Probabilistic Development Factor Models with Applications to Loss Reserve Variability, Prediction Intervals, and Risk Based Capital

by Ben Zehnwirth

1.0 INTRODUCTION AND SUMMARY

The present paper aims to present a statistical modelling framework and environment for conducting loss reserving analysis. The modelling framework and approach affords numerous advantages including increased accuracy of estimates and modelling of loss reserve variability. Since the loss reserve is likely to be the largest item in the insurer's balance sheet and is subject to much uncertainty, modelling of loss reserve variability is an integral component of assessing insurer solvency and assessment of risk based capital.

The paper is organised as follows:

Forecasting and some modelling concepts are introduced in Section 2. The salient features of the data that ought to be captured by a model are discussed and arguments in favour of probabilistic models are presented. It is emphasised that the only way to assess loss reserve variability is through probabilistic models. The statistical MODELLING FRAMEWORK is introduced where each model in the framework has four components of interest. The first three involve trends in the three directions, development year, payment/calendar year and accident year and the fourth component is the random fluctuations (distributions) about the trends.

In Section 3 we begin by discussing trend adjustments to a univariate time series and illustrate how analogous adjustments to loss reserving data cannot be handled by graph and ruler, mainly as a consequence of the projection of the payment/calendar year trends onto the development year and accident year directions. Two deterministic models Cape Cod (CC) and Cape Cod with constant inflation (CCI) are discussed. Age-to-age development factors are defined as trend parameters.

A rich class of deterministic development factor models is introduced in Section 4 where each model in the framework contains the three trend components of interest. It is shown how as a result of the projection of calender year (trends), a very simple trend model causes very different development year trends (development factors) for different accident years. Standard actuarial techniques based on age-to-age link ratios of the cumulative payments cannot capture the payment/calendar year trends in the payments.

In Section 5 the class (or family) of deterministic development factor models that only contain trend components in the three directions is extended to include random fluctuations. The resulting models in the rich Development Factor Family (DFF) are probabilistic models that relate the distributions of 'payments' in the various cells in the triangle by trend parameters. It is emphasised that one of the principal uses of regression is the estimation (or fitting) of distributions. Estimation of a model belonging to the DFF involves the fitting of distributions to the cells in the loss development array. Data based on a simple DFF model are generated (simulated) and it is demonstrated how the development year patterns are invariably complex. The trends cannot be determined from the age-to-age link ratios nor from graphs. For readers who are sceptics and may argue "But this is simulated data" should read Section 12 where we analyse real life data involving a line written by a larger insurer for which the age-to-age link ratios on the cumulative payments are relatively smooth. HOWEVER, there are major shifts in payment/calendar year trends in the payments that are quite alarming.

We use regression for a number of purposes:

* Estimation of trends.
* Estimation or fitting of distributions.

In Section 6 we demonstrate how regression can also be employed to adjust data for trends. We state as a THEOREM that the only way to separate payment/calendar year trends from development year trends is by application of regression. Practical applications of regressions involving real life data sets are given in Sections 12 and 13.

In Section 6 we also present a number of tests that we believe any sound loss reserving statistical framework should pass. It is shown that standard actuarial techniques based on age-to-age link ratios fail these minimum tests.

As a result of the dependence of the payment/calendar year direction on the other two directions, many of the models in the DFF that contain many parameters cannot be estimated in a spreadsheet or statistical package and some that can be estimated may contain much parameter uncertainty. This phenomenon, known as multicollinearity, is discussed in Section 7 and motivates the introduction of varying parameter, dynamic or credibility models. Varying parameters or stochastic parameters can also be regarded as proxies for the myriad of variables that affect the complex claims generating process.

In Section 8 we show how the (fixed) parameter regression models may be estimated in a spreadsheet or statistical package and how an estimated model may be employed in producing forecast distributions of (incremental) payments. The forecast (estimated) distributions provide information required for the assessment of risk based capital and solvency.

Additional modelling concepts including parsimony, Akaike Information Criterion and distributional assumptions are discussed in Section 9. Moreover, we describe the importance of the twin concepts of stability and validation analysis and show how data with unstable trends (in the payments) are less predictable (subject to greater uncertainty) than data with stable trend (and some random fluctuations). Parameter *uncertainty (or instability) can reduce predictability much more than process* uncertainty.

Accuracy of forecast distributions is also discussed. We emphasise that the "optimal" statistical model, when trends are unstable, may not be the best for producing forecasts and discuss what assumptions may be appropriate for the future, especially in the light of analysing other data types. Instability in trends in the more recent payment years in the incremental payments requires more actuarial judgment about future trends.

The model building strategy and selection of appropriate assumptions about the future are discussed in Section 10. It is stressed that the model building strategy is necessarily an iterative cycle of model specification, estimation and testing. If trends in the more recent payment/calendar years are unstable, the nature of the instability and possible explanation for the instability is relevant information in deciding on assumptions for the future. This typically may require analysis of other data types employing the advocated modelling framework. We conclude in Section 10 with a discussion of time series models versus explanatory (or casual) models and offer arguments for the superiority of the former over the latter.

Section 11 discusses how prediction intervals may be derived from the forecast distributions and how they are relevant to the assessment of risk based capital and solvency. Prediction intervals computed from the forecast distributions are conditional on the assumptions made about the future remaining true.

The preliminary diagnostic analysis and the model building strategy are illustrated with two real life examples. Project 1 of Section 12 is concerned with real data of a large company. In terms of standard age-to-age link ratio techniques the data and ratios are relatively smooth and it does not appear that there are any problems. HOWEVER, there are major shifts in payment/calendar year trends in the payments that are alarming especially since the new high trend cannot be explained by a corresponding increase in speed of closure of claims. Project 2 of Section 13 also involves real data. Here the link ratios are relatively irregular, yet trends are stable, so that three years earlier estimation of the same model would have forecast the distributions of payments in the cells of the last three payment/calendar years and moreover would have produced the same outstanding reserve estimates.

In Section 14 we remark about an important extension of the DFF MODELLING FRAMEWORK that makes the family of models infinitely richer.

The paper concludes with summary remarks in Section 15.

Throughout the paper we also hope to dispel a number of pervasive loss reserving myths regarding data, age-to-age link ratios, volume, credibility, sources of information, actuarial judgment (when and where required), business knowledge, statistical probabilistic modelling and forecasting.

2.0 STATISTICAL FORECASTING

The best way to suppose what may come, is to remember what is past.

George Savile, Marquis of Halifax.

In this section we discuss a number of fundamental statistical forecasting concepts including which salient features of the data should be "*remember what is past*".

2.1 FORECASTING

Indeed it (forecasting) has been likened to driving a car blindfolded while following directions given by a person looking out the back window. Nevertheless, if this is the best we could do, it is important that it should be done properly, with the appreciation of the potential errors involved. In this way it should at least be possible to negotiate straight stretches of road without a major disaster.

Andrew C. Harvey [9]

In the loss reserving context the '*straight stretches*' are the stable trends in the (incremental) payments. If the trends have been stable in past years, we are confident in supposing the same trends in the future.

2.2 WHY A PROBABILISTIC OR STOCHASTIC MODEL?

There are extremely compelling reasons as to why we should be using probabilistic models to model insurance data, whether for the purpose of loss reserving, rate making or any other purpose.

According to Arthur Bailey's [2] paper ***Sampling Theory in Casualty Insurance***, any insurance data can only be regarded as "*an isolated sample* ...". See top of page 8 of the text book ***Foundations of Casualty Actuarial Science*** [5]. Bailey is basically saying that any insurance data can only be regarded as a sample (path) from perhaps a very complex process.

If a fair coin is tossed 100 times. the mean number of heads is 50, but the probability of observing 50 heads is only 0.08. If a fair dice numbered 1 to 6 is rolled, the mean is 3.5, yet the probability of observing 3.5 is zero. (The variability inherent in coin tossing in known as process uncertainty).

So, the probability of observing the mean in most, if not all, insurance processes is zero. Given, that we do not observe the mean, we need to compute more than just the mean. The mean on its own is not terribly informative. We need to also compute the standard deviation, so that we have some idea of how 'far' our (future) observations will be from the mean. The best, of course, is to compute the whole distribution. From the computed distribution we can derive the moments, percentiles and prediction (confidence (*sic*)) intervals.

Returning to the text book ***Foundations of Casualty Actuarial Science*** [5], the introductory chapter 1, top of page 2, says "*The mention of probabilities reminds us to state the obvious, that probability theory (whether classical or Bayesian) forms the basis of actuarial science. If the actuaries hadn't probability theory, they would have to invent it.*" Indeed, the author also believes that statistical probabilistic methods are essential to actuarial studies, and it is principally by the aid of such methods that these studies may be raised to the rank of sciences.

2.3 MODELLING FRAMEWORK

The models considered in the present paper are relatively simple. They have four components of interest that have a straightforward interpretation.

The first three components are the trends in the three directions, **development year, accident year** and **payment/calendar year**. The fourth component is the random fluctuations about the trends. The random fluctuations component is just as important as the three trend components and is necessarily an integral part of the model. The data or transform thereof are decomposed thus:

DATA = TRENDS + RANDOM FLUCTUATIONS

The concept of **trends** and **random fluctuations** about trends is over two hundred years old. These concepts have been widely used in analysing (and forecasting) any univariate time series such as sales, stock market prices, interest rates, consumption, energy and so on.

The principal aim of analysing a loss development array is to obtain a sensible description of the data. The trends in the past, especially in the payment/calendar year direction, are determined and the random fluctuations about the trends are quantified, so that it can be best judged which assumptions should be used for future trends (and random fluctuations). The models are probabilistic (equivalently, stochastic) since the probability distributions of the random fluctuations 'about' the trends are identified. Probabilistic models are testable and can also be validated. They also afford numerous other advantages including computation of risk margins required for the assessment of **risk based capital.**

IF THE TRENDS ARE STABLE THEN THE MODEL WILL VALIDATE WELL AND BE STABLE. If the trends are unstable then the decision about future trends is no longer straightforward. Instability in trends with little random variation about the trends makes data less predictable then stable trends with much random fluctuation. See Sections 9.6, 10.2 and 10.3. The same principles apply to the modelling of a univariate time series.

The 'best' identified model contains assumptions (equivalently, information). All the assumptions must be tested to ensure they are supported by the data (experience).

As we proceed through the model identification strategy we are extracting information (about trends and stability thereof and the amount of random variation) and we 'hope' that the 'best' identified model tells us that the calendar year trend is stable (especially more recently). If trends are not stable then we may not necessarily use the optimal statistical model for forecasting. See Section 9.6.

None of the numerous models contained in the MODELLING FRAMEWORK actually represent explicitly the underlying claims generating processes. The multitude of variables involved in generating the claims are invariably complex. What we attempt to achieve is the identification of a parsimonious model in terms of the simple components of interest for which all the assumptions inherent in the (probabilistic) model are supported by the data. It is subsequently argued that the experience (data) can be regarded as a sample (path) from the identified probabilistic model. The multitude of variables that are the determinants of the claims processes are proxied by the TRENDS and the (residual) variance of the RANDOM FLUCTUATIONS. Another classical modelling example in insurance where the same kind of modelling concepts are used is when a Pareto distribution, say, is fitted to loss sizes. It is not assumed that the Pareto distribution represents the underlying generating process. Whatever is driving the claims is very complex and depends on many variables. All that is assumed is that the experience (sample) can be regarded as a realisation from the estimated Pareto distribution. Subsequently the estimated Pareto distribution is used to estimate probabilities of very large claims including those exceeding the maximum observed claim in our sample and most importantly it is used to quantify probabilistically the variability in loss sizes.

The principal advantage of an explicit statistical model is that it makes the assumptions clear. Other advantages include improved accuracy and quantification of variability required for assessment of risk based capital and testing of solvency.

3.0 THE GEOMETRY OF TRENDS AND AGE-TO-AGE DEVELOPMENT FACTORS

In this section we show that loss development arrays possess only two independent directions, not three, and define age-to-age development factors as development year trends.

3.1 TREND ADJUSTMENTS TO A UNIVARIATE SERIES

In one dimension, or equivalently for any univariate series, trend concepts are intuitive and natural.

Consider the series log P_t where P_t is the price of gasoline in year t. Figure 3.1.1 below depicts the log P_t series (dark line segments) over a 20 year period.

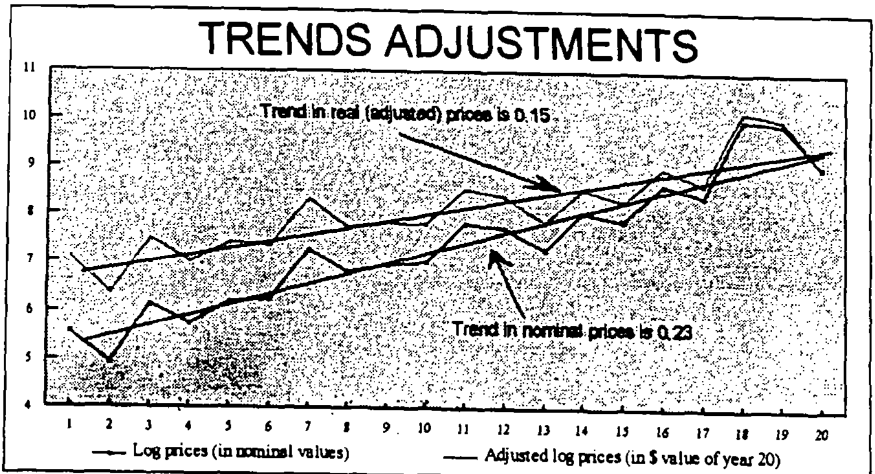


Figure 3.1.1

It appears that there is a constant average trend in the nominal prices. The least squares estimate of the trend is 0.23, say. So prices have been growing at an average rate of 23%. However, 23% is the nominal growth, since there has been economic inflation over the 20 year period. Suppose economic inflation has been 8% continuous rate for the whole 20 year period. The light line segments represent the log prices adjusted to the $ value of year 20.

The trend in the adjusted prices is 23% - 8% = 15%. If instead, one was only given the nominal prices and the adjusted prices (without knowing the adjustment), the 8% adjustment could be determined by estimating the difference in trends in the two series. Trends (on a log scale) are additive.

So, REGRESSION as an approach to estimating trends and adjusting data, immediately suggests itself.

3.2 TREND PROPERTIES OF LOSS DEVELOPMENT ARRAYS

Since a model is suppose to capture the trends in the data, it behoves us to discuss the geometry of trends in the three directions, víz., **development year** (or delay), **accident year** and **payment** (or calendar) **year**.

Development years are denoted by d; d=0,1,2,...,s-1; accident years by w; w=1,2,...,s; and payment years by t; t=1,...,s.

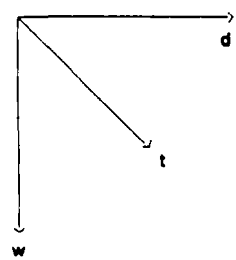


Figure 3.2.1

The payment year variable t can be expressed as t = w + d. This relationship between the three directions implies that there are only two 'independent' directions.

The two directions, delay and accident year, are orthogonal, equivalently, they have zero correlation. That is, trends in either direction are not projected onto the other. The payment year direction t however, is not orthogonal to either the delay or accident year directions. That is, a trend in the payment year direction is also projected onto the delay and accident year directions. Similarly, accident year trends are projected onto payment year trends.

In order to aid the exposition we shall assume, without loss of generality, that the numbers in the loss development array are incremental payments. **It is emphasised that all the arguments and concepts presented apply to all loss development arrays including incurreds, counts, averages and so on.**

We now illustrate the geometric properties of trends of a loss development array with some data.

Consider the following triangle of incremental paid losses:

Triangle One

100	200	150	100	80	60	40	20
100	200	150	100	80	60	40	
100	200	150	100	80	60		
100	200	150	100	80			
100	200	150	100				
100	200	150					
100	200						
100							

This triangle will be said to satisfy the Cape Cod assumptions, viz., homogeneity of age-to-age development factors across accident years and homogeneity of levels across accident years. Each accident year has the same initial starting value, that is, same value in delay 0.

Suppose we subject the payments to a 10% yearly inflation across the payment years. We obtain the next triangle:

Triangle Two

100	220	182	133	117	97	71	39
110	242	200	146	129	106	78	
121	266	220	161	142	117		
133	293	242	177	156			
146	322	266	195				
161	354	292					
177	390						
195							

To obtain the t^{th} diagonal of the second triangle, we multiply each payment in the t^{th} diagonal of triangle one by $(1.1)^{t-1}$.

We observe the following:

1. For triangle two, age-to-age development factors are homogeneous across accident years but are 10% higher than in triangle one.

2. In triangle two there is a 10% accident year trend.

Observations 1 and 2 imply that triangle two could be obtained from one by the two successive (and commutative) operations: subject triangle one to 10% per year trend in accident year direction to obtain:

Triangle Three

100	200	150	100	80	60	40	20
110	220	165	110	88	66	44	
121	242	182	121	97	73		
133	266	200	133	106			
146	293	220	146				
161	322	242					
177	354						
195							

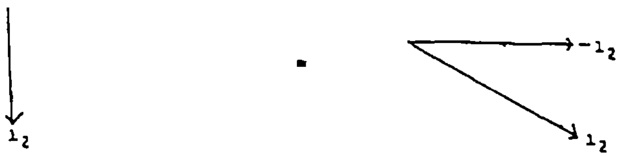
and then subject triangle three to 10% trend in the development year direction to obtain:

Triangle Four

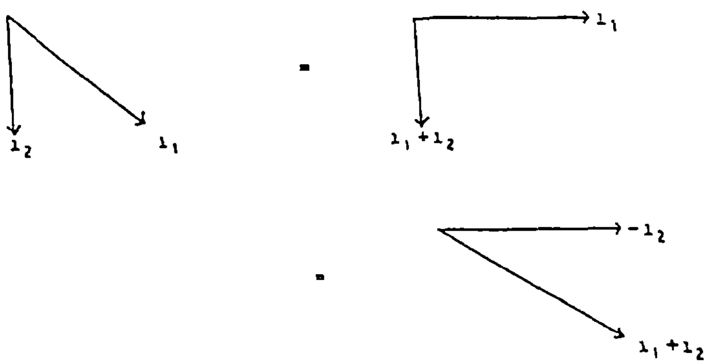
100	220	182	133	117	97	71	39
110	242	200	146	129	106	78	
121	266	220	161	142	117		
133	293	242	177	156			
146	322	266	195				
161	354	292					
177	390						
195							

Triangle four is the same as triangle two. A loss development array depicted by triangle two (or four) is said to satisfy the Cape Cod with constant payment year inflation assumptions.

The following displays demonstrate the equivalence of trends in general.



The above equivalence relations are exemplified by the relationships between the four triangles. We also have,



It is important that the reader understands the relationship and difference between Cape Cod (CC) data and Cape Cod with constant inflation (CCI) data.

CC data have accident years that are completely homogeneous (homogeneity of level or values at development year zero and homogeneity of age-to-age factors). CCI data can be obtained from CC data by subjecting the payment years to a constant trend. If we remove the constant payment year trend from the CCI data we will have CC data.

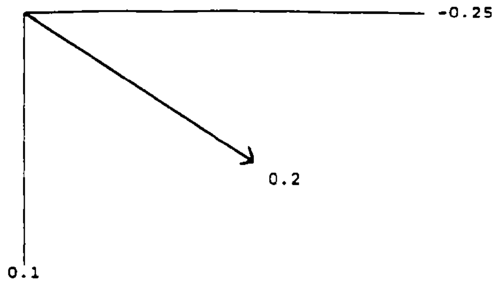
So, the difference between CCI data and CC data is a calendar year trend adjustment. If we did not know how the CCI data were created from the CC data, how would we determine the (simple) difference?

With the univariate series considered in Section 3.1 the difference between the nominal prices and adjusted prices can be determined by estimating the trend, using eye and ruler, for each series. Estimating trend using eye and ruler can be regarded as a form of crude regression. With the loss reserving data CC and CCI, it also makes sense to estimate the payment year trends and subsequently conclude that the difference in the two loss development arrays resides in the difference in the two trends. But how do we estimate the trends? Given the dimensionality of the data, eye and ruler are not useful. Moreover, given the geometry of trends, we need to separate the trends in the three directions. Equivalently, we need to determine the payment year trends after adjusting for development year trends.

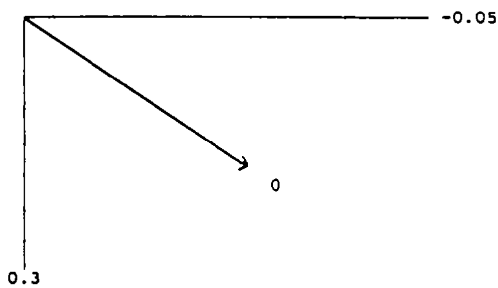
Accordingly, formal regression is suggested as the only way of separating the trends.

A number of words of caution. In actual fact the "true" trends in the three directions are non-identifiable. It is only the resultant trends that are identifiable.

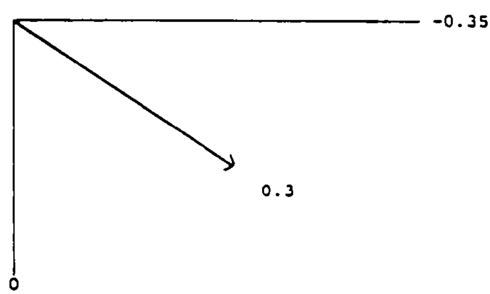
Here is an example. Consider a CC triangle for which the (continuous) trend across development years is constant and is -0.25. Suppose to this CC triangle we introduce a continuous calendar year trend of 0.2 and a continuous accident year trend of 0.1. The adjusted triangle can be represented thus:



Alternatively, it can be represented as:



or,



All three trend triangles are the same and would produce the same projections for the completion of the rectangle. We have three directions (or variables) but only two independent equations.

3.3 DETERMINISTIC AGE-TO-AGE DEVELOPMENT FACTORS

Consider, at first, only one accident year (say, the first) that takes the value p(d) at development year d and let y(d) = log p(d).

Define: $\alpha = \log p(o)$

and

$\gamma = y(j)-y(j-1)$.

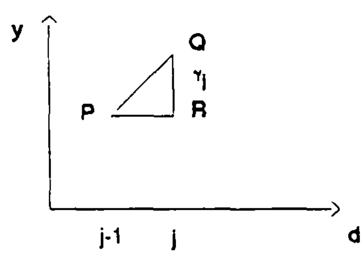


Figure 3.3.1

The parameter α (alpha), denotes the initial value, or intercept, or level whereas the parameter γ_j represents the trend, on a logarithmic scale, from development year j-1 to development year j.

The parameter γ_j is a difference on a log scale and since the length of PR in Figure

3.3.1 is 1, γ_j is the slope of the line PQ, and hence is the trend between development years j-1 and j.

Now,

$$y(d) = y(o) + y(1) - y(o) + \dots + y(d) - y(d-1)$$

$$= \alpha + \sum_{j=1}^{d} \gamma_j \quad . \qquad (3.3.1)$$

That is, y(d) can be expressed as the initial value plus the sum of the differences to development year d. The differences can also be regarded as trends. Indeed,

$$\gamma_j = y(j) - y(j-1)$$

$$= \log p(j) - \log p(j-1)$$

$$= \log \left[\frac{p(j)}{p(j-1)}\right] \quad . \qquad (3.3.2)$$

One of the principal reasons for taking logarithms of the data is because the difference of two logarithms is equivalent to analysing trends and approximately equivalent to analysing percentage changes.

The trend parameter γ_j is the log of the ratio p(j)/p(j-1). The latter ratio is an age-to-age development factor. So, γ_j can also be interpreted as a log of a development factor. Indeed, in what follows we shall refer to it as a development factor (on a log scale).

Consider the following monotonically increasing series $\{p(j)\}$ for which the trends are depicted in the Figure 3.3.2 below.

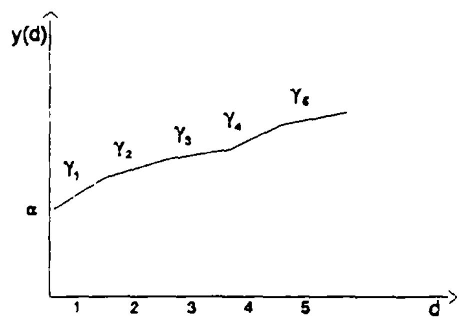


Figure 3.3.2

The γ's represent both the differences in y values and the trends depicted by the straight line segments.

Accordingly, development factors on a log scale form a curve comprising of straight line segments (trends).

4.0 DETERMINISTIC DEVELOPMENT FACTOR MODELS

In this section we develop the mathematical description of the two models corresponding to triangles one and two respectively of Section 3.2.

Let p(w,d) denote the value in the loss development array corresponding to accident year w and development year d and set y(w,d)=logp(w,d).

4.1 CAPE COD (CC)

Consider triangle one of Section 3.2. Each accident year has the same α value, viz., α =log100 and each accident year has the same development factors $\gamma_1, \gamma_2, \ldots, \gamma_6$ (γ_7). For example, γ_3=log(100/150).

So, we can write

$$y(w,d)=\alpha+\sum_{j=1}^{d}\gamma_j \quad . \tag{4.1.1}$$

Equation (4.1.1) describes the deterministic CC model.

4.2 CAPE COD WITH CONSTANT INFLATION (CCI)

Consider now triangle two of Section 3.2. It was obtained from triangle one by subjecting it to a constant trend in the payment year direction.

Let's denote the payment year trend on a logarithmic scale by the Greek letter, ι (called iota). For triangle two ι =log 1.1.

The value y(w,d) that lies in payment year w + d is inflated by ι (w + d - 1).

So, for triangle two,

$$y(w,d) = \alpha + \sum_{j=1}^{d} \gamma_j + \iota \cdot (w + d - 1) \ . \qquad (4.2.1)$$

The last equation may be re-cast.

$$y(w,d) = \alpha + \iota \cdot w - \iota + \sum_{j=1}^{d} (\gamma_j + \iota) \ . \qquad (4.2.2)$$

The two foregoing equations are identical and represent the CCI deterministic model. The latter equation tells us that the level parameter for accident year w is $\alpha + \iota + W - \iota$, so that there is an ι trend along the accident years and that the development factor from delay j-1 to j is $\gamma_j + \iota$. This is just a mathematical verification that the payment year trend ι projects on the other two directions.

4.3 CC FAMILY AND CCI FAMILY

There are other CC models for which the CC assumptions viz., homogeneity of accident years, apply.

For example, it may be that $\gamma_3 = \gamma_4 = \ldots = \gamma_8$, so that the trends from development year two to eight are constant as depicted below:

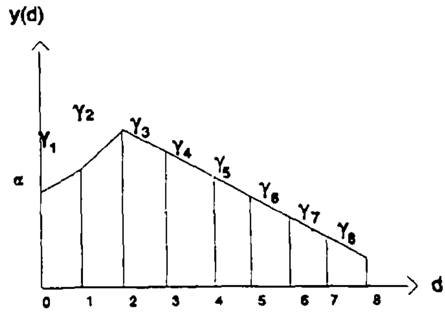


Figure 4.3.1

Another possibility is that all development factors γ_1, γ_2, ..., are equal to γ say, so that we could write:

$$y(w,d) = \alpha + \gamma d \quad . \qquad (4.3.1)$$

This model we call the single development factor (SDF) model. It is a straight line curve on a log scale and exponential curve on the $ scale. It is the same curve for each accident year.

So, we can construct many variants of the CC model (4.1.1.). In the sequel, anytime we refer to CC without an added qualification we shall mean model (4.1.1) with distinct γ's.

Similarly, depending on the "relationships" in the γ's in the CCI model, we can construct many variants of the CCI model.

4.4 A CC MODEL WITH THREE INFLATION PARAMETERS

The data in Appendix A1 to Appendix A4 are generated as follows.

First we create payments based on formula:

$$p(w,d) = \exp(alpha - 0.2*d).$$

So this is deterministic SDF data (where the accident years are homogeneous). See Appendix A1.

On a log scale we introduce a 10% trend from 1978-82, 30% trend from 1982-83 and 15% trend from 1983-91. See Appendix A2.

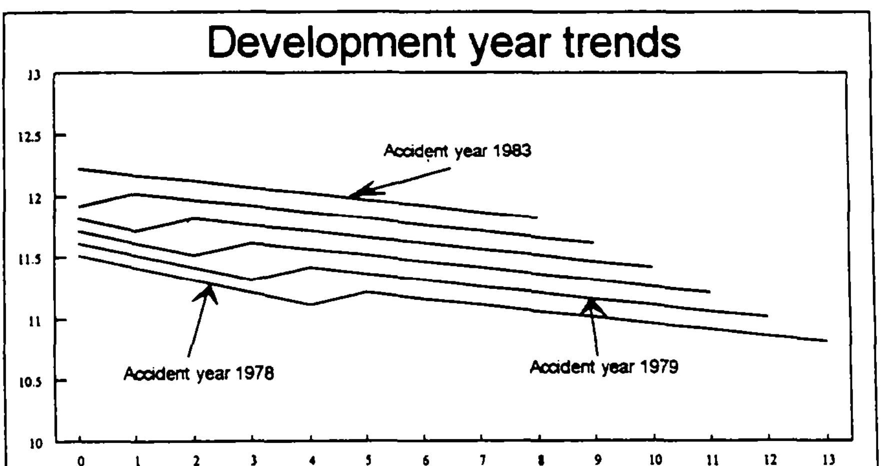


Figure 4.4.1

Figure 4.4.1 displays the graph of the log data versus development year for the first six accident years. The reader can reproduce this graph in a spreadsheet.

Observe how calendar year trends project onto development years and accident years.

Consider the first accident year 1978. The 10% calendar year trend projects onto the development year, so that the resultant trend from development year 0 to development year 4 is -0.2 (the gamma) + .1 (the iota)= -.1. The 30% trend between calendar years 1982 and 1983 also projects onto the development year so that the trend between development year 4 and 5 is +.1=-0.2+.3. Thereafter the trend is -.2+ 15=-.05. Since .15 is larger than 1, the decay in the tail is less rapid (-.05>-.1).

Consider the next accident year 1979. First up to development year 3, this accident year is 10% higher than the previous one since the 10% calendar year trend also projects onto the accident years. The 10% upward trend is one development year earlier than in previous accident year since the 30% trend is a calendar year change.

So, changing calendar year trends can cause some interesting development year patterns. The pattern is different for each accident year. The calendar year trends cannot be determined by the link ratios (Appendix A4).

The patterns became much more complicated in the presence of random fluctuations superimposed on the trends. See Section 5 for a discussion of the current example including random fluctuations.

The model describing the data we have constructed can be represented pictorially thus:

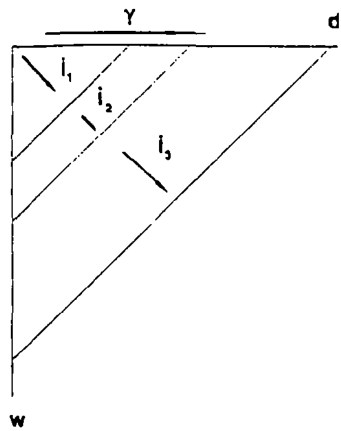


Figure 4.4.2

where γ = -0.2, ι_1=0.1, ι_2=0.3 and ι_3=0.15.

Writing the equations explicitly is not necessary. Indeed, it is too complicated.

We note that the resultant trend (age-to-age development factor) between development years j-1 and j is the (base) development factor γ between the two development years plus the payment year trend ι (iota) between the two corresponding payment years.

The above model can be described succinctly in terms of the five parameters, α, γ, ι_1, ι_2 and ι_3. We could create a slightly more involved model by adding accident year trends (more α's).

4.5 CHAIN LADDER (CL)

The chain ladder (CL) statistical model is described in Christofides [4]. It is a two-way ANOVA model where accident years and development years are two factors at various levels. The CL statistical model is the direct statistical extension of the standard age-to-age development factor technique. See Christofides [4] for details. It is written (omitting the random fluctuations).

$$y(w,d) = \alpha_w + \sum_{j=1}^{d} \gamma_j \qquad (4.5.1)$$

The parameter α_w corresponding to accident year w represents the effect of accident year w and the parameter $\gamma_j - \gamma_{j-1}$ (difference in trends) represents the effect of development year j. The number of parameters in the model is 2s-1.

The CC model assumes complete accident year homogeneity, that is, same α and same γ_j's. For the CL model we assume homogeneity of development factors (γ_j's), but heterogeneity of levels (α's).

The principal deficiency of the CL model is that it does not relate the calendar years in terms of trends.

If we do not have an estimate of trends in the past, how do we know what assumptions we can make about the future trends? See comments by George Savile at beginning of Section 2.0 and the discussion in Section 9.6.

HOWEVER, the CL model is an extremely powerful interpretive tool as we shall see in Section 6 and more impressively in an application to a real life example in Section 12.

4.6 THE SEPARATION MODEL (SM)

The separation method separates the base systematic run-off pattern (assumed homogeneous across accident years) from exogenous influences, viz., payment year inflation (or effects). The deterministic model is usually expressed (parametrized) as

$$p(w,d) = e(w)\, b_d\, \lambda_{w+d} \quad ,$$

where the { e (w) } are the exposures, proportional to number of claims incurred. $\{ b_d \}$ are the development factors and the parameter λ_{w+d} expresses the 'effect' of payment year t = w + d.

The *corresponding model in our framework* is written (parametrized) as

$$y(w,d) = \alpha + \sum_{j=1}^{d} \gamma_j + \sum_{t=2}^{w+d} \iota_t \quad , \qquad (4.6\ 1)$$

where the parameters $\{ \gamma_j \}$ are the base systematic development factors and ι_t is the force of inflation from payment year t - 1 to payment year t.

The model has 2s - 1 parameters.

Note that this model necessarily assumes that there are significant changes in inflation rates (trends) between every two contiguous payment years and, moreover that *there are significant changes in base development factors between every two* development years.

Refer to the discussion of Section 9.6 where we show that if trends are indeed unstable then the payments are not terribly well predictable.

4.7 DETERMINISTIC DEVELOPMENT FACTOR FAMILY

Let's reconsider the model of Section 4.4. It can be described succinctly as a version of CC (viz., SDF) subjected to three payment year trends. If we remove the three payment year trends, we are back to SDF. On this model we could also superimpose (add) accident year trends.

So, any determistic development factor model (DFF) can be described as some version of CC subject to payment year trends and accident year trends.

Mathematically, the family of development factor models is

$$y(w,d) = \alpha_w + \sum_{j=1}^{d} \gamma_j + \sum_{t=2}^{w+d} \iota_t \quad . \tag{4.7.1}$$

A model has a level parameter α_w for accident year w - it represents the effect or level or exposure of the accident year. Between every two development years, we have a development factor or trend parameter γ_j (the factor from delay j-1 to j) and between every two payment years we have a trend (or inflation) parameter ι_t, the inflation from payment year t-1 to t.

All models considered thus far belong to the development factor family. For example, CC is written as:

$$y(w,d) = \alpha + \sum_{j=1}^{d} \gamma_j \quad . \tag{4.7.2}$$

So for CC type model $\alpha_w = \alpha$ (for each w) and $\iota_t = 0$ for each t.

There is no need to memorise the equation representing the family of models. All that needs to be understood is that the parameters of a model comprise (i) trends (development factors) in the development year direction (the γ's), (ii) levels (exposures) for each accident year (the α's) and (ii) trends (inflation) in the payment year direction (ι 's). Furthermore, any payment year trend projects on the other two directions.

5.0 STOCHASTIC DEVELOPMENT FACTOR MODELS

In this section the class of deterministic DFF models (4.7.1) that only contain trend components is extended to include random fluctuations.

Consider one accident year only for which the deterministic model is

$$y(d) = \alpha + \sum_{j=1}^{d} \gamma_j \quad . \tag{5.1}$$

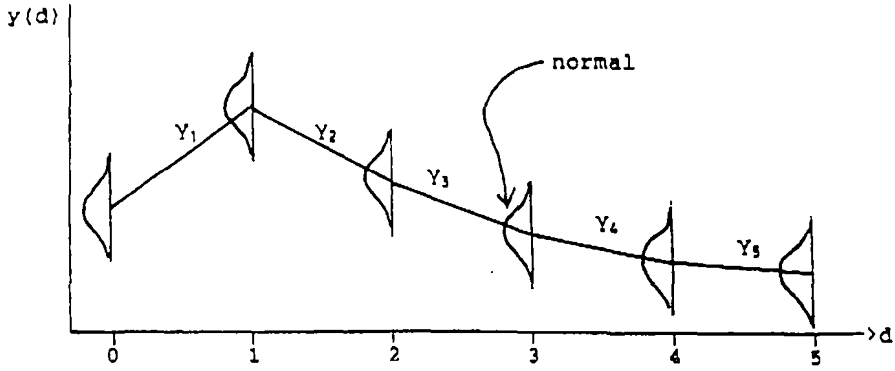
This model says that at delay o we can only observe one (log) value, viz α. Similarly, for the other delays. Between any two delays we can only observe one trend, the trend corresponding to the development factor.

We now assume that around the trends there are random fluctuations. We write

$$y(d) = \alpha + \sum_{j=1}^{d} \gamma_j + \varepsilon \quad , \tag{5.2}$$

where ϵ the error term, has a normal distribution with mean 0 and variance σ^2. In actuarial parlance σ^2 is known as the process uncertainty. Given that the errors are random variables, the dependent variable y is also a random variable.

The probabilistic (stochastic or regression) model is depicted below.



For the stochastic model, α is no longer the value of y observed at delay 0. It is the mean of y(o). Indeed, y(o) has a normal distribution with mean α and variance σ^2.

Similarly, γ_j is not the observed trend between delay j-1 and j, but rather it is the mean trend.

The parameters of the stochastic model represent means of random variables. Indeed, the model (on a log scale) comprises a normal distribution for each development year where the means of the normal distributions are related by the parameter α and the trend parameters $\gamma_1, \gamma_2, \ldots, \gamma_5$.

From equation (5.2) we have

$$y(d) - y(d-1) = \gamma_d + \varepsilon_d - \varepsilon_{d-1} \, , \qquad (5.3)$$

where ε_d is the 'error' at delay d.

Accordingly,

$$E\left[\log\frac{p(d)}{p(d-1)}\right] = \gamma_d \quad . \tag{5.4}$$

That is, the development factor γ_d is the mean of the log of the ratio on the $ scale. A development factor is a parameter.

Based on model (5.2), the random variable p(d) has a lognormal distribution with,

Median = $\exp[\alpha + \sum_{j=1}^{d} \gamma_j]$, (5.5)

Mean = mean $\cdot \exp[0.5 \ \sigma^2]$, (5 6)

and

Standard

Deviation = mean $\cdot \sqrt{\exp[\sigma^2] - 1}$. (5.7)

Since, y(d) - y(d-1) ~ $N(\gamma_d , 2\sigma^2)$, we have

$$E\left[\frac{p(d)}{p(d-1)}\right] = \exp[\gamma_d + \sigma^2], \tag{5 8}$$

so that the development factor on the $ scale (the mean of a ratio) is given by the last equation.

The stochastic model for p(d) comprises a lognormal distribution for each development year where the medians of the lognormal distributions are related by equation (5.5) and the means are related by equation (5.6). So, in fitting or estimating the model (Section 8) we are essentially fitting a lognormal distribution to each development year. The curve (on a log scale) comprising straight line segments is only one component of the model. The principal component comprises the distributions.

As another example, we consider the stochastic CC model, viz.,

$$y(w,d) = \alpha + \sum_{j=1}^{d} \gamma_j + \varepsilon \quad . \tag{5.9}$$

In this model we assume, for example, that y(1,0),...,y(s,0) are observations from a normal distribution with mean α and variance σ^2.

The assumptions contained in the model must be tested to ensure that they are not violated by the data.

The stochastic development factor family **(DFF)** is written as:

$$y(w,d) = \alpha_w + \sum_{j=1}^{d} \gamma_j + \sum_{t=2}^{w+d} \iota_t + \varepsilon \quad . \tag{5.10}$$

Note that the mean trend between cells (w,d-1) and (w,d) is $\gamma_d + \iota_{w+d}$ and the mean trend between cells (w,d) and (w+1,d) is $\alpha_{w+1} - \alpha_w + \iota_{w+1+d}$.

A model belonging to the DFF of (stochastic) models relates the lognormal distributions of the cells in the triangle. On a log scale the distribution for each cell is normal where the means of the normal distributions are related by the "trends" equation belonging to the family (4.7.1).

Another deficiency of the CL probabilistic model is that it contains the explicit assumption that the errors for the youngest accident year and the last development year are both zero. The chance of that, is zero!

We now return to the deterministic development factor model of Section 4.4.

To all the log "payments" in the triangle we add random numbers from a normal distribution with mean zero. Equivalently, to the trends depicted in Figure 4.4.1, we add random numbers from a normal distribution displayed in Appendix A5. The sum of trends (Appendix A2) plus random fluctuations (Appendix A5) is displayed in Appendix A6.

The graph of the first six accident years of the data in Appendix A6 is given in the Figure 5.2 below.

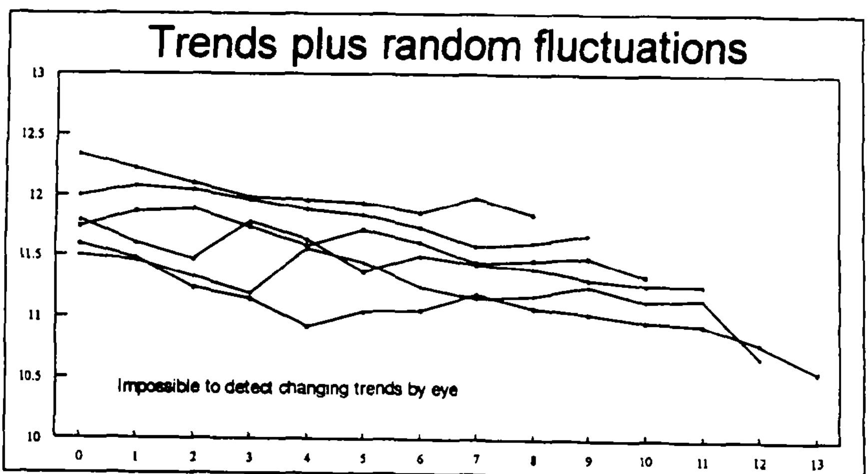


Figure 5.2

NOTE that it is impossible to determine the trends and/or change in trends by eye or from the age-to-age link ratios of the cumulative payments (Appendix A9). See Appendices A7 - A9. THE TRENDS CAN ONLY BE DETERMINED BY USING REGRESSION.

Notwithstanding the fact that the DFF modelling framework can be applied to any loss development array, much of the remainder of the discussion will involve analysis of the incremental payments for the following reasons:

* the geometry of trends;
* simplicity and parsimony;
* *distributions of future payments is relevant information for financial* statements.

Other reasons are given in Sections 10.3 and 10.4.

6.0 REGRESSION AS A FORM OF ADJUSTMENT AND MINIMUM TESTS

Hitherto we have applied regression for two related purposes. Estimation of trends in the 'payments' and estimation of the distribution of payments in each cell. The estimated trends relate the means of the distributions on a log scale.

For example, if the CC model is an appropriate model, then the 'payments' come from lognormal distributions and the means of the log 'payments' lie on the surface:

$$y(w,d) = \alpha + \sum_{j=1}^{d} \gamma_j \quad .$$

6.1 REGRESSION AS A FORM OF ADJUSTMENT

Regression is also a very powerful approach to adjusting data, especially in the loss reserving context.

In view of the fact that payment/calendar year trends project onto the other two directions, a graph of the data in one direction gives no indication of the trends. See for example, the simulated data with three payment year trends discussed in Section 5, and in particular, Figure 5.2.

We define a residual by

$$\hat{\varepsilon} = y - \hat{y} \quad .$$

That is, a residual is an observed value minus its fitted value.

Residuals can be interpreted as the data adjusted for what has been fitted. Let's consider a number of examples.

Suppose we simulate (generate) a triangle based on a CC model. The model generating the data can be written

CC DATA = CC TRENDS + ERROR (randomness)

If to the data we estimate the CC model, then the residual is

residual = CCDATA - FITTED CC TRENDS

= estimate of error,

that is, the residuals represent the data after we take away (subtract) what we fitted, alternatively, the residuals represent the data adjusted for what we fit. Here we subtract the estimates of the trends we used to create the data, so residuals should represent what is left, which is "randomness" in the three directions. "Random" residuals versus payment years are depicted in Figure 6.1.1.

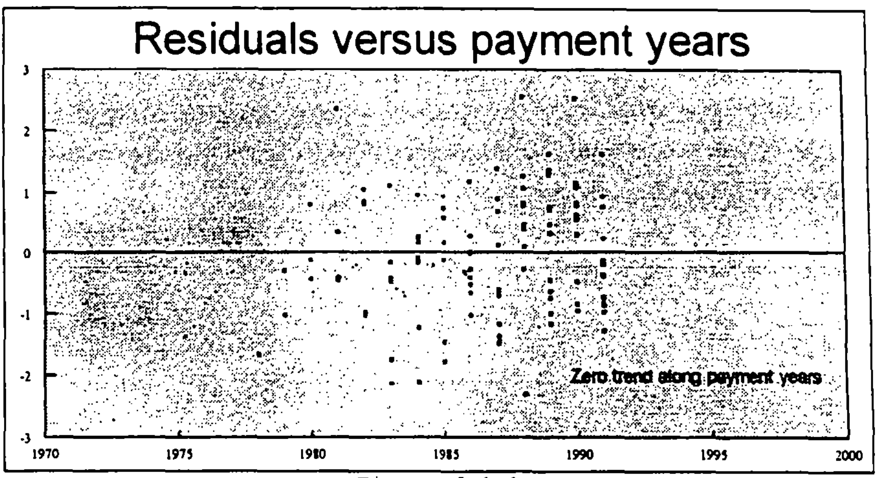


Figure 6.1.1

Suppose we now generate,

DATA = CC data + 10% calendar year trend

If we fit the CC model to this data the residual is

residual = DATA - fitted CC TRENDS

= estimate of error + 10% calendar year trend

So here residuals versus payment/calendar years will exhibit a straight upward trend (+ randomness) as depicted in Figure 6.1.2. After removing the CC trends from the data, there still remains the 10% calendar year trend plus the random fluctuation.

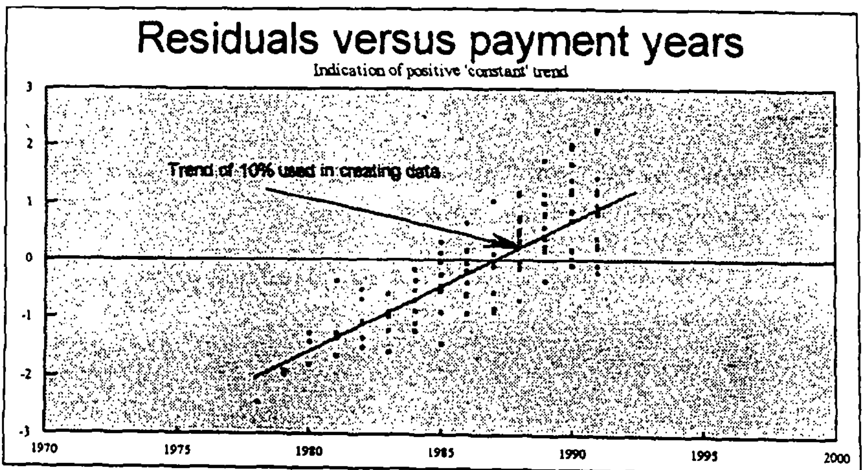


Figure 6.1.2

If you estimate the average trend in these residuals in a spreadsheet you would obtain an estimate of approximately 10% (the trend introduced into the data).

If we estimate the CCI model to the data, we are essentially estimating a trend parameter through the payment year residuals (Figure 6.1.2) of the previous CC model.

Now the residuals versus payment years should be random as we have removed (subtracted) all the (estimated) trends we introduced into the data.

Consider now data created as follows

DATA = CC data + 10% trend (calendar years 1978-85)
+ 20% trend (calendar years 1985-91)

If we fit the CC model to this data the residual is

residual = DATA - fitted CC TRENDS

= estimate of error + 10% (78-85)+15% (85-91)

The residuals versus payment/calendar years exhibit two trends, one from 1978-85 and sharper trend from 1985-91. See Figure 6.1.3 below.

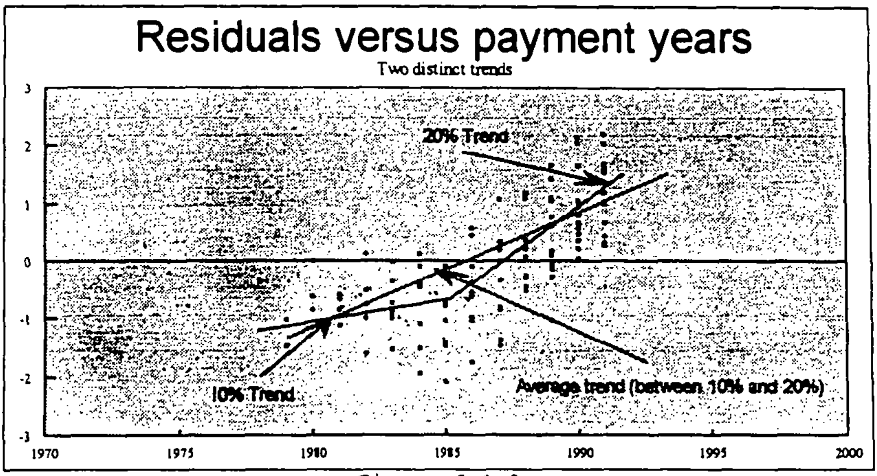


Figure 6.1.3

In now estimating the CCI model to the data, we are essentially estimating a trend parameter through the payment year residuals of Figure 6.1.3. The average trend is between 10% and 20%. The residuals versus payment years are now 'v-shaped'. See Figure 6.1.4 below.

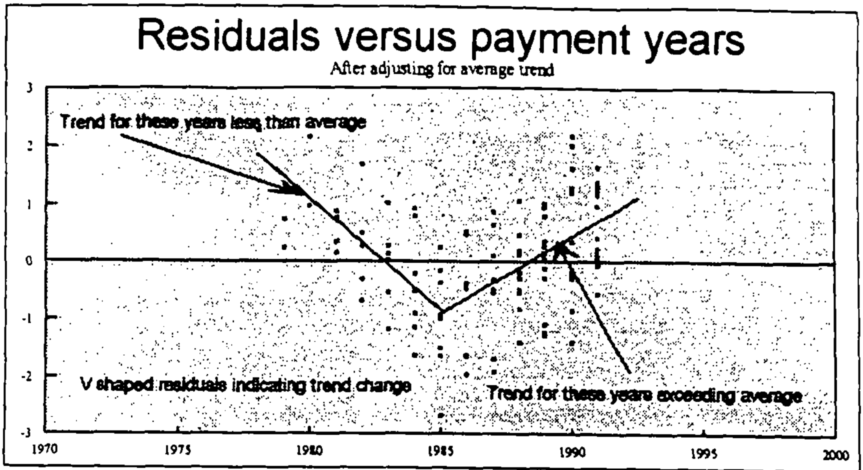


Figure 6.1.4

We are now led to estimate the two trends.

In view of the fact that calendar year trends project onto the other two directions, we can only obtain an indication of payment year trends, after we first remove the development year trends from the data (and vice versa).

REGRESSION IS A VERY POWERFUL TECHNIQUE FOR SEPARATING THE TRENDS IN THE THREE DIRECTIONS FROM RANDOM FLUCTUATIONS

In Section 12 we analyse a real life example that possesses relatively smooth age-to-age link ratios, yet there are major shifts in calendar year trends that are quite alarming.

6.2 MINIMUM TESTS

The author believes that a sound loss reserving statistical modelling framework should pass a number of very simple basic fundamental tests.

Turning to the univariate (log price) series of Section 3.1, if the (average) trend in the nominal prices is zero, that is, the prices are random about a zero trend then this feature in the data could be determined informally by examining the graph with eye and ruler and formally in a spreadsheet by estimating the trend, showing that it is insignificant and testing the residuals for randomness. Hence,

Test 1: If the (incremental) payments in a loss development array are random observations (from a lognormal distribution), and accordingly there are no trends in each of the three directions, then a sound loss reserving methodology should determine this.

We illustrate with an example. Appendix B1 contains incremental payments drawn at random from the same lognormal distribution. Note the variability. The mean forecast or fitted value for each cell is the same. Indeed, estimation of the CC model, for example, to the data would yield insignificant γ's, as they should be. Application of the DFF modelling framework will allow us to identify the salient features of the data extremely fast.

The age-to-age link ratios are displayed in Appendix B3 and do not appear to convey much relevant information. (Compare with age-to-age link ratios in Appendix B5. What can you tell?)

For those readers who feel that random data (no trends) represents a pathological case, should analyse a number of Lloyd's Syndicates data.

Returning to the univariate series of Section 3.1, it is rather straightforward to identify both informally and formally the difference between the nominal prices and the adjusted prices. A second loss reserving test is suggested.

Test 2: Consider any real life incremental paid loss development array. Create from this array a second array by subjecting it to a number of trends, for example, a 10% trend (say) in the first five calendar years (say), and a 15% trend (say) in the subsequent calendar years, then a sound loss reserving methodology will allow for a quick determination of the simple difference between the two loss development arrays.

The DFF modelling framework passes Test 2 with flying colors. The reader will find that by applying Test 2 to standard age-to-age link ratio techniques they fail it. That is because standard techniques do not satisfy the necessary and simple property of additivity of trends.

In order to dispel the myth that smooth age-to-age link ratios imply stability of trends we analyse in Section 12 a real life array with smooth factors and find major trend instability that is quite alarming and in order to dispel the converse myth that rough age-to-age link ratios imply trend instability, we analyse in Section 13 a real life array with rough ratios and find stability so that had we used the same model estimated three years earlier, it would have accurately predicted the distributions for the last three calendar years and would have given the 'same' outstanding estimates.

To further illustrate the impact of randomness of payments on age-to-age link ratios, Appendix B4 contains an array generated by an SDF probabilistic model with constant 20% calendar year trend. The link ratios are presented in Appendix B5 and appear relatively rough. Yet, the same model estimated four years earlier would have predicted the distributions of the payments of the last four years and would have produced the 'same' completion of the rectangle!

It is interesting to also observe that even though the data in Appendix B4 has a 20% calendar year (and accident year) trend, as you step down a column (development year), sometimes the numbers decrease rather than increase (by 20%).

For example, (1989, 1) to (1991, 1) the payment reduces from 767664 to 350789. This is explained by the random fluctuations component of the model. Examine now Figure 3.1.1 and note that even though the mean trend in nominal prices is 23%, prices from one year to the next do not necessarily increase. This is due to the random fluctuations. So, the same phenomenon applies to loss reserving data.

Consider now the unusual large value of 1317425 corresponding to (1985,6). It is not unusual. It comes from the tail of the lognormal distribution. Given that the lognormal is skewed to the right, values greater than the median tend to be 'far' from the median, whereas values less than the median tend to be relatively close to the median.

7.0 VARYING PARAMETER, DYNAMIC OR CREDIBILITY MODELS

7.1 MULTICOLLINEARITY

Many of the models within the family (5.10) cannot be estimated in a spreadsheet or any statistical package. Models that contain "many" iotas, alphas and gammas suffer from a problem known as multicollinearity. This problem is explained as follows.

To estimate the Ordinary Least Squares line for the simple linear regression:

$$y_i=\alpha+\beta x_i+\varepsilon_i \quad , \tag{7.1.1}$$

we estimate the intercept α and slope β by minimising the error sum of squares,

$$SS=\sum(y_i-\alpha-\beta x_i)^2 \quad .$$

Taking partial derivatives of the last equation with respect to α and β, and setting them to zero we obtain:

$$-2\sum(y_i-\alpha-\beta x_i) \quad = \quad 0 \tag{7.1.2}$$

and

$$-2\sum x_i(y_i-\alpha-\beta x_i) \quad = \quad 0 \tag{7.1.3}$$

Equivalently,

$$\bar{y}-\alpha-\beta\bar{x} \quad = \quad 0 \tag{7.1.4}$$

and

$$\sum x_i y_i - n\alpha\bar{x} - \beta\sum x_i^2 \quad = \quad 0 \quad . \tag{7.1.5}$$

The two linear homogeneous equations are known as the normal equations and their solution yields the least squares estimates of α and β.

For a model having P parameters in the DFF family, a spreadsheet (or a statistical package) sets up P linear homogeneous equations in order to solve for P unknowns. However, as a result of the non-orthogonality of the payment year direction with the other two directions, some of the equations in the normal equations are redundant, e.g.,

$$\alpha - \beta \quad = \quad 2$$

and

$$2\alpha - 2\beta \quad = \quad 4$$

So, there is no unique solution.

If there are some equations that are almost redundant, e.g.,

$$\alpha - \beta \quad = \quad 2$$

and

$$2\alpha - 2\beta \quad = \quad 4.00001,$$

then the estimates will have high standard errors, so that the resulting model will be unstable.

7.2 OVERCOMING MULTICOLLINEARITY

The phenomenon of multicollinearity associated with fixed parameter models can be interpreted in terms of information. There is not sufficient information in one loss development array to estimate many payment year parameters and accident year parameters (especially, for more recent accident years). Another interpretation is that the independent variables in the regression are not really independent. We showed in Section 3 that calendar year trends are related to development year trends and accident year trends.

If we include another α parameter for the last accident in our model we are using one single datum to estimate that parameter. That is, we assign full credibility to the last accident year's datum and zero credibility to previous years in respect of the estimation of the additional α parameter. A better approach may be to assign some credibility to the previous years data and less than full credibility to the last year's datum.

We are motivated to introduce exponential smoothing/varying parameter/credibility models, as a result of multicollinearity. Multicollinearity can lead to fixed parameter regression models that (i) are unstable and (ii) have large prediction errors.

The technique of exponential smoothing has received widespread use in the context of forecasting a time series. It originated more than 40 years ago without any reference to an underlying model that makes the technique optimal.

We first present heuristic arguments for exponential smoothing and varying parameter models. The following illustrations and arguments may be viewed from two different perspectives. The data may be regarded as either

(1) sales data over time, or
(2) incremental paid losses for delay 0 across accident years.

(i) Constant mean level (one parameter)

Suppose we have a sequence of time series observations $y_1, y_2, \ldots, y_n$ such that

$$y_t = \alpha + \varepsilon_t \quad , t = 1, \ldots, n$$

where α is a constant mean level and ε_t is a sequence of uncorellated errors with constant variance. Figure 7.2.1 below depicts such a series.

The model describing the data is the simplest regression model.

Our model has only one parameter. so that the years are completely homogeneous (stable!).

If α is known, the best forecast of a future observation $y_{(n)+1}$, based on information up to time n, is

$$\hat{y}_{(n)+1} = \alpha .$$

If the parameter a is unknown, we estimate it from the past data $(y_1 \ldots y_n)$ by its ordinary least squares estimate,

$$\hat{\alpha} = \Sigma y_i/n,$$

so that the one-step-ahead forecast of $y_{(n)+1}$ is now

$$\hat{y}_{(n)+1} = \bar{y} .$$

We can now write,

$$\hat{y}_{(n+1)+1} = \hat{y}_{(n)+1} + \frac{(y_{n+1} - y_{(n)+1})}{n+1}$$

The last equation indicates how a forecast from time origin n+1 can be expressed as a linear combination of the forecast from time origin n and the most recent observation. This is the simplest credibility formula, due to Gauss [8], used when updating sample averages. Since the mean level a is assumed constant, each observation contributes equally to the forecast.

The above formula for updating sample averages is an experience rating (credibility) formula in the context of adjusting a premium, assuming the risk (parameter) does not change from year to year.

In computing $\hat{\alpha}$ $(=\bar{y})$ we assign the same weight to each observation. From the loss reserving perspective, we are assuming that the accident years are completely homogeneous. In order to estimate the next years premium, we use all the accident years' data!

We now turn to another example.

(ii) Unstable mean level (each year its own parameter)

Here,

$$y_t = \alpha_t + \varepsilon_t$$

where the mean level α_t changes dramatically in successive time periods. Each year t has its own parameter α_t. Figure 7.2.2 depicts a series of y_t values that may be generated by this model.

Figure 7.2.2

Here, the best we could do, is forecast $y_{(n)+1}$ by

$$\hat{y}_{(n)+1} = y_n \;.$$

We are assigning zero weight to the past and full weight to the current observation. From the loss reserving perspective, accident years are completely heterogeneous, so that each accident year's individual parameter is estimated by that year's individual experience.

(iii) Locally constant mean level exponential smoothing and credibility

Often situations present themselves where the mean is approximately constant locally. Assigning equal weights to the past would be too restrictive and assigning zero weight would result in **loss of information**. It would be more reasonable to choose weights that decrease (geometrically) with the age of the observations.

We could have

$$\hat{y}_{(n)+1} = Ky_n + K(1-K)y_{n-1} + K(1-K)^2 y_{n-2} + \ldots$$

For n sufficiently large this may be written

$$\hat{y}_{(n)+1} = \hat{y}_{(n-1)+1} + K(y_n - \hat{y}_{(n-1)+1})$$

$$= (1-K)\,\hat{y}_{(n-1)+1} + Ky_n \,. \qquad (7.2.1)$$

This is also a credibility formula.

Muth [12] showed that the exponential smoothing formula (7.2.1) is an optimal forecast for the following model:

$$y_t = \alpha_t + \varepsilon_t : Var[\varepsilon_t] = \sigma_\varepsilon^2$$

$$\alpha_t = \alpha_{t-1} + \eta_t : Var[\eta_t] = \sigma_\eta^2 \qquad (7.2.2)$$

Here the mean level α_t process is a random walk. If $\sigma_\eta^2 = 0$, then we have the constant mean level situation (i) and if σ_η^2 is large we have the unstable mean level situation (ii). The parameter σ_η^2 should be chosen as small as possible at the same time ensuring that the trend in the data is captured.

Choosing σ_η^2 (relative to σ_ε^2) that minimises the SSPE yields the maximum likelihood estimates of σ_η^2.

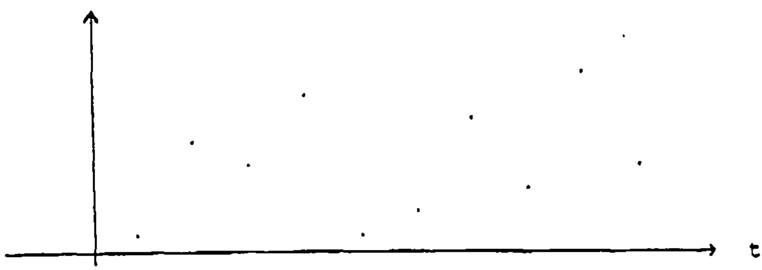


Figure 7.2.3

The exponential smoothing formula (7.2.1) formally credibility weights all the observations. It is an experience rating formula for a risk (parameter) that changes. If in the situation depicted in Figure 7.2.3, one were to assign zero weight to the past in place of using formula (7.2.1), then much information would be potentially lost.

We illustrate the methodology of formula (7.2.1) in the loss reserving context.

Suppose, for the sake of argument, there are only two accident years (but more than three development years), and the γ and ι parameters are zero.

We have,

$$y(1,d) = \alpha_1 + \varepsilon(1,d);\ d=0,1,2,\ldots,n_1\text{-}1\text{(say)} \tag{7.2 3}$$

and

$$y(2,d) = \alpha_2 + \varepsilon(2,d);\ d=0,1,2,\ldots,n_2\text{-}1\text{(say)} \tag{7.2.4}$$

The first accident year has n_1 observations and the second n_2 observations. Denote the sigma-squared assigned to observations by σ^2. Accordingly, Var[ε (1,d)] = Var[ε (2,d)] = σ^2.

The relation between α_2 and α_1 is given by

$$\alpha_2 = \alpha_1 + \eta : \text{Variance}(\eta) = \sigma_\eta^2 . \tag{7.2.5}$$

Substituting equation (4.4) for α_1 into (4.3) yields:

$$y(2,d) = \alpha_1 + \eta + \varepsilon(2,d) . \tag{7.2.6}$$

Combining the last equation with (4.2) we have,

$$y(1,d) = \alpha_1 + \varepsilon(1,d)$$

with (7.2.7)

$$y(2,d) = \alpha_1 + \eta + \varepsilon(2,d)$$

Since, conditional on α_1 the observations y(2,0), y(2,1), ... are <u>correlated</u>, we reduce by sufficiency to obtain:

$$\bar{y}_1 = \alpha_1 + \varepsilon_1$$

and

$$\bar{y}_2 = \alpha_1 + \varepsilon_2$$

where $$Var[\varepsilon_1] = \sigma^2/n_1 \quad , \quad Var[\varepsilon_2] = \sigma^2/n_2 + \sigma^2_\eta$$

and $$\bar{y}_1 = \sum_{d=0}^{n_1 - 1} y(1,d)/n_1 \quad , \quad \bar{y}_2 = \sum_{d=0}^{n_2 - 1} y(2,d)/n_2 \ .$$

The estimate of α_1 minimises the weighted error sum of squares

$$w_1(\bar{y}_1 - \alpha_2)^2 + w_2(\bar{y}_2 - \alpha_1)^2 \ ,$$

where

$$w_1^{-1} = Var[\varepsilon_1] = \sigma^2/n_1 \ ,$$

and

$$w_2^{-1} = Var[\varepsilon_2] = \sigma^2/n_2 - \sigma^2{}_\eta \ .$$

/

Similarly, the estimate of α_2 is obtained by minimising,

$$w_1(\bar{y}_2 - \alpha_2)^2 + w_2(\bar{y}_1 - \alpha_2)^2 \;,$$

where now $\qquad w_1^{-1} = \sigma^2/n_2 \qquad$ and $\qquad w_2^{-1} = \sigma^2/n_1 + \sigma_\eta^2$

The estimates of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are given by respectively,

$$\hat{\alpha}_1 = (1 - z_1)\bar{y}_2 + z_1\bar{y}_1$$

and

$$\hat{\alpha}_2 = (1 - z_2)\bar{y}_1 + z_2\bar{y}_2$$

where,

$$z_1 = \frac{\dfrac{n_1}{\sigma^2}}{\dfrac{n_1}{\sigma^2} + \dfrac{n_2}{\sigma^2 + n_2\,\sigma_\eta^2}} \;, \text{ and } \qquad z_2 = \frac{\dfrac{n_2}{\sigma^2}}{\dfrac{n_2}{\sigma^2} + \dfrac{n_1}{\sigma^2 + n_1\sigma_\eta^2}}$$

Both $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are credibility estimators.

The smaller σ_η^2 is (relative to σ^2), the more information is being pooled across the two years in estimating α_1 and α_2. We are credibility weighting the two years' data.

For a description of general recursive credibility formulae, see Zehnwirth [14].

We conclude this section by remarking that even in the absence of multicollinearity, varying parameter models are more stable and validate better than the 'corresponding' fixed parameter regression models. Moreover, according to A.C. Harvey's [9] modern book on forecasting, explanatory variables are "proxied by a stochastic trend".

8.0 PARAMETER ESTIMATION AND FORECASTING OF DISTRIBUTIONS

In the present section we describe how the (fixed parameter) regression models may be set up in a spreadsheet (or a statistical package) for the twofold purpose of estimating the model parameters and forecasting the distributions of future (incremental) payments.

A practical illustration of this procedure for the chain ladder statistical model is given by Christofies [4] in the second volume of the Institute of Actuaries Loss Reserving Manual [11].

8.1 ESTIMATION

In order to estimate a regression model in a spreadsheet we need to create, corresponding to each dependant observation y, the values of the (row) design vector containing the values of the independent variables.

Let y(w,d) = log p(w,d) and let β' be a row vector holding the parameters of the model, that is,

$$\underline{\beta}'=(\alpha_1,\alpha_2,\ldots,\alpha_k,\ \gamma_1,\ldots,\gamma_p\ \ \iota_1,\ldots,\iota_m)\quad .$$

The model has (i) k distinct α parameters where α_1 represents the level for accident years 1,2. ., w_1 (say); α_2 represents the level of accident years w_1 + 1, ... w_2 (say), and so on, (ii) l distinct γ parameters where γ_1 is trend along development years 0, 1, ... d_1; γ_2 is trend along development years d_1, $d_1 + 1$, ... d_2 and so on and (iii) m distinct iota parameters where ι_1 represents the trend along payment years 0,1,2,...;t_1; ι_2 represents the trend along payment years t_1, ..., t_2, and so on.

The arguments k, l and m may take the value 0.

The corresponding design vector is

$$\underline{x}'(w,d) = (\delta_{11}, \delta_{12}, .., \delta_{1k}, \quad \delta_{21}, ..., \delta_{2l}, \delta_{31}, ... \delta_{3m})$$

where each δ is a variable defined as follows

$$\delta_{1j} = 1 \text{ if } w_{j-1} + 1 \le w \le w_j \ (w_0 = 1)$$

$$= 0, \quad \text{otherwise} \ ;$$

$$\delta_{21} = 1$$

and $$\delta_{2j} = d - d_{j-1}, \text{ if } d \ge d_{j-1} + 1 \ (j \ge 2)$$

$$= 0 \ , \quad \text{otherwise} \ ;$$

and

$$\delta_{3j} = w + d - t_{j-1}, \text{ if } w + d \ge t_{j-1}$$

$$= 0 \ , \quad \text{otherwise.}$$

We now stack the y observations to form a column vector

$$\underline{y} = (y(1,0), \ldots y(1,s-1), y(2,1), \ldots, y(2,s-2), \ldots, \ldots y(s,0)) \quad .$$

and corresponding design vectors to form a design matrix,

$$X = (\underline{x}'(1,0), \ldots\ldots, \underline{x}'(s,0)) \quad .$$

The observation equation can now be written

$$\underline{y} = X\underline{\beta} + \varepsilon \quad ,$$

where ε contains independent errors from a normal distribution with mean zero and variance σ^2.

To estimate a DFF model in a spreadsheet, one needs to specify the column vector $\underline{y}$ and the columns of X as the independent variables.

The spreadsheet will create $\hat{\underline{\beta}}$, the ordinary least squares estimator of $\underline{\beta}$, and some other statistics including R^2, S^2 and standard errors of parameters.

The estimate of the variance - covariance matrix of $\hat{\underline{\beta}}$ is given by

$$V(\hat{\underline{\beta}}) = \hat{\sigma}^2(X'X)^{-1} \quad .$$

Some statistical packages such as MINITAB will produce the variance - covariance matrix as explicit output. Residuals and standardised residuals are straightforward to compute.

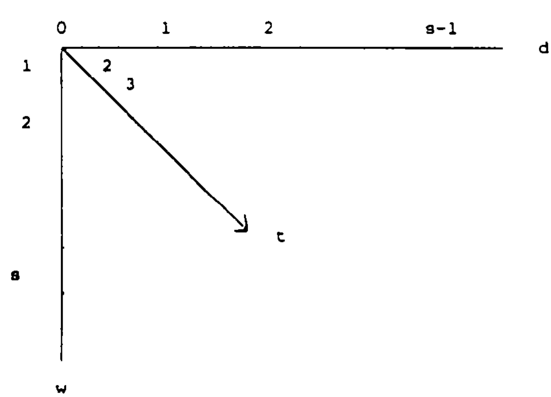
A lucid exposition of multivariate regression theory is given in Chatterjee and Price [3].

8.2 FORECASTING (PREDICTION) OF DISTRIBUTIONS

We have stressed repeatedly that a regression model is a probabilistic model and that the models contained in our rich DFF framework relate the normal distributions of the log payments of the cells in the loss development array by (trend) parameters.

We now would like to obtain estimates of normal distributions for payment years exceeding s.

That is, for calendar years beyond the evaluation year.



Consider a cell (w,d) for which w+d>s and d≤s-1.

Suppose we assume that the mean trend along payment years $\geq$ s is $\hat{\iota}_s$, the estimate of trend from payment year s-1 to s. (If ι_s is not a parameter in the model then $\hat{\iota}_s = 0$). We also assume that the standard deviation of the trend is se ($\hat{\iota}_s$), the standard deviation of the estimate. We stress emphatically that the larger se ($\hat{\iota}_s$) is, the mean trend $\hat{\iota}_s$ being the same, the larger the (mean) payments.

The vector of parameter estimates now contains the $\hat{\alpha}'$s, $\hat{\gamma}'$s but only one iota estimate, viz. $\hat{\iota}_s$.

The (design) independent value in the design vector $\underline{x}'(w,d)$ corresponding to $\hat{\iota}_s$ is now (w+d-s) = number of payment years from s to w+d. The other parameters contain the same design elements as in the estimation stage. The forecast $\hat{y}$ of y corresponding to cell (w,d) is given by:

$$\hat{y}(w,d) = \underline{x}'(w,d)\,\hat{\underline{\beta}} \quad .$$

We can now stack all forecasts $\hat{y}$ into a vector $\hat{\underline{Y}}$ and design vectors $\underline{x}'$ into a matrix X.

The estimate of the variance - covariance matrix of $\hat{\underline{Y}}$ is

$$V(\hat{\underline{Y}}) = X'\,V(\hat{\underline{\beta}})\,X + \hat{\sigma}^2 I \quad ,$$

where I is the identity matrix.

The quantity $\hat{\sigma}^2$ is the estimate of the process variance (uncertainty), whereas

$$X' \, V(\hat{\underset{\sim}{\beta}}) X$$

is a function of the variance of $\hat{\underset{\sim}{\beta}}$, representing the parameter uncertainty.

Since $V(\hat{\underset{\sim}{\beta}})$ is a function of $\hat{\sigma}^2$, the estimates of parameter uncertainty and process uncertainty are related. Quite often the smaller σ^2 is (relatively speaking), *the smaller the parameter uncertainty.*

Using Fisher's fiducial approach we can argue that our forecast for the distribution of y(w,d) is normal with mean $\hat{y}(w,d)$ and variance $V(\hat{y}(w,d))$, the diagonal element of $V(\hat{\underset{\sim}{y}})$ corresponding to y(w,d).

Indeed, $\underset{\sim}{y}$ has a multivariate normal distribution with mean $\hat{\underset{\sim}{y}}$ and variance covariance $V(\hat{\underset{\sim}{y}})$.

So, by applying standard regression theory we can compute our estimate of the multivariate normal distribution of the y values in the lower right of the rectangle.

Each estimate $\hat{y}$ of the corresponding y variable is best in the sense that it minimises the mean square error.

$$E[(y-f(\underset{\sim}{y}))^2],$$

over all statistics f(.), where f(.) is a function of the data $\underset{\sim}{y}$.

In order to obtain the distributions (multivariate) of the (incremental) payments and accident year and payment year sums, we employ the relationship between the multivariate lognormal and the multivariate normal distributions and standard statistical theory involving variances of sums. The means of the lognormal distributions are best estimates of the corresponding incremental payments.

We remark that our forecast distributions can also be argued for from a Bayesian viewpoint. The forecasts are Bayes with respect to a noninformative prior.

The reader will appreciate that to write a macro in a spreadsheet for a particular model in the modelling framework would be extremely prohibitive in terms of time. let alone writing a macro for each model

For readers that are interested, the author can make available a Lotus worksheet containing some of the models discussed in the real life study of Section 13.

9.0 MODELLING CONCEPTS

9.1 INTRODUCTION

The mechanisms by which claim severities, frequencies and delays are generated are invariably complex. When a model is constructed, it is not intended to be an accurate description of every aspect of the claims processes. The aim is to simplify the underlying processes in such a way that the essential features are brought out. According to Milton Friedman [7]: *'A hypothesis is important if it 'explains' much by little...'*. Similar views are expressed by Popper [13]: *'Simple statements... are to be prized more highly than less simple ones because they tell us more; because their empirical content is greater, and because they are better testable.'*

The "essential features" of the data in the loss reserving context are the trends and the random fluctuations about the trends. We decompose the data thus:

Log 'payments' = Trends + Random Fluctuations

Another way of thinking of this statistical model is to regard the Trends as a mathematical description of the main features of the data and the Random Fluctuations (or error or noise component) as all those characteristics not 'explained' by the Trends. All the complex mechanisms involved in generating the data are implicitly included in the model as creating the Trends plus the residual variance in the Random Fluctuations. See also Section 7 on varying parameter models.

The final identified model that 'explains' the data does not represent explicitly the underlying generating process. The model has probabilistic properties for which the data may be regarded as a sample (path) from it. Another classical modelling example in insurance where the same kind of modelling concepts are used is when we fit a Pareto distribution, say, to loss sizes. We do not assume that the Pareto distribution represents the underlying generating process. Whatever is driving the claims is very complex and depends on many factors. All we are saying is that our experience (sample) can be regarded as a random sample from the estimated Pareto distribution. The estimated Pareto distribution describes the variability in the loss sizes.

By way of summary, in order to take account of variables (or factors) not included in the Trends, we consider probabilistic models. See also Section 7 on varying parameter models.

There are a number of criteria for a good model with high predictive power:

* Ockham's Razor - parsimony;
* goodness of 'fit';
* validation and stability.

9.2 OCKHAM'S RAZOR - PARSIMONY

Ockham's razor, also known as the principle of parsimony, says that in a choice among competing models, other things being equal, the simplest is preferable. Accordingly, a parsimonious model that provides a description of the salient features of the data may be preferable to a complicated one for which the residual variance in the error is smaller (and so R-squared is larger). See also Section 10.4.

We stress R-squared (or adjusted R-squared) does not measure the predictive power of a model.

Consider two data generating models, Model 1 is,

$$y_t = \mu + \varepsilon_t \quad , \tag{9.2.1}$$

where $\varepsilon_t \sim N(o, \sigma^2)$ and the signal to noise ratio μ/σ^2 is large. Here, R-squared = 0 and since σ^2 is "small" predictions based on samples from this model will be relatively accurate.

For Model 2,

$$y_t = \alpha + \beta t + \varepsilon_t \quad , \tag{9.2.2}$$

where $\epsilon_t \sim N(0, \sigma^2)$. Suppose σ^2 is relatively large and R-squared is 85%. Predictions based on samples from this model will have larger errors than predictions in the first model. The forecasting errors are not a function of R-squared.

The consequences of adopting an inappropriate model will depend on its relationship to the 'true' model.

Underparametrisation - it imposes invalid constraints on the 'true' model.

Overparametrisation - the model is more general than is necessary.

Overparametrisation has different consequences to underparametrisation. Overparametrisation leads to high errors of prediction. The forecasts are extremely sensitive to the random component (in contrast to the trends) in the observations. Indeed, overfitting can be disastrous in certain circumstances. Overfitting a model is equivalent to including randomness as part of the (systematic) trend (component). Underparametrisation, on the other hand, tends to lead to bias rather than instability.

The dangers of overparametrisation are illustrated with a simple example. Imagine we have some yearly sales figures, as depicted below in Figure 9.2.1, and generated by

$$Y_t = 1 + 2t + 3t^2 + \varepsilon_t \quad ,$$

say, where the ε_t's are random from $N(0,\sigma^2)$, and Y_t represents the number of sales in year t.

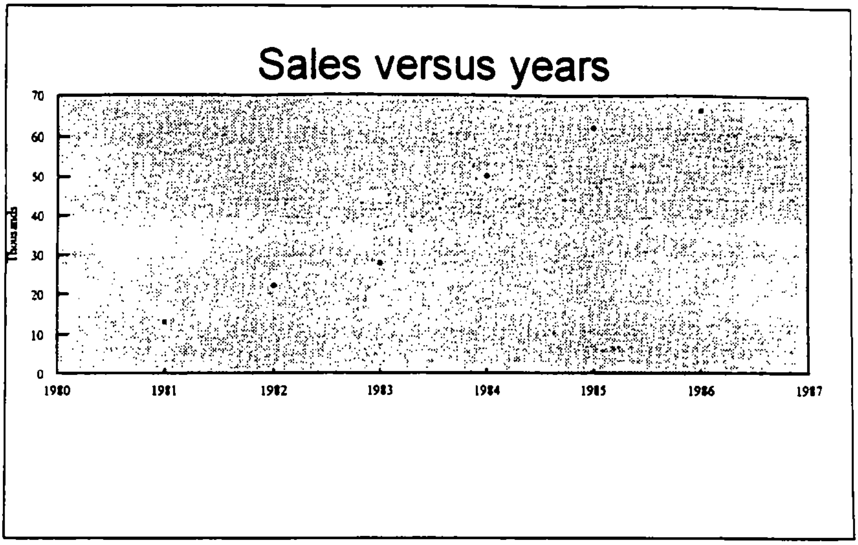


Figure 9.2.1

We wish to forecast sales for 1987. We could estimate a straight line model:

$$Y_t = \beta_0 + \beta_1 * t + \varepsilon_t \quad . \qquad (9.2.3)$$

This model produces residuals that are not random and is therefore rejected. The quadratic model,

$$Y_t = \beta_0 + \beta_1 * t + \beta_2 * t^2 + \varepsilon_t \quad , \qquad (9.2.4)$$

on the other hand, produces residuals that appear random. Moreover, R-squared is higher and parameters are significant.

We could try a fifth degree polynomial, viz.,

$$Y_t = \beta_0 + \beta_1 * t + \beta_2 * t^2 + \ldots + \beta_5 * t^5 + \varepsilon_t \quad . \qquad (9.2.5)$$

This model will produce zero residuals, that is, it will go through every data point and the R^2 = 100%. However, it is <u>useless</u> from the point of view of forecasting. Why? If we change only one data point marginally, the forecast will change to a very large degree. Moreover, if we use the model at year end 1986 to forecast sales for 1988, re-estimate the model at year end 1987 to update our forecast for 1988, the two forecasts would be completely different. The data are **NOT** unstable. **IT IS THE MODEL THAT IS UNSTABLE**. The model is incredibly sensitive to the random component in the data. It should only be sensitive to the systematic trend. Incidentally, standard techniques based on calculation of age-to-age link ratios suffer from the same defect.

9.3 AKAIKE INFORMATION CRITERION AND INFORMATION

It has been emphasised that in comparing the goodness of 'fit' of various models, an appropriate allowance should be made for parsimony. This has a good deal of appeal, especially where the model may be based primarily on pragmatic considerations.

Akaike Information Criterion (AIC) is both a function of S^2 and the number of parameters in the model. It is an information theoretic criterion that can be used for discriminating between any two models, even if they are non-nested. It originated with the work of Akaike.

In general the AIC is given by

$$AIC = -2\log(\text{likelihood}) + 2P$$

For DFF models it reduces to

$$AIC = N\log[2\prod S^2(MLE)] - N + 2P,$$

where

(i) N = Number of observations,

(ii) s^2 (MLE) is the maximum likelihood estimator of σ^2,

and (iii) P denotes the number of parameters.

The aim is to select a model with a minimum (relative) AIC. Note that the AIC can be used to discriminate between any two models, irrespective of whether they have any parameters in common.

9.4 RECURSIVE RESIDUALS AND SSPE

Consider a time series $z_1, z_2, \ldots, z_n$ where $\hat{z}_{t+1}(t)$ denotes a forecast of z_{t+1} based on the data $z_1, z_2, \ldots, z_t$. That is, the forecast is based on the information up to time t only. The one-step-ahead forecast (prediction) error is given by

$$\hat{\varepsilon}_t(1) = Z_{t+1} - \hat{Z}_{t+1}(t) \quad .$$

The notation $\hat{\varepsilon}_t(1)$ expresses the fact that it is the one-step-ahead prediction error that is calculated from past data up to and including time t. The estimates of the parameters of the model are only based on the data $Z_1, Z_2, \ldots, Z_t$.

In order to compute the errors $\{\hat{\varepsilon}_t(1)\}$ the model has to be estimated many times.

The sum of the squared one-step-ahead prediction errors, denoted by SSPE, is given by

$$\text{SSPE} = \sum_{t=t_0}^{n} \hat{\varepsilon}^2_t(1) \quad .$$

The time t_0 is chosen so that it exceeds the maximum number of parameters amongst the models being considered; by at least one.

Computation of the SSPE may take much time even with a good spreadsheet program, as the model has to be estimated for sub-samples, $\{Z_1,..., Z_t\}$; $t=t_0, t_0+1,...,n-1$.

Readers familiar with exponential smoothing will note that the optimal smoothing constant of exponential smoothing is determined by minimising the SSPE. See Abraham and Ledholter [1] for a lucid exposition of exponential smoothing.

By way of summary of the quality of 'fit' statistics, consider the quadratic polynomial example of Section 9.2, and suppose there are at least twenty data points. The relative magnitudes of R^2, AIC and SSPE as we fit polynomials of order one to six (say) are:

. R^2 increases with more parameters;

. AIC decreases from polynomial of order one to polynomial of order two, subsequently increasing as degree of polynomial increases (for most samples);

. SSPE behaves in much the same way as AIC.

Accordingly, a polynomial of degree exceeding two would have performed worse in a forecasting context than a polynomial of degree two, had we used them each year.

A relatively 'low' SSPE is preferable to a high SSPE. Naturally, there are other aspects of testing, including significance of parameters, distributional assumptions, residual displays and the number of parameters.

The 'tests' should be seen as complementary rather than competitive.

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9.5 OUTLIERS, SYMMETRIC DISTRIBUTIONS AND NORMALITY

Outliers are data points with large standardised residuals. Observations classified as outliers have residuals that are large relative to the residuals for the remainder of the observations.

Estimates of parameters and supporting summary statistics may be sensitive to outliers. Residual displays provide information on outliers. Moreover, if omission of outliers from the regression affects the output, then that provides more evidence that the omitted observations are in fact outliers.

An outlier may be a result of a coding error, in which case it should be assigned zero weight, or it may be a genuine observation that is unusual and accordingly has a large influence on the estimates, unless it is assigned reduced weight.

To detect outliers routinely, we need a rule of thumb that can be used to identify them. A **Box plot** is a schematic plot devised by J.W. Tukey. The following steps summarise the general procedure for constructing a Box plot.

. Order the data.

. Find the median (M), lower quartile (LQ), upper quartile (UQ) and mid-spread (MS), where MS = UQ - LQ. [1]

. Find the upper and lower boundaries defined by

LB = LQ - 1.5*MS

UB = LQ + 1.5*MS.

[1] Footnote: LQ and UQ are actually the lower and upper hinges. They are only approximately the quartiles.

List all outliers. An outlier is defined as any observation above the upper boundary or below the lower boundary.

Construct a Box plot as follows:

(a) Draw a horizontal scale;

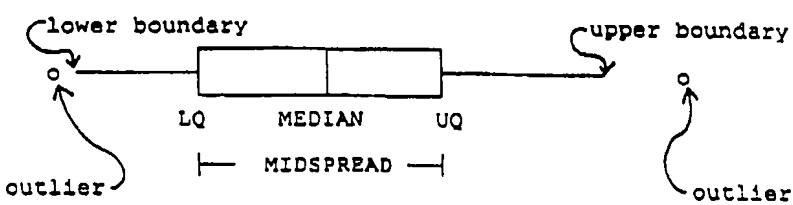
(b) Mark the position of the median using " | ";

(c) Draw a rectangular box around the median, with the right side of the box corresponding to the UQ and the left side corresponding to the LQ. The length of the box is equal to the MS. The median divides the box into two boxes;

(d) Find the largest and smallest observations between the boundaries and draw straight horizontal lines from the UQ to the largest observation below the upper boundary and from the LQ to the smallest observation above the lower boundary;

(e) Mark all observations (outliers) outside the boundaries with hollow circles (o). If an outlier is repeated, mark the number of times it is repeated.

Box Plot



We can also conclude (diagnostically) that a distribution is symmetric if the median is approximately half way between the LQ and the UQ.

A DFF model assumes that the weighted standardised residuals come from a normal distribution. Accordingly a normal probability plot should appear approximately linear. That is, the plot of weighted residuals against normal scores should have points that fall close to a straight line. This means that the correlation should be close to unity.

9.6 VALIDATION AND STABILITY

The important question is whether the estimated model can predict outside the sample. It is therefore important to retain a subset (the most recent one or two payment years) of observations for post-sample predictive testing. This post-sample prediction testing is called **VALIDATION**.

VALIDATION of the last payment year, or any payment year, is also related to the concept of **STABILITY**. If we don't use the last payment years' data to estimate the model, the ultimate losses should not differ from that obtained by using the last years' data by more than one standard error. We would like to identify a model that delivers **STABILITY** of reserves from year to year (only if trends are stable).

9.6.1 VALIDATION

Consider the triangle of incremental paid losses depicted below.

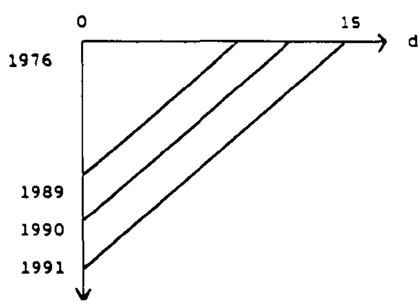


Figure 9.6.1.1

We have model that has been identified and estimated using all the data, up to 1991.

If the same model were estimated at year end 1988, would it predict accurately the incremental payments for 1989, 1990 and 1991? And what do we mean by 'predict accurately'?

Let's illustrate with a fair coin. If a fair coin is to be tossed 100 times we can 'predict accurately' the distribution of the number of heads. The exact distribution is Binomial (100, 0.5). The distribution details the probabilities of all the possible outcomes. If instead, we had a mutilated coin and we required a future prediction based on a sample data then our predicted distribution is Binomial (100, $\hat{p}$) where $\hat{p}$ is an optimal estimate of the true probability p of a head occurring, based on the sample.

We now return to our triangle. At year end 1988, we would estimate the parameters of the same model using the smaller sample and we would predict a distribution for each of the log 'payments' in 1989, 1990 and 1991. See Section 8.2 on forecasting of distributions.

So, one of the most important validation tests is to determine whether the observed log 'payments' in 1989, 1990 and 1991 can be regarded as a sample from the predicted distributions.

More specifically, let $\hat{y}$ be a prediction of a log 'payment' y for a cell in payment year 1989, 1990 or 1991. We call,

$$\tilde{\varepsilon} = y - \hat{y} \quad ,$$

the validated residual or the prediction error.

We test the validated residuals for (i) randomness in the three directions **delay**, **accident year** and **payment year**; (ii) randomness versus predicted values $\hat{y}$ and (iii) most importantly, normality.

9.6.2 STABILITY

Returning to our example of the foregoing section, we ask the question whether at year end 1988 our completion of the rectangle should be materially different to our completion at year end 1991. The answer is in the negative if trends (especially in the payment year direction) are stable.

We illustrate with four examples. (There are numerous others that occur in practice.)

Example 1: Suppose payment year trends (after adjusting for trends in the other two directions) are as depicted in Figure 9.6.2.1 below. The trend is stable and suppose its estimate is 10% $\pm$ 2%. How do we know that the trend is stable? Well, as we remove the more recent payment years from the estimation, the estimates of trends do not change (significantly). For example, after removing 1990 and 1991, the estimate of trend is 9.5% $\pm$ 2.1%, say. Alternatively, we could estimate a new trend parameter from 1989-1991 and examine whether the trend has changed significantly.

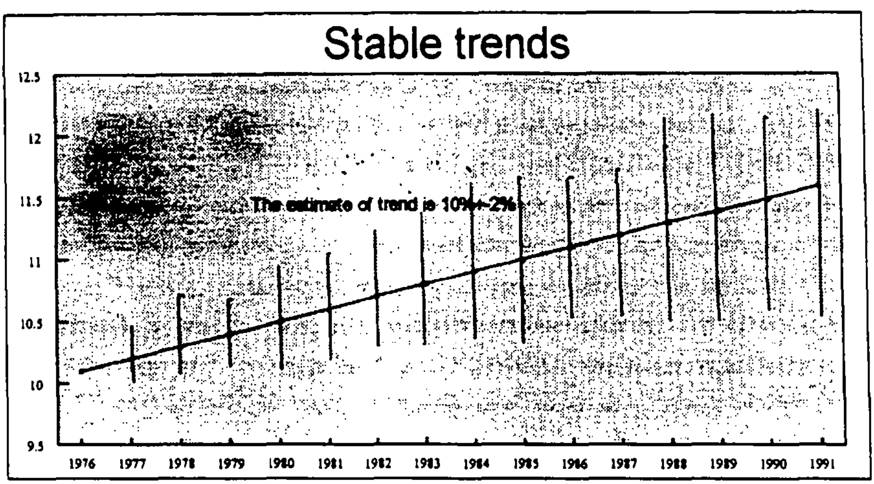


Figure 9.6.2.1

Typically, if the payment/calendar year trend is stable, the model will also validate well. Here our estimates of outstanding payments do not change significantly as we omit recent years.

Example 2: Consider the payment year trends depicted in Figure 9.6.2.2 below.

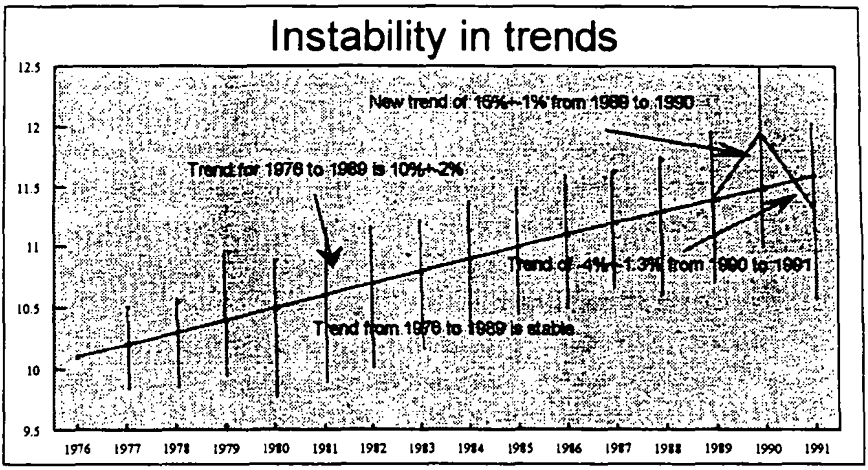


Figure 9.6.2.2

The trend in the years 1976 to 1989 is relatively stable. Its estimate is 10% $\pm$ 2%, say. However, the trend from 1989 to 1990 is higher at 15% ($\pm$ 1%) and from 1990 to 1991 it is -4%.($\pm$ 1.3%), say. This information is extracted from the "optimal" statistical model. The shifts in trends is a property of the data (determined through the model). A question now emerges as to which trend assumption do we make for the future, first in the absence of any other information. It would be foolhardy to assume the estimate between the last two years of -4% $\pm$ 1.3%. The most reasonable assumption (for the future) is a mean trend of 10% with a standard deviation of 2%, that which was estimated for the years 1976-1989.

Suppose we also have access to another data type, the number of closed claims development array. See Sections 10.2 and 10.3. We find utilising our DFF modelling framework that the additional 5% above the 10% between 1989 and 1990 can be explained by a corresponding increase in speed of closures of claims and the -15% from 1990 to 1991 below the 10% from 1976-1989 can be explained by a corresponding decrease in the speed of closures of claims. What assumption about *future trends* in payments should we adopt then? I would still recommend 10% ± 2% for the future. That's a decision based on my judgement and experience. The instability in trends in the last few years means that the model will not validate well. At year end 1990, we would not have forecast the distributions for 1991, for example.

Example 3: It is possible to have a transient change in trend. Consider Figure 9.6.2.3. The business has been moving along 10% ± 2% but between the last two calendar years 1990 and 1991 the trend increases to 20% ± 3%. What do we assume for the future? Well, that depends on the explanation for the increase in trend. Suppose its a "transient" change that can be explained by a new level of benefits that apply retrospectively. Then it is reasonable to assume 10% ± 2% for the future. Suppose instead that subsequent to analysis of claims closed triangle, the trend change is explained by increase in severities. That's a problem, because this means that it is now more likely that the new trend will continue.

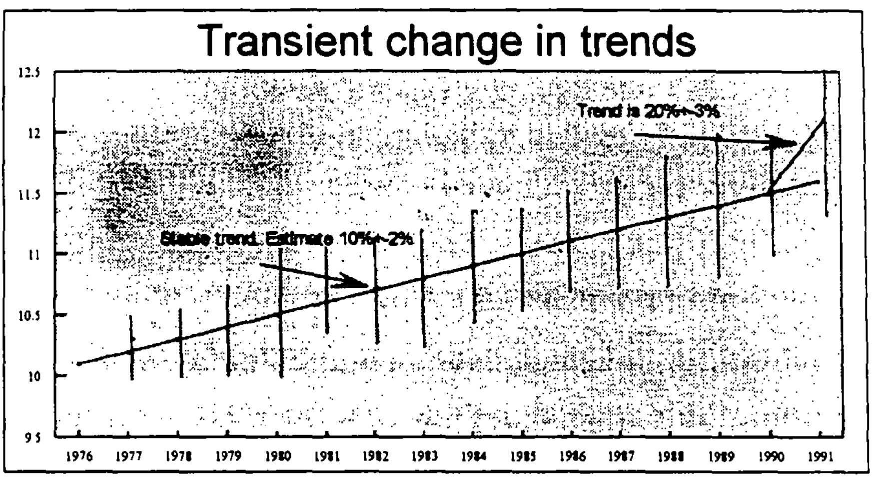


Figure 9.6.2.3

So the decision making process about the future becomes more complicated when trends are unstable. We are talking about trends in the (incremental) payments not age-to-age link ratios.

The last example illustrates an 'unpredictable' loss development array.

Example 4: The payment year trends are depicted in Figure 9.6.2.4 below.

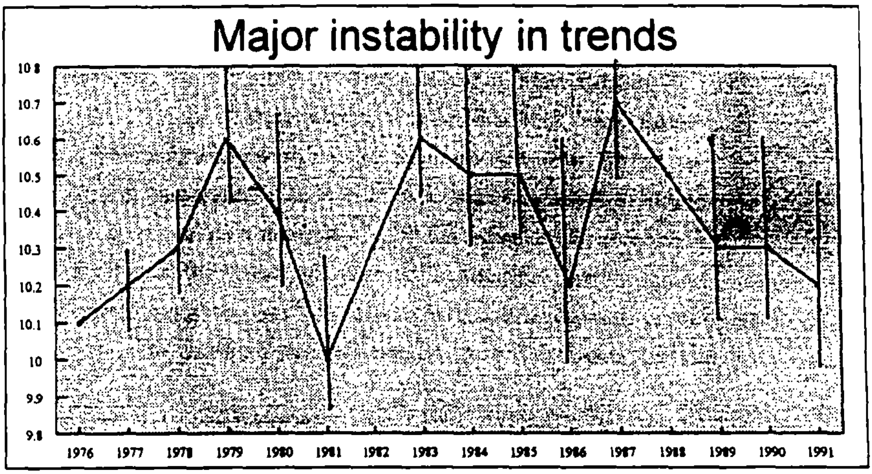


Figure 9.6.2.4

Note instability in trends. At year end 1989, would anyone be able to predict a flat trend for the next year and a downward trend for the following year?

Here, maybe, one could calculate a $\hat{\iota}$, a weighted average of trends estimated in the past with a weighted variance $\hat{\sigma}^2$ and assume for the future a mean trend of $\hat{\iota}$ with standard deviation of trend $\hat{\sigma}$. Since $\hat{\sigma}$ will be relatively large, mean forecasts will be well above the median forecasts and the standard deviation of the distributions relatively large. See Section 8.2

It is instructive to relate the foregoing discussion with the quote from A.C. Harvey [9] given at the beginning of Section 2.1.

9.7 POST-SAMPLE PREDICTIVE TESTING AND MODEL MAINTENANCE

Once a model has been identified for year end 1991, and assumptions about the future are made, the model is stored.

One year later, in 1992, on receipt of additional information (diagonal), there is no need to analyse the (augmented) triangle from the start. We already have a model for which we now conduct post-sample predictive testing and model updating and maintenance.

Has the model at year end 1991, predicted the distributions for 1992? This question is answered by restoring the model, assigning zero weight to "payments" in 1992 and validating the year. We also test for stability of parameters. If the model estimated at year end 1991 does not predict 1992 accurately, we know which parameter is the culprit and accordingly may have to amend the model (slightly).

For example, consider Example 2 of the preceding sub-section. If the 1992 data do not lie on the 10% $\pm$ 2% trend, then we have more evidence of changes in trends and our assumption of 10% $\pm$ 2% becomes pretty suspect.

Typically, once a model is identified for an incremental paid loss development array, the same model (with occasional minor amendments) is used in every subsequent year.

There is no way that a statistical method can automatically determine the "best" model and assumptions to be adopted for the future Rather, this decision is based on the model identification strategy (that may include analysis of other data types) and considerable judgment, especially if trends in the incremental payments are unstable.

Of course, any information about the nature of the business (especially change in business) may be critical in determining the assumptions for the future.

For example, in a number of loss development arrays of Lloyd's Syndicates analysed by the author, asbestos and pollution claims are not covered by policies written after 1978, say. This means that the calendar year effects of asbestos and pollution claims only apply to accident years prior to 1978. So, the iota estimates applying to accident years prior to 1978, do not apply to accident years post 1978.

For loss development arrays where the forecast uncertainties are relatively large, analysis of "similar" arrays within the company or analysis of industry wide arrays, for the purpose of formally credibility adjusting the parameter (estimates) could prove very useful. Incidentally, credibility is not just a function of volume. It is a myth that if claim numbers are "small" or incremental paids are small, or the triangle dimensions are small, then random fluctuations necessarily swamp the pattern (trends). The noise to signal ratio, equivalently, process uncertainty, may be very small even with small volume. Of course large volume and little process uncertainty does not mean that standard actuarial techniques will pick up the changing trend. See Section 12 for a study of a real life example involving (very) large volume and alarming calendar year shifts that cannot be detected using standard actuarial techniques.

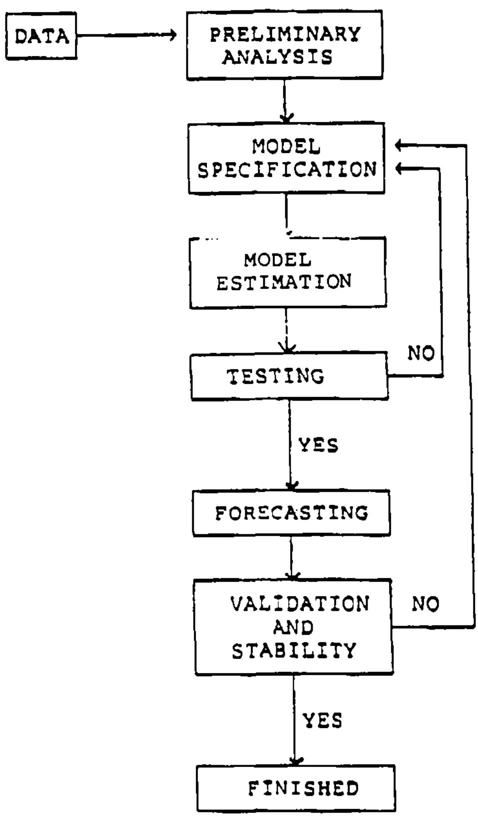
On every subsequent evaluation date post-sample predictive testing is conducted and the model is updated. Since data are recorded sequentially over time, updating procedures that can be applied routinely and that avoid re-analysis of the history are very desirable. See Section 9.6.2.

criterion is not satisfied, the model may have to be re-specified and the identification cycle repeated.

Step 6: Assumptions about the future based on Step 5 involving possibly analysis of other data types (Sections 10.2 and 10.3), are decided and forecasts and standard errors are produced. The final model is stored.

Step 7: Finished.

STEPS IN MODELLING



10. MODEL IDENTIFICATION AND ASSUMPTIONS ABOUT THE FUTURE

The aim is to identify a parsimonious model that separates the (systematic) trends from the random fluctuations and moreover determine whether the trend in the **payment/calendar year** direction is stable.

Recall that models contain information and accordingly the 'best' identified model *conveys information about the loss development array being analysed.*

For example, CCI (with constant development in the tail) indicates that the calendar year trend has been stable. This model should validate well and produce 'stable' outstanding estimates as recent calendar years are added or removed from the estimation. See preceding Sections 9.6.1 and 9.6.2.

10.1 MODEL IDENTIFICATION

The identification of the 'optimal' statistical model involves a number of iterative steps.

Step 1: Preliminary analysis facilitates the diagnostic identification of the heterogeneity in the data. The types of heterogeneity are also diagnostically identified.

Step 2: Based on step 1 a (preliminary) model is specified.

Step 3: The specified model is estimated.

Step 4: The model is checked to ensure that all assumptions contained in the model are satisfied by the data. If the model is inadequate, it has to be re-specified (step 2), and the iterative cycle of model specification - estimation - checking must be repeated.

Step 5: The best identified model is validated and tested for stability. If either

10.2 ASSUMPTIONS ABOUT THE FUTURE

We demonstrated in Section 9.6 that if payment/calendar year trend has been stable, especially in the more recent years, then the assumption about the future trend is relatively straightforward. For example, if the estimate in the last five years has been $\hat{\iota} \pm$ s.e. ($\hat{\iota}$), then we assume for the future a mean trend of $\hat{\iota}$ with a standard deviation of trend of s.e. ($\hat{\iota}$). We do not assume that trend in the future is constant. Our model does include the variability (uncertainty) in trend in the future.

If on the other hand, payment/calendar year trend has been unstable as is illustrated in examples 2 and 3 of Section 9.6, assumptions about future trends are not so obvious and may depend on analysis of other data types.

In Section 10.1 we also cited a practical example where special knowledge about the business is a contributory factor in making decisions about the future. But, that special knowledge is combined with what we found in the past experience.

10.3 OTHER DATA TYPES AND METHODS

Hitherto much emphasis has been placed on the importance of analysing and predicting distributions for (incremental) paid loss development arrays. Reasons given include:

- the geometry of trends;
- simplicity and parsimony;
- distributions of future payments is relevant information for financial statements.

We now discuss other data types and methods.

10.3.1 PAYMENTS PER CLAIM CLOSED

Let the "series" $\{p_t\}$ denote the payments loss development array and the series $\{n_t\}$ denote the closed claims development array.

We shall say that $\{n_t\}$ causes $\{p_t\}$, if taking account of past values of n_t leads to improved predictions of future values of p_t. (This is know as Granger causality.)

Typically, an actuary analyses $z_t = p_t/n_t$ and obtains predictions $\hat{z}_t$ of future values of z_t. The analysis of $\{n_t\}$ leads to predictions $\hat{n}_t$ of future values of n_t.

The future values of p_t are then predicted by $\hat{p}_t = \hat{n}_t \; \hat{z}_t$.

So, is the forecast $\hat{p}_t$ better than the forecast $\tilde{p}_t$ that only depends on past values of p_t. A forecast is better if its mean square error is less. That is, $\hat{p}_t$ is better than $\tilde{p}_t$ if

$$E[(\hat{p}_t - p_t)^2] < E[(\tilde{p}_t - p_t)^2] \quad .$$

The author believes that $\tilde{p}_t$ is better than $\hat{p}_t$. That is, there is no reduction in forecast error with respect to the given information set { $\hat{z}_t$, $\hat{n}_t$, $\hat{p}_t$ }. However, this does not rule out the possibility that when there is an instability in calendar year trends in $\{p_t\}$ as described in Section 9.6, analysis of $\{n_t\}$ will not lead to improved accuracy of predicting future values of $\{p_t\}$. The information extracted from the analysis of $\{n_t\}$ <u>may</u> improve the actuary's judgment in respect of which assumptions to use for future trends of p_t.

10.3.2 INCURRED LOSSES AND CASE RESERVES

Analysis of incurred losses (paid to date plus case reserves) does not provide information about what is still to be paid. We have given sufficient reasons why any analysis of cumulative data is unsound. And adding case reserves to cumulative paids reduces the information (not increases the information).

Incremental paid losses and case reserves should be analysed separately. That is the best way to determine the information contained in each data type and any relationships that may exist between the two data types.

For example, if there is a trend shift in the incremental paids between calendar years 1984 and 1985 and a corresponding shift in the case reserves one year later, between 1985 and 1986, then we know that the case reserves are lagging the payments.

If instead we found that case reserves are leading the payments then a change in trend in the case reserves between the last two calendar years, for example, may suggest an increase in trend in payments one year later (in the future). See Sections 10.1 and 10.2.

For a small dimensional triangle of a long tail line, case reserves for the early accident years will be helpful in determining the development year trend (γ) in the future.

There are ways of determining whether case reserves have been "accurate" in forecasting subsequent payments. See the paper by Fisher and Lange [6].

Perhaps we should also remark that case reserves vary between and within claims personnel and due to changing reserving philosophy of the company.

10.4 TIME SERIES MODELS VERSUS EXPLANATORY (OR CAUSAL) MODELS

The rich modelling framework advocated by the author contains essentially time series models. The only "causal" variable is time, equivalently payment year, accident year and development year. The past values of the incremental payments are used to forecast future values of the payments.

There is an alternative approach to forecasting in statistics called explanatory or causal models. These models make an attempt to discover the factors (or variables) affecting the behaviour of the claims process.

There are many reasons for preferring time series models to explanatory models.

- Causality based on the definition given in Section 10.3.1 is hard to prove, especially since the causal variables need to also be forecast.

- Simplicity and parsimony discussed in Sections 9.1 and 9.2.

- The claims process is complex and is unlikely to be understood and even if it were understood, it may be extremely difficult to determine the relationships that govern the behaviour of claims. Moreover, its likely the relationship changes with time. This last reason is part motivation for varying parameter models. (See Section 7).

- Explanatory models are difficult to validate and test for stability and when they don't work it may be hard to determine the reason.

By way of summary, we advocate the use of the DFF of models applied primarily to the incremental payments and applied to "related" data types, especially for the case in which calendar year trend instabilities are found in the incremental payments.

11.0 PREDICTION INTERVALS, RISK BASED CAPITAL AND RELATED ISSUES

11.1 INTRODUCTION

Loss reserves often constitute the largest single item in an insurer's balance sheet. An upward or downward 10% movement of loss reserves could change the whole financial picture of the company.

The current paper is not meant to focus on risk based capital and solvency issues, but mainly to stress that these are necessarily probabilistic concepts. The paper's principal intention is to show how the distributions (or variability) of loss reserves may be derived from sample data. It is the variability or uncertainty of loss reserves that is relevant to risk based capital and solvency considerations.

11.2 PREDICTION INTERVALS

We have given persuasive arguments for the use of probabilistic models, especially in assessing the variability or uncertainty inherent in loss reserves. The probability that the loss reserve, carried in the balance sheet, will be realised in the future, is necessarily zero, even if the loss reserve is the best estimate. See Sections 8.0 and 10.3 for definition of best.

Future (incremental) paids may be regarded as a sample path from the forecast (estimated) lognormal distributions. The estimated distributions include both process risk and parameter risk.

The forecast distributions are accurate provided the assumptions made about the future will remain true. For example, if it is assumed that future calendar year trend (inflation) has a mean of 10% and a standard deviation of 2%, and in two years time it turns out that inflation is 20%, then the forecast distributions are far from accurate.

Accordingly, any prediction interval computed from the forecast distributions is conditional on the assumptions about the future remaining true.

Suppose $\hat{p}$ is a mean of a forecast lognormal distribution corresponding to payment p. Both $\hat{p}$ and p are random variables.

Let u = log p, μ = E [u] and σ^2 = Var [u]. A 100 (1-α)% prediction interval for u (a random variable) is given by

$$\mu \pm \sigma Z\ (\alpha/2) \quad ,$$

where Z (α/2) is the 1-α/2 percentage point of the standard unit normal distribution.

A 100 (1-α)% prediction interval for p (=log u) is

$$\exp\ [\mu \pm \sigma\ Z\ (\alpha/2)\ ].$$

The latter interval is non-symmetric about $\hat{p}$ since the lognormal distribution is skewed (to the right). The parameters μ and σ are computed from the mean and standard deviation of p, and the relationship between the lognormal and normal distributions.

The limits of the interval can be interpreted as follows. Suppose repeated samples of the rectangle are taken (from the estimated probabilistic model), then the proportion of times the observed p value will lie in the observed interval (in the long run) is 1-α. Bear in mind that p is a random variable.

The distribution of sums, for example, accident year outstanding payments, is the distribution of a sum of lognormal variables that are correlated. The exact distribution of the sum can be obtained by generating (simulating) samples from the estimated multivariate lognormal distributions. Alternatively, one can approximate the distribution of the sum by a lognormal. Indeed, the lognormal would be the riskiest.

If there are 'many' components in the sum, then the Central Limit Theorem could be invoked, especially if the lognormal distributions of the paids are not terribly skewed. See Section 13 for a real life example.

Insurer's risk can be defined in many different ways. Most definitions are related to the standard deviation of the risk, in particular a multiple of the standard deviation.

If an insurer writes more than one long tail line and aims for a $100(1-\alpha)\%$ security level on all the lines combined, then the risk margin per line decreases the more lines the company writes. This is always true, even if there exists some dependence (correlation) between the various lines.

Consider a company that writes n independent long tail lines. Suppose that the standard error of loss reserve L(j) of line j is se(j). That is, se(j) is the standard error of the loss reserve variable L(j). The standard error for the combined lines L(1)+...+L(n) is

$$se(Total) = [se^2(1)+\ldots+se^2(n)]^{0.5}$$

If the risk margin for all lines combined is k*se(Total), where k is determined by the level of security required, then the risk margin for line j is

$$k*se(Total)*se(j)/[se(1)+\ldots+se(n)]$$

$$<kse(j).$$

The last inequality is true even when se(Total) is not given by the above expression.

If as a result of analysing each line using the DFF modelling framework we find that for some lines trends change in same years and the changes are of the same order of magnitudes, then the paid losses are not independent. (There may also be some probabilistic model, derived from the company's experience, that describes the particular line for that company. In the hundreds of arrays that the author has analysed, no one model described more than one loss development array.

The approach the author is advocating allows the actuary to determine the relationships within and between companies experiences and their relationships to the industry in terms of simple well understood features of the data.

In establishing the loss reserve, recognition is often given to the time value of money by discounting. The absence of discounting implies that the (median) estimate contains an implicit risk margin. But this implicit margin may bear no relationship to the security margin sought. The risk should be computed before discounting (at a zero rate of return).

correlations between the residuals).

In that situation, line i and j are correlated, say, then one should use se(i)+se(j) as the upper bound of the standard error of L(i)+L(j).

We now return to an important modelling concept or 'law of payments'.

Suppose we assume for the future payment/calendar years a <u>mean</u> trend of ($\hat{\iota}$) with a standard deviation (standard error) se ($\hat{\iota}$). Specifically we are saying that the trend ι, a random variable, has a normal distribution with mean $\hat{\iota}$ and standard deviation se ($\hat{\iota}$). Recognition of the relationship between the lognormal and normal distributions tells us that the mean payment increases as se ($\hat{\iota}$) increases (and $\hat{\iota}$ remains constant). The greater the uncertainty in a parameter (the mean remaining constant), the more money is paid out.

The foregoing arguments apply to each parameter in the model.

11.3 RISK BASED CAPITAL

The author understands that the NAIC is drafting regulations where part of the risk based capital requirements will be based on loss reserves. In the article by Laurenzano [10], page 50, the loss reserve component of the risk based capital formula "*selects the worst reserve development ...*".

The approach advocated by the NAIC is flawed for many reasons including:

* The uncertainty in loss reserves (for the future) should be based on a probabilistic model (for the future) that may bear no relationship to reserves <u>carried</u> in the past;

* The uncertainty for each line for each company should be based on a

12.0 ANALYSIS OF PROJECT 1

12.1 INTRODUCTION AND SUMMARY

The principal objectives of the analysis of real life data in this section are to demonstrate that:

1. Age-to-age link ratios based on the cumulative paid losses give no indication about the trends and random fluctuations in the (incremental) payments.

2. Smooth data may have major shifts in calendar year trends.

3. Regression as an approach to adjusting data and determining trends and changes thereof is very powerful.

4. Large company's run-off payments are not necessarily stable in respect of calendar year trends, even though the payments may be extremely smooth (with very little random fluctuations about the trends).

12.2 DATA AND AGE-TO-AGE LINK RATIOS

The data (save a multiplicate factor in order to preserve confidentiality) come from a large insurer and are given in Appendix C1. Accident year exposures, (from memory), represent earned premium (relativities). As we shall see in the next section, the exposures are not that important.

The age-to-age link ratios presented in Appendix C2 are relatively smooth. For the early development years they tend to decrease slightly in the middle accident years and then increase in the latter payment years.

12.3 ANALYSES

We define a normalised payment as the (incremental) paid divided by the corresponding accident year exposure and apply the MODELLING FRAMEWORK to the normalised payments.

If p(w,d) is the incremental payment corresponding to accident year w and development year d, and e(w) is the accident year exposure, then the normalised payment is p(w,d)/e(w) and we define,

$$y(w,d) \quad = \quad \log\,[p(w,d)/e(w)]$$

Figure C3 (in Appendix C3) represents a graph of the normalised payments versus delay for the first two accident years in the triangle. Observe that the run-off development for both years is remarkably smooth.

The chain ladder (CL) statistical model is given by,

$$y(w,d) \quad = \quad \alpha_w + \sum_{j=1}^{d} \gamma_j + \varepsilon$$

Since the exposures e(w) are absorbed into the parameters α_w, the estimates of the development trends γ_j do not depend on the exposure base used. Indeed, there are other statistics that are invariant (for CL) with respect to exposure base including, AIC, residuals, S-squared, normality testing and forecasts. The chain ladder model adjusts for the different levels (α's) of each accident year.

The estimates of the CL parameters and associated regression table are presented in Appendix C4. R-squared is high and S-squared is small. Hence, the random fluctuations are small. Now, the CL model adjusts the data for development year trends and accident year trends (or levels). Many parameter differences are insignificant but that is not important since we are not trying to identify a parsimonious model here but rather show how some of the models in the FRAMEWORK may be used for fast identification of payment/calendar year trend changes.

So, the residuals represent the data adjusted (after removing) for the average development year trends and the average accident year trends.

Residuals versus development years (Figure C5.1) and accident years (Figure C5.2) are the "best" we can obtain since we have removed the trends in these two directions. In Figure C5.1, the sum of residuals for any one development year is zero and in Figure C5.2, the sum of residuals for any one accident year is zero. HOWEVER, residuals versus payment years (Figure C5.3) exhibit a very strong V shape AND THIS IS FOR SMOOTH DATA OF A LARGE COMPANY. So, after removing accident year and development year trends from the data we observe major shifts in calendar year trends. (Compare this with the simulated data of Sections 4.4 and 5). There appears to be a change in trend in 1984 and definitely a change in trend in 1985.

We now estimate the CC model. It adjusts the data for the average development year trends. Appendix C6 presents the regression output and Figure C7 is a graph of residuals versus payment years that indicates an upward trend (positive inflation). It is hard to tell from this graph whether there is a major shift in trends.

In order to estimate a trend parameter through the residuals of Figure C7, we estimate the CCI model to the data. The regression output is presented in Appendix C8 and the residuals versus payment years are displayed in Figure C9. The average payment year trend is 12.1% ($\pm$ 0.53%). The V shape in residuals is distinct, suggesting very strongly the change in trends.

Our final model introduces another two payment year trend parameters. One from

1984-1985 and one from 1985-1987. The regression output is given in Appendix C10. Note shift in trend from 9.85% to 19.52%. This is quite alarming, especially if it cannot be explained by an increase in speed of finalisations of claims. See Section 10.2, for a discussion of assumptions to be applied for the future.

We now graph in Appendix C11 the lognormalised payments versus delay for the first two accident years. Since 19.52% is much higher than 9.85%, observe that the trend in the tail increases for both accident years, and for accident year 1978 the change is one development year earlier than in accident year 1977. That is because the trend change is a calender year change.

So there is overwhelming statistical evidence of a major shift in calendar year trends in the last two calendar years. What assumptions do we make about the future trends? We could analyse the number of claims closed development array and determine whether the substantial increase in trend in the payments is due to a corresponding increase in trend in the number of closed claims. If the answer is in the negative, then the trend increase must be due to increase in severities which would then be a major concern for the company. See Section 10.2.

In this section we have not identified a parsimonious model for the data. Instead the objective was to demonstrate how some of the models in the MODELLING FRAMEWORK may be used for quick determination of major calendar year shifts (in data that are relatively smooth and do not appear problematic if we are to employ the standard actuarial approaches based on link ratios).

The reader will appreciate that our modelling approach is interactive and terribly computer intensive. In order to identify the calendar year trend changes we have had to estimate four models. To set up each model in a spreadsheet is extremely time consuming. See Section 8.

13.0 ANALYSIS OF PROJECT 2

13.1 INTRODUCTION AND SUMMARY

In the present section we analyse a real life loss development array for which the age-to-age link ratios of the cumulative paids are relatively unstable, yet the trends in the paids are stable.

The "best" identified model is essentially a version of CC with two additional iotas (payment year trend parameters) that are used to adjust for "low" payments in one payment year. The model (and so the trend in the data) is stable and validates very well. Had the model been employed three years earlier, it would have yielded the "same" outstanding payments and would have forecast the distributions of (incremental) payments for the last three years extremely accurately.

13.2 DATA AND PRELIMINARY ANALYSIS

The incremental paid loss development array and accident year exposures are displayed in Appendix D1. The exposures are estimates of the number of ultimate claims incurred in each accident year. We define a normalised payment as the paid divided by the corresponding accident year exposure and identify a DFF model for the normalised payments.

The first step in the preliminary (diagnostic) analysis is to graph the data. Figure D2.1 displays a graph of normalised payments versus development year for all accident years combined. It exhibits a band whose width (variability) increases as the normalised payments get larger.

On the other hand, the graph of the lognormalised payments depicted in Figure D2.2 exhibits a band whose width is relatively constant. That is, % variability is constant with development year suggesting a lognormal distributions for the normalised payments.

The graph in Figure D2.2 also gives us a preliminary idea of a parsimonious number of γ's (development year trend parameters) that may be required in the model.

It appears we require one γ from delay 0-1, one from delay 1-2 (that turns out to be insignificant to zero), one from delay 2-4 and one from delay 4-8.

13.3 MODEL IDENTIFICATION

In this sub-section we implement the model selection strategy discussed in Section 10.

Model 0 and 1: Estimate a CC model with the four γ parameters suggested by the preliminary diagnostic analysis. It turns out that the parameter γ_2 is insignificant from zero, as was anticipated from the graphs. Set γ_2 to zero and re-estimate the model. Regression tables and residual displays are given in Appendix D3 and Appendix D4, respectively.

Residuals versus delay and accident years suggest that the trends in these two directions have been captured well. This diagnostic test can be formalised by adding more parameters and testing for significance of parameters and their differences.

Since we have estimated a CC model, the residuals may be interpreted as the data adjusted for the development year trends.

Residuals versus payment years (Figure D4.3) suggest (i) zero trend from 1975-1979, (ii) low payments in 1974 and (iii) perhaps zero trend from 1969-1973. So we next estimate.

Model 2: This model is the previous CC model with four iota parameters. The first iota represents the trend from 1969-1973, the second iota the trend from 1973-1974 the third iota represents the trend from 1974-1975 and the fourth iota represents the trend from 1975-1979. We find that both the first and fourth iota are insignificant, and the first being less significant than the fourth.

Model 3: Previous model with first iota set to zero. We find that fourth iota is still insignificant.

Model 4: *Previous model with fourth iota set to zero. We find all parameters and* their differences are significant. Moreover, SSPE and AIC are the lowest amongst the four models. Outlier analysis indicates that the observation in accident year 1972, delay 7 is an outlier.

So our final identified model (before conducting validation and stability analysis) has three gammas (0-1, 2-4 and 4-8), two iotas (1973-1974 and 1974-1975) and one alpha, and it also assigns zero weight to (1972,7).

The regression tables and various statistical displays are given in Appendices D5 to D7.

Figure D7.5 of Appendix D7 displays a normal probability plot where r^2 (correlation squared) between the normal scores and ordered residuals is 0.993. The P-value is in excess of 0.5.

So we have shown that the log incremental payments in the cells of the loss development array can be regarded as observations from normal distributions whose means are related by the (trend) parameters given in Appendix D5.

Forecasts, standard errors and % errors based on the model are presented in Appendices D8 and D9, respectively.

Appendix D8

This appendix presents:

(i) each observed payment (OBS);

(ii) each expected model payment (EXP), that is a mean of a lognormal distribution;

(iii) forecasts for each accident year subdivided according to development year (right side of stair-case corresponding to EXP row);

(iv) standard errors of each individual forecast (below each forecast);

(v) total forecast (outstanding) for each accident year and associated standard error (right hand column);

(vi) total forecast (payment) to be made in each future payment year in respect of all the accident years and associated standard errors (bottom row). This is the future liability stream with corresponding uncertainties that may prove useful for asset/liability matching;

(vii) total outstanding with associated standard error (bottom right hand corner).

Expected values and forecasts are estimates of means of lognormal distributions. Standard errors are estimates of standard deviations of lognormal distributions.

Appendix D9

Here we present a quality of fit table comparing the original observed payments with the model expected payments. For each accident year and for each payment year we compute the ratio of the difference in total observed and total expected to the total observed. The quality of fit is high.

13.4 VALIDATION AND STABILITY ANALYSIS

We now re-estimate the same model and assign zero weight to the last three calender years (1979, 1978 and 1977). We aim to determine (i) whether the model estimated at year end 1976, would have forecast the distributions of payments in years 1977-1979 and (ii) are the parameter estimates of the model and the forecasts based on the model stable.

Appendix D10 presents the parameter estimates as of year end 1976. Compare these estimates with those obtained at year end 1979 (Appendix D5). Note that none of the parameter estimates have changed significantly. The estimate of the tail. -0.5544 ($\pm$ 0.0753) at year end 1976, is slightly higher than the estimate -0.6749 ($\pm$ 0.0390), at year end 1979, hence the higher forecasts in the tail. The estimates of iotas 1973-1974 and 1974-1975 are very close (and so stable).

Appendix D11 represents "All" residuals displays. All residuals include those corresponding to observations used in the estimation (1969-1976), and the validated residuals (1977-1979) corresponding to observations not included in the estimation. All displays are great.

In particular, Figure D11.3 shows the distribution of the validated residuals (prediction errors) for 1977-1979 relative to residuals corresponding to years used in the estimation.

Appendix D12 presents displays of the validated residuals (only those corresponding to years 1977-1979). All displays are in good shape.

Most importantly, Appendix D12.4 presents a test whether the lognormalised payments in 1977-1979 come from the forecasted distributions as at year end 1976. The squared correlation between normal scores and validated residuals is 0.959 with a P-value of 0.313.

By way of summary, there is very strong statistical evidence that the model at year end 1976 would have predicted accurately the <u>distributions</u> of 'payments' for 1977-1979.

Let's now compare the forecasts, Appendix D13 (validation model) with Appendix D8.

Total outstanding beyond 1979, based on estimated model at year end 1976 is 12,620,833 $\pm$ 1,072,089 compared with estimated model at year end 1979 of 12,948,473 $\pm$ 1,030,808. No difference.

So, we could have obtained the same answers three years ago (that is, without the last three years information). All other forecasts compare extremely favourably.

Note that in Appendix D13 the expected values corresponding to payment years 1977-1979 actually represent mean forecasts based on estimated model at year end 1976.

From Appendix D14 we see that had we reserved mean forecasts at year end 1976 (for years 1977-1979) we would have underforecast 1977 and 1978 by 13% and 1% respectively, and overforecast 1979 by 5%.

Our findings using probabilistic models have shown that:

* calendar year trends are essentially stable, save the dip in the year 1974;
* the model used three years earlier would have predicted accurately the distributions of payments for the last three years;

and

* rough (irregular) age-to-age link ratios, especially in the early development years, give no indication of stability of trends.

The author has analysed numerous data sets with rough (or irregular) age-to-age link ratios for which the payment/calendar year trends are stable. Conversely, smooth age-to-age link ratios does not mean stability of trends.

We conclude this section by showing how to compute a prediction interval for the total outstanding payments, using the discussion of Section 11.2.

From Appendix D8, the mean outstanding is given by

$$m \quad = \quad \text{mean} = 12{,}948{,}473$$

and the standard deviation (or standard error) by

$$sd = 1,030,808$$

We assume that the total reserve (or liability) L is a random variable with mean m and standard deviation sd and moreover the distribution of L is lognormal.

Put y = log L, then y has a normal distribution with mean μ and standard deviation σ, say.

Therefore,

$$m = \exp [\mu + 0.5 \sigma^2]$$

and

$$sd = m [\exp (\sigma^2) -1]^{0.5}$$

Solving the last two equations for μ and σ we obtain,

$$\mu = 16.37332$$

and

$$\sigma = 0.079482$$

Employing Section 11.2, a 100 $(1-\alpha)$% prediction interval for the random variable L is given by

$$\exp [16.37648 \pm 0.079482 Z (\alpha/2)]$$

where Z (α /2) is the 1 - α /2 percentage point of the standard unit normal distribution

The median of the distribution of L is exp [μ] = 12,907,636 which is very close to the mean of 12,948,473. Since σ^2 is small the lognormal distribution is not terribly skewed, so that were we to assume that the distribution of L is normal (rather than lognormal), the prediction intervals would be almost the same.

14. EXTENSION OF THE DFF MODELLING FRAMEWORK

We observed that a fruitful extension of the DFF modelling framework was the introduction of varying parameter (dynamic) models in Section 7.

Another important extension is related to the distributional assumption of normality. Hitherto, we have assumed that the variances of the y values, denoted by σ^2, are identical (constant)

The variance on a log scale can be interpreted as % variability. So constant σ^2 implies constant % variability. For many loss development arrays this assumption is not valid. For some arrays, % variability increases in the tail, for some others, % variability is higher in the early development years. When σ^2 is not constant and varies with development years we need to also model the σ^2's. That is, we introduce a secondary equation.

This is outside the scope of the present paper.

15. CONCLUSIONS

We have argued that the four components of interest regarding a loss development array are the trends in the three directions and the distributions (random fluctuations) about the trends.

A MODELLING FRAMEWORK was introduced where each model contained therein possesses the four components of interest. The modelling approach offers the actuary a way of fitting (estimating) distributions to the cells in a loss development array and predicting (forecasting) distributions for future years that affords numerous advantages including:

- simplicity;
- clarity of assumptions;
- testing of assumptions;
- assessment of loss reserve variability;
- asset/liability matching;
- model maintenance and updating.

We showed how the identified optimal statistical model for the (incremental) payments conveys information about the loss experience to date. In applying the model to predicting distributions of future payments the actuary may (need to) adjust some of the parameters to reflect knowledge about the business and to incorporate his view of the future. View of the future may be based on analysis of other data types, especially if there are instabilities in the payments in the recent calendar years.

A prediction interval computed from the forecast distributions is conditional on the assumptions made about the future remaining true.

In passing we have debunked a number of pervasive loss reserving perceptions concerning data types, age-to-age link ratios, stability, forecasting and regression.

Methods based on age-to-age link ratios do not (and cannot) separate trends from random fluctuations and moreover do not satisfy the basic fundamental property of additivity of trends.

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Appendix A1

Model is p = exp(alpha-.2d) with no error or randomness
alpha = 11.51293

Year/delay

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1978	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	7427
1979	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	
1980	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080		
1981	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534			
1982	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530				
1983	100000	81873	67032	54881	44933	36788	30119	24660	20190					
1984	100000	81873	67032	54881	44933	36788	30119	24660						
1985	100000	81873	67032	54881	44933	36788	30119							
1986	100000	81873	67032	54881	44933	36788								
1987	100000	81873	67032	54881	44933									
1988	100000	81873	67032	54881										
1989	100000	81873	67032											
1990	100000	81873												
1991	100000													

Appendix A2

y=log(p) plus .1 inf. from 1978-82, .3 inf. from 1982-83 and .15 inf. from 1983-91

Year\delay

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1978	11.5129	11.4129	11.3129	11.2129	11.1129	11.2129	11.1629	11.1129	11.0629	11.0129	10.9629	10.9129	10.8629	10.8129
1979	11.6129	11.5129	11.4129	11.3129	11.4129	11.3629	11.3129	11.2629	11.2129	11.1629	11.1129	11.0629	11.0129	
1980	11.7129	11.6129	11.5129	11.6129	11.5629	11.5129	11.4629	11.4129	11.3629	11.3129	11.2629	11.2129		
1981	11.8129	11.7129	11.8129	11.7629	11.7129	11.6629	11.6129	11.5629	11.5129	11.4629	11.4129			
1982	11.9129	12.0129	11.9629	11.9129	11.8629	11.8129	11.7629	11.7129	11.6629	11.6129				
1983	12.2129	12.1629	12.1129	12.0629	12.0129	11.9629	11.9129	11.8629	11.8129					
1984	12.3629	12.3129	12.2629	12.2129	12.1629	12.1129	12.0629	12.0129						
1985	12.5129	12.4629	12.4129	12.3629	12.3129	12.2629	12.2129							
1986	12.6629	12.6129	12.5629	12.5129	12.4629	12.4129								
1987	12.8129	12.7629	12.7129	12.6629	12.6129									
1988	12.9629	12.9129	12.8629	12.8129										
1989	13.1129	13.0629	13.0129											
1990	13.2629	13.2129												
1991	13.4129													

Appendix A3

Cumulative data (on a $ scale) derived from Appendix A2

100000	190484	272357	346439	413471	487552	558021	625053	688816	749469	807164	862045	914250	963908
110517	210517	301001	382874	473358	559428	641302	719182	793263	863732	930764	994527	055180	
122140	232657	332657	443174	548302	648302	743425	833908	919979	001852	1079732	1153814		
134986	257126	392112	520515	642655	758838	869355	974482	1074482	1169605	1260089			
149182	314055	470886	620068	761975	896961	1025363	1147504	1263687	1374204				
201375	392929	575141	748467	913339	1070170	1219352	1361259	1496245					
233965	456519	668219	869594	1061148	1243360	1416685	1581557						
271828	530399	776359	1010324	1232878	1444578	1645954							
315819	616236	902001	1173829	1432400	1678360								
366930	715964	1047976	1363795	1664212									
426311	831831	1217574	1584504										
495303	966450	1414619											
575460	1122855												
668589													

Appendix A4

Age-to- age link ratios of the cumulative losses of Appendix A3

1.904837	1.429816	1.272002	1.193488	1.179170	1.144535	1.120124	1.102011	1.088054	1.076981	1.067992	1.060558	1.054316
1.904837	1.429816	1.272002	1.236327	1.181830	1.146351	1.121440	1.103008	1.088834	1.077607	1.068505	1.060986	
1.904837	1.429816	1.332224	1.237213	1.182381	1.146726	1.121712	1.103213	1.088994	1.077736	1.068611		
1.904837	1.524979	1.327463	1.234652	1.180786	1.145639	1.120925	1.102618	1.088529	1.077362			
2.105170	1.499375	1.316812	1.228856	1.177152	1.143152	1.119119	1.101248	1.087456				
1.951229	1.463726	1.301361	1.220279	1.171712	1.139400	1.116378	1.099162					
1.951229	1.463726	1.301361	1.220279	1.171712	1.139400	1.116378						
1.951229	1.463726	1.301361	1.220279	1.171712	1.139400							
1.951229	1.463726	1.301361	1.220279	1.171712								
1.951229	1.463726	1.301361	1.220279									
1.951229	1.463726	1.301361										
1.951229	1.463726											
1.951229												

Appendix A5

Random error random from Normal with mean 0

Year\delay

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1978	0.083	0.075	-0.076	-0.065	-0.188	-0.164	-0.101	0.078	0.021	0.029	0.005	0.03	-0.073	-0.241
1979	-0.113	-0.049	-0.086	-0.123	0.148	0.09	-0.06	-0.099	-0.032	0.096	0.028	0.1	-0.331	
1980	0.086	-0.007	-0.037	0.17	0.071	-0.138	0.047	0.022	0.036	0.003	0.004	0.058		
1981	-0.071	0.147	0.067	-0.028	-0.132	0.049	0	-0.117	-0.042	0.026	-0.078			
1982	0.081	0.059	0.073	0.048	0.025	0.029	-0.023	-0.133	-0.044	0.066				
1983	0.117	0.059	-0.017	-0.081	-0.051	-0.024	-0.048	0.124	0.033					
1984	-0.024	-0.026	0.134	0.214	0.071	0.193	-0.022	0.012						
1985	0.022	0.015	0.076	-0.028	-0.004	0.155	0.032							
1986	-0.043	0.181	0.184	-0.192	-0.16	-0.048								
1987	0.07	0.106	0.144	0.032	-0.102									
1988	0.056	-0.195	0.032	0.041										
1989	0.145	0.187	-0.159											
1990	0.001	-0.153												
1991	-0.142													

Appendix A6

Sum of data in Appendices A2 and A5 to produce trends + randomness

Year\delay

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1978	11.5959	11.4879	11.2369	11.1479	10.9249	11.0489	11.0619	11.1909	11.0839	11.0419	10.9679	10.9429	10.7899	10.5719
1979	11.4999	11.4639	11.3269	11.1899	11.5609	11.4529	11.2529	11.1639	11.1809	11.2589	11.1409	11.1629	10.6819	
1980	11.7989	11.6059	11.4759	11.7829	11.6339	11.3749	11.5099	11.4349	11.3989	11.3159	11.2669	11.2709		
1981	11.7419	11.8599	11.8799	11.7349	11.5809	11.7119	11.6129	11.4459	11.4709	11.4889	11.3349			
1982	11.9939	12.0719	12.0359	11.9609	11.8879	11.8419	11.7399	11.5799	11.6189	11.6789				
1983	12.3299	12.2219	12.0959	11.9819	11.9619	11.9389	11.8649	11.9869	11.8459					
1984	12.3389	12.2869	12.3969	12.4269	12.2339	12.3059	12.0409	12.0249						
1985	12.5349	12.4779	12.4889	12.3349	12.3089	12.4179	12.2449							
1986	12.6199	12.7939	12.7469	12.3209	12.3029	12.3649								
1987	12.8829	12.8689	12.8569	12.6949	12.5109									
1988	13.0189	12.7179	12.8949	12.8539										
1989	13.2579	13.2499	12.8539											
1990	13.2639	13.0599												
1991	13.2709													

Appendix A7

Incremental paids derived from Appendix A6

1978	108651	97529	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023
1979	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	
1980	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504		
1981	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692			
1982	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054				
1983	226364	203191	179136	159835	156670	153108	142187	160637	139511					
1984	228411	216837	242050	249422	205644	220996	169549	166858						
1985	277868	262472	265375	227499	221660	247187	207918							
1986	302519	360015	343485	224336	220334	234427								
1987	393525	388054	383425	326081	271278									
1988	450855	333667	398276	382277										
1989	572576	568013	382277											
1990	576021	469724												
1991	580068													

Appendix A8

Cumulative paids from Appendix A7

1978	108651	206180	282059	351477	407019	469894	533591	606059	671173	733609	791592	848143	896671	935694
1979	98706	193922	2769473	49343	454257	548431	625534	696072	767819	845386	914320	984787	1028347	
1980	133106	242849	339214	470207	583067	670175	769873	862367	951591	1033708	1111898	1190402		
1981	125731	267209	411545	536399	643433	765448	875962	969479	1065364	1162990	1246682			
1982	161765	336653	505357	661871	807366	946320	1071800	1178727	1289906	1407960				
1983	226364	429555	608691	768526	925196	1078304	1220491	1381128	1520639					
1984	228411	445248	687298	936720	1142364	1363360	1532909	1699767						
1985	277868	540340	805715	1033214	1254874	1502061	1709979							
1986	302519	662534	1006019	1230355	1450689	1685116								
1987	393525	781579	1165004	1491085	1762363									
1988	450855	784522	1182798	1565075										
1989	572576	1140589	1522866											
1990	576021	1045745												
1991	580068													

Appendix A9

Age-to-age factors (link ratios) of the cumulative payments

1978	1.897635	1.368023	1.246111	1.158024	1.154476	1.135556	1.135811	1.107438	1.093025	1.079038	1.071439	1.057216	1.043519
1979	1.964642	1.428136	1.261407	1.300318	1.207314	1.140588	1.112764	1.103074	1.101022	1.081541	1.077070	1.044232	
1980	1.824478	1.396810	1.386166	1.240021	1.149396	1.148764	1.120141	1.103464	1.086294	1.075640	1.070603		
1981	2.125243	1.540161	1.303378	1.199541	1.189631	1.144378	1.106759	1.098903	1.091636	1.071962			
1982	2.081123	1.501121	1.309709	1.219823	1.172107	1.132597	1.099763	1.094321	1.091521				
1983	1.897629	1.417026	1.262588	1.203857	1.165487	1.131861	1.131616	1.101012					
1984	1.949328	1.543629	1.362902	1.219536	1.193454	1.124361	1.108850						
1985	1.944592	1.491125	1.282356	1.214534	1.196981	1.138421							
1986	2.190057	1.518441	1.222993	1.179081	1.161597								
1987	1.986097	1.490577	1.279896	1.181933									
1988	1.740076	1.507667	1.323197										
1989	1.992030	1.335157											
1990	1.815463												
1991													

One cannot determine changing calendar year trends from the age-to-age link ratios.

APPENDIX B1

Random incremental paids from (same) lognormal distribution

ACC. YEAR	DELAY 0	1	2	3	4	5	6	7	8	9
1976	10266	3419	3724	9606	8152	8175	3958	3030	1733	351
1977	1767	2454	6580	2819	1957	2150	3677	4751	2832	
1978	6232	5143	2667	4278	2289	6215	6273	4905		
1979	4597	3591	5909	5156	4013	3557	1961			
1980	2483	3805	3995	6315	3480	3486				
1981	1643	2077	5101	1907	3274					
1982	3270	7230	1853	4158						
1983	3161	2065	5890							
1984	5305	6078								
1985	6127									

APPENDIX B2

Cumulative payments

ACC. YEAR	DELAY 0	1	2	3	4	5	6	7	8	9
1976	10266	13685	17409	27015	35167	43342	47300	50330	52063	55574
1977	1767	4221	10801	13620	15577	17727	21404	26155	28987	
1978	6232	11375	14042	18320	20609	26824	33097	38002		
1979	4597	8188	14097	19253	23266	26823	28784			
1980	4248	8053	12048	18363	21843	25329				
1981	1643	3720	8821	10728	14002					
1982	3270	10500	12353	16511						
1983	3161	5226	11116							
1984	5305	11383								
1985	6127									

APPENDIX B3

Age-to-Age Link Ratios

DELAY

ACC. YEAR	0/1	1/2	2/3	3/4	4/5	5/6	6/7	7/8	8/9
1976	1.333041	1.272122	1.551783	1.301758	1.232462	1.091320	1.064059	1.034432	1.067437
1977	2.388794	2.558872	1.260994	1.143685	1.138024	1.207423	1.221967	1.108277	
1978	1.825256	1.234461	1.304657	1.124945	1.301567	1.233857	1.148200		
1979	1.781161	1.721665	1.365751	1.208435	1.152884	1.073108			
1980	1.895715	1.496088	1.524153	1.189511	1.159593				
1981	2.264150	2.371238	1.216188	1.305182					
1982	3.211009	1.176476	1.336598						
1983	1.653274	2.127057							
1984	2.145711								
1985									

APPENDIX B4

Incremental paids generated by SDF model with 20% calendar year trend

DELAY

YEAR	0	1	2	3	4	5	6	7	8	9	10	11	1
1978	53275	66971	121278	292065	86300	79271	240147	86269	73645	225638	218708	72438	861
1979	31912	85884	42106	150200	88290	82798	230017	346594	169950	113715	48703	82441	1689
1980	24964	96951	208159	697227	213581	251802	489886	387322	524382	133462	206570	76440	
1981	82867	117837	279958	469997	577054	378084	438640	556884	338201	173980	161958		
1982	41268	252181	101806	219303	283631	352082	748704	727854	147742	299994			
1983	32190	491133	239252	228226	375903	494626	323417	482001	157137				
1984	231651	401780	626068	496230	388360	395640	653268	535755					
1985	31273	409563	433997	831822	572787	468844	1317425						
1986	92728	342040	246087	530327	837381	694392							
1987	147772	208578	389162	602683	743423								
1988	146151	209864	1827396	1391050									
1989	81526	767664	1042474										
1990	206885	350789											
1991	559279												

APPENDIX B5

Age-to-age link ratios

DELAY

	0/1	1/2	2/3	3/4	4/5	5/6	6/7	7/8	8/9	9/10	10/11	11/12
1978	2.26	2.01	2.21	1.16	1.13	1.34	1.09	1.07	1.21	1.17	1.05	1.05
1979	3.69	1.36	1.94	1.28	1.21	1.48	1.49	1.16	1.09	1.04	1.06	1.11
1980	4.88	2.71	3.11	1.21	1.20	1.33	1.20	1.22	1.05	1.07	1.02	
1981	2.42	2.39	1.98	1.61	1.25	1.23	1.24	1.12	1.05	1.05		
1982	7.11	1.35	1.55	1.46	1.39	1.60	1.36	1.05	1.10			
1983	16.26	1.46	1.30	1.38	1.36	1.17	1.22	1.06				
1984	2.73	1.99	1.39	1.22	1.18	1.26	1.17					
1985	14.10	1.98	1.95	1.34	1.21	1.48						
1986	4.69	1.57	1.78	1.69	1.34							
1987	2.41	2.09	1.81	1.55								
1988	2.44	6.13	1.64									
1989	10.42	2.23										
1990	2.70											
1991												

Note that link ratios do not tell us that we have a constant stable calendar year trend

APPENDIX C1

	DELAY										
ACC. YEAR	0	1	2	3	4	5	6	7	8	9	10
1977	153638	188412	134534	87456	60348	42404	31238	21252	16622	14440	12200
1978	176536	226412	158894	104686	71448	47990	35576	24818	22662	18000	
1979	210172	259168	188388	123074	83380	56086	38496	33768	27400		
1980	11448	253482	183370	131040	78994	60232	45568	38000			
1981	219810	266304	194650	120098	87582	62750	51000				
1982	205654	252746	177506	12952	96786	82400					
1983	197716	255408	194648	142328	105600						
1984	239784	329242	264802	190400							
1985	326304	471744	375400								
1986	420778	590400									
1987	496200										

ACCI EXPOSURES

YEAR	
1977	2.20
1978	2.40
1979	2.20
1980	2.00
1981	1.90
1982	1.60
1983	1.60
1984	1.80
1985	2.20
1986	2.50
1987	2.60

APPENDIX C2

AGE LINK RATIOS OF CUMULATIVE PAYMENTS

DELAYS

	0/1	1/2	2/3	3/4	4/5	5/6	6/7	7/8	8/9	9/10
1977	2.226337	1.393316	1.183505	1.106992	1.067912	1.046848	1.030445	1.023109	1.019622	1.016259
1978	2.268158	1.392381	1.185665	1.106873	1.064853	1.045149	1.030135	1.026712	1.020665	
1979	2.233123	1.401389	1.187119	1.106787	1.064900	1.041831	1.035220	1.027606		
1980	2.198791	1.394403	1.202128	1.101360	1.070173	1.049607	1.039413			
1981	2.211519	1.400420	1.176416	1.109359	1.070629	1.053616				
1982	2.228986	1.387229	1.203681	1.126446	1.095567					
1983	1.291792	1.429568	1.219719	1.133653						
1984	2.373077	1.465360	1.228344							
1985	2.445719	1.470397								
1986	2.403115									

APPENDIX C3

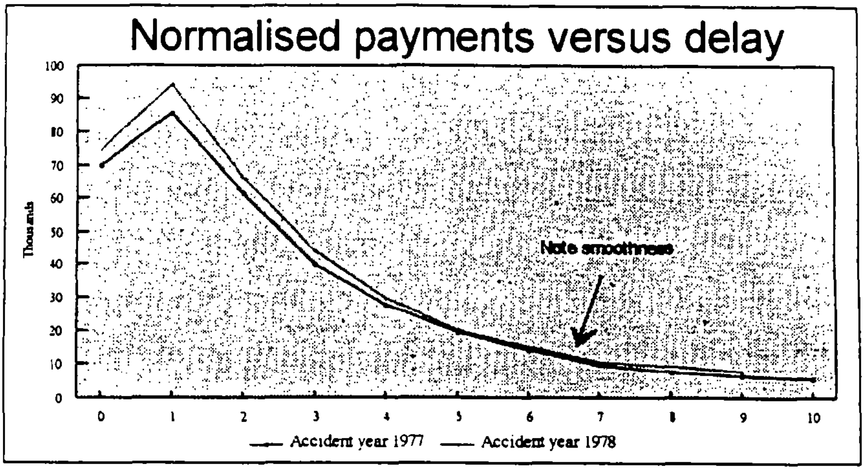

Figure C3

APPENDIX C4 - (Statistical Chain Ladder)

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	0.2511	0.0370	6.79			
2	-0.3069	0.0385	-7.97	-0.5580	0.0650	-8.59
3	-0.3928	0.0406	-9.68	-0.0859	0.0682	-1.26
4	-0.3803	0.0432	-8.81	0.0124	0.0723	0.17
5	-0.3402	0.0464	-7.34	0.0401	0.0773	0.52
6	-0.3384	0.0505	-6.71	0.0018	0.0835	0.02
7	-0.2908	0.0559	-5.20	0.0476	0.0917	0.52
8	-0.2248	0.0637	-3.53	0.0660	0.1030	0.64
9	-0.2152	0.0763	-2.82	0.0095	0.1202	0.08
0	-0.1893	0.1030	-1.84	0.0259	0.1526	0.17

NOT ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1977	11.0484	0.0380	290.75			
1978	11.1402	0.0380	293.17	0.0918	0.0370	2.48
1979	11.3935	0.0385	295.97	0.2533	0.0385	6.58
1980	11.5218	0.0393	293.10	0.1283	0.0406	3.16
1981	11.6001	0.0405	286.71	0.0783	0.0432	1.81
1982	11.7939	0.0420	280.55	0.1938	0.0464	4.18
1983	11.7979	0.0442	266.67	0.0040	0.0505	0.08
1984	11.9095	0.0474	251.04	0.1115	0.0559	1.99
1985	12.0116	0.0524	229.08	0.1022	0.0637	1.60
1986	12.0774	0.0613	196.88	0.0657	0.0763	0.86
1987	12.1592	0.0827	147.00	0.0818	0.1030	0.79

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX C4

(REGRESSION OUTPUT CONTINUED)

S =	0.0827	S-SQUARED =	0.0068	S-SQUARED(SCI) =	0.0449
S(B) =	0.0827	S(B)-SQUARED =	0.0068	DELTA =	0.0000
R-SQUARED =	99.5 PERCENT	N =	66	P =	21.0
SSPE =	0.948	WSSPE =	0.948	AIC = -124.97	AIC(SCI) = -52.18

APPENDIX C5

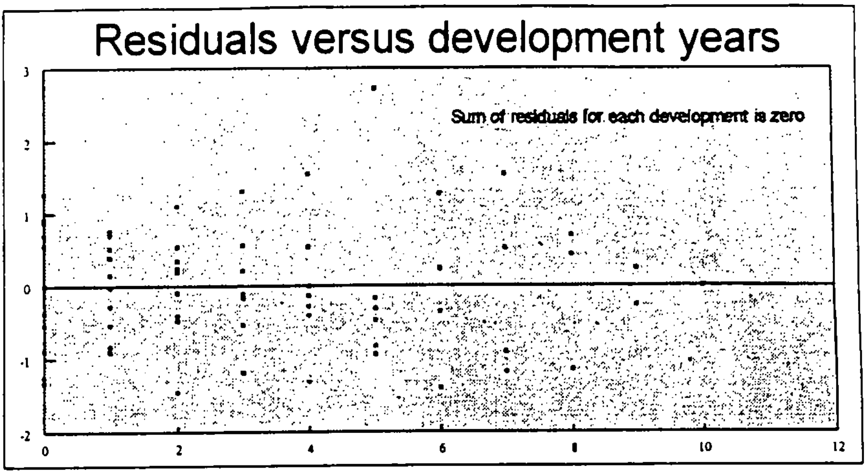

Figure C5.1

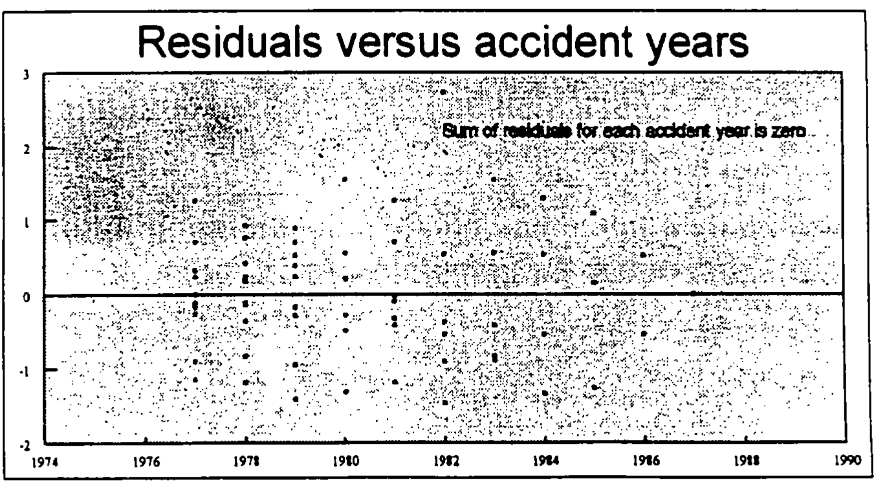

Figure C5.2

APPENDIX C5

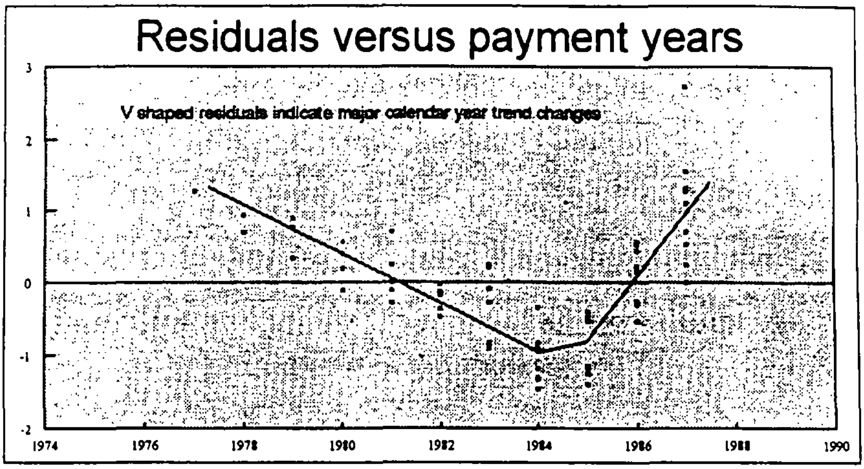

Figure C5.3

APPENDIX C6 - Cape Cod

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	0.2029	0.1416	1.43			
2	-0.3567	0.1489	-2.40	-0.5596	0.2514	-2.23
3	-0.4468	0.1574	-2.84	-0.0901	0.2651	-0.34
4	-0.4352	0.1677	-2.59	0.0116	0.2814	0.04
5	-0.3947	0.1803	-2.19	0.0404	0.3010	0.13
6	-0.4139	0.1962	-2.11	-0.0192	0.3256	-0.06
7	-0.3556	0.2174	-1.64	0.0583	0.3574	0.16
8	-0.3067	0.2475	-1.24	0.0489	0.4012	0.12
9	-0.3150	0.2958	-1.06	-0.0083	0.4677	-0.02
10	-0.2352	0.3968	-0.59	0.0797	0.5916	0.13

NOT ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1977	11.6776	0.0977	119.53			
1978	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1979	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1980	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1981	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1982	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1983	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1984	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1985	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1986	11.6776	0.0977	119.53	0.0000	0.0000	0.00
1987	11.6776	0.0977	119.53	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX C6

(REGRESSION OUTPUT CONTINUED)

S = 0.3240 S-SQUARED = 0.1050 S-SQUARED(SCI) = 0.0449

S(B) = 0.3240 S(B)-SQUARED = 0.1050 DELTA = 0.0000

R-SQUARED = 91.1 PERCENT N = 66 P = 11.0

SSPE = 7.433 WSSPE = 7.433 AIC = 48.51 AIC(SCI) = -52.18

APPENDIX C7

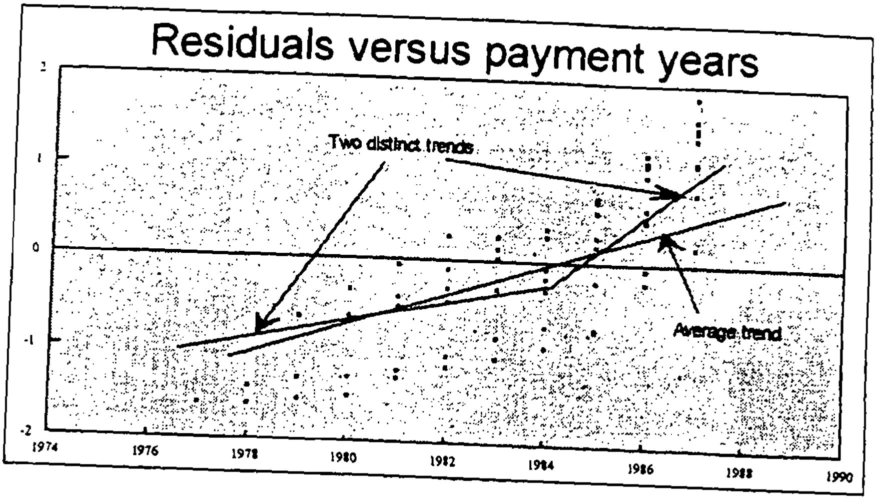

Figure C7

APPENDIX C8 - Cape Cod with constant inflation

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	-RATIO
1	0.1424	0.0439	3.24			
2	-0.4172	0.0462	-9.03	-0.5596	0.0779	-7.19
3	-0.5072	0.0488	-10.39	-0.0901	0.0821	-1.10
4	-0.4956	0.0520	-9.53	0.0116	0.0871	0.13
5	-0.4552	0.0559	-8.14	0.0404	0.0932	0 43
6	-0.4744	0.0608	-7.80	-0.0192	0.1008	-0.19
7	-0.4161	0.0674	-6.18	0.0583	0.1107	0.53
8	-0.3672	0.0767	-4.79	0.0489	0.1243	0.39
9	-0.3754	0.0917	-4.10	-0.0083	0.1449	-0.06
10	-0.2957	0.1230	-2.41	0.0797	0.1832	0.44

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1977	11.0728	0.0403	275.09			
1978	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1979	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1980	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1981	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1982	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1983	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1984	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1985	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1986	11.0728	0.0403	275.09	0.0000	0.0000	0.00
1987	11.0728	0.0403	275.09	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

PMNT YEAR	IOTA	S.E.	T-RATIO	DIFFERENCE IN IOTA	S.E.	T-RATIO
1978	0.1210	0.0053	22.79			
1979	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1980	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1981	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1982	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1983	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1984	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1985	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1986	0.1210	0.0053	22.79	0.0000	0.0000	0.00
1987	0.1210	0.0053	22.79	0.0000	0.0000	0 00

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX C8

(REGRESSION OUTPUT CONTINUED)

S = 0.1004 S-SQUARED = 0.0101 S-SQUARED(SCI) = 0.0449

S(B) = 0.1004 S(B)-SQUARED = 0.0101 DELTA = 0.0000

R-SQUARED = 99.2 PERCENT N = 66 P = 12.0

SSPE = 1.176 WSSPE = 1.176 AIC = -105.40 AIC(SCI) = -52.18

APPENDIX C9

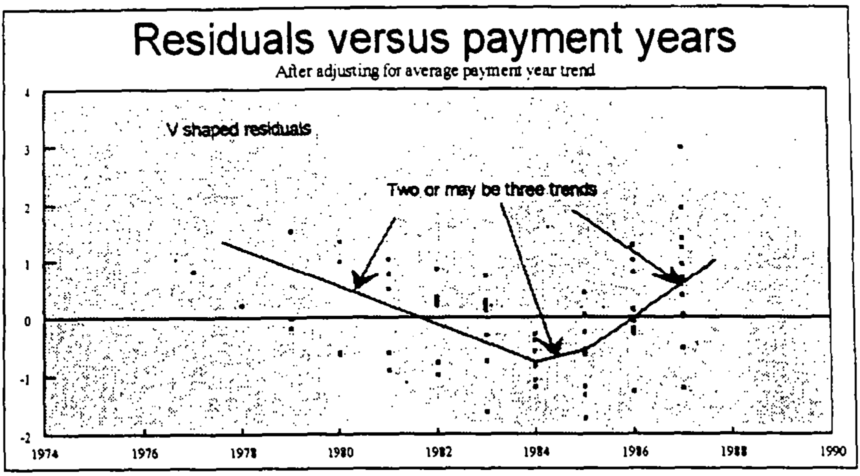

Figure C9

APPENDIX C10 - Cape Cod with three payment year parameters
(1977-84, 1984-1985 and 1985-1987)

R E G R E S S I O N   T A B L E

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	-0.1505	0.0371	4.05			
2	-0.4098	0.0390	-10.50	-0.5603	0.0657	-8.52
3	-0.5008	0.0413	-12.14	-0.0910	0.0693	-1.31
4	-0.4906	0.0439	-11.17	0.0102	0.0736	0.14
5	-0.4522	0.0472	-9.58	0.0384	0.0787	0.49
6	-0.4748	0.0514	-9.24	0.0225	0.0851	-0.26
7	-0.4222	0.0569	-7.41	0.0526	0.0935	0.56
8	-0.3849	0.0651	-5.91	0.0373	0.1050	0.36
9	-0.4126	0.0780	-5.29	-0.0277	0.1229	-0.23
10	-0.3329	0.1042	-3.19	0.0797	0.1547	0.52

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1977	11.1536	0.0400	278.91			
1978	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1979	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1980	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1981	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1982	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1983	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1984	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1985	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1986	11.1536	0.0400	278.91	0.0000	0.0000	0.00
1987	11.1536	0.0400	278.91	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

PMNT YEAR	IOTA	S.E.	T-RATIO	DIFFERENCE IN IOTA	S.E.	T-RATIO
1978	0.0985	0.0077	12.74			
1979	0.0985	0.0077	12.74	0.0000	0.0000	0.00
1980	0.0985	0.0077	12.74	0.0000	0.0000	0.00
1981	0.0985	0.0077	12.74	0.0000	0.0000	0 00
1982	0.0985	0.0077	12.74	0.0000	0.0000	0.00
1983	0.0985	0.0077	12.74	0.0000	0.0000	0.00
1984	0.0985	0.0077	12.74	0.0000	0.0000	0.00
1985	0.1174	0.0343	3.42	0.0189	0.0385	0.49
1986	0.1952	0.0197	9.91	0.0778	0.0484	1.61
1987	0.1952	0.0197	9.91	0.0000	0.0000	0.00

APPENDIX C10

(REGRESSION OUTPUT CONTINUED)

S = 0.0847 S-SQUARED = 0.0072 S-SQUARED(SCI) = 0.0449

S(B) = 0.0847 S(B)-SQUARED = 0.0072 DELTA = 0.0000

R-SQUARED = 99.4 PERCENT N = 66 P = 14.0

SSPE = 1.000 WSSPE = 1.000 AIC = -126.26 AIC(SCI) = -52.18

APPENDIX C11

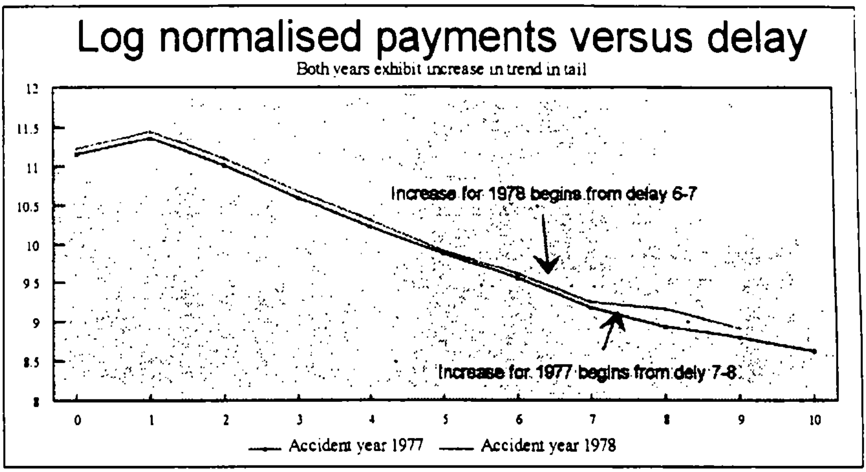

Figure C11

APPENDIX D1

INCREMENTAL PAID LOSSES

DELAY

ACCI. YR	0	1	2	3	4	5	6	7	8
1969	193013	1584331	1151882	778980	475203	143352	128612	70845	25077
1970	376473	1541950	1719509	1032570	289305	382508	270087	108354	23133
1971	568891	1579158	1277822	734670	680369	217221	147800	57099	64829
1972	428753	970640	955898	1095771	510072	491853	242995	299845	
1973	458252	989072	1417606	953222	881133	278778	197156		
1974	355229	948807	1292900	748003	547288	274367			
1975	282419	688332	1158793	903450	629983				
1976	267600	1044790	1216437	527644					
1977	560307	940002	1185899						
1978	360171	1011773							
1979	445545								

ACCI YR	EXPOSURES
1969	523.00
1970	643.00
1971	676.00
1972	673.00
1973	809.00
1974	669.00
1975	513.00
1976	543.00
1977	622.00
1978	703.00
1979	743.00

APPENDIX D2

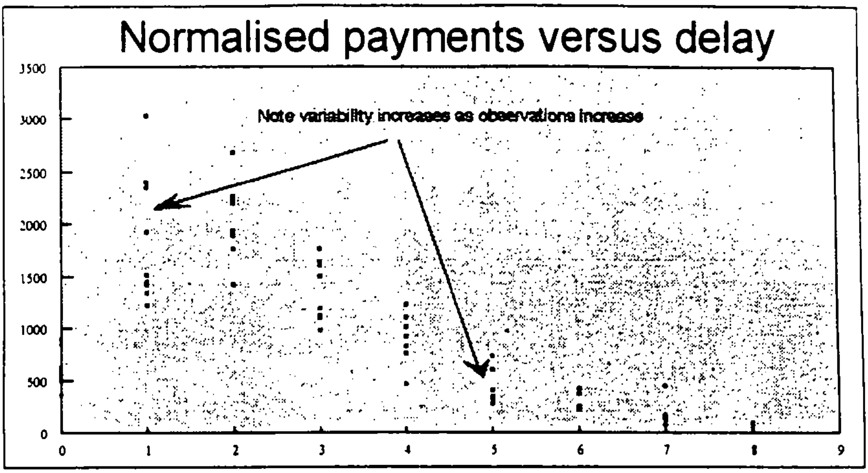

Figure D2.1

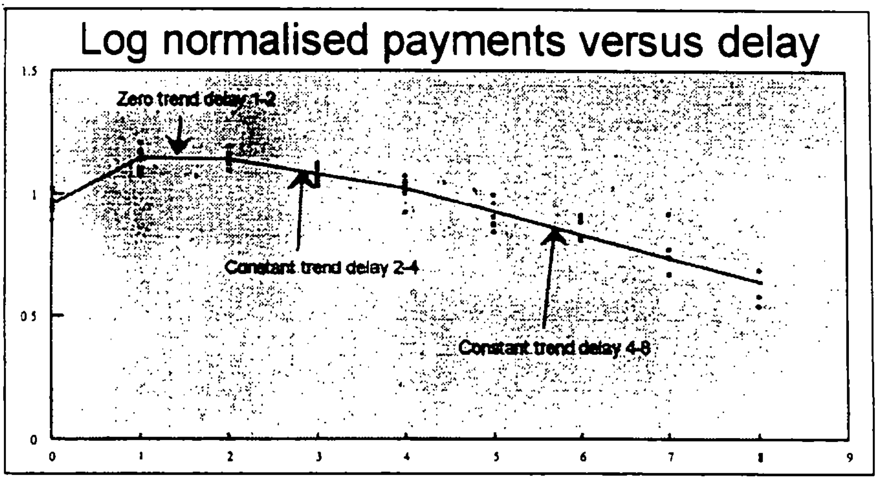

Figure D2.2

APPENDIX D3

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	1.1647	0.1234	9.44			
2	0.0000	0.0000	0.00	-1.1647	0.1234	-9.44
3	-0.3769	0.0631	-5.98	-0.3769	0.0631	-5.98
4	-0.3769	0.0631	-5.98	0.0000	0.0000	0.00
5	-0.6226	0.0466	-13.35	-0.2457	0.0985	-2.49
6	-0.6226	0.0466	-13.35	0.0000	0.0000	0.00
7	-0.6226	0.0466	-13.35	0.0000	0.0000	0.00
8	-0.6226	0.0466	-13.35	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1969	6.3672	0.0997	63.84			
1970	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1971	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1972	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1973	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1974	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1975	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1976	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1977	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1978	6.3672	0.0997	63.84	0.0000	0.0000	0.00
1979	6.3672	0.0997	63.84	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

PMNT YEAR	IOTA	S.E.	T-RATIO	DIFFERENCE IN IOTA	S.E.	T-RATIO
1970	0.0000	0.0000	0.00			
1971	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1972	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1973	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1974	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1975	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1976	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1977	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1978	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1979	0.0000	0.0000	0.00	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX D4

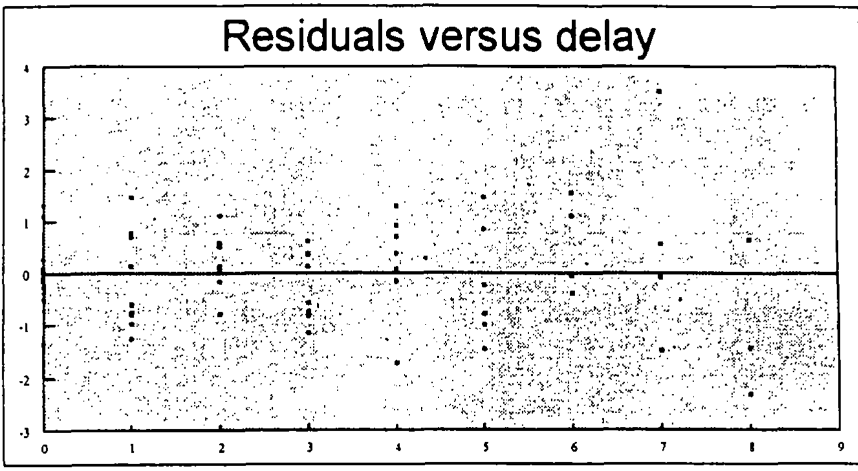

Figure D4.1

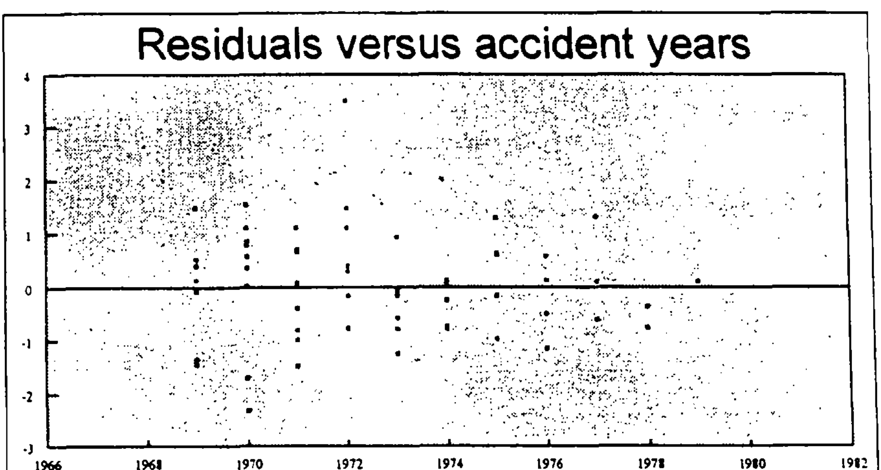

Figure D4.2

APPENDIX D4

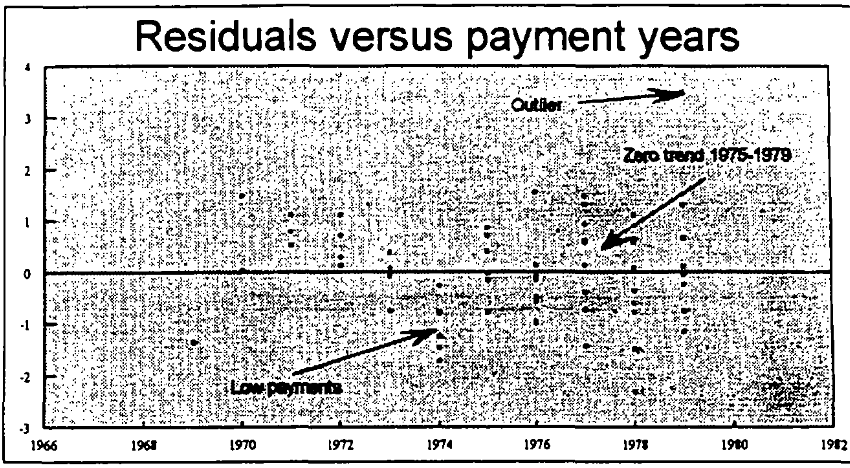

Figure D4.3

APPENDIX D5

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	1.1777	0.0993	11.86			
2	0.0000	0.0000	0.00	-1.1777	0.0993	-11.86
3	-0.3478	0.0519	-6.70	-0.3478	0.0519	-6.70
4	-0.3478	0.0519	-6.70	0.0000	0.0000	0.00
5	-0.6749	0.0390	-17.32	-0.3270	0.0803	-4.07
6	-0.6749	0.0390	-17.32	0.0000	0.0000	0.00
7	-0.6749	0.0390	-17.32	0.0000	0.0000	0.00
8	-0.6749	0.0390	-17.32	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1969	6.4594	0.0927	69.68			
1970	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1971	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1972	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1973	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1974	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1975	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1976	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1977	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1978	6.4594	0.0927	69.68	0.0000	0.0000	0.00
1979	6.4594	0.0927	69.68	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

PMNT YEAR	IOTA	S.E.	T-RATIO	DIFFERENCE IN IOTA	S.E.	T-RATIO
1970	0.0000	0.0000	0.00			
1971	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1972	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1973	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1974	-0.4792	0.1306	-3.67	-0.4792	0.1306	-3.67
1975	0.3723	0.1182	3.15	0.8515	0.2330	3.65
1976	0.0000	0.0000	0.00	-0.3723	0.1182	-3.15
1977	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1978	0.0000	0.0000	0.00	0.0000	0.0000	0 00
1979	0.0000	0.0000	0.00	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX D6

(REGRESSION OUTPUT CONTINUED)

S = 0.2654 S-SQUARED = 0.0704 S-SQUARED(SCI) = 0.5469

S(B) = 0.2654 S(B)-SQUARED = 0.0704 DELTA = 0.0000

R-SQUARED = 93.5 PERCENT N = 62 P = 6.0

SSPE = 7.360 WSSPE = 7.360 AIC = 17.13 AIC(SCI) = 43.81

APPENDIX D7

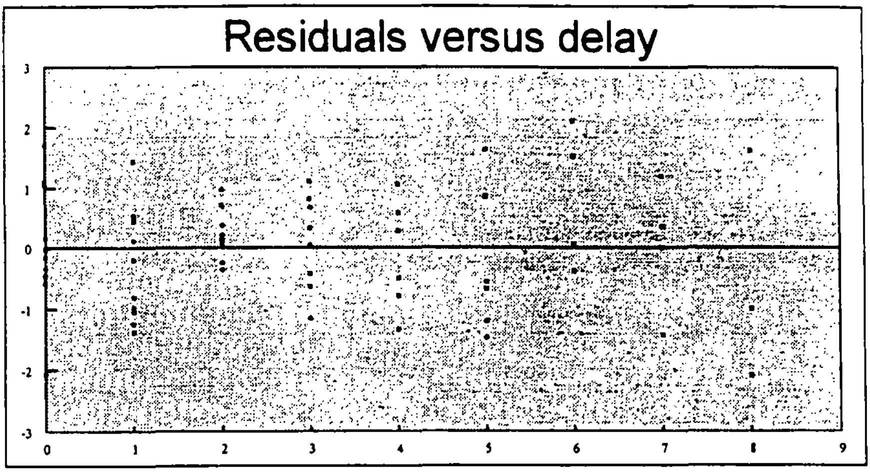

Figure D7.1

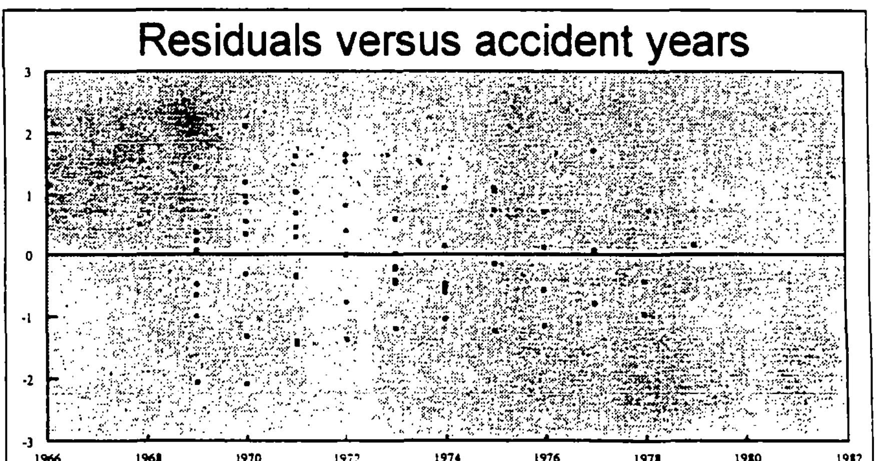

Figure D7.2

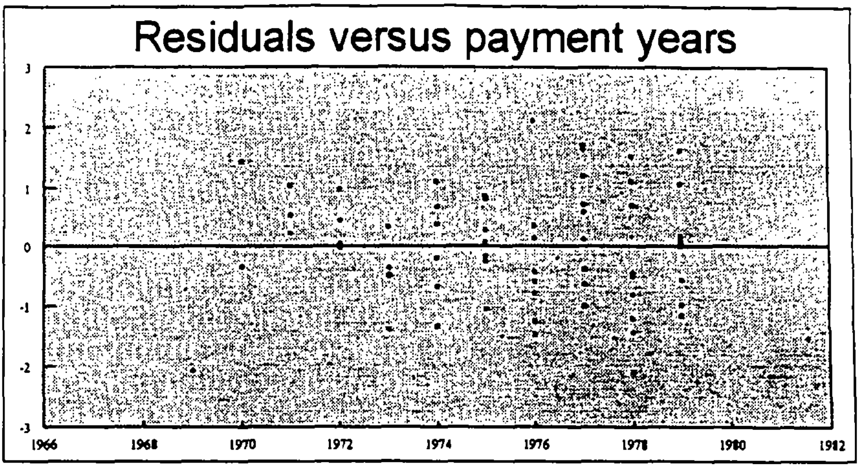

Figure D7.3

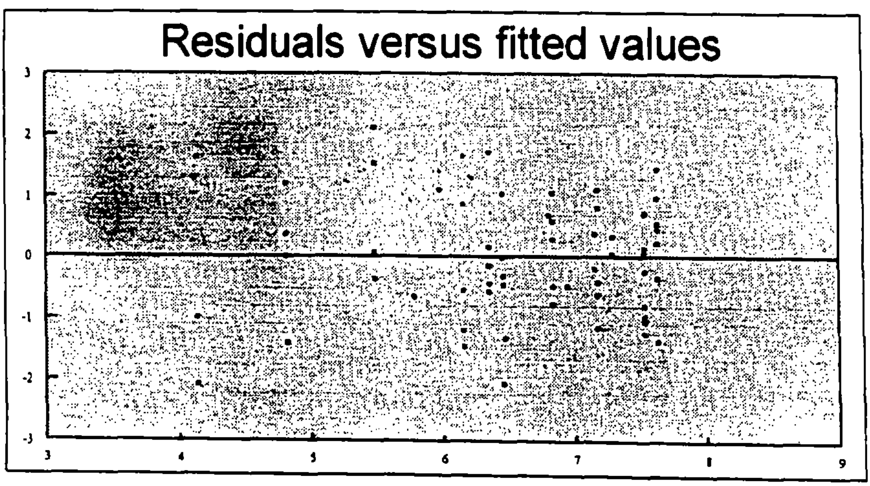

Figure D7. 4

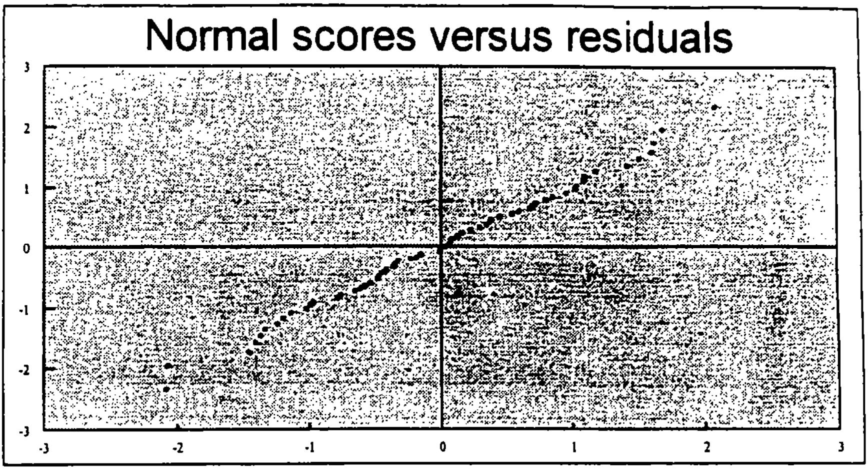

Figure D7.5

FORECASTING OUTPUT

ASSUMED FUTURE INFLATION = 0.0000
STANDARD ERROR = 0.0000

EXPECTED PAYMENTS/OBSERVED PAYMENTS +--------+ **FORECAST MEAN PAYMENTS/STANDARD ERRORS**

(PAYMENTS IN $1 S)

EXP:	346295	1123112	1123112	793172	561671	177321	130329	66469	33951	0
OBS:	193013	1584331	1151882	778980	475203	143352	128612	70845	25077	0
EXP:	425750	1380806	1380806	975161	428440	314654	160233	81720	41741	0
OBS:	376473	1541950	1719509	1032570	289305	382508	270087	108354	23133	0
EXP:	447601	1451671	1451671	636733	650595	330803	168456	85914	43883	0
OBS:	568891	1579158	1277822	734670	680369	217221	147800	57099	64829	0
EXP:	445614	1445229	898519	915576	647708	329335	167709	85533	43689	43689
OBS:	428753	970640	955898	1095771	510072	491853	242995	299845	12280	12280
EXP:	535664	1080091	1559962	1100596	778597	395887	201599	102817	52517	155334
OBS:	458252	989072	1417606	953222	881133	278778	197156	27673	14762	32822
EXP:	275565	1290006	1290006	910134	643858	327378	166712	85024	43429	295165
OBS:	355229	948807	1292900	748003	547288	274367	43841	22884	12207	53323
EXP:	305191	989197	989197	697906	493721	251038	127837	65198	33302	477376
OBS:	282419	688332	1158793	903450	629983	65999	33618	17548	9361	79132
EXP:	323039	1047045	1047045	738719	522593	265719	135313	69011	35249	1027886
OBS:	267600	1044790	1216437	527644	140549	69858	35584	18574	9908	167258
EXP:	370037	1199377	1199377	846194	598624	304378	155000	79051	40378	2023625
OBS:	560307	940002	1185899	221766	160997	80022	40761	21276	11350	300456
EXP:	418225	1355566	1355566	956389	676580	344016	175185	89345	45636	3642717
OBS:	360171	1011773	360708	250646	181963	90443	46069	24047	12827	502218
EXP:	442022	1432697	1432697	1010807	715077	363590	185152	94429	48233	5282681
OBS:	445545	381231	381231	264907	192317	95589	48690	25415	13557	674135
PAYMENT YRS:		4721306	3518808	2235705	1316405	653075	314876	140065	48233	12948473
RD ERRORS:		623018	504462	345451	223516	111688	57849	29752	13557	1030808

APPENDIX D9

TABLE OF OBSERVED AND EXPECTED BY YEAR

ACC. YEAR	EXPECTED (PAYMENTS IN $1'S)	OBSERVED (PAYMENTS IN $1'S)	DIFFERENCE	%ERROR	PMNT YEAR	EXPECTED	OBSERVED (PAYMENTS IN $1's)	DIFFERENCE (PAYMENTS IN $1's)	%ERROR
69	4355433	4551295	195862	4	69	346295	193013	-153282	-79
70	5189312	5743889	554577	9	70	1548863	1960804	411941	21
71	5267328	5327859	60531	1	71	2951519	3262723	311204	9
72	4849689	4695982	-153707	-3	72	4071263	4506400	435137	9
73	5652397	5175219	-477178	-9	73	4969396	4214487	-754909	-17
74	4736946	4166594	-570352	-13	74	3496670	3467526	-29144	0
75	3475212	3662977	187765	5	75	5166314	4936092	-230222	-4
76	3155847	3056471	-99376	-3	76	4908050	4270279	-637771	-14
77	2768792	2686208	-82584	-3	77	4708472	5166110	457638	8
78	1773792	1371944	-401848	-29	78	4697662	4569353	-128309	-2
79	442022	445545	3523	0	79	4802265	4337196	-465069	-10

NDI)

FORECASTING OUTPUT

ASSUMED FUTURE INFLATION = 0.0000
STANDARD ERROR = 0.0000

EXPECTED PAYMENTS/OBSERVED PAYMENTS +--------+ FORECAST MEAN PAYMENTS/STANDARD ERROR

(PAYMENTS IN $1 S)

EAR											
69	EXP:	346295	1123112	1123112	793172	561671	177321	130329	66469	33951	
	OBS:	193013	1584331	1151882	778980	475203	143352	128612	70845	25077	
70	EXP:	425750	1380806	1380806	975161	428440	314654	160233	81720	41741	
	OBS:	376473	1541950	1719509	1032570	289305	382508	270087	108354	23133	
71	EXP:	447601	1451671	1451671	636733	650595	330803	168456	85914	43883	
	OBS:	568891	1579158	1277822	734670	680369	217221	147800	57099	64829	
72	EXP:	445614	1445229	898519	915576	647708	329335	167709	85533	43689	4368
	OBS:	428753	970640	955898	1095771	510072	491853	242996	299845	12280	1228
73	EXP:	535664	1080091	1559962	1100596	778597	395887	201599	102817	52517	15533
	OBS:	458252	989072	1417606	953222	881133	278778	197156	27673	14762	3282
74	EXP:	275565	1290006	1290006	910134	643858	327378	166712	85024	43429	29516
	OBS:	355229	948807	1292900	748003	547288	274367	43841	22884	12207	5332
75	EXP:	305191	989197	989197	697906	493721	251038	127837	65198	33302	47737
	OBS:	282419	688332	1158793	903450	629983	65999	33618	17548	9361	7913
76	EXP:	323039	1047045	1047045	738719	522593	265719	135313	69011	35249	102788
	OBS:	267600	1044790	1216437	527644	140549	69858	35584	18574	9908	16725
77	EXP:	370037	1199377	1199377	846194	598624	304378	155000	79051	40378	202362
	OBS:	560307	940002	1185899	221766	160997	80022	40761	21276	11350	30045
78	EXP:	418225	1355566	1355566	956389	676580	344016	175185	89345	45636	364271
	OBS:	360171	1011773	360708	250646	181963	90443	46069	24047	12827	50221
79	EXP:	442022	1432697	1432697	1010807	715077	363590	185152	94429	48233	528268
	OBS:	445545	381231	381231	264907	192317	95589	48690	25415	13557	67413
)T.FOR PAYMENT YRS:			4721306	3518808	2235705	1316405	653075	314876	140065	48233	1294847
TANDARD ERRORS:			623018	504462	345451	223516	111688	57849	29752	13557	1030800

598

APPENDIX D10

VALIDATION

REGRESSION TABLE

PARAMETER ESTIMATES

DEV. YEAR	GAMMA	S.E.	T-RATIO	DIFFERENCE IN GAMMA	S.E.	T-RATIO
1	1.2468	0.1076	11.58			
2	0.0000	0.0000	0.00	-1.2468	0.1076	-11.58
3	-0.4024	0.0639	-6.29	-0.4024	0.0639	-6.29
4	-0.4024	0.0639	-6.29	0.0000	0.0000	0.00
5	-0.5544	0.0753	-7.37	-0.1520	0.1213	-1.25
6	-0.5544	0.0753	-7.37	0.0000	0.0000	0.00
7	-0.5544	0.0753	-7.37	0.0000	0.0000	0.00
8	-0.5544	0.0753	-7.37	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

ACCI YEAR	ALPHA	S.E.	T-RATIO	DIFFERENCE IN ALPHA	S.E.	T-RATIO
1969	6.4278	0.0922	69.72			
1970	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1971	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1972	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1973	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1974	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1975	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1976	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1977	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1978	6.4278	0.0922	69.72	0.0000	0.0000	0.00
1979	6.4278	0.0922	69.72	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

PARAMETER ESTIMATES

PMNT YEAR	IOTA	S.E.	T-RATIO	DIFFERENCE IN IOTA	S.E.	T-RATIO
1970	0.0000	0.0000	0.00			
1971	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1972	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1973	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1974	-0.4798	0.1208	-3.97	-0.4798	0.1208	-3.97
1975	0.3087	0.1203	2.57	0.7886	0.2196	3.59
1976	0.0000	0.0000	0.00	-0.3087	0.1203	-2.57
1977	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1978	0.0000	0.0000	0.00	0.0000	0.0000	0.00
1979	0.0000	0.0000	0.00	0.0000	0.0000	0.00

ALL PARAMETERS ARE SIGNIFICANT

APPENDIX D11

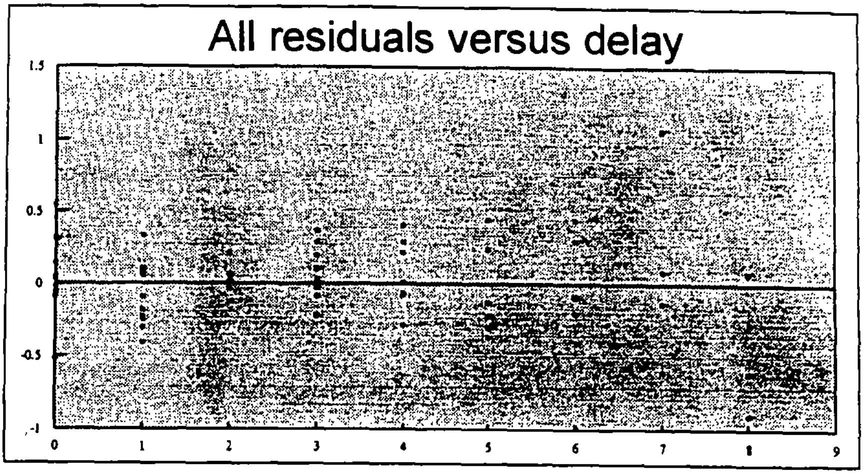

Figure D11.1

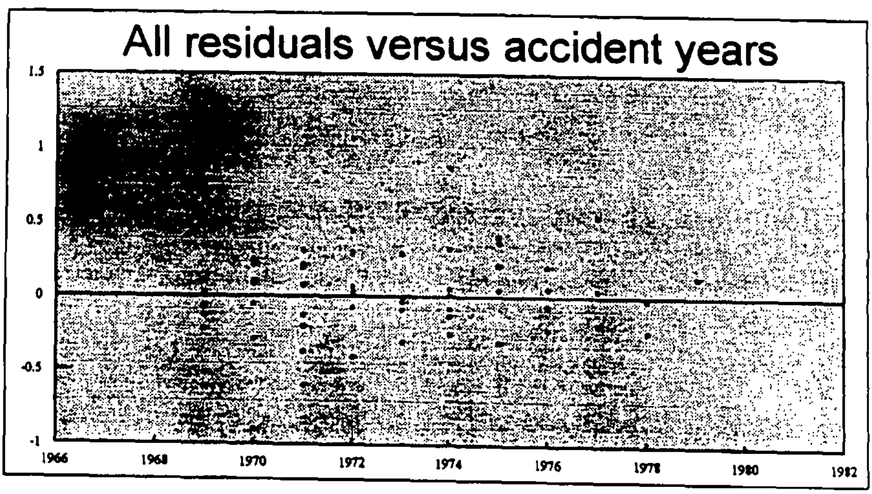

Figure D11.2

APPENDIX D11

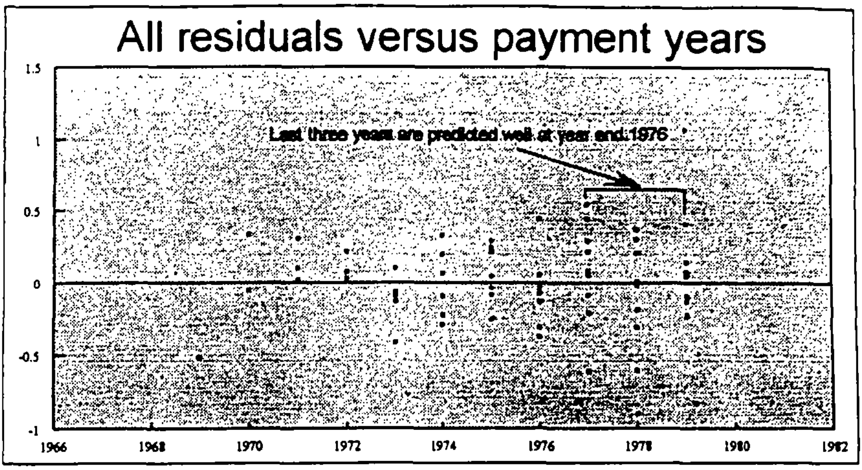

Figure D11.3

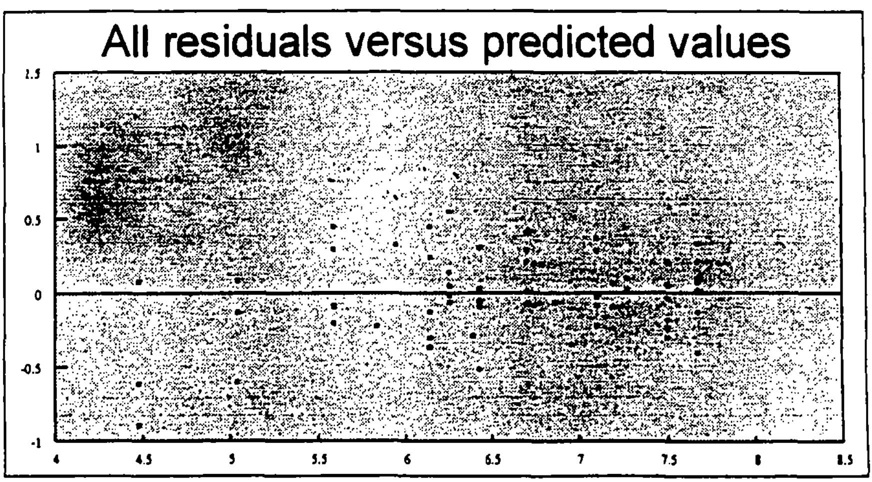

Figure D11.4

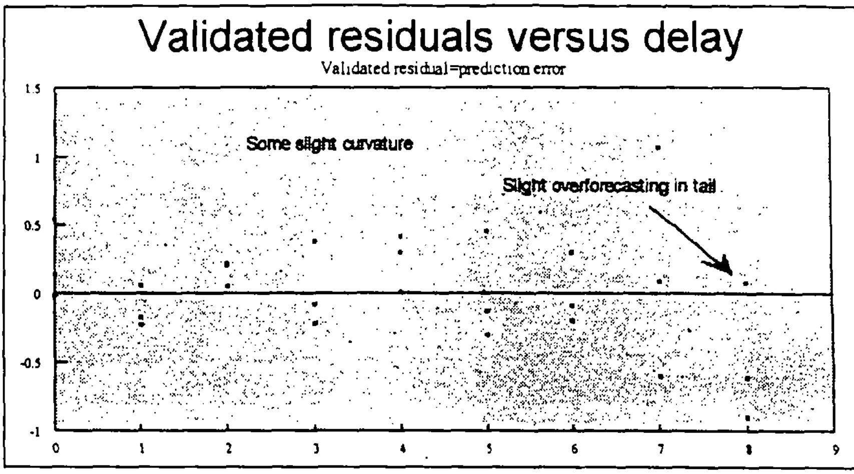

Figure D12.1

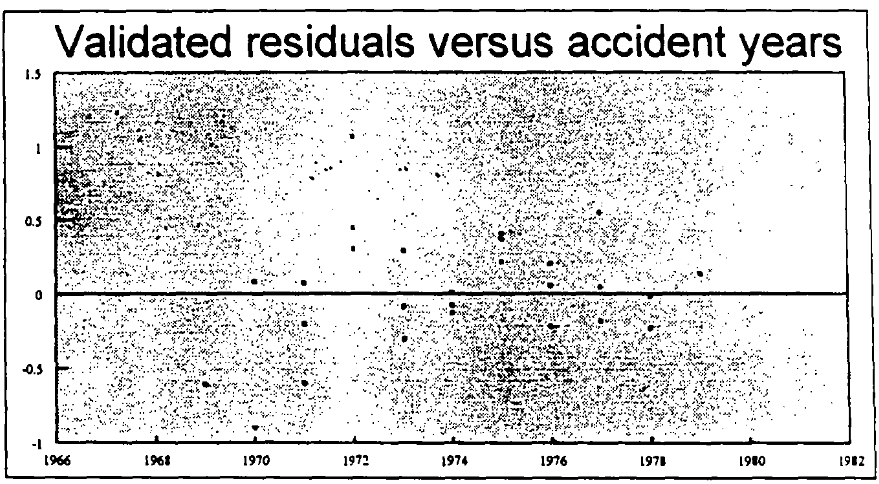

Figure D12.2

APPENDIX D12

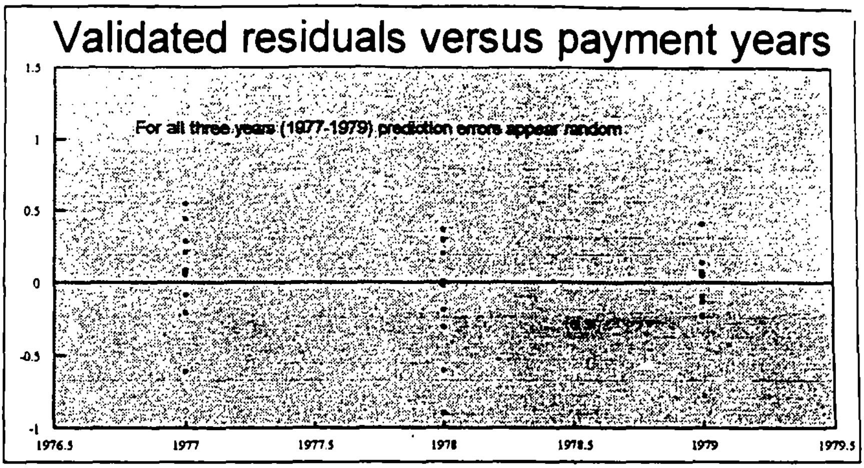

Figure D12.3

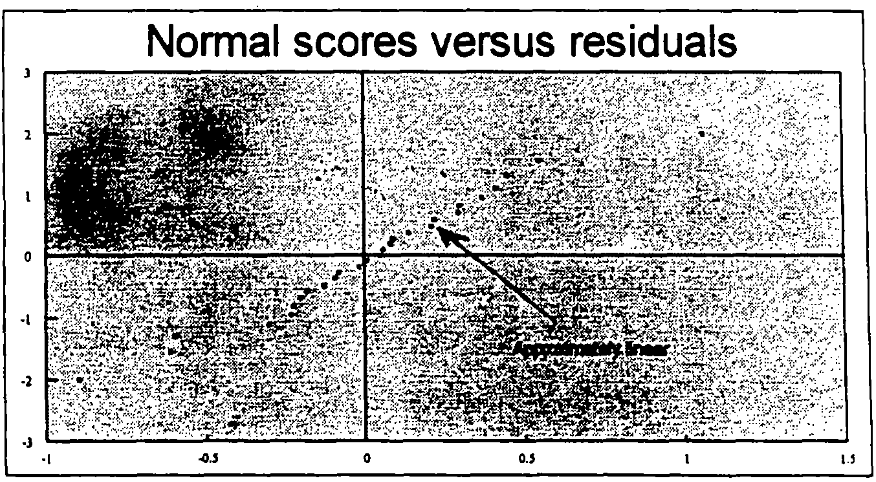

Figure D12.4

FORECASTING OUTPUT

.IDATION MODEL

ASSUMED FUTURE INFLATION = 0.0000

STANDARD ERROR = 0.0000

EXPECTED PAYMENTS/OBSERVED PAYMENTS +-------+ **(PAYMENTS IN $1 S)** **FORECAST MEAN PAYMENTS/STANDARD ERROR:**

.R											
9	EXP:	333078	1157384	1157384	774069	519825	184730	144344	83624	48721	0
	OBS:	193013	1584331	1151882	778980	475203	143352	128612	70845	25077	0
)	EXP:	409501	1422941	1422941	951676	395746	308064	177464	102811	59900	0
	OBS:	376473	1541950	1719509	1032570	289305	382508	270087	108354	23133	0
	EXP:	430518	1495969	1495969	620480	565416	323874	186571	108087	62974	0
	OBS:	568891	1579158	1277822	734670	680369	217221	147800	57099	64829	0
	EXP:	428607	1489330	925009	839313	562907	322437	185743	107607	62695	62695
	OBS:	428753	970640	955898	1095771	510072	491853	242995	299845	20985	20985
	EXP:	515220	1111935	1510497	1008922	676660	387595	223278	129353	75364	204717
	OBS:	582529	989072	1417606	953222	881133	278778	197156	36802	25226	52836
	EXP:	264794	1249101	1249101	834325	559561	320521	184639	106968	62322	353929
	OBS:	355229	948807	1292900	748003	547288	274367	46231	30433	20860	72717
	EXP:	275840	957831	957831	639774	429081	245780	141584	82025	47790	517179
	OBS:	282419	688332	1158793	903450	629983	58502	35451	23336	15996	85810
	EXP:	291971	1013844	1013844	677187	454173	260153	149864	86821	50584	1001596
	OBS:	267600	1044790	1216437	527644	113657	61923	'37524	24701	16931	147670
	EXP:	334450	1161346	1161346	775710	520250	298003	171668	99453	57944	1923027
	OBS:	560307	940002	1185899	184214	130192	70932	42983	28295	19395	262653
	EXP:	378003	1312583	1312583	876727	588000	336810	194023	112404	65489	3486036
	OBS:	360171	1011773	318349	208204	147147	80169	48581	31980	21920	452821
	EXP:	399511	1387267	1387267	926612	621456	355974	205063	118800	69216	5071655
	OBS:	445545	336463	336463	220050	155519	84731	51345	33799	23168	625691
FOR PAYMENT YRS:			4552200	3368314	2106825	1264545	700034	375411	184289	69216	12620833
NDARD ERRORS:			578766	453677	299060	193348	117101	77112	47562	23168	1072089

A IDI>

TABLE OF OBSERVED AND EXPECTED BY YEAR

ACC. YEAR	EXPECTED (PAYMENTS IN $1'S)	OBSERVED	DIFFERENCE	%ER	PMNT YEAR	EXPECTED (PAYMENTS IN $1's)	OBSERVED	DIFFERENCE	%ER
69	4403160	4551295	148135	3	69	333078	193013	-140065	-72
70	7251043	5743889	492846	8	70	1566886	1960804	393918	20
71	5289859	5327859	38000	0	71	3010843	3262723	251880	7
72	4753347	4695982	-57365	-1	72	4121587	4506400	384813	8
73	5434107	5175219	-258888	-5	73	4972020	4214487	-757533	-17
74	4477402	4166594	-310808	-7	74	3502693	3467526	-35167	-1
75	3260356	3662977	402621	10	75	4892575	4936092	43517	0
76	2996847	3056471	59624	1	76	4655693	4270279	-385414	-9
77	2657142	2686208	29066	1	77	4477648	5166110	688462	13
78	1690586	1371944	-318642	-23	78	4493854	4569353	75499	1
79	399511	445545	46034	10	79	4586481	4337196	-249285	-5