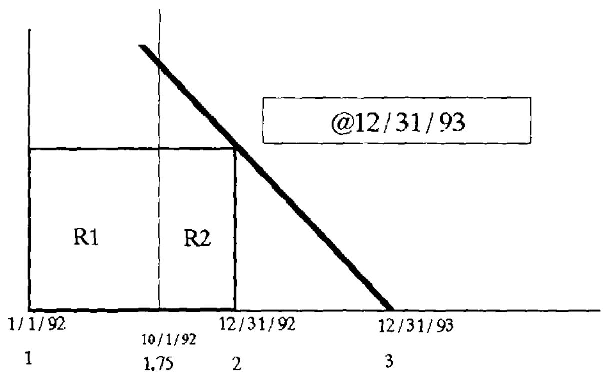


Fig. E.1 : R = R1 + R2 on Level

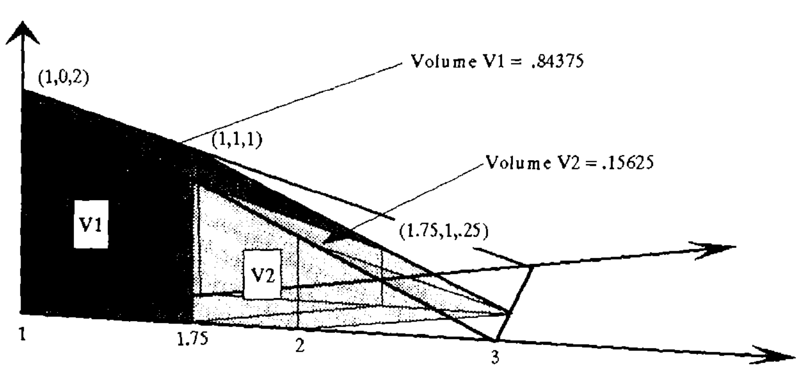


Fig. E.2: V = V1 + V2 on Level [1]

[1] Volumes derived using Table F.2 of Example F.

EXAMPLE F

Recall that for a (2.0) valuation we may apply a uniformity assumption over R as well as over V. We provide average points, areas and volumes under the two assumptions. Tables F.1 and F.2 were derived using (4.1.d), (4.1.e), (4.2.d) and (4.2.e). Both tables were determined using Figure F, below.

From Table F.1 we have the rule, for policy and accident periods, that x^* and y^* are at the middle of their respective coordinates ranges. For a policy / accident period, average coordinates are 1 / 3 the way into coordinate ranges.

As an example, recall Figure 3.1B. Consider Accident Year 1991 (to which the $65 million level was assigned). We estimated average points of (1 / 1 / 91, 6) and (11 / 1 / 90, 6, 4) under uniformity over R and V, respectively, for valuation as of date 12 / 31 / 91.

Let 1 / 1 / 91 and 12 / 31 / 91 correspond to 0 and 1, respectively, so that h=w=t=1 in Figure F. R is region A+B. Using Table F.1, we derive area 1 square year or 144 square months and average point, (0, 1 / 2), which corresponds to (1 / 1 / 91, 6). Using Table F.2, we derive volume 1 / 2 cubic years or 864 cubic months and point, (–1 / 6, 1 / 2, 1 / 3), corresponding to the average point (11 / 1 / 90, 6, 4).

We have suggested that the average valuation date be at the middle of a calendar year. Suppose that "ultimate" is reached at valuation lag 10 years and the plane z=10 years intersects the figure in Fig. 3.1B in the shape of region Accident Year 1991. For this "calendar period", it can be shown that the average valuation date under uniformity is 7 / 1 / 91.

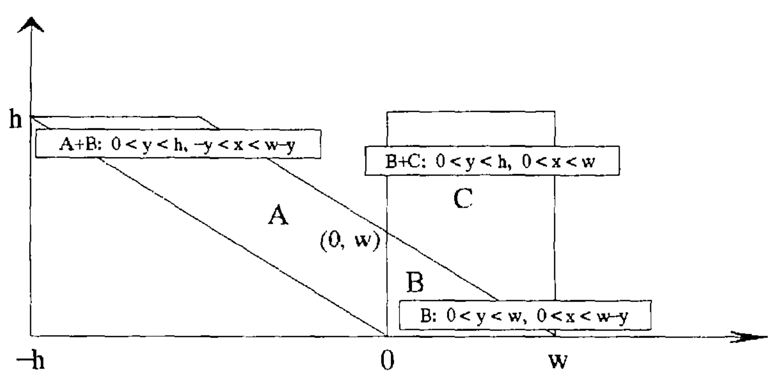


Fig. F : Basic Regions Definitions

Region R	Type (2.0) Valuation	Area	x^*	y^*
A+B	all	wh	$(w-h)/2$	$h/2$
B+C	all	wh	$w/2$	$h/2$
B	all	$w^2/2$	$w/3$	$w/3$

Table F.1 : Areas and Average Points of R [1] – Uniform Level

Region R	Type (2.0) Valuation	Set V Volume	x^*	y^*	z^*
A+B	$t > w$	$thw-hw^2/2$	$\frac{tw/2 - th/2 - w^2/3 + hw/4}{t - w/2}$	$h/2$	$(t-x^*-y^*)/2$
	$t2 > t1 > w$	$(t2-t1)wh$	$(w-h)/2$	$h/2$	$(t2+t1)/2-x^*-y^*$
	Lag c	cwh	$(w-h)/2$	$h/2$	$c/2$
B+C	$t > h+w$	$thw-hw^2/2-h^2w/2$	$\frac{tw/2 - hw/4 - w^2/3}{t - h/2 - w/2}$	$\frac{th/2 - hw/4 - h^2/3}{t - h/2 - w/2}$	$(t-x^*-y^*)/2$
	$t2 > t1 > w+h$	$(t2-t1)wh$	$w/2$	$h/2$	$(t2+t1)/2-x^*-y^*$
	Lag c	cwh	$w/2$	$h/2$	$c/2$
B	$t > w$	$tw^2/2-w^3/3$	$\frac{tw/6 - w^2/8}{t/2 - w/3}$	$\frac{tw/6 - w^2/8}{t/2 - w/3}$	$(t-x^*-y^*)/2$
	$t2 > t1 > w$	$(t2-t1)w^2/2$	$w/3$	$w/3$	$(t2+t1)/2-x^*-y^*$
	Lag c	$cw^2/2$	$w/3$	$w/3$	$c/2$

Table F.2 : Volumes and Average Points of V [1] – Uniform Level

[1] See Figure F for the definitions and limits of integration for regions A+B, B+C, and B.
The area of R is $R \mid 1$ and the average point is: $(x^*, y^*) = (R \mid x / R \mid 1,\ R \mid y / R \mid 1)$.
The volume of V is $V \mid 1$ and the average point is: $(x^*, y^*, z^*) = (V \mid x / V \mid 1,\ V \mid y / V \mid 1,\ V \mid z / V \mid 1)$.
Thus, use the following results to determine Tables F.1 and F.2.

Region	$R \mid 1$	$R \mid x$	$R \mid y$	$R \mid xy$	$R \mid x^2$	$R \mid y^2$
A+B	wh	$w^2h/2-wh^2/2$	$wh^2/2$	$w^2h^2/4-wh^3/3$	$w^3h/3-w^2h^2/2+wh^3/3$	$wh^3/3$
B+C	wh	$w^2h/2$	$wh^2/2$	$w^2h^2/4$	$w^3h/3$	$wh^3/3$
B	$w^2/2$	$w^3/6$	$w^3/6$	$w^2w^2/24$	$w^2w^2/12$	$w^2w^2/12$

Valuation	$V \mid 1$	$V \mid x$	$V \mid y$	$V \mid z$
Date t	$tR \mid 1 - R \mid x - R \mid y$	$tR \mid x - R \mid xy - R \mid x^2$	$tR \mid y - R \mid xy - R \mid y^2$	$(tV \mid 1 - V \mid x - V \mid y)/2$
t1 to t2	$(t2-t1) \cdot R \mid 1$	$(t2-t1) \cdot R \mid x$	$(t2-t1) \cdot R \mid y$	$(t2+t1)\frac{1}{2} \cdot V \mid 1 - V \mid x - V \mid y$
Lag c	$c \cdot R \mid 1$	$c \cdot R \mid x$	$c \cdot R \mid y$	$c/2 \cdot V \mid 1$