CASUALTY ACTUARIAL SOCIETY FORUM

Spring 1994

Volume One

Including Selected Papers from the
1994 Variability in Reserves Prize Program

CASUALTY ACTUARIAL SOCIETY
ORGANIZED 1914

CASUALTY ACTUARIAL SOCIETY

Date: March 1994

To: CAS Readership

Re: *The Forum*, 1994 Special Edition

This special edition of *The Forum* includes Stephen Philbrick's "Accounting for Risk Margins," a paper that was commissioned by the Committee on Reserves. Mr. Philbrick's paper is preceded by a brief introduction by Paul O'Connell, explaining the committee's charge in funding this paper.

The main body of this issue is devoted to ten papers submitted for the Committee on the Theory of Risk prize on how to measure the variability of loss reserves. Gary Venter gives an introduction and summary of the ten Theory of Risk papers. He ends his introduction by listing outstanding issues that merit further research.

As always, any submissions, question or comments may be directed to me, or anyone on the Committee on *The Forum.*

Very Truly Yours,

Joel Kleinman

Joel Kleinman
Chairperson, The Committee on *The Forum*

The Casualty Actuarial Society *Forum*

The Casualty Actuarial Society *Forum* is a non-refereed journal printed by the Casualty Actuarial Society. The viewpoints express in it do not necessarily reflect the views of the Casualty Actuarial Society.

The *Forum* is edited by the committee for the Casualty Actuarial Society Forum. The committee for the *Forum* invites all members of the CAS to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but should have content of interest to the CAS membership.

The *Forum* is printed on a periodic basis, based on the number of articles submitted. Its goal is to publish two editions during the calendar year.

Comments or questions about the *Forum* may be directed to the committee for the Casualty Actuarial Society Forum.

Stephen W. Philbrick's paper "Accounting for Risk Margins" was edited by the CAS Committee on Reserves.

The Theory of Risk call papers were edited by the CAS Committee on the Theory of Risk

VOLUME ONE OF TWO VOLUMES

Correction

The paper "A Quantification of Snader's Deductible Safety Factor," by John Rollins and Monty J. Washburn, which appeared in the 1994 edition of the *Forum, Including the 1994 Ratemaking Call Papers*, is a copyrighted paper. The copyright notice was inadvertently deleted in printing the *Forum.*

The copyright notice should have read as follows:

> Copyright 1993, National Council on Compensation Insurance. All rights reserved.
>
> The material contained in this paper represents the ideas of the authors and not necessarily those of the National Council on Compensation Insurance.

We regret the error, and apologize to all affected parties.

Spring 1994 CAS Forum

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Introduction

by Paul G. O'Connell

The CAS Committee on Reserves is pleased to present a funded research paper titled "Accounting for Risk Margins", authored by Stephen W. Philbrick. The committee's charge to Mr. Philbrick was to explore possible ways to adapt statutory and GAAP accounting to reflect formal existence of margins for adverse deviations in loss reserves. The focus was not to be on methods for calculating margins, but rather on proper accounting treatment for the calculated margin. In his paper he has accomplished this and more.

Mr. Philbrick demonstrates the conflict between profit recognition and the true economic reality of the insurance transaction under both current accounting principles and in an environment where losses are discounted at a risk-free rate. Through his research in this area, he has advanced a theoretical framework that addresses the appropriate accounting technique for reflecting and amortizing a risk margin, which when used with discounted loss reserves, results in a more accurate formula for profit recognition.

Mr. Philbrick's paper is a valuable addition to casualty actuarial literature. It is certain to prompt debate among actuaries, accountants and others as well as to inspire additional research on the appropriate method or methods for calculating risk margins.

The current and former members of the CAS Committee on Reserves who assisted on this project are as follows:

Neil A. Bethel (Chairperson 1991-1993)
Paul G. O'Connell (Chairperson 1994)
Raja R. Bhagavatula
Charles F. Cook
Linda A. Dembiec
Todd J. Hess
Russell T. John
Jeffrey P. Kadison
Stephen P. Lowe
Steven D. Marks
Richard J. Roth, Jr.
Mark J. Sobel
Chris M. Suchar
Ronald Swanstrom
John P. Tierney
Ronald F. Wiser

Accounting for Risk Margins

By Stephen W. Philbrick

Introduction

The importance of risk margins is growing rapidly. Not long ago, the subject of risk margins was not considered a burning issue within the actuarial profession, much less the insurance community at large. The recent insolvencies in the industry and the attendant search for causes and solutions, however, have led to heightened interest in risk margins.

Risk margins, whether in pricing or in loss reserves, have always been easy to understand superficially but difficult to pin down precisely. It is well known that case reserves and IBNR reserves are estimates of *unknown* future loss payments. Actual results will differ from estimated amounts, and the concept of risk margin reflects that fact.

A risk-averse individual or corporation would prefer a fixed liability of $100 to a liability whose expected value is $100, but whose actual payout amount is uncertain. "The greater the uncertainty, the larger the risk margin." Beyond that statement, however, there is little universal agreement. Methods vary not only for calculating uncertainty, but also for determining risk margins from uncertainty measures.

Furthermore, assuming a risk margin has been calculated, it is not obvious how that risk margin should be incorporated into statutory accounting, and it is arguable whether it should be incorporated at all.

The purpose of this paper is to explore how such a risk margin should be incorporated in statutory accounting. Rather than researching methods of calculating risk, this paper will assume that a satisfactory method for calculating risk margins will be separately developed.

Nevertheless, a discussion of the accounting treatment of risk margins can hardly proceed without a clear understanding of what we mean by the term "risk margin". Unfortunately, the actuarial profession not only needs to develop methods to calculate a risk margin, but it also needs to agree on a common definition. There are three common situations where the term "risk margin" is used: undiscounted loss reserves, loss portfolio transfers, and self-insurance trust funds. Each of these situations will be discussed briefly.

Undiscounted Loss Reserves

The term risk margin is commonly used in the observation that stating the loss reserve at nominal (rather than discounted) values provides an implicit risk margin. It is clear that the amount of the risk margin in this circumstance is the difference between undiscounted and discounted reserves. This observation, however, does not provide much insight into the purpose or definition of a risk margin. Compounding the problem is the fact that there is no general agreement regarding which discount rate should be used in such a calculation; thus, the specific value of the implicit risk margin is not clearly defined.

Loss Portfolio Transfers/Commutations

There is a market (albeit tiny) for loss reserves. A company retiring from business may sell a portfolio of reserves to another company. In theory, the amount of the purchase price less the present value of the expected payments represents a risk margin. Unfortunately, this does not provide a good empirical source for risk margin data because most of these transactions involve other factors, such as tax considerations and stop loss agreements. In concept, however, this provides one measure of a risk margin. Conceptually, a risk margin represents the difference between the expected (discounted) value of the reserves and the certainty equivalent of the reserves, where certainty equivalent means the amount certain one would accept (or pay) now in exchange for a stream of payments in the future whose amount and timing is uncertain. A lump sum payment in exchange for a portfolio of reserves will represent a certainty equivalent, if no extraneous items (such as taxes, expenses, side agreements or default risk) are involved.

A commutation is a special case of a loss portfolio transfer, where the loss reserves are transferred back to the ceding company. In most cases, one of the parties to the transaction may be in financial difficulty, which will distort the value agreed upon. If both parties are financially strong (and the commutation is not simply the unwinding of a financial reinsurance agreement), the terms of the commutation may provide insight into the value associated with the riskiness of the loss reserves.

Self-Insurance Trust Funds

The use of risk margins is fairly well developed in the case of Self-Insurance Trust Funds. From a financial structure standpoint, one notable difference between a traditional insurance company and a trust fund is that a trust fund typically does not have a capital or surplus account. Rather, it establishes a funding requirement such that the available funds correspond to the p^{th} percentile of the aggregate loss distribution, where p is typically 75% or 90%. The risk margin is the recommended funding requirement less the expected present value of the reserves. It is typical for an actuarial funding study to explicitly show the amount of the risk margin in the report. It is important to note, however, that this risk margin is not intended to represent the difference between the expected value of the reserves and the certainty equivalent.

Thus, while an actuary has a tool kit of methods to arrive at a best estimate of loss reserves, the goal of all such methods is the same. The actuary also has a variety of methods to calculate risk margins, however, the goals of the various methods are not all the same although all are termed risk margins.

Risk Margins in Canada

In developing recommendations for the calculation and accounting of risk margins, it may be helpful to review developments in other countries.

The Canadian Institute of Actuaries (CIA) has adopted a standard of practice which requires a provision for adverse deviations. The mechanics of the standard are quite different from the methodology outlined in this paper. The CIA methodology is heavily judgmental, requiring the actuary to select margins relating to three variables:

-claims development

-reinsurance recovery

-interest rate

Each margin is selected from a range (with options for selecting outside the range) based on a qualitative list of considerations. The claims development margin range is 0-15%, the reinsurance recovery is 0-25% and the interest rate margin is a downward adjustment, (which can vary by line of business) to the interest rate used for discounting. The range is from 50 basis points to 200 basis points.

The overall provision for adverse deviations is added to discounted liabilities.[1]

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[1]Memorandum to Fellows and Property/Casualty Actuaries of the Canadian Institute of Actuaries from the Committee on Property and Casualty Financial Reporting May 5, 1993. 1

Types of Risk Margin

It is critically important to recognize the potential differences in the type of risk margin that might be proposed in any attempt to standardize risk margin calculations. At least four different possibilities could occur.

1. **A risk margin based on a certainty equivalent concept**. Under this concept, a risk margin would be calculated such that, when added to a present-value reserve, it produces an estimate of the certainty equivalent value; that is, the amount of cash immediately payable to transfer the liability. This concept corresponds to loss portfolio transfers.

2. **A risk margin based on a theory of ruin concept**. Under this concept, a risk margin would be calculated such that the probability of insolvency or the expected cost of insolvency is reduced to an acceptable level. This concept corresponds to risk theoretic discussions of insurance enterprises.

3. **A risk margin based on probability intervals.** Under this concept, a probability, such as 75% or 90%, is specified. A risk margin is calculated such that the actual loss amount is less than or equal to the expected loss plus the risk margin in the specified proportion of times. These intervals are sometimes referred to as confidence intervals.[2] This concept is commonly used in trust fund analyses. A reasonable question is

[2]Technically, confidence intervals are intervals around parameters, while prediction intervals are intervals around results.

whether the probability intervals should be applied to discounted or undiscounted losses. The use of probability intervals is discussed in Loss Reserving for Solvency (David A. Arata [1983], PCAS LXX, P. 1). Although that article does not explicitly discuss discounting, it appears that the calculations are performed with undiscounted losses. This concept is also discussed in a CLRS presentation by Robin Harbage (1989 CLRS Transcripts, p. 1075). Again, it is not explicitly stated whether discounted values are used, but the context of the discussion implies that undiscounted values are used.

4. **A risk margin intended to simply provide a *relative* measure of risk.** It is conceivable that the actuarial profession may conclude that the calculation of a risk margin satisfying the goals of methods 1 or 2 is beyond current capabilities. Alternatively, the profession may decide that it is possible to design a measure of *relative* risk. For example, some fixed percentage of the aggregate loss variance might be proposed as a risk measure. This value would be higher for companies with more risk, thereby providing a relative measure. The absolute value of the measure, however, might not have a precise meaning. This is analogous to the concept of utility functions, which attempts to *rank* preferences, but not necessarily to ascribe a meaning to the absolute level of the utility function value. The implicit margin in undiscounted loss reserves corresponds to this concept, because the absolute amount of the risk margin does not result from risk theory, but lines of business considered to be riskier (i.e., long-tail) will tend to have relatively larger implicit risk margins.

Relevant Accounting Issues

Accounting for risk margins will be dependent on the concepts underlying the calculation. The implications for balance sheet and income statement are different for the different choices. In particular, a measure following the fourth concept of risk margin may not easily be transferable to a balance sheet and might have to be accounted for in a separate schedule.

Accounting for risk should be largely coincident with accounting for profit. An entrepreneur (or group of entrepreneurs) starts an insurance company with the intention and expectation of earning a profit.[3] An entrepreneur wishing to earn a profit without taking risk will find few opportunities. In order to earn a profit substantially in excess of rates available in the United States government securities, an entrepreneur must assume some risk.

[3]Exceptions arguably exist; there are some nonprofit organizations formed primarily for the purpose of providing difficult-to-obtain insurance. However, mutual insurance companies do operate as profit-maximizing firms, despite the blurred distinction between entrepreneurs and customers.

Types of Risk

An enterprise engaged in the business of insurance faces a wide variety of types of risks. Some of these risks include:

Underwriting risk—This is often defined as the risk that actual losses and expenses will exceed premiums. For the purpose of this paper, underwriting risk represents the possibility that discounted actual losses exceed the loss portion of the premium.

Investment risk—This term encompasses all risks related to the returns on invested assets. This risk is often subdivided into:

Interest rate risk–Possibility that asset values may drop due to a change in market yield rates

Default risk–Possibility of non-payment of interest and/or principal

Reinvestment risk–shortfall in investment income due to lower available yields for reinvested assets

Market risk–Generally referring to stocks or real estate; possible reductions in asset values due to changes market prices

Timing risk—the risk that the actual payout pattern of losses will differ from expected.

Reinsurance risk—The risk that reinsurance placed by an enterprise may not be collectible.

Credit risk—The risk that an insured might not pay all premiums due.

Risks are also classified as process risk or parameter risk. Process risk represents the possibility that actual results differ from expected results while parameter risk represents the possibility that the estimated expected differs from the true expected.

This paper will concentrate on underwriting risk. The assumption will be made that assets are invested in risk-free securities such as T-bills, where the durations are closely matched to the expected. This is not an optimal investment strategy, nor does it eliminate all investment risk. Some timing risk remains. However, there is an important interrelationship between timing risk and underwriting risk (as defined here). Sometimes a company can settle a claim for a smaller amount by settling it earlier that "expected". If the reduction in nominal costs equaled the discount associated with the length of time, this would produce no net change in underwriting risk calculated on a discounted basis. Conversely, a settlement for more than the "expected" amount arising from a protracted settlement might also have no effect on underwriting risk. In reality, these amounts will not precisely offset, however, it should be clear that at least some portion of timing risk is mitigated by the possibility that nominal settlements will be dependent on the timing of settlement. Major changes in timing can still have a deleterious effect on investment results when a portfolio has been precisely duration-matched. However, this is beyond the scope of this paper.

This paper will incorporate process risks and some aspects of parameter risks. Parameter risk, by its very nature, cannot be precisely estimated. Some aspects of the estimation process, such as model selection, are considered by many to be parameter risk, but are beyond the scope of this paper. Other significant issues, such as the possibility of a government takeover of the workers compensation system, might be considered parameter risk, but are almost certainly not incorporated into the pricing decision and are excluded from consideration in this paper.[4]

The largest single source of profit for a typical insurance enterprise is the assumption of underwriting risk.[5] Some companies earn income for operations that are not strictly labeled as underwriting. For instance, companies that provide fronting services earn profits for the assumption of credit risk. Other companies offer "unbundled services" and may sell services such as claims-handling and loss prevention without incurring underwriting risk.

The fact that "underwriting income" has been negative for the industry as a whole for many years does not mean that companies are not engaging in underwriting risk. The definition of underwriting income (excess of premium over expenses and losses in nominal dollars) is an anachronism, determined when the time value of money was a much smaller component of income (both because interest rates were lower than today and because the length of time between premium receipt and loss payment was shorter).

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[4]While it may appear obvious that our pricing mechanism does not formally incorporate the possibility of a government takeover of some portion of the insurance business, it is not as obvious as it sounds. To the extent that investors are truly worried about such an event, capital will be less likely to flow into the industry and the remaining capital may be able to command a higher rate of return that otherwise.

[5]Operations such as Berkshire Hathaway may be exceptions because of their large investment amounts relative to insurance operations.

To the extent that companies invest in securities with significantly higher risk characteristics than Treasury bills, they are truly earning income from their assumption of investment risks. For the purposes of this discussion, underwriting returns will be defined as the excess of premium over the sum of *discounted* expenses and losses, where expenses and losses are discounted at risk-free rates of return, over the time period between receipt of premium and payment of expenses or losses. Under this definition, the largest single component of insurance companies' returns will arise from underwriting returns.

Accounting rules—whether statutory, tax, or GAAP—do not explicitly state how the accounting for either the profit or risk margin should take place. Rather, accounting rules specify how to account for the various components of the insurance transaction (premium, expenses, losses, investment income) that drive the accounting treatment of the profit component. It should not be inferred, though, that the resulting accounting treatment of profit is of minor relevance. In fact, with respect to GAAP accounting, the determination of rules for accounting of the various elements is guided in part by whether the resulting income statement bears a close resemblance to "reality."

Accounting is needed whenever the timing of cash flows associated with a particular transaction occur at different times. In a classic goods manufacturing company, the initial cash flow is the capital inflow to the company. This is followed by cash outflow for capital equipment, which is followed by cash outflows for labor and materials. Finally, there is a cash inflow as the customer buys the product. Because the time frame typically encompasses several years, accounting principles were developed in order to more closely match revenues and expense. For example, the cash outflow for capital equipment is not expensed in the year purchased, but capitalized over some time frame, effectively charging a portion of the total cost to each year it provides service.

Similarly, an insurance company has an original cash inflow of capital, followed by cash inflows of premium and the cash outflows of expenses and losses. Accounting conventions, including such items as loss reserves, exist so that balance sheets and income statement more accurately reflect the economic reality of the corporation.

In any industry, accounting conventions are not expected to perfectly reproduce economic reality. Instead, the goal of closely approximating economic reality is balanced with the desire for reasonable simplicity, consistency and efficiency of performing the accounting. We might argue therefore that the theoretically correct way to depreciate an item of capital equipment is to precisely measure its life time and its yearly contribution to the business. These calculations would be expensive and are subject to dispute and manipulation, so accounting conventions exist to prorate the original cost of capital equipment over some fixed length of time which only approximates the actual useful lifetime. When significant changes occur to the environment (e.g., new classes of equipment), accounting conventions must be devised or revised to reflect the new situation. In any such situation, the goal is to promulgate accounting conventions such that the resulting accounting statements are a reasonable reflection of economic reality.

In the property-casualty insurance industry, accounting conventions have generally dictated that loss reserves should be established on a nominal basis, that is, without any reduction for the time value of money These accounting standards were established at a time when:

- reserves were smaller (relative to premiums) than today
- interest rates, and therefore the potential amount of discount, were much lower

The decision to carry reserves on a nominal basis was not justified on theoretical grounds, but rather, pragmatic ones. As additional evidence, it should be noted that life insurance has always formally incorporated the time value of money in its accounting. The length of time associated with life contracts has always been long enough that the simplicity arising from nominal reserves is far overshadowed by the material distortions which would result.

This situation has changed in three significant ways:

- The property-casualty industry has migrated from a predominantly property (i.e., short-tail) to a predominantly casualty (longer tail) book of business
- Individual lines of business have experienced a lengthening of the payment tail
- Interest rates, while lower today than a decade ago, are still well above rates prevalent over the first half of this century.

Each of these three changes has combined to increase the financial impact of the time value of money. Appendix A provides an analysis of the accounting of a single policy, starting with a low-interest scenario on short-tail business, and gradually changing assumptions to a level consistent with today's marketplace. The conclusions of that analysis are:

- if very short-tail business is written in a low interest environment, the timing of profit recognition arising from accounting rules roughly mirrors the pattern that corresponds to economic reality.
- if longer-tail business is written in a higher interest rate environment, accounting conventions significantly delay the recognition of profit. The accounting of a single policy implies that the business loses money in the year it is written, and profits are earned in subsequent years.

Pricing and Risk Margins

Before directly addressing the accounting for loss reserve risk margins it will be helpful to review basic assumptions associated with the pricing of a policy. Risk margins for loss reserves should not be considered separately from premium pricing issues, but rather as a different point on a continuum of a policy from inception to final loss payment.

A block of business is normally priced at a level intended to provide a sufficient profit after paying expected losses and expenses. Premium levels will be affected by many external events, but over long periods of time for the industry as a whole, it is reasonable to assume that the profit margins will be related to the amount of risk assumed by the company.

An insurance enterprise must be financially able to withstand actual loss payments in excess of expected payments. There are two ultimate sources of funds to provide for this contingency:[6]

1. Surplus[7] from investors, and
2. Profit margin from insureds

The term capital is sometimes used to refer to the original amount of assets provided by the investors (or subsequent infusions) as distinct form retained earnings. This paper will use the term surplus to refer to the entire amount of policyholders surplus, including original contributed capital, subsequent capital or surplus infusions and any retained earnings.

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[6]The absence of investment income is deliberate. For the purposes of this paper, we will assume that investment income does accrue to the insurance enterprise but that the amount is not under the control of the insurance company. Thus, if an insurance enterprise determines that it faces higher potential losses than contemplated in its current financial structure, it may raise capital, raise rates (or some combination) but it cannot choose to raise investment income rates.

[7]Theoretically, a third potential source is debt. Because debt is so rarely used in the insurance industry, it will not be treated in this discussion.

Appendix B contains a more in-depth discussion of the nature of the surplus and profit margin components, including a discussion of how the relative amounts are determined.

In the pricing context, there is little confusion about the term risk margin. While there are two sources of funds which pay losses in excess of expected losses, one of these sources, the profit margin, is generally considered to be the risk margin, and the other source, surplus, is not. When we turn to loss reserves, the situation is not as clear.

Loss Reserves and Risk Margins

The actuarial profession has not yet settled on a methodology to determine risk margins for loss reserves. It is possible, however, to consider conceptually what such a provision means without necessarily specifying the calculation procedure. A loss reserve margin is an amount needed over and above the expected (discounted) reserves to reflect the inherent riskiness of the reserves. While this description is obviously imprecise, it is difficult to refine it without specifying, or implying, a calculation methodology. For example, if we define loss reserve margin as that amount, which when added to the reserves, provides a total amount sufficient to pay actual losses with probability x% (where x might be 99 or 99.5), then we have essentially adopted a ruin theory approach to loss reserve margins.

Despite the vagueness of the definition, there is an important conclusion that can be drawn. Specifically, the normal use of the term "risk margin" in the context of loss reserves does not provide for a distinction between the two ultimate sources of fund—the insured and the investor. Indeed, some actuaries argue that a loss reserve margin should be merely an earmarked surplus item which is equivalent to implying that the source of the amount is the investor.

Risk Margin Calculation

For the purpose of this paper, it will be necessary to divide the total risk margin into two components:

- a loss reserve risk margin arising from the original profit margin
- an earmarked surplus amount

To avoid confusion between two distinct terms with the same name, the first of these two items will be referred to as the "narrow risk margin" (NRM) and the second of the two items will be referred to as the "surplus risk margin" (SRM)[8]. The sum of the two amounts will be referred to as the "broad risk margin" (BRM). The term NRM will be defined to mean that portion of the total risk margin which belongs "above the line", that is, the portion which theoretically should be considered a liability of the company rather than any part of surplus. The term SRM is defined to be that

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[8]This choice of terminology is deliberately based on the convention used to distinguish between two overlapping definitions of IBNR.

portion of the total risk margin which belongs in the surplus section of the balance sheet.

Calculation of these three elements will probably proceed in one of two ways: either the NRM and SRM values will be separately calculated directly, or the total BRM value will be calculated and apportioned into the two components. A methodology to directly calculate the NRM value might start by exploring what it would cost to "sell" the liabilities.

Assume that an efficient secondary market exists for loss reserves. The difference between the sale price and the best estimate reserves on a discounted basis would represent the narrow risk margin. In this case, this amount would identically be the amount that a company should carry as a liability (that is, above the line) in the normal case that it does not sell its reserves.

Several caveats should be noted. First, actual sales of loss reserves typically include aggregate limits on the amount that the assuming company will pay. The appropriate narrow risk margin must be calculated without any such limits.[9] Second, actual transactions often reflect a different tax situation between the ceding and assuming company. The narrow risk margin should be calculated exclusive of tax considerations. Third, this "thought experiment"doesn't specify how such a sales price should be calculated.

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[9]Obviously, existing policy limits and aggregates should be incorporated into the calculations.

Alternatively, industry risk margin calculations may, instead, directly calculate the broad definition, BRM. If the methodology is based on aggregate loss distributions or ruin theory approaches, it is likely that the resulting amount will be either the total amount (including loss reserves) needed as available assets to ensure the viability of the insurance enterprise or the amount needed in excess of the best estimate loss reserves. It is highly likely that in the second case, the amounts will be relative to *discounted* loss reserves. If the resulting margin is to be added to *nominal* loss reserves, an additional calculation will be required.

Once the broad risk margin is calculated, then the two components, NRM and SRM, can be calculated using formulas outlined in a later section.

Another possible approach to risk margin calculations is an adjusted discount rate calculation where the best estimate loss reserves are discounted at a rate less than prevailing market rates.[10] In this case, the difference between the reserves discounted at the adjusted rate and discounted at market rates will represent the risk margin. This calculation normally produces a narrow risk margin.

Many questions still need to be addressed. Subsequent sections will discuss:

- Transition from the current situation to the proposed situation
- Proper handling of the change in risk margin arising from a consistent application of the methodology

[10]In the case of property, it is conceivable that the appropriate adjustment to interest rates may produce "discounted" loss reserves slightly larger than the nominal amount.

- Proper handling of the change in risk margin arising from a change in the methodology used to calculate risk margins

  However, before we go into those issues, we should discuss specific examples with a proposed methodology for the calculation of the BRM and its components, NRM and SRM.

Assume that an insurance company has the opportunity to write a volume of business with expected nominal losses of $300.00. For simplicity, we will treat the entire business as if it consists of a single policy. This amount is chosen to keep the numbers in the exhibits manageable. The reader is invited to think of this as a surrogate for a more realistic number such as 30,000,000 or 300,000,000.[11]

We will examine three different scenarios, starting with an over-simplified example and moving toward a more realistic example.

In the first example, only one loss payment is made (at the middle of the policy period), the premium is paid at the beginning of the policy period, and only a single policy is written. This will be referred to as the SINGLE PERIOD model.

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[11]It would be preferable to examine a single policy added to an existing company. However, exploration of this alternative suggests that proper handling of a single policy requires analysis of the covariance of the individual policy with the remainder of the portfolio. It was felt that this complication would detract from the central theme of this paper, so the decision has been made to analyze the company as a whole. The correct treatment of covariance is important to the issue of pricing an individual contract, but less important to the accounting of the risk margin of a contract which has already been priced.

In the second example, loss payments are made at the middle of three calendar periods following the inception date. The premium is paid at the beginning of the policy year. Only one policy is issued; this will be referred to as the THREE PERIOD model.

Finally, we will assume that a policy identical to the one in the second example is written each year for three years. This allows the company to reach a steady state. Expected loss reserve reductions from expected payments exactly offset additions from the new policy. This will be called the STEADY STATE model.

Several assumptions will be common to all models:

- Policy period - One year
- Policy inception date - January 1
- Risk-free interest rate - 6%
- Company's desired rate of return on equity - 15%
- Expenses - none
- Taxes (federal and premium) - none

SINGLE PERIOD MODEL

Assume that a policy is issued whose undiscounted expected losses are \$300.00. All losses are paid at the middle of the year, so the present value of expected losses is \$291.39 ($300/1.06^{.5}$). The present value of *actual* losses may turn out to be less than or greater than this amount. This variability will be quantified by assuming that the aggregate distribution of losses is modeled by a lognormal distribution with mean \$291.39 and coefficient of variation (CV) equal to .128. To put this value in perspective, the probability that actual losses could 23% higher than expected is approximately 5%. In other words, there is roughly a 95% probability that actual losses will be less than \$359 (291 x 1.23).

The insurance company must provide for an amount of assets sufficiently large so that regulators and policyholders will be satisfied that the company is highly likely to pay any losses which arise under this policy. Obviously, it would be desirable if the company could be *certain* that it could pay losses under any scenario. However, with an unlimited potential for loss, no finite amount of assets can guarantee payment in all circumstances. Therefore, regulators and policyholders must be satisfied that non-performance is reduced to an acceptable level.

Non-performance can be measured in two important ways. The most common way is to measure the probability of non-performance, which is the probability that the available assets of the company are insufficient to pay the actual losses. This approach is often referred to as the "probability of ruin" approach. We could specify the acceptable probability of ruin, for example, 1%, and solve for the amount of assets necessary to cover the 99th percentile of the aggregate loss distribution.

Another way to measure non-performance is to measure the total cost of non-performance, which is the expected losses in excess of available assets. This approach will be referred to as the "expected deficit" approach. We specify the acceptable deficit (as a percentage of expected discounted losses) and solve for the amount of assets. If we specify an expected deficit of 2%, then we need to find the amount of assets such that the area of the aggregate loss distribution above the asset amount is 2% of \$291.39 or \$5.83. This asset amount is \$359.42[12] . Thus, the insurance company needs to provide \$359.42 in additional assets in order to write this policy.

The company can get some of these assets from the policyholder and some from the investors in the company.

The policyholder will provide assets by paying a premium. The amount of the premium will be equal to expected losses plus a risk premium. The risk premium will be called the narrow risk margin (NRM). The investor will supply surplus, which will be called the surplus risk margin (SRM). The sum of the NRM and the SRM produces the broad risk margin (BRM). The BRM is that amount of assets needed in addition to

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[12]The actual calculation involves solving the equation for expected deficit directly $\int_A^\infty (z - A)dF = .003$. In this equation, Z is a random variable representing the possible aggregate losses of the company, F is the cumulative distribution of aggregate losses, and A is the desired amount of assets required to satisfy the expected deficit criterion. In words find the value A such that the sum of losses in excess of A is .3% of the total. Alternatively, solve the following equation which represents the proportion of losses which can be covered by a company with assets equal to A.

$$\int_A^\infty z dF + A[1-F(A)] = 997$$

In both cases, F is the aggregate distribution of losses incorporating both process and parameter risks, and in this example, is a lognormal distribution with CV = .128.

expected losses to satisfy policyholders and regulators that the company is financially sound. Consequently, the BRM is equal to the total asset need ($359.42), less the expected losses ($291.39) which produces a BRM value of $68.03.

We now need to determine the relative contributions of the policyholder and the investor to the BRM. How is the BRM apportioned between NRM and SRM? The answer is that we solve for a value of NRM sufficiently large to provide the required rate of return (ROR). The solution (for the single period model) is:

$$NRM = \frac{(ROR - i) \times BRM}{1 + ROR}$$

Although this is the formula we would use in practice for a one-period model, this formula will be more understandable in a slightly different form. Using the relationship BRM = SRM + NRM, the formula for NRM can be written in terms of SRM as follows:

$$NRM = \frac{(ROR - i)SRM}{1 + i}$$

In words, the investor will supply the amount SRM and expects to earn a return on this amount at a rate of ROR. The SRM amount can be invested during the year at rate i, so the policyholder must supply the difference ROR-i. This amount is needed at the end of the year, so the amount required at the beginning of the year is discounted by ones year's interest (1+i).

Solving, we find in this specific example:

NRM = $5.32

SRM = \$62.71

BRM = \$68.03

Now we will examine how to handle the statutory accounting for this policy. Exhibit 1 contains the suggested handling of this policy in terms of the effect on existing lines in the balance sheet and income statement pages, as well as required additional lines. Exhibit 2 contains the identical information as Exhibit 1, as well as some supplemental information which will be used to better understand how the process works. This discussion will concentrate on Exhibit 2 for ease of explanation, but the conclusions apply to Exhibit 1.

In Exhibit 2, the relevant lines of the balance sheet and income statement are shown. The shaded lines contain the supplemental, explanatory material not intended to be included in actual financials. (A quick glance at Exhibits 1 and 2 will verify that the only difference between them is the shaded lines.) The lines in bold print are lines not currently in statutory financials that would be required if we incorporate discounting and loss reserve margins into the statement. The annual statement line numbers are shown on the left side of the exhibits. Four-digit numbers corresponding to write-in lines are used to indicate where the newly required information should reside if a current statement format is used. Obviously, if these recommendations are accepted, the organization and line numbering of these exhibits may change.

One-Period Model

Exhibit 1

An St Line #'s	Balance Sheet	At 12/31/X-1	At 1/1/X	At 12/31/X
	Assets			
1.	*T-Bills*	$62.71	$359.42	$0.00
	Liabilities			
1.	*Undiscounted Loss Reserve*		$0.00	$0.00
9.	*UPR*	0	$291.39	$0.00
2101.	***Reserve for Discount***		$0.00	$0.00
2101a.	***Discounted Loss Reserve***	0	$0.00	$0.00
2102.	***Risk Margin***	0	$5.32	$0.00
22.	*Total Liabilities*	0	$296.71	$0.00
26.	*Surplus*	$62.71	$62.71	$0.00

	Statement of Income	Year X
1.	*Premium*	$296.71
2.	*Undiscounted Incurred Loss*	($300.00)
8.	*Investment Income*	$12.70
0501.	***Change in Risk Margin***	$0.00
0502.	***Change in Reserve for Discount***	$0.00
16.	*Net Income*	$9.41
	Capital and Surplus Account	0
17.	*Surplus, Dec. 31 previous year*	$62.71
	Gains And (Losses) in Surplus	
18.	*Net Income*	$9.41
25(a).	*Surplus Adjustments: Paid in*	($62.71)
27.	*Investor Dividend*	($9.41)
31.	*Changes in Surplus for the year*	($62.71)
32.	*Surplus, Dec. 31 current year*	$0.00

NEWEX.XLS

One-Period Model

Exhibit 2

With Supplemental Information

An St Line #'s	**Balance Sheet**	*At 12/31/X-1*	*At 1/1/X*	*At 12/31/X*
	Assets			
1.	*T-Bills*	$62.71	$359.42	$0.00
	Liabilities			
1.	*Undiscounted Loss Reserve*		$0.00	$0.00
9.	*UPR*	0	$291.39	$0.00
2101.	***Reserve for Discount***		$0.00	$0.00
2101a.	***Discounted Loss Reserve***	0	$0.00	$0.00
2102.	***Risk Margin***	0	$5.32	$0.00
22.	*Total Liabilities*	0	$296.71	$0.00
26.	*Surplus*	$62.71	$62.71	$0.00

	Statement of Income		*Year X*
	Loss Portion of Premium		$291.39
	Risk Margin Portion of Premium		$5.32
1.	*Premium*		$296.71
2.	*Undiscounted Incurred Loss*		($300.00)
	Inv. Inc. (Discounted Loss Reserves)		$8.61
	Inv. Inc. (Risk Margin and Surplus)		$4.08
8.	*Investment Income*		$12.70
0501.	***Change in Risk Margin***		$0.00
0502.	***Change in Reserve for Discount***		$0.00
16.	*Net Income*		$9.41
	Capital and Surplus Account		
17.	*Surplus, Dec. 31 previous year*		$62.71
	Gains And (Losses) in Surplus		
18.	*Net Income*		$9.41
25(a).	*Surplus Adjustments: Paid in*		($62.71)
	Narrow Risk Margin Component		$5.32
	Inv. Inc. Component		$4.08
27.	*Investor Dividend*		($9.41)
31.	*Changes in Surplus for the year*		($62.71)
32.	*Surplus, Dec. 31 current year*		$0.00
	Investor Return	(line 18/line17)	15.00%

NEWEX.XLS

Balance Sheet

We presume that the required surplus must be in the company *before* writing the new policy. With an effective date of 1/1/X, this means we need to include the $62.71 of required surplus in the company on the previous day, 12/31/X-1. Before the policy is written, the balance sheet will contain only this surplus amount.[13]

Financial statements are typically calculated at year-ends, but we have shown a balance sheet on 1/1/X, immediately following the writing of this policy, to help follow the transactions. The premium of $296.71 is received by the company on January 1 and invested in Treasury bills. Consequently, the asset side of the balance sheet shows the surplus as well as the total premium, invested in Treasury bills. This total is $359.42. Technically, this entire amount of the premium should be established as an unearned premium reserve on that date. However, it will be difficult to trace the flow of the narrow risk margin if it is buried in the UPR, so we have placed the expected loss portion of the premium into the UPR and the narrow risk margin of $5.32 into a risk margin reserve.

[13]The mathematics would be slightly "cleaner" if we could assume that the risk margins are contributed on a discounted basis. However, it is unlikely that regulators would accept such a concept, so a convention is adopted where expected loss amounts are provided on a discounted basis while surplus and risk margins are provided on a nominal basis.

Income Statement

Now let us examine the income statement. The premium line (line 1) contains the entire premium of \$296.71. The shaded lines show that this total is comprised of an expected loss portion and a narrow risk margin. In the expected case, actual losses of \$300.00 are incurred and paid halfway through the year (line 2). The company earns investment income of \$12.70 during the year (line 8). The shaded lines show that this total is comprised of two amounts, 4.08 and 8.61. The first value arises from the 6% rate applied to the narrow risk margin and surplus amounts, which are held for the full year ((62.71 + 5.32) x .06 = 4.08). The second arises from investment income on assets supporting loss reserves.

These assets are only held for six months before the loss is paid, so the asset amount, \$291.39, earns 6% for six months or \$8.61. Note that the \$8.61, added to the \$291.39 precisely provides enough money to pay the \$300.00 loss. The net income (line 16) is the sum of these values, or \$9.41 (line 16).

Capital and Surplus Account

The remaining part of the exhibit shows the reconciliation of the surplus account. It starts at \$62.71 (line 17) at the previous year-end. The net income of \$9.41 (line 18) is an addition. In this example, the policy is now completed so the company can return the surplus to the investor. This is shown as a negative paid-in amount of \$62.71 (line 25(a)). At the end of the year, we would issue a stockholder dividend to the investor in the amount of the net income of \$9.41 (line 27). The shaded lines show that this total is comprised of the \$4.08 investment income earned on the surplus and narrow risk margin during the year, and the \$5.32 provided by the policyholder in the form of a narrow risk margin. If we then calculate the ratio of the net income to the surplus at risk, we find that the investor has earned a 15% return on the investment. This exhibit should help provide an understanding of the rationale behind the formula for determining the narrow risk margin. The investor can earn a "safe" return of 6% in Treasury bills. Instead the investor risks the surplus in an insurance enterprise with the expectation of a higher reward. The required return is 15% on the surplus. The surplus can be invested at rate *i*, so the remaining requirement (ROR-i) must be provided by the policyholder.

At the end of the year, all losses are paid and all surplus returned to the investor, so that balance sheet contains all zeros.

Of course, actual loss experience might be less than or greater than the expected loss experience. The excess amount, if actual losses are less than expected, will result in a higher rate of return. The short-fall, if actual losses exceed expected, will result in a reduced return or a need to use surplus to pay losses. The risk that the latter may occur is, of course, the reason that the investor can expect a return in excess of a risk-free rate.

While this example illustrates some of the dynamics of the situation, note that the lines for change in risk margin (line 0501) and change in reserve for discount (line 0502) contain zeros. This is correct in this example, because the reserves for these items are zero before the policy is written and zero after the last loss is paid, so the year-end changes are zero. We will next examine a policy with payments over a three-year period so we can better understand the dynamics of these items.

THREE PERIOD MODEL

In this example, assume a single policy is written on 1/1/X. The expected nominal losses are $600.00. The expected payout of these losses is $300.00, $200.00, and $100.00, with payments taking place at the middle of years X, X+1, and X+2. The expected losses, discounted at 6%, are $561.09.

Again assume that the actual nominal losses can be described by a lognormal model with mean of $561.09 and CV of .128. We will make the simplifying assumption that the overall amount is variable, but the timing of the payments is fixed. As calculated before, the total assets required is found by multiplying the expected losses by 1.233. This produces:

$$Total\ Asset\ Need = \$561.09 \times 1.233 = \$692.09$$

Arguably, not all of these assets are needed presently because some assets are needed for payments in the future. It is unrealistic, however, to assume that regulators would permit surplus to be promised at some future date. We will assume that all surplus must be supplied prior to the policy inception date.

As before, we expect the policyholder to pay a premium consisting of the expected losses $561.09 plus the NRM, while the investor will supply the SRM. The total asset need less the expected losses produces the sum of the NRM and SRM which we call the BRM:

$$BRM = \$692.09 - \$561.09 = \$131.00$$

The calculation of the apportionment between NRM and SRM is now more complicated. The investor must commit surplus, not simply for the upcoming twelve month period, but over the entire life of the policy, which extends until the last loss is paid. Similarly, regulators and policyholders require that assets over and above expected losses need to remain in the company until the last loss is paid. Denote L_t to be the expected unpaid losses at time t. The policy inception date will correspond to t=0, but subsequent values of t will correspond to year-end points in time. Make the simplifying assumption that the same aggregate loss distribution can be used to describe the unpaid losses at any point in time[14]. Then the total asset need at any time will be calculated the same way as at the beginning of the policy period:

$$\textit{Asset need at time } t = 1.233 \times L_t$$

For simplicity we will assume that the evaluation of asset need does not take place continuously, but at year-end points in time. For each time t, we can also calculate a value we denote BRM_t, which will be the total asset need, less the expected losses:

$BRM_t = (1.233 \times L_t) - L_t = .233L_t$

Now we have a sequence of future BRM_t values, each of which can be decomposed into an NRM_t and SRM_t. The formula for calculating the initial, as well as all subsequent NRM_t is:

$$NRM_t = (ROR - i)\sum \frac{BRM_t}{(1+ROR)^t}$$

In words, the future BRM_t values are discounted at rate ROR. The total of the discounted values is multiplied by (ROR - i), reflecting the fact that the BRM_t will earn investment income at rate i, and the policyholder must supply the remaining amount in order to pay the investor the required rate of return.

[14] This assumption will be discussed further in a later section.

Applying these formulas produces the following values:

BRM_0 = .233

L_0 =\$131.00

NRM_0 = \$16.14

SRM_0 = \$114.87

Now let us examine the financial statements corresponding to this policy. As before, two exhibits are provided. Exhibit 3 contains the proposed accounting. Exhibit 4 contains the identical information, as well as supplemental information in the shaded lines.

Balance Sheet (Year X)

On the day before the policy is issued, (12/31/X-1) the required surplus of \$114.87 is supplied (Exhibit 4 Line 1). On the following day, the total premium is paid, consisting of expected losses of \$561.09 and narrow risk margin of \$16.14, for a total premium of \$577.23. When added to the surplus, this produces total assets on 1/1/X of \$692.09. On that same day, a UPR reserve is established in the amount of \$561.09 (line 9) and a risk margin reserve of \$16.14 is established (line 2102).

During the year, a \$300.00 loss is paid. By year-end, the premium is earned, so the UPR reserve drops to zero. The remaining unpaid losses, on an undiscounted basis are

Three-Period Model

Exhibit 3

An St Line #'s	Balance Sheet	At 12/31/X-1	At 1/1/X	At 12/31/X	At 12/31/X+1	At 12/31/X+2
	Assets					
1.	*T-Bills*	$114.87	$692.09	$352.64	$119.81	$0.00
	Liabilities					
1.	*Undiscounted Loss Reserve*		$0.00	$300.00	$100.00	$0.00
9.	*UPR*	0	$561.09	$0.00	$0.00	$0.00
2101.	***Reserve for Discount***		$0.00	($14.11)	($2.87)	$0.00
2101a.	***Discounted Loss Reserve***	0	$0.00	$285.89	$97.13	$0.00
2102.	***Risk Margin***	0	$16.14	$6.77	$1.77	$0.00
22.	*Total Liabilities*	0	$577.23	$292.65	$98.90	$0.00
26.	*Surplus*	$114.87	$114.87	$59.98	$20.90	$0.00

	Statement of Income	Year X	Year X+1	Year X+2
1.	*Premium*	$577.23	$0.00	$0.00
2.	*Undiscounted Incurred Loss*	($600.00)	$0.00	$0.00
8.	*Investment Income*	$32.66	$15.25	$4.23
0501.	***Change in Risk Margin***	($6.77)	$4.99	$1.77
0502.	***Change in Reserve for Discount***	$14.11	($11.24)	($2.87)
16.	*Net Income*	$17.23	$9.00	$3.14
	Capital and Surplus Account			
17.	*Surplus, Dec. 31 previous year*	$114.87	$59.98	$20.90
	Gains And (Losses) In Surplus			
18.	*Net Income*	$17.23	$9.00	$3.14
25(a).	*Surplus Adjustments: Paid in*	($54.88)	($39.08)	($20.90)
27.	*Investor Dividend*	($17.23)	($9.00)	($3.14)
31.	*Changes in Surplus for the year*	($54.88)	($39.08)	($20.90)
32.	*Surplus, Dec. 31 current year*	$59.98	$20.90	$0.00

NEWEX.XLS

Three-Period Model

Exhibit 4

With Supplemental Information

An St Line #'s	Balance Sheet	At 12/31/X-1	At 1/1/X	At 12/31/X	At 12/31/X+1	At 12/31/X+2
	Assets					
1.	*T-Bills*	$114.87	$692.09	$352.64	$119.81	$0.00
	Liabilities					
1.	*Undiscounted Loss Reserve*		$0.00	$300.00	$100.00	$0.00
9.	*UPR*	0	$561.09	$0.00	$0.00	$0.00
2101.	***Reserve for Discount***		$0.00	($14.11)	($2.87)	$0.00
2101a.	***Discounted Loss Reserve***	0	$0.00	$285.89	$97.13	$0.00
2102.	***Risk Margin***	0	$16.14	$6.77	$1.77	$0.00
22.	*Total Liabilities*	0	$577.23	$292.65	$98.90	$0.00
26.	*Surplus*	$114.87	$114.87	$59.98	$20.90	$0.00

	Statement of Income	Year X	Year X+1	Year X+2
	Loss Portion of Premium	$561.09		
	Risk Margin Portion of Premium	$16.14		
1.	*Premium*	$577.23	$0.00	$0.00
2.	*Undiscounted Incurred Loss*	($600.00)	$0.00	$0.00
	Inv. Inc. (Discounted Loss Reserves)	$24.80	$11.24	$2.87
	Inv. Inc. (Risk Margin and Surplus)	$7.86	$4.00	$1.36
8.	*Investment Income*	$32.66	$15.25	$4.23
0501.	***Change in Risk Margin***	($6.77)	$4.99	$1.77
0502.	***Change in Reserve for Discount***	$14.11	($11.24)	($2.87)
16.	*Net Income*	$17.23	$9.00	$3.14
	Capital and Surplus Account			
17.	*Surplus, Dec. 31 previous year*	$114.87	$59.98	$20.90
	Gains And (Losses) in Surplus			
18.	***Net Income***	$17.23	$9.00	$3.14
25(a).	*Surplus Adjustments: Paid in*	($54.88)	($39.08)	($20.90)
	Narrow Risk Margin Component	$9.37	$4.99	$1.77
	Inv. Inc. Component	$7.86	$4.00	$1.36
27.	*Investor Dividend*	($17.23)	($9.00)	($3.14)
31.	*Changes in Surplus for the year*	($54.88)	($39.08)	($20.90)
32.	*Surplus, Dec. 31 current year*	$59.98	$20.90	$0.00
	Investor Return (line 18/line17)	15.00%	15.00%	15.00%

NEWEX.XLS

$300.00 (line 1). The present value of these losses at year-end is $285.89, so the difference between the discounted and undiscounted amounts is established as a reserve for discount of $ -14.11 (line 2101). The discounted loss reserve is shown as the sum of line 9 and line 2101 in line 2101a.

The required year-end risk margin and surplus are calculated as before—the expected losses are now $285.89, so the BRM_1 is:

BRM_1 = .233 x $285.89 = $66.75

NRM_1 = $6.77

SRM_1 = $59.98

Thus, we need to maintain $6.77 in the risk margin reserve and $59.98 in the surplus account.

Income Statement (Year X)

Looking at the income statement (Exhibit 4), we see that income includes a premium of $577.23 (line 1) comprised of the loss portion of the premium $561.09, plus the risk margin portion, $16.14. Incurred losses are shown on an discounted basis, and are equal to the total of payments made during the year, $300.00 and all expected future payments. This results in a total incurred loss amount of $600.00 (line 2). The company earns investment income of $32.66 during the year (line 8). This is comprised of two items, the amount earned on the assets underlying the loss reserves and the investment income earned on the risk margin and surplus. These two amounts are $24.80 and $7.86 (shaded lines).

In our previous example, the investment income associated with loss reserves added to the loss reserve precisely paid off the loss payment. In this example, the situation is slightly more complicated. The total discounted losses at policy inception of $561.09 is comprised of two components: $291.39 for the expected payment of $300.00 in the first year, and $269.70 for subsequent loss payments. Consequently, $8.61 of the investment income is combined with the $291.39 to make the midyear loss payment of $300.00. The remaining investment income, $24.80 - $8.61 = $16.19 is added to the $269.70 value to produce the needed loss reserve of $285.89 (line 2101a).

As a check, either confirm that $285.89 represents the correct expected value of the remaining two payments, or note that the original discounted amount of $269.70 should be multiplied by 1.06 to bring it forward one year.

The change in risk margin is $-6.77 (line 0501). This amount may appear puzzling at first. It is important to note the entire effect of the risk margin on the income statement. The entire risk margin of $16.14 is included in the income because it is a part of the earned premium (see shaded line above line 1). The $-6.77 ensures that not all of the original risk margin is included in income in this year. The total of these two items, \$16.14 - \$6.77 = \$9.37, represents the portion of the total risk margin that is included in the first year's income.[15]

The change in reserve for discount is \$14.11 (line 0502), representing the fact that the prior year-end reserve was zero, and the required reserve of \$-14.11 (line 2101) must be established. The sum of all the income items produces the net income for the year of \$17.23 (line 16).

Capital and Surplus Account (Year X)

The previous year-end surplus \$114.87 (Exhibit 4, line 17) is carried down from line 26 of the balance sheet. To this amount we will add the net income for the year, calculated above to be \$17.23 (line 16 as well as line 18). Now that some of the losses are paid, the remaining obligations require less total assets and less surplus to support possible fluctuation in actual payments. The new surplus requirement (SRM_1) is \$59.98. Thus, the difference between the surplus on hand, and the required surplus, is returned to the investor as a negative paid-in surplus amount[16]. This amount, \$114.87 - \$59.98 = \$54.88, is shown on line 25(a).

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[15] If we had assumed that the policy has an effective date of 12/31/X-1, the individual line entries would be different, although the total effect would be the same. This effective date of 1/1 was deliberately chosen to make sure the accounting treatment for this situation would be understood. The subsequent years treatment should be reviewed for understanding.

[16] In practice, the investor would leave the surplus in to support a *new* policy written in the second year.

We also paid the entire net income amount out to the investor as a stock dividend[17] (line 27). The shaded lines show that this dividend is comprised of the portion of the narrow risk margin that has been taken into income $9.37, and the investment income earned during the year on the surplus and risk margin, $7.86.

The bottom shaded line shows that the investor return, $17.23, divided by the surplus at risk during the year, $114.87, produces a return on surplus of 15%.

Balance Sheet (Year X+1)

In the second year, the loss payment of $200.00 is made, leaving an undiscounted loss reserve equal to the single remaining payment of $100.00 (Liabilities, line 1). The present value of this payment of $97.13 (line 2101a) leaving a reserve for discount of $-2.87 (line 2101). The total required assets are calculated as before:

$$Required\ Assets\ = 1.233\ x\ \$97.13\ = \$119.81$$

The BRM_1 is calculated as before:

$$BRM_1\ = .233\ x\ \$97.13\ = \$22.68$$

which is apportioned into its components:

NRM_1 = $1.77

SRM_1 = $20.90

Thus, the risk margin on the liability side becomes $1.77 (line 2101) and the surplus amount becomes $20.90 (line 26).

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[17]In practice, this amount, or a portion of this amount would be left in the company as retained earnings, to support growth in premium writings.

Income Statement (Year X+1)

No premium is received in this year, and our estimate of ultimate losses for prior years does not change, so the incurred losses are zero. Our assets earn a total of $15.25 during the year (line 8). This is comprised of $11.24 earned on assets supporting loss reserves and $4.00 earned on the surplus and risk margin (shaded lines). Once the losses are paid, we will need a smaller broad risk margin, hence a smaller narrow risk margin and a smaller surplus risk margin. Consequently, we can release $4.99 of the risk margin amount into income (line 0501). This is calculated as the difference between the required risk margin at the prior year-end, $6.66 and the amount needed at the current year-end, $1.77.

Similarly, we have to charge against income the need to increase the reserve for discount from $-14.11 to $-2.87. This increase in reserves of $11.24 is a charge to income of $-11.24 (line 0502). The net income for the year is the total of these items, or $9.00 (line 16)

Finally, we reconcile the surplus account. It started with $59.98 (line 17). We earned $9.00 in net income (line 18), returned the surplus not needed for the next year's operations, $-39.08 (line 25(a)), and paid an investor dividend equal to total net income of $9.00 (line 27). This amount is comprised of the $4.99 of the original narrow risk margin released into earnings this year plus the investment income on the remaining risk margin and surplus (shaded lines above line 27). The sum of these changes is the total change in surplus for the year, \$-39.08 (line 31), yielding the year-end surplus of \$20.90 (line 37), which is precisely the amount needed to support the remaining loss reserves. The investor dividend of \$9.00 divided by the surplus throughout the year of \$59.98, provides the investor with a return on invested surplus of 15%.

Balance Sheet (Year X+2)

In the third year, the final loss is paid, so at the end of the year, there is no need for reserves or surplus, and therefore no need for assets. Consequently, all values in the balance sheet are zero.

Income Statement (Year X+2)

There is no premium in this year. Loss payments exactly equal as expected, so incurred losses are zero. Investment income of \$4.23 is earned during the year (line 8). This is comprised of the \$2.87 earned on the assets supporting loss reserves of \$97.13, and the \$1.36 earned on the risk margin and surplus (Exhibit 4, shaded lines above line 8). Note that the sum of the investment income on loss reserve assets plus the reserve of \$97.13 provides enough money to pay the loss of \$100.00.

The required risk margin at the end of the year is zero, so the change in the risk margin of \$1.77 flows into income (line 0501). The \$-2.87 corresponds to the fact that the reserve for discount becomes zero (line 0502). The total of the income items is the net income for the year of $3.14 (line 16).

Finally, we reconcile the surplus account. The beginning surplus is $20.90 (line 17). Net income of $3.14 is added (line 18). All remaining surplus can be returned to the investor, $-20.90 (line 25(a)), and we pay a stock holder dividend to the investor equal to the net income (line 27). This dividend is comprised of the release of the remaining portion of the narrow risk margin $1.77, and the investment income earned during the year on the risk margin and surplus, $1.36 (shaded lines above line 27). The total dividend of $3.14, divided by the surplus invested throughout the year $20.90, provides the investor with a 15% return.

STEADY STATE

In this example, a policy is written every year. Each policy will be identical to the policy in the three period model. The nominal losses will be $600.00, paid out over three calendar years with $300.00 in the current year, $200.00 in the second year, and 100.00 in the third year. Because there are no losses outstanding after the third year, any point in time in the steady state model needs only to consider the current year, the prior year and the second prior year.

The results are summarized in Exhibit 5, (This exhibit is laid out in the same format as Exhibits 2 and 4.)

Three-Period Model

Exhibit 5

Steady State

An St Line #'s	Balance Sheet	At 12/31/X-1	At 1/1/X	At 12/31/X	At 12/31/X+1	At 12/31/X+2
	Assets					
1.	*T-Bills*	$603.45	$1,164.54	$603.45	$603.45	$603.45
	Liabilities					
1.	*Undiscounted Loss Reserve*	$400.00	$400.00	$400.00	$400.00	$400.00
9.	*UPR*	0	$561.09	$0.00	$0.00	$0.00
2101.	***Reserve for Discount***	($16.98)	($16.98)	($16.98)	($16.98)	($16.98)
2101a.	***Discounted Loss Reserve***	$383.02	$383.02	$383.02	$383.02	$383.02
2102.	***Risk Margin***	$24.68	$24.68	$24.68	$24.68	$24.68
22.	*Total Liabilities*	$407.70	$968.79	$407.70	$407.70	$407.70
26.	*Surplus*	$195.75	$195.75	$195.75	$195.75	$195.75

	Statement of Income		Year X	Year X+1	Year X+2
	Loss Portion of Premium		$561.09	$561.09	$561.09
	Risk Margin Portion of Premium		$16.14	$16.14	$16.14
1.	*Premium*		$577.23	$577.23	$577.23
2.	*Undiscounted Incurred Loss*		($600.00)	($600.00)	($600.00)
	Inv. Inc. (Discounted Loss Reserves)		$38.91	$38.91	$38.91
	Inv. Inc. (Risk Margin and Surplus)		$13.23	$13.23	$13.23
8.	*Investment Income*		$52.13	$52.13	$52.13
0501.	***Change in Risk Margin***		$0.00	$0.00	$0.00
0502.	***Change in Reserve for Discount***		$0.00	$0.00	$0.00
16.	*Net Income*		$29.36	$29.36	$29.36
	Capital and Surplus Account				
17.	*Surplus, Dec. 31 previous year*		$195.75	$195.75	$195.75
	Gains And (Losses) in Surplus				
18.	*Net Income*		$29.36	$29.36	$29.36
25(a).	*Surplus Adjustments: Paid in*		$0.00	$0.00	$0.00
	Narrow Risk Margin Component		$16.14	$16.14	$16.14
	Inv. Inc. Component		$13.23	$13.23	$13.23
27.	*Investor Dividend*		($29.36)	($29.36)	($29.36)
31.	*Changes in Surplus for the year*		$0.00	$0.00	$0.00
32.	*Surplus, Dec. 31 current year*		$195.75	$195.75	$195.75
	Investor Return	(line 18/line17)	15.00%	15.00%	15.00%

NEWEX.XLS

Most of the values in the exhibits can be calculated as the sum of the appropriate amounts in Exhibit 4 associated with each of the three years. For example, the undiscounted loss reserve is \$400.00 at each year-end. This is the sum of the unpaid amount of \$300.00 on the previous year's policy, and the remaining unpaid of \$100.00 on the second prior year's policy. (The current policy does not contribute to loss reserves, but rather to the UPR account.) The narrow risk margin is \$24.68 at each year end, corresponding to the sum of the three narrow risk margins in Exhibit 4.

Note that surplus adjustments (line 25(a)) are always zero. On December 31, there is a potential for a return of surplus, arising from the fact that losses have been paid during the year and therefore do not require surplus to support the loss reserves. However, a new policy will be written the following day, which will require surplus. In the steady state model, this return of surplus and the required additional surplus exactly offset. Of course, a company which is growing would require either additional surplus paid in, or (more likely) an increase in surplus by paying out to the investor less than the profit earned in the year (i.e., retained surplus).

ADVERSE RESULTS

To this point, the assumption has been made that actual loss emergence follows expected loss emergence. If actual experience always matched expected, there would be no need for risk margins (or, for that matter, insurance). It will be helpful to analyze what happens if actual experience deviated from the expected. This paper will only analyze adverse departures from expected. Positive deviations can, and do, occur but the proper accounting should be obvious from the following discussion. The discussion will focus on two types of deviation:

1. Actual payments not equal to expected payments
2. A change in the estimated outstanding reserve

Of course, these two types of changes often occur simultaneously, but they will be separated for the purposes of discussion.

Actual paid not equal to expected paid

This is the easiest type of deviation to handle. In the three-period model (Exhibit 4), expected payments are \$300, \$200, and \$100 for the three calendar periods. If actual paid losses differ from that amount, but *this does not affect the estimate of outstanding*, the only adjustments are that:

- Actual incurred (line 2) will be different, which will flow dollar for dollar through to
- Net Income (line 16), which will flow through to
- Investor Dividend (line 27) and will change the
- Investor Return (bottom line)

If the adverse results are large enough, i.e., greater than income, then it would require a surplus contribution (line 25(a)).

Risk margins are unaffected whenever the loss reserves are unchanged. It might seem reasonable that investment income should be affected, but note that assets are not reduced because investor dividends are reduced by the excess payments dollar for dollar. (Technically, there is a small impact, if loss payments are made mid-year and investor dividends and surplus replenishments are made at year-end.)

Change in estimated outstanding reserves

The more complicated situation occurs when there is a change in the estimate of outstanding reserves[18]. As noted above, the expected payments for the three-period model are \$300, \$200, and \$100. Let us suppose that, sometime during the second year, we determine that the best estimate of third year payments is \$200, rather than \$100. Exhibit 6 summarizes how this change flows through accounting statements. A comparison of Exhibits 4 and 6 will reveal that some items are different. Each entry in Exhibit 6 which differs from its counterpart in Exhibit 4 has a box around it.

Not surprisingly, there are no changes in any of the values up through 12/31/X. During year X + 1, the determination is made that nominal loss reserves should be set at \$200. This creates a change in the Undiscounted Loss Reserve (line 1). The Reserve for Discount and Discounted Loss reserves (line 2101 and 2010a) are similarly affected. The formula for risk margin is applied to the revised amounts. In this simplified situation (only one remaining payment), the required risk margin (line 2102) increases directly in proportion to the loss reserve change. The required surplus (line 26) is also increased. Changes in the loss reserve and required risk margin also change the total liability entry (line 22)).

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[18] Of course, this does not include reductions in reserves due to the payment of losses originally as expected, or reserve changes arising from a change in discount. It only covers changes due to a change in future expected payments.

Three-Period Adverse Model

Exhibit 6

With Supplemental Information

An St Line #'s	**Balance Sheet**	*At 12/31/X-1*	*At 1/1/X*	*At 12/31/X*	*At 12/31/X+1*	*At 12/31/X+2*
	Assets					
1.	*T-Bills*	$114.87	$692.09	$352.64	$239.61	$0.00
	Liabilities					
1.	*Undiscounted Loss Reserve*		$0.00	$300.00	$200.00	$0.00
9.	*UPR*	0	$561.09	$0.00	$0.00	$0.00
2101.	***Reserve for Discount***		$0.00	($14.11)	($5.74)	$0.00
2101a.	***Discounted Loss Reserve***	0	$0.00	$285.89	$194.26	$0.00
2102.	***Risk Margin***	0	$16.14	$6.77	$3.55	$0.00
22.	*Total Liabilities*	0	$577.23	$292.65	$197.81	$0.00
26.	*Surplus*	$114.87	$114.87	$59.98	$41.81	$0.00

	Statement of Income		*Year X*	*Year X+1*	*Year X+2*
	Loss Portion of Premium		$561.09		
	Risk Margin Portion of Premium		$16.14		
1.	***Premium***		$577.23	$0.00	$0.00
2.	***Undiscounted Incurred Loss***		($600.00)	($100.00)	$0.00
	Inv. Inc. (Discounted Loss Reserves)		$24.80	$11.24	$5.74
	Inv. Inc. (Risk Margin and Surplus)		$7.86	$4.00	$2.72
8.	*Investment Income*		$32.66	$15.25	$8.46
0501.	***Change in Risk Margin***		($6.77)	$3.22	$3.55
0502.	***Change in Reserve for Discount***		$14.11	($8.37)	($5.74)
16.	*Net Income*		$17.23	($89.91)	$6.27
	Capital and Surplus Account				
17.	*Surplus, Dec. 31 previous year*		$114.87	$59.98	$41.81
	Gains And (Losses) in Surplus				
18.	*Net Income*		$17.23	($89.91)	$6.27
25(a).	*Surplus Adjustments: Paid in*		($54.88)	$71.73	($41.81)
	Narrow Risk Margin Component		$9.37	$3.22	$3.55
	Inv. Inc. Component		$7.86	$4.00	$2.72
27.	*Investor Dividend*		($17.23)	$0.00	($6.27)
31.	*Changes in Surplus for the year*		($54.88)	($18.18)	($41.81)
32.	*Surplus, Dec. 31 current year*		$59.98	$41.81	$0.00
	Investor Return	(line 18/line17)	15.00%	-149.89%	15.00%

NEWEX.XLS

Note carefully that we can simply "change" our liabilities by calculating new values. However, simply calculating a new required surplus amount, which also affects the required asset amount (line 1) does not, by itself, change the surplus or assets. The other required changes will be discussed after the income statement changes.

The undiscounted incurred loss (line 2) is changed by the \$100 change in reserve estimate. Investment income *during this year* is unaffected, because we won't change any cash items prior to year end. A change in risk margin still flows into income, but the amount is less than in Exhibit 4. (We only get credit for a reduction in the nominal reserve from \$300 to \$200, rather than the anticipated change from \$300 to \$100). Similarly, the reserve for discount account is increased, but by a smaller amount.

Net income is now a loss of \$89.91, compared to a gain of \$9.00 in Exhibit 4. Now we inform our investor that we will be unable to pay a dividend this year (line 27). In fact, we do not have enough surplus to support our liabilities, so we need a surplus infusions of \$71.73 (line 25(a)).

The investor has earned a negative return in this year of almost -150% (bottom line). Note that, in the subsequent year, investment income amounts are higher, because the assets are higher. The investor earns exactly 15% on surplus in the final year, although this policy represents an overall loss.

Practical Considerations

Although the methodology for the calculation of risk margins outlined in this paper is straight forward, there are several practical reasons for choosing computationally simple algorithms. The most complex step of the process involves the determination of the aggregate distribution of losses and the solution of an equation which produces the BRM value. It is likely that a company would not determine an aggregate distribution for its entire portfolio of business directly, but would determine distributions for individual blocks or lines of business. It is reasonable to assume that aggregate distributions for a particular line of business do not change materially in a short period of time. Thus, it may be reasonable to determine an aggregate distribution for a particular line of business as a one time project and use the resulting factors for some period of years. Moreover, it may also be reasonable to assume that aggregate distributions for a particular line of business do not vary materially among companies writing similar business. Thus it may be possible for an industry wide effort to generate aggregate distributions for lines of business with individual companies either using these results or making adjustments based on their own specific characteristics.

It might be necessary to calculate different factors for different ages. However, I believe that we will find that the ratio of BRM values to expected losses does not vary materially by age. If so, then the same factor can be used for any year of outstanding, and in particular, for the entire block of outstanding losses combined. Alternatively, we may find that the factors vary over the first one or two years and then stabilize. In which case, we would have separate factors for the first year or two, and then an all-subsequent factor. A bit more problematic is the fact that aggregate distributions are likely to vary considerably based on the retentions kept by the company. It may be possible to specify an aggregate distribution applying to unlimited losses and an algorithm to adjust this distribution to various underlying retention and limit profiles. It will take some additional research to determine how robust these factors are with respect to the characteristics of the various lines of business, but it may be possible to generate a table of factors which could be applied to the outstanding reserves of a particular company.

Not only would this approach simplify the calculations of risk margins of individual companies, it would also allow/facilitate a comparison of risk margins among companies. If Company A has approximately the same amount of business as Company B, but has a higher risk margin on its financial statements, this may be due to the fact that Company A writes lines of business that require a higher risk margin. A better comparison would be to look at the specific factor that Company A uses for its auto liability outstanding reserves vs. the comparable factor for Company B.

GAAP ACCOUNTING

The discussion to this point has concentrated on the appropriate statutory accounting for risk margins. An obvious question is how GAAP accounting should incorporate risk margins.

With minor exceptions, the proposals for the statutory treatment of risk margins should apply equally to the appropriate GAAP treatment. This conclusion arises from the fact that the derivation of the proposed statutory accounting treatment was motivated by the GAAP accounting principle of matching revenues and expenses. GAAP theory suggests that the value of uncertainty, i.e. risk margins, should be incorporated into financial statements.

The fact that GAAP does not currently incorporate risk margins or discounting into property-casualty financial statements does not arise from theoretical but, rather, practical considerations. There are a number of practical issues, including calculation methodologies, which need to be resolved before the accounting profession decides to incorporate both elements into GAAP accounting.

The main difference between GAAP and statutory accounting with respect to risk margins is likely to be related to the handling of discounting. This paper suggests that statutory accounting should continue to deal with loss reserves on an undiscounted basis with a contra liability line reflecting the amount of discount. The reason for this is to preserve the existence of undiscounted liabilities in order to best track runoff of liabilities on a basis consistent with history. This amount will also allow continued calculation of traditional formulas and ratios such as are incorporated in the IRIS tests. GAAP accountants may conclude that it would be preferable to show the discounted liabilities as a single line item, with undiscounted liabilities disclosed in the notes to the financial statements, if these amounts are needed.

Alternative Treatment of the Risk Margin

The proposed treatment of the loss reserve risk margin is based upon several assumptions believed to be reasonable. However, it is possible to put forth arguments for a different set of assumptions, which might lead to different recommended accounting treatments. Alternatively, it is possible to accept the overall assumptions, yet reach a different conclusion regarding accounting treatment based upon criteria such as simplicity or ease of presentation. This section will explore some of these alternatives.

Earmarked Surplus

A key assumption is that an insured purchases an insurance policy to receive a service—the specific service being the elimination (or at least reduction) of the risk associated with the unknown losses covered by the policy. This paper argues that the performance of that service is not complete until the final loss is paid. Consequently, the earnings should be recognized over the same period. However, some may argue that the performance of the service is complete when the policy period has ended. Indeed, the concept of earned premium is consistent with that theory. Under this theory, the entire NRM should be included in income by the time the policy is fully earned.

But a different opinion on the time frame associated with the service provided does not change the fact that uncertainty in loss reserves exists after the end of the policy period. The NRM amount is still a necessary component of an adequate financial structure. Under this scenario, it would be a required component of surplus, rather than a liability item.

We have already determined that the SRM values belong in the surplus account. (In fact, we have presumed that they are identical to the surplus account. However, this paper does not treat other sources of risk such as asset risk. Inclusion of asset risk would require surplus amounts other than those supporting loss reserves.) If only the risk associated with loss reserves is included, then the NRM value and the SRM value can be recombined into the total BRM. This amount could be shown as an earmarked surplus item.

In summary, if one subscribes to the theory that the service under an insurance policy is complete by the end of the policy year, and agrees that discounting of reserves is appropriate, the following accounting treatment would follow:

- Start with the accounting treatment suggested in this paper, including the establishment of an undiscounted loss reserve and a contra-liability for discount (calculated at risk-free rates of return)

- Earn the NRM, along with the remainder of the premium, over the policy period

- Earmark the entire amount BRM_t within surplus, not just SRM_t, which recognizes that the asset needs are not reduced under this scenario. The NRM amounts are still required for financial adequacy, but they will be part of surplus rather than as a liability item.

Identification of an earmarked surplus amount may make sense in a scenario where risk- based capital concepts are incorporated into the financial statements. Such a treatment might include an earmarked surplus item for each category of risk-based surplus. In theory, the surplus account for an insurer would contain an earmarked amount for reserve risk and asset risk, with the remainder in a "free" surplus account. Presumably, regulators could put restrictions on the ability to pay dividends if the amounts would exceed the "free" surplus amounts.

Footnote or Alternative Schedule

The accounting treatment proposed in this paper would require only two items in a balance sheet - the total NRM value shown in the liability section and the SRM amount contained in the surplus account (optionally, this does not even need to be shown as a separate item). The alternative treatment discussed above only requires one item, an optional earmarked surplus account. It may be that regulators and other users of financial statements may wish to have additional information. This additional information should be included as a footnote or as a separate schedule, depending on the complexity of the additional information desired. For example it may be desirable to show NRM and SRM values for each accident year. Such a schedule might show the current totals of the NRM and SRM values as the sum of individual components, as well as an historical registry of values at prior valuation periods. This schedule would be especially useful if the methodology used to calculate the values changed over time, or if critical parameters such as ROR changed over time.

Other categorizations may also be desirable:

- Ongoing versus runoff business
- Gross, net and ceded business
- Special situations, such as environmental coverage

Transition Issues

This paper has outlined a methodology to account for loss reserve risk margins. An obvious question is how to get from here to there. Two methods of conversion are possible. One method is a complete restatement of the annual statement at a particular year-end. It may be necessary to restate the beginning year balances so that income for the year is stated correctly. The difference between the prior year end balance sheet and the restated beginning year would be reported as an extraordinary gain (or loss) due to an accounting change.

Another method would be to phase in the new accounting rules prospectively. This could be done on an accident year or policy year basis, although accident year would be preferred. Presumably, after some period of time (five to ten years), the remaining prior years would also be restated to avoid carrying multiple accounting conventions forever.

There are advantages and disadvantages to both methods. The first method, complete conversion, is "cleaner", but to the extent that some companies have understated their undiscounted liabilities by implicitly recognizing some time value of money, they would be forced to either book inadequate discounted reserves or admit openly that the undiscounted reserves were deficient. The phase-in approach would mitigate this problem, but would mean that annual statement accounting would be a hybrid for years to come. More research into the advantages and disadvantages of these alternatives (as well as possible other approaches) is needed.

In addition to transition issues, there are other reasons why the actual implementation of the recommended procedures does not have to precisely follow the theory. This paper has outlined an accounting treatment consistent with a theoretically correct calculation of NRM and SRM values at each point in time. It is important to keep the calculation of these values in context. Once the initial (NRM) value is calculated, the subsequent calculations of NRM_0 values dictates how the original amount is released into earnings. A different method for calculating NRM_t, $t \geq 1$ would only affect the timing of the release into earnings, not the absolute amount. As an analogy, consider the purchase of capital equipment. In general, the entire purchase price is not taken as an expense immediately. Instead, a depreciation schedule is established. The purpose of this schedule is not the depreciation amounts per se, but the time frame over which the expense is recognized. In theory, we might wish to create a schedule which shows how the equipment contributes to the corporation over its life time, and charge expenses on a proportional basis. Instead, an arbitrary lifetime is ascribed to the equipment, and the original purchase price is spread over this time period according to specified rules (e.g. pro-rata, double declining balance). For NRM values, we might decide to apply similar procedures. This would mean assigning periods of time to lines of business, and an associated schedule which might approximate a payout schedule. For example, we might assign a 10 year life to general liability, and select an exponential pattern. This schedule would be applied to the original NRM values at all future points in time.

Alternatively, a study might be completed which calculates total NRM values under a variety of circumstances. It might be possible to analyze the results and conclude that selected factors applied to discounted reserves will approximate the theoretically correct amounts.

Non-constant Variability of Loss Reserves

One assumption used in this paper is the assumption that an aggregate loss distribution appropriate at the time of pricing is also appropriate for the unpaid losses at future points in time. Mathematically, let L_o be a random variable which represents unpaid losses at time zero (i.e. the entire loss amount). Now consider the unpaid losses at some future time, L_t. Of course $E(L_t) < E(L_o)$, so the distributions cannot be identical. But if we assume that the unpaid loss amounts are scaled up to the same level as the original loss distribution, how would we expect the two distributions to differ? First, we might anticipate that some of the smaller, more predicable losses have been paid, leaving larger, potentially more variable O/S claim. With additional information we should anticipate a "tighter", less variable distribution. In total, the unpaid loss distribution will probably differ from the original, but it may not be obvious that the riskiness is materially different.

More important, while we might find that the shape of the unpaid distribution is significantly different than the original distribution, the important question is whether the expected deficit, as a ratio to the expected, has changed materially.

If the answer to these questions is that the expected deficits of the unpaid loss distributions are not materially different than the original, then it may be acceptable to use a simplifying assumption that the same distribution, or at least the same critical ratio values can be used for all ages. Keeping in mind that the goal is simply to determine the timing of the release of the original NRM value, we might conclude that expected deficit values as a ratio to expected amounts could vary as much as the age.

For example, the critical value used to calculate BRM_0 is 1.233. Suppose the corresponding value for BRM_t is a sequence of values each greater than 1.233. The formula for calculating NRM_0 still works. It will produce an NRM value somewhat higher than the original problem. In words, higher critical ratios arising from more variability of loss reserves means that more surplus is needed at future points in time to support the loss reserves. In order to produce our target ROR, we will need a higher NRM_0 value. In other words, if the line of business is significantly more volatile over its entire life span than the original example, the company is subject to more risk and can command a larger price in the marketplace.

Note carefully that insureds with more risk do not automatically imply a higher required ROR. Instead, the higher risk may require a higher surplus commitment with the same ROR. The target ROR will be related to the expected deficit, rather than to the risk of the individual insured.

SUMMARY

Current statutory and GAAP accounting rules performed adequately when investment income was modest and the length of time until loss payment was reasonably short. However, with high interest rates or longer payout patterns, current accounting rules provide a poor match between recognition of profit and the true economic reality of the insurance transaction. Specifically, current accounting rules artificially defer the recognition of profit.

A change in accounting rules to reflect loss reserves discounted at risk-free rates of return would over-correct. Specifically, this change would accelerate the recognition of profit (in the expected case) entirely into the initial policy year, despite the fact that risk exists with respect to loss reserves.

This paper outlined a procedure which provides a more appropriate balance between these two incorrect alternatives. It specifies a formula to determine how much of an overall asset requirement consistent with regulatory goals should be established as a narrow risk margin as a liability on the balance sheet. This formula provides for the proper release of the original profit margin into earnings over time.

This paper discussed specific methodologies for the calculation of margins (e.g., based on expected deficit). Other methodologies may exist or may be developed. More research is necessary to determine the proper choice of a methodology for the calculation of margins. Once that research is complete, an acceptable method for calculating and accounting for risk margins will exist.

Appendix A

It will be instructive to review the accounting treatment of the profit margin for a variety of assumptions. This discussion will trace how the timing of profit recognition differs under various assumptions, which will roughly correspond to changes in the insurance environment over time. This discussion will form a background and framework for a discussion of how accounting for loss reserve margins ought to be handled.

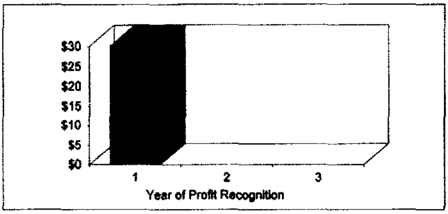
The first example considers a situation where investment income is earned at a rate of 3%. This forms a benchmark starting point. The following examples will also ignore expenses. While the treatment of expenses plays some part in the evolution of accounting rules, it is not particularly relevant to the issue of risk margins and therefore will not be discussed.

In Case 1, we assume that a policy is written with expected losses of $600 and a premium of $611.65. Interest is assumed to be 3%. Losses are paid at the end of the policy period.

In the first example, the premiums shown were calculated by selecting a paid amount of $600.00, an expected profit of 5% (or $30.00), and calculating the required premium. A $30.00 profit at the end of the year represents a 4.9% return on premium.

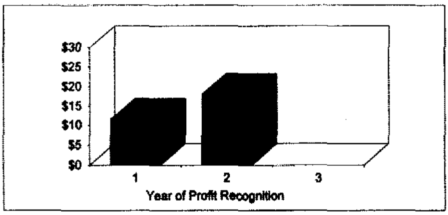
Case 1

Premium	$611.65		
Inv Inc.	$18.35		
Paid	$600.00		
Δ O/S	0.00		
Incurred	$600.00		
Profit	**$30.00**		



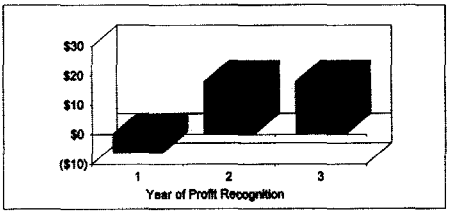
Case 2

Premium	$593.84	$0.00	
Inv Inc.	$17.82	$18.00	
Paid	$0.00	$600.00	
Δ O/S	600.00	(600.00)	
Incurred	$600.00	$0.00	
Profit	**$11.65**	**$18.00**	



Case 3

Premium	$576.54	$0.00	
Inv Inc.	$17.30	$18.00	$18.00
Paid	$0.00	$0.00	$600.00
Δ O/S	600.00	0.00	(600.00)
Incurred	$600.00	$0.00	$0.00
Profit	**($6.16)**	**$18.00**	**$18.00**



All other examples were constructed such that the expected profit (discounted to the end of Year 1 where necessary) is 4.9% of premium.

The policy is written on January 1, premium is earned evenly over the next twelve months, and losses are paid at the end of the year. As a consequence, the net income (i.e., profit) is earned proportionately throughout the policy period. The company has entered into a risk-taking enterprise. The result of the insurance policy could be a no-claim situation or a situation with a large claim. This first example will assume that, once the loss occurs, the amount is known with certainty. The risk is taken during the policy year and no risk exists at the end of the policy period. In this example, the incidence of risk undertaken by the insurance company and the emergence of profit precisely track each other.

In case 2, we continue the 3% interest rate, but we now assume that the losses are paid out over a longer time frame. Specifically, losses are paid 24 months after they are incurred. This represents the evolution of the industry from predominately property coverage to an increased proportion of longer tailed liability coverage.

When we inspect the graphs that identify when the profit is recognized, we see that a large proportion of the profit is recognized during the policy year, but now a fairly significant amount of the profit is recognized subsequent to the end of the policy period. This fact should not be troubling. The reserves could be too high or too low at any point up to the final payment period. Thus, the insurance company which has taken assumed risk from the insured for a fee finds itself with risk on its books until the last day of payment. It earns a profit over this period of time and statutory accounting has recognized profit over this entire period of time. It is possible to argue whether the exact heights of the bars in the graph precisely conform to the exact amount of profit that should be earned in each period, but we should all be in agreement that the general shape of the profit recognition curve conforms to what we think is the true economic reality of this company.

However, the last two or three decades have not conformed to assumptions such as this. Interest rates are higher and the average length of time to loss payout is longer. Let's relax these assumptions one at a time. In Case 3, we will assume that the loss payment period is lengthened to thirty-six months on average, but the interest rate remains at 3%. Under this scenario, the premium necessary to generate a reasonable profit will be is set at $576.54.

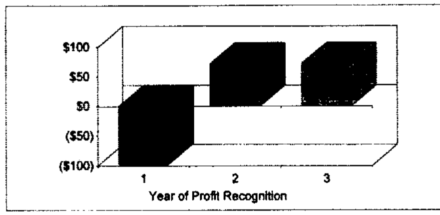
Now when we examine the graphs that show profit recognition, they begin to depart from our assumptions about the reality of the insurance transaction. Statutory accounting would generate a loss during the policy period followed by gains in the three subsequent years. While the overall profit is finally recognized, statutory accounting rules cause the profit to be earned much later than would conform to economic reality. We would believe that an insurance policy whose final loss payment does not occur for thirty-six months earns at least some of its profit in the two years after the policy period ends, but we would be hard pressed to argue that no profit, indeed even a loss, is incurred during the policy period. Obviously, losses can be incurred on particular policies that end up with losses for the entire period, but this example is intended to represent a situation that is profitable for its lifetime.

The situation becomes even more extreme if we increase the interest rate to 12%. Case 4 summarizes the situation where loss payments occur thirty-six months after occurrence, but interest is assumed at a 12% annual rate. This rate obviously exceeds rates available today, but is consistent with rates available in the early 80's. The policy generates significant losses during the policy period, which are offset by significant gains in the subsequent years. Of course, it should be recognized that overall <u>calendar</u> year results for an entire company would not look like this. Any particular calendar year would contain a mixture of new policies with a negative contribution to profit and older policies whose artificially high profits are running off. Eventually, a steady state situation might be reached, but in a situation where interest rates were growing and loss payment lags were increasing and the overall business volume was increasing, statutory accounting produced an understatement of income. Even if a steady state situation were reached, statutory accounting produces a poor match between reality and the accounting for any particular set of policies.

The insurance industry had a mixed reaction to this situation. On the one hand, there is a desire to report as high an earnings result as possible to satisfy the stockholders (in the case of stock companies) and to portray the financial strength of their companies to the policyholders, (in the case of mutual companies). On the other

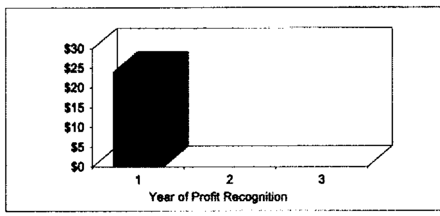
Case 4

Premium	$448.42	$0.00	
Inv Inc.	$53.81	$72.00	$72.00
Paid	$0.00	$0.00	$600.00
Δ O/S	600.00	0.00	(600.00)
Incurred	$600.00	$0.00	$0.00
Profit	($97.77)	$72.00	$72.00



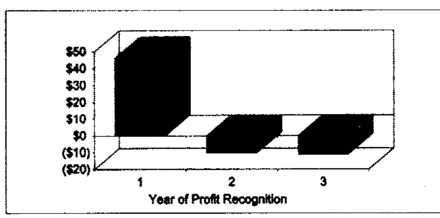
Case 5

Premium	$448.42	$0.00	
Inv Inc.	$53.81	$57.40	$64.29
Paid	$0.00	$0.00	$600.00
Δ O/S	478.32	57.40	(535.71)
Incurred	$478.32	$57.40	$64.29
Profit	$23.92	$0.00	$0.00



Case 6

Premium	$528.96	$0.00	
Inv Inc.	$31.74	$30.86	$33.33
Paid	$0.00	$0.00	$600.00
Δ O/S	514.40	41.15	(555.56)
Incurred	$514.40	$41.15	$44.44
Profit	$46.29	($10.29)	($11.11)



hand, a delayed recognition of profit meant a reduced, or at least deferred, tax bill to the extent that the Internal Revenue Service accepted statutory income as tax income. (Income for tax purposes was not identical to statutory income, but the magnitude of the other adjustments is rather small compared to the discounting issue.)

Not surprisingly, the Internal Revenue Service did not accept this situation quietly. For many years, the definition of income for tax purposes was virtually the same as statutory income. However, as the length of time to loss payment increased and interest rates increased, the use of nominal reserves artificially depressed statutory income to such a degree that the Internal Revenue Service decided to make changes. They decreed that reserves should be established on a discounted basis. (The precise formula for the discount is beyond the scope of this paper, but it was roughly based on T-bill rates.) The imposition of discounting dramatically changed the profit incidence. As Case 5 shows, under discounting almost all of the income for a policy is recognized in the first twelve months and only modest amounts of income are recognized in the following three years. While some might argue that the premium has been fully earned by the end of the policy period, it is certainly a fact that much uncertainty still remains at the end of the policy period and prior to the loss payments. If any one questions the existence of uncertainty, the following thought experiment should suffice.

Consider a company issuing a policy for $448.47 with expected losses of $600.00. This premium is set at a level which will generate a profit over its lifetime. If the company wished to sell this policy, that is, sell off the loss reserves at the end of the premium earning period, would they be able to sell it at a rate such that they would recoup the entire profits? I submit that they would only be able to earn a modest amount of profit by that period and the premium for transferring the remaining uncertain loss liabilities would be sufficiently high that a significant amount of the original profit would need to be transferred to the reinsuring company.

Finally, we look at Case 6. This also follows the tax accounting rules, except it approximates the situation in place right now where the discount rate for tax purposes exceeds the amount of interest a company can earn on T-bills. (This arises because the discount rate used for IRS calculations is based on 60-month average rates, so when rates are dropping rapidly the average rate exceeds the current rate.) In this situation, we see that, while the overall profit of the contract is unchanged, the tax accounting rules actually "recognize" more than all of the profit by the end of the policy period and then recognize losses in subsequent periods. An insurance company is in the unenviable position of having an artificially high income calculated for the purposes of determining income tax, but artificially low statutory accounting income to report to stockholders.

At low interest rates with short-tail lines of business, the true incidence of profit and the incidence of profit arising under statutory accounting are approximately the same. As the length of time between occurrence of the claim and payment of the claim increases, the incidence of reported profit and the incidence of true profit (that is, the price paid to transfer risk) remain approximately in sync. However, as the length of time gets fairly long, the comparison starts to get out of sync and at modestly high interest rates, statutory accounting rules generate a profit recognition picture which bears almost no resemblance to reality.

Tax accounting attempted to correct this distortion, but it has over-corrected. Discounting essentially forces the profit to be fully recognized during the policy period, despite the fact that an insurance company remains at risk substantially after the last dollar of premium is earned.

In summary, the failure to formally incorporate risk margins and recognition of discount produced only modest distortions between statutory recognition of profit and the true economic earning of profit when interest rates were low and most business was short-tailed. However, the situation is very different today. The recognition of profit under statutory and GAAP accounting is artificially deferred, and under tax accounting is artificially accelerated relative to the true economic incidence of profit. Simultaneous recognition of discounting and risk margins could correct these problems.

Appendix B

The theoretical justification for a loss reserve margin arises because the loss liabilities are uncertain. Economic values of assets are reduced when uncertainty is introduced; the economic value (or cost) of a liability is increased because of uncertainty. Before analyzing the uncertainty associated with loss reserves, it will be helpful to examine the treatment of uncertainty in the pricing situation. Then, by analogy, the reserving situation can be analyzed.

Consider an entrepreneur in the process of establishing a brand-new insurance company. Assume that the entrepreneur has identified a portfolio of insureds, whose losses can be described by a random variable $\mathbf{z}$, with distribution $\mathbf{F(z)}$. In order to obtain regulatory approval to start the company, as well as convince potential insureds to become customers, the company will have to show evidence of its ability to meet its financial obligations. This is normally done by starting the company with a certain amount of capital, C.

A ruin theory approach to solvency would require that the company have sufficient assets such that the probability of being unable to satisfy its obligation, i.e., probability of ruin is q, where q might be .01 or .005. Mathematically, we need to solve the following equation for A where A represents the total assets of the company (which, of course, equals the expected liabilities plus surplus):

$$\int_A^\infty dF(z) = q$$

Alternatively, we might specify an expected deficit requirement. Rather than simply being interested in the *probability* that a company is unable to meet some of its obligations, we may be more interested in the *cost* of insolvency. In this case, we want the sum of all losses in excess of total assets. An expected deficit requirement consistent with actual industry insolvency costs would be approximately .5% of premium or .3% of expected losses. Let ***d*** equal .003 times all expected unpaid losses.

Mathematically, we need to solve the following equation for A

$$\int_A^\infty (z - A)dF(z) = d$$

(For convenience, we will ignore operating expenses of the company and include LAE with loss.)

The company will charge a premium, ***P***. At start-up, the company will have assets equal to the premium plus the start-up capital:

$$\boldsymbol{A} = \boldsymbol{P} + \boldsymbol{C}$$

$\boldsymbol{P}$ is supplied by the policyholder and $\boldsymbol{C}$ is supplied by the stockholder. To determine these values, we need another equation. Note that the premium can be decomposed into two components — the expected losses, $\boldsymbol{E(z)}$, and a risk margin, $\boldsymbol{R}$ (ignoring expenses). Thus:

$$P = E(z) + R$$

Substituting:

$$A = E(z) + R + C$$

In words, the total assets of the company include an amount necessary to cover the expected losses, plus an amount necessary to cover the possibility that losses exceed the expected amount. On average (or over the long term), the company will pay $\boldsymbol{E(z)}$, leaving profits of $\boldsymbol{R}$ on capital of $\boldsymbol{C}$. Thus:

$$\frac{R}{C} = \textit{return on capital}$$

We can presume that the market will determine the acceptable level of profit for a company with this level of risk. Call this amount $\boldsymbol{ROE}$. In summary, the value $\boldsymbol{ROE}$ is fixed by the market, and the value $\boldsymbol{A}$ is fixed by the characteristics of the portfolio of risks. The two equations

$$A = E(Z) + R + C$$

$$ROE = \frac{R}{C}$$

simultaneously determine the total amount needed to start up the company, and the relative contribution needed from policyholder and stockholder. (It should be noted that ROE in the above discussion is a total return on capital, not an annual rate of return, unless the company expects to pay all its losses by year-end. The appropriate calculations reflecting annualized rates of return are handled in the main text of the paper.)

In the pricing context, there is no confusion regarding the term "margin." Although both $\boldsymbol{R}$ and C are needed to meet the solvency requirements, only $\boldsymbol{R}$ is considered the pricing margin.

To examine the reserving situation, we need to add some notation. Define $\boldsymbol{L_t}$ as the random variable representing losses unpaid at age $\boldsymbol{t}$. For example, $\boldsymbol{L_{24}}$ represents losses unpaid 24 months after the inception of the policy period. By definition, $\boldsymbol{L_o}$ represents the unpaid losses at the beginning of the period, so $\boldsymbol{L_o} = \boldsymbol{L}$. If we let $\boldsymbol{u}$ be the date the last loss is paid (ultimate), then $\boldsymbol{L_u} = \boldsymbol{0}$. For every value of $\boldsymbol{t}$, we have a random variable $\boldsymbol{L_t}$ and a distribution associated with that variable. We are typically interested in the inverse distribution at the value $\boldsymbol{q}$; that is, the dollars of loss unpaid at time $\boldsymbol{t}$ corresponding to the $\boldsymbol{q^{th}}$ percentile of the aggregate loss distribution. For notational convenience, we will define $\boldsymbol{G(t)}$ to be the inverse of the distribution function associated with the random variable $\boldsymbol{L_t}$ and cumulative probability $\boldsymbol{q}$. In most cases, we will not need to distinguish between various values of $\boldsymbol{q}$. If needed, we can extend the notation to $\boldsymbol{G(t,q)}$ to allow for varying values of $\boldsymbol{q}$.

Assume the new company starts with capital C and premium P. Assume no new business is written and we examine our company at time t. Some of the losses have been paid — the remaining unpaid are represented by the random variable L_t. Assuming that our criteria for solvency is unchanged, we would require total assets of $A_t = G(t)$ which, as before, can be written as:

$$A_t = E(L_t) + R_t + C_t$$

In words, the assets required for an insurance company at time t are the expected value of the liabilities plus an amount $(R_t + C_t)$ such that the probability of insolvency is sufficiently low. This latter amount is comprised of two quantities. The first, R_t, is the portion of the original profit margin which must still remain in the company. The second quantity, C_t, is the proportion of original capital required to remain in the company. We can presume that the amount $R - R_t$ has been returned to the stockholder in the form of a dividend and the amount $C - C_t$ has been returned as a return of capital. If we assume that the market rate of return is unchanged, then it is reasonable to assume that

$$\frac{R_t}{C_t} = ROE$$

which means that

$$\frac{R_t}{R} = \frac{C_t}{C}$$

or that the release of profit has been proportional to the reduction in the capital requirements.

It is important to note what assumptions are being made and what assumptions are not being made, at least at this time. If a block of business is priced at inception, use the following relationship to determine the overall required assets and the relative contributions from owner and policyholder:

$$A_o = E(L_o) + R_t + C_t$$

At time t, use the following relationship to determine the required assets which must remain in the insurer if the outstanding liabilities are retained or the amount of assets needed by another company if the liabilities are fully reinsured:

$$A_t = E(L_t) + R_t + C_t$$

Suppose that, at time t, the expected remaining losses are some proportion of the original expected. For concreteness, assume that this value is 25%. That is:

$$\frac{E(L_t)}{E(L_o)} = .25$$

We are not making the assumption that $G(t)$ is 25% of $G(0)$. Nor are we making the assumption that A_t is 25% of A_o. More importantly, we are not making the assumption that $(R_t + C_t)$ is 25% of $(R_o + C_o)$.

We might examine the characteristics of the business being written and conclude that the outstanding and IBNR and time t are more volatile than the overall block of business. In this case, we would conclude that, while

$$\frac{E(L_t)}{E(L_o)} = .25$$

the ratio of the required assets is greater than 25%:

$$\frac{A_t}{A_o} + \frac{G(t)}{G(0)} > .25$$

In this case, the amounts in excess of expected losses would also, in total, have a ratio greater than 25%:

$$\frac{R_t + C_t}{R_o + C_o} > .25$$

Additionally, from the formula for assets, we note that:

$$\frac{A_t}{A_o} = \frac{G(t)}{G_o} > \frac{R_t + C_t}{R_o + C_o}$$

Alternatively, if we conclude that the additional information provided by the paid data and outstanding case reserve information sufficiently improves our ability to estimate the ultimate cost of the unpaid liabilities, such that:

$$\frac{A_t}{A_o} = \frac{G_t}{G_o} > .25$$

The relationship between $R_t + C_t$ and $R_o + C_o$ will be determined by the characteristics of the business written. As we have seen, this ratio might be greater than or less than the ratio of the expected unpaid losses to the original expected losses.

On the other hand, the determination of the individual components of $R_t + C_t$ will depend on the market rate of return for runoff business of this type. Unfortunately, this market is very thinly traded, so empirical evidence may not be available to determine the appropriate conclusions. Instead, we will have to use more theoretical approaches.

It is tempting to conclude that, if the unpaid liabilities are viewed as more "risky" than the original business, the required ***ROE*** is higher. Equivalently, it is tempting to conclude that, if solid information on outstanding claims reduces the "riskiness" of the unpaid liabilities, the required ***ROE*** would be lower. This does not necessarily follow. If the outstanding liabilities are "risky," then the value of ***G(t)*** will be relatively high, which means a relatively large amount of assets is required, <u>but this can be satisfied by requiring more capital and leaving the ***ROE*** the same</u>. That is, riskier business either requires a higher return on a fixed amount of capital, or higher capital. If we conclude that we establish our company (either the original company writing the business or a company formed to reinsure the runoff) using a ruin criteria concept, that is, we fix the required probability of solvency q, then, to a high degree of approximation, we are assuming that riskier business requires more capital rather than a higher return on capital. Thus, it will be reasonable to assume that the ROE for a runoff situation will be the same as the original business and:

$$\frac{R_o}{C_o} = \frac{R_t}{C_t} = m$$

I suggest that we should adopt the convention that the term "loss reserve margin" should refer to R_t, not to the total of R_t and C_t. One reason for this suggestion is that we should not combine two very dissimilar concepts (a portion of the original premium provided by the policyholder, and a portion of the original capital supplied by the owner). Unfortunately, there is some precedence which is inconsistent with this convention. In many hospital trust funds, it is typical to establish a funding requirement consistent with the formula

$$A_t = E(L_t) + R_t + C_t.$$

This formula specifically refers to the funding requirement for one particular historical accident year (specifically, the one at age t). Conceptually, this formula is calculated for all open years plus possibly an amount for the upcoming year:

$$A_o = E(L_o) + R_o + C_o$$

In practice, all years other than the upcoming year may be handled as a group. More importantly, there are two related issues that distinguish this situation from the classic insurance company example.

First, the value of q is typically set at .75 or .90 rather than a value very close to one. This decision arises from the second distinction, which is that the policyholder and the owner are often identical. Because of this identity, a value of q can reasonably be chosen lower than that of a stand-alone insurance company. Also, because of the identity, the distinction between funds provided as a policyholder (R_t) and funds provided as an owner (C_t) is often not made. As a consequence, the fact that the term "loss reserve margin" is used to refer to the entire amount $R_t + C_t$ is not surprising.

The appropriate theoretical accounting for loss reserve margins is quite straightforward, given these assumptions. An insurance company would write premiums equal to P in year zero. This premium amount contains R_o of expected profit. The company would need to have capital C_o to support this business. At time $t = 1$, the end of the year, the company should book actual incurred losses (including an appropriate amount for IBNR). The company conceptually can return $C_1 - C_o$ to the owners, although in practice, this amount will be "rolled over" to support new writings in the next year. The company can "release" $R_1 - R_0$ into earnings. If actual loss experience exactly matches expected loss experience, then booked incurred losses will exactly equal $E(L_0)$, and a profit of $R_1 - R_0$ will be reported. To the extent that actual experience is better or worse than expected, so will the reported results. The company will maintain a "reserve" at year-end of R_1, which it will label the loss reserve margin. It will be carried above the line as a liability, not below the line as a part of surplus.

The introduction of this paper stated that the term "risk margin" requires careful definition because the term is used to refer to fundamentally different concepts. This appendix has laid a theoretical framework for terminology. As a result, issues in the introduction can now be stated in precise terms: When we refer to risk margins, it is important to clarify whether we are discussing only $\boldsymbol{R}_1$, or the larger quantity $\boldsymbol{R}_1 + \boldsymbol{C}_1$ (or perhaps some other quantity). In the terminology of the main text, $\boldsymbol{R}_1$ corresponds to the narrow risk margin and $\boldsymbol{R}_1 + \boldsymbol{C}_1$ corresponds to the broad risk margin.

Selected Papers from the Variability in Reserves Prize Program

Introduction

by Gary G. Venter

Selected Papers from Variability in Reserves Prize Program

This volume contains a selection of the papers submitted for the Committee on the Theory of Risk prize on how to measure the variability of loss reserves. Due to various constraints, not all of the submitted papers are included. Several of the excluded papers contained good analyses of reserving, but did not specifically address measuring variability. Others had some promising ideas not fully worked out into calculations. Hopefully these will be refined and submitted for publication in other venues.

Introduction

Several types of variation need to be accounted for to get a representative distribution of loss liabilities. Random fluctuation of the data around the expected value is generally called "process risk." Possible errors arising from estimating the mean, process variance, or parameters of any fitted model can be called "parameter risk." The standard error combines these two elements into a variance measure, and this is calculated in several of the papers. "Model risk" (sometimes called "specification risk") is an additional element of uncertainty arising from the possibility that the model assumptions themselves may be incorrect. A few papers attempt to quantify this as well.

The papers included here fell into three categories: Methods based on variance of link ratios; methods based on the collective risk model; and methods based on parametric models of development.

Methods Based on Variance of Link Ratios

Each age-to-age factor is a mean of several observed factors, so a variance can be calculated as well. Adding an assumption that the observed factors are samples from a lognormal, and that the ages are independent of each other, make the age-to-ultimate factors also lognormal, with readily computable variances. Both assumptions are possible to check, and adjustments can be made if they are too far off. The result is a distribution for the estimated liability for each accident year. Independence is important in that the product of the expected values is the expected value of the product for independent factors.

To get the distribution for the entire reserve, the distributions for the different accident years can be added by simulation or by matching moments of the sum. Normal, lognormal, and shifted gamma distributions are possible candidates for the summed distribution. Another one, not discussed in these papers but used by at least one committee member, is the shifted loglogistic $G(x - x_0) = F(x) = x^2/[x^2 + b^2]$. The moments for F are given by $E(X^n) = b^n(n/a)!(-n/a)!$. The variance and coefficient of skewness from these are the same for G, being unaffected by the shift x_0, and so they can be used to match the parameters a and b. Then x_0, is computed as needed to get the right mean for G.

Measuring the Variability of Chain Ladder Reserve Estimates, Thomas Mach

This paper tied for second prize in the competition. It contains a detailed discussion of what assumptions underlie the development factor (often called "chain ladder") method; i.e., the assumptions that make this method optimal, and how to test if they hold. This includes a test for correlation of age-to-age factors as well as for other assumptions of the chain ladder method. Standard errors are measured without assuming age-to-age factors are lognormal, but age-to-ultimate factors are assumed to be lognormal in any case. The version of the chain ladder here uses all observed factors to compute mean age-to-age factors, but the formulas can be converted to apply to using only the last n diagonals by just using the last n terms of those sums indexed from 1 to I-k.

Unbiased Loss Development Factors, Daniel M. Murphy

Variances of link ratios are derived from loss development triangle data using regression statistics measuring both process and parameter risk. Regression is presented as a generalized procedure which can be used to model age-to-age factors from loss development triangles. Many techniques currently in use can be viewed as types of regression models.

Murphy describes some of the main regression assumptions and illustrates how these assumptions can be tested and used to select an appropriate model. He then describes a recursive calculation of variances of ultimate losses based on the regression statistics. Although the introduction discusses three models frequently used to estimate loss development factors (weighted average development, simple average development and geometric average development), the calculation of variances is presented in detail for only two models: The least squares linear and the least squares multiplicative models. In actual practice, actuaries generally use the weighted average development or simple average development to estimate age-to-age factors. Using the paper's approach, the variances for the models more commonly used can be derived also, but the reader may need to derive the formulas from basic principles or refer to formulas (i.e. for weighted average factors) contained in an appendix.

Murphy presents the calculation of variances for multiple as well as single accident year ultimates. His formulas assume independence between development ages. Finally, he invokes the t-distribution to derive confidence intervals for the ultimate and the outstanding losses. In order to support the use of the t-distribution, he requires a further assumption that the variances are constant across development ages, which would need to be checked against actual data.

In addition to providing variance and confidence interval formulas, Murphy also uses a simulation procedure introduced by Stanard (*PCAS* 1986) to evaluate the bias and variance of eight development factor estimates. It would also have been informative if the simulations had been used to test the accuracy of the confidence interval estimates.

Correlation and the Measurement of Loss Reserve Variability, Randall D. Holmberg

An important issue for the development factor approach is potential correlation of link ratios. If they are correlated, the product of the age-to-age factors is not an unbiased estimate of the age-to-ultimate development, and the variance of the age-to-ultimate factor is understated. This paper provides a method to measure and adjust for correlation. The author suggests a simplified model in which the correlation ρ between a given age-to-age factor and the subsequent age-to-ultimate factor is constant for all ages. He then shows how this correlation can be estimated, and how it affects the reserve mean and variance. For the latter, an assumption on the distribution of the factors is made to simplify the computation, and here the uniform distribution is assumed. However, it would not be difficult to change to another distribution, just by plugging its density function and domain of definition into two integrals. The significance of the single ρ assumption is difficult to evaluate, and this area needs further support. The sensitivity to the distributional assumption would also be useful to know. It may in fact be possible to avoid the distributional assumption by using moment formulas for correlated variables, e.g.,

$E(AB) = E(A)E(B) + Cov\ (A,B)$, $V(AB) = V(A)V(B) + E(A)^2V(B) + E(B)^2V(A) + Cov\ (A^2,B^2) - Cov(A,B)^2 - 2E(A)E(B)Cov(A,B)$.

The paper at least touches upon several other important issues in the variance calculation. For the variance of tail factors it raises the possibility of expressing the standard deviation of a development factor as a linear function of |factor-1|, and applying this to the last actual factor and the tail factor. Correlation among accident years and among lines of insurance is also estimated.

Variability of Loss Reserves, Robert L. Brown

The effects of parameter risk and correlation among companies are illustrated in this paper, which looks at historical variability in loss reserves for a large sample of companies. Reserve adequacy for the entire sample showed a cyclical variation over time that would not be observed for a like sample of independently fluctuating companies. Reserve adequacy was found to vary by size of company as well, but the largest identifiable influence was consistent variation among companies: Some tended to be more adequately reserved than others over considerable periods of time, even taking into account all other impacts.

Methods Based on the Collective Risk Model

The basic idea of using the collective risk model to measure variability in loss reserves, as outlined in Roger Hayne's paper in the 1989 *PCAS*, is to estimate frequency and severity distributions for outstanding claims, and combine these to get an aggregate loss distribution for those claims. Hayne originally did this separately for reported and unreported claims.

A Method to Estimate Probability Levels for Loss Reserves, Roger M. Hayne

The earlier work by Hayne is expanded to include parameter uncertainty. This is broadly defined to include not just uncertainty about the parameters of a given model, but also the variability that can arise from using different modelling approaches. Significantly greater uncertainty in the reserves is found when this is taken into account. Hayne presents a detailed illustration of his procedure using a professional liability data set from the Berquist-Sherman 1975 *PCAS* paper and for an auto liability data set from the Advanced Case Study of the 1992 CLRS. These present straightforward techniques for estimating parameters of the claim count and severity distributions and require only a modest amount of data. The severity distribution parameters here are determined somewhat judgmentally.

It should be noted that the use of individual claim information would produce superior parameter estimates, although such information often is not available. It should also be noted that the selected lognormal coefficients of variation appeared to be low for this line of business. (However, Hayne notes that his example illustrates only one of many ways of selecting parameters and he provides some reasoning for his parameter selections). The most innovative contribution of this paper is the use of the results of different methods of estimating reserves to derive the mixing parameter for the severity distribution. This allows the actuary to incorporate specification error into the estimation of loss reserve variability. Once the claim count, severity, contagion and mixing parameters are estimated, the Heckman-Meyers procedure is used to compute the aggregate probability distribution for the loss reserves. Simulation could also be used to implement this approach.

A Note on Simulation of Claims Activity for Use in Aggregate Loss Distributions, Daniel K. Lyons

This paper suggests using severity distributions for both paid and case incurred losses at different valuations and annual probabilities of claims moving from one severity class to another (a transition matrix) to project claim movement over time. The severity distributions are incorporated into a simulation which 1) simulates the number of claims for a year, 2) simulates the report lag for each claim, 3) simulates the movement in each claim's value over time until an ultimate value is reached, and 4) works backward from the ultimate value of the claims to simulate their paid value. By simulating many years of data distributions of paid, incurred and outstanding losses can be produced. The procedure described in the paper could be used to approximate the process which underlies loss development when the losses are aggregated.

The author illustrates his method using severity distributions and transition matrices which have been judgmentally selected; i.e., not based on real data. To actually apply this technique, one would have to construct actual severity distributions and transition probabilities using techniques not described in the paper. The author's example applies to outstanding losses at the beginning of an accident or policy period, before any losses have been reported. While he mentions that in real life reserving situations, the actuary would need to determine outstanding loss severity distributions on a conditional basis, he does not describe how to do this.

Methods Based on Parametric Models of Development

The chain ladder method is characterized by having a separate level for each accident year and a separate degree of development for each lag. Thus models that have a parameter for each accident year and for each lag are regarded as statistical versions of the chain ladder. This can now be seen to be somewhat of a misnomer, as the assumptions of the usual chain ladder, as outlined in the Mack paper, are significantly different from these models. The logarithms of the losses in an incremental claims triangle (paid in year, for example) may meet the assumptions of regression analysis, which then can be used to estimate model parameters and provide variances.

Statistical Methods for the Chain Ladder Technique, Richard J. Verrall

This paper, which took first prize in the competition, gives a comprehensive presentation of the use of regression models to estimate loss development. It also lays out an interesting approach to adjusting lognormal maximum likelihood estimators for bias, and shows how to construct some Bayesian estimators relevant to the model. The paper does not note, however, that adjusting the MLE of the lognormal mean for bias involves some controversy, with different authors advising upward or downward or no adjustment. The Bayesian estimates discussed include estimation of runoff, estimates for the analysis of variance model, and relation to credibility theory. Relations to the chain ladder method are also discussed, and an excellent list of references is provided.

Probabilistic Development Factor Models with Applications to Loss Reserve Variability, Prediction Intervals and Risk Based Capital, Ben Zehnwirth

Loglinear versions of chain ladder, Cape Cod, and separation method ar all outlined. The paper, which tied for second prize, also addresses models that allow for changing parameters over time or smoothing of parameters to avoid multicollinearity. It contains a general discussion of statistical forecasting methods, and sufficient detail is given that many of the examples presented can be reproduced by the reader.

While many of the assumptions made are explicitly listed, there are a number of assumptions that are either unstated, or appear to be statements of fact. For example, in Section 5.0 there is a statement that the logarithm of paid losses at the earliest stage of development has a normal distribution with a mean α and variance σ^2. This is an assumption implicit in the main model here, but is not necessarily true in general. The assumptions about inflation also need to be carefully evaluated. Inflation is assumed to affect all payments in a given calendar year equally, but in fact losses at different stages of settlement might be affected differently.

Even though the model assumptions may not apply for every data set, this paper gives a comprehensive discussion of methods for fitting a regression model to development data and the testing of such a model for goodness of fit.

IBNR Reserve under a Loglinear Location-Scale Regression Model, Louis Doray

Most authors who use regression to model loss development assume that the initial data (incremental paid losses for example) is lognormally distributed. They take the logarithm of the initial data, and fit linear regression models to the logged data. The logged data is then normally distributed, and the error term (the difference between the fitted values and the logged data) is also normally distributed—hopefully with a reduced variance and zero mean.

The main thrust of this paper is to explore four possible distributions of the error term other than the normal distribution. In each case it presents the mathematics needed and tests the model against a common data set. Maximum likelihood estimation is used to estimate the regression coefficients. Because of the complexity of the various distributions used, and the need for various second derivatives of the log-likelihood function, the use of a computer algebra system would help implementation. Various issues regarding goodness of fit and bias of estimates are discussed. Possible bias of the maximum likelihood estimates is not discussed. The paper does suggest incorporating interest rate risk in presenting interval estimates for discounted reserves.

A comparison of estimates made by a regression method to estimates made using the chain ladder method shows that if the regression model is correct, the chain ladder method underestimates reserve needs. The correctness of the regression model is not verified, however. The issue of moving back to estimates of the unlogged data is also not addressed.

A Generalized Framework for the Stochastic Loss Reserving, Changsoeb Joe Kim

This approach measures variability by using goodness of fit from time series (ARIMA) models. It may, however, require a great deal of stability across accident years. The author uses a two dimensional auto-regressive procedure to estimate future incremental payments on a loss development triangle. The application of the procedure assumes that a constant auto-regressive parameter(s) applies to all accident years (which appears to be reasonable) and a constant parameter(s) applies to all development ages (which may not be reasonable). Thus, the payment at age 10 is assumed to be the same proportion of the age nine payment as the payment at age two is of the age one payment. This technique does not address the "tail" problem, or the estimation of payments at development ages greater than that in the historical data. Because the number of observations in most triangles is relatively small and time series techniques generally require a large number of points, the author uses standard loss development factors to convert the triangle into a matrix and derive initial values for the fitting process. Formulas presented for the n-year-ahead variance of the two dimensional auto-regressive process can be used to compute confidence intervals, presumably by using the standard normal distribution, but this is not explicitly stated. (It should be noted that the formula given for the one year ahead variance appears to actually be the formula for the two year ahead forecast variance).

Outstanding Issues

Several issues are still not addressed and could benefit from further research:

1. What techniques are appropriate for which situations and what kinds of data? For instance the regression techniques seem to require relatively stable, homogenous data. The development factor methods require enough observations in each column for a reasonable estimate of the variance of factors in the column. The collective risk model methods require estimates of claim count and severity parameters and these can best be derived from individual claim data. When these parameters are selected based on aggregate data or judgment, does the aggregate probability model reflect the additional uncertainty contributed by these less rigorous parameter estimates?

2. More work is needed on the "tail" problem. How does the actuary quantify the variability for development ages beyond the last observation in the data? The uncertainty associated with the tail can be substantial.

3. The impact of correlation needs further analysis. This includes correlation within the development triangle, among lines of business, with inflation, and with interest rates, especially for discounted reserved.

4. How does the actuary realistically reflect the uncertainty in reserve estimates for companies or lines of business with little or no data, or with recent changes in the data? It is reasonable to assume that the variability of such reserve estimates should be higher than for a company or line of business with abundant data. What about when different data sets are combined (company/industry, external indexes, etc.)?

5. What kind of testing is needed to truly validate the use of these models? Tests based on the triangle and fitted data can invalidate models, but failure to invalidate does not necessarily give much comfort for forecasting. An understanding of the assumptions used, and reflecting on their reasonableness may always be necessary, regardless of the fit provided.

6. How can the regression models be enhanced to incorporate a finite probability of no losses paid in a future period for given accident years? For small companies this is a realistic possibility, and should be reflected in prediction intervals.

Gary G. Venter

Measuring the Variability of Chain Ladder Reserve Estimates

by Thomas Mack

MEASURING THE VARIABILITY

OF CHAIN LADDER RESERVE ESTIMATES

Thomas Mack, Munich Re

Abstract:

The variability of chain ladder reserve estimates is quantified without assuming any specific claims amount distribution function. This is done by establishing a formula for the so-called standard error which is an estimate for the standard deviation of the outstanding claims reserve. The information necessary for this purpose is extracted only from the usual chain ladder formulae. With the standard error as decisive tool it is shown how a confidence interval for the outstanding claims reserve and for the ultimate claims amount can be constructed. Moreover, the analysis of the information extracted and of its implications shows when it is appropriate to apply the chain ladder method and when not.

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1. Introduction and Overview

The chain ladder method is probably the most popular method for estimating outstanding claims reserves. The main reason for this is its simplicity and the fact that it is distribution-free, i.e. that it seems to be based on almost no assumptions. In this paper, it will be seen that this impression is wrong and that the chain ladder algorithm rather has far-reaching implications. These implications also allow it to measure the variability of chain ladder reserve estimates. With the help of this measure it is possible to construct a confidence interval for the estimated ultimate claims amount and for the estimated reserves.

Such a confidence interval is of great interest for the practitioner because the estimated ultimate claims amount can never be an exact forecast of the true ultimate claims amount and therefore a confidence interval is of much greater information value. A confidence interval also automatically allows the inclusion of business policy into the claims reserving process by using a specific confidence probability. Moreover, there are many other claims reserving procedures and the results of all these procedures can vary widely. But with the help of a confidence interval it can be seen whether the difference between the results of the chain ladder method and any other method is significant or not.

The paper is organized as follows: In Chapter 2 a first basic assumption underlying the chain ladder method is derived from the formula used to estimate the ultimate claims amount. In Chapter 3, the comparison of the age-to-age factor formula used by the chain ladder method with other possibilities leads to a second underlying assumption regarding the variance of the claims amounts. Using both of these derived assumptions and a third assumption on the independence of the accident years, it is possible to calculate the so-called standard error of the estimated ultimate claims amount. This is done in Chapter 4 where it is also shown that this standard error is the appropriate measure of variability for the construction of a confidence interval. Chapter 5 illustrates how any given run-off triangle can be checked using some plots to ascertain whether the assumptions mentioned can be considered to be met. If these plots show that the assumptions do not seem to be met, the chain ladder method should not be applied. In Chapter 6 all formulae and instruments established including two statistical tests set out in Appendices G and H are applied to a numerical example. For the sake of comparison, the reserves and standard errors according to a well-known claims reserving software package are also quoted. Complete and detailed proofs of all results and formulae are given in the Appendices A - F.

The proofs are not very short and take up about one fifth of the paper. But the resulting formula (7) for the standard error is very simple and can be applied directly after reading the basic notations (1) and (2) in the first two paragraphs of the next chapter. In the numerical example, too, we could have applied formula (7) for the standard error immediately after the completion of the run-off triangle. But we prefer to first carry through the analysis of whether the chain ladder assumptions are met in this particular case as this analysis generally should be made first. Because this analysis comprises many tables and plots, the example takes up another two fifths of the paper (including the tests in Appendices G and H).

2. Notations and First Analysis of the Chain Ladder Method

Let C_{ik} denote the accumulated total claims amount of accident year i, $1 \leq i \leq I$, either paid or incurred up to development year k, $1 \leq k \leq I$. The values of C_{ik} for $i+k \leq I+1$ are known to us (run-off triangle) and we want to estimate the values of C_{ik} for $i+k > I+1$, in particular the ultimate claims amount C_{iI} of each accident year i = 2, ..., I. Then,

$$R_i = C_{iI} - C_{i,I+1-i}$$

is the outstanding claims reserve of accident year i as $C_{i,I+1-i}$ has already been paid or incurred up to now.

The chain ladder method consists of estimating the ultimate claims amounts C_{iI} by

(1) $$\hat{C}_{iI} = C_{i,I+1-i} \cdot \hat{f}_{I+1-i} \cdot \ldots \cdot \hat{f}_{I-1}, \qquad 2 \leq i \leq I,$$

where

$$(2) \qquad \mathbf{f_k} = \sum_{j=1}^{I-k} C_{j,k+1} \Big/ \sum_{j=1}^{I-k} C_{jk}\,, \qquad 1 \le k \le I-1,$$

are the so-called age-to-age factors.

This manner of projecting the known claims amount $C_{i,I+1-i}$ to the ultimate claims amount $\mathbf{C_{iI}}$ uses for all accident years $i \ge I+1-k$ the same factor $\mathbf{f_k}$ for the increase of the claims amount from development year k to development year k+1 although the observed individual development factors $C_{i,k+1}/C_{ik}$ of the accident years $i \le I-k$ are usually different from one another and from $\mathbf{f_k}$. This means that each increase from C_{ik} to $C_{i,k+1}$ is considered a random disturbance of an expected increase from C_{ik} to $C_{ik}f_k$ where f_k is an unknown 'true' factor of increase which is the same for all accident years and which is estimated from the available data by $\mathbf{f_k}$.

Consequently, if we imagine to be at the end of development year k we have to consider $C_{i,k+1}, \ldots, C_{iI}$ as random variables whereas the realizations of $C_{i1}, \ldots, C_{ik}$ are known to us and are therefore no longer random variables but scalars. This means that for the purposes of analysis every C_{ik} can be a random variable or a scalar, depending on the development year at the end of which we imagine to be but independently of whether C_{ik} belongs to the known part $i+k \le I+1$ of the run-off triangle or not. When taking expected values or variances we therefore must always also state the development year at the end of which we imagine to be. This will be done by explicitly indicating those variables C_{ik} whose values are assumed to be known. If nothing is indicated all C_{ik} are assumed to be unknown.

What we said above regarding the increase from C_{ik} to $C_{i,k+1}$ can now be formulated in stochastic terms as follows: The chain ladder method assumes the existence of accident-year-independent factors $f_1, \ldots, f_{I-1}$ such that, given the development $C_{i1}, \ldots, C_{ik}$, the realization of $C_{i,k+1}$ is 'close' to $C_{ik}f_k$, the latter being the expected value of $C_{i,k+1}$ in its mathematical meaning, i.e.

(3) $$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k\ , \quad 1 \le i \le I, \quad 1 \le k \le I-1.$$

Here to the right of the '|' those C_{ik} are listed which are assumed to be known. Mathematically speaking, (3) is a conditional expected value which is just the exact mathematical formulation of the fact that we already know $C_{i1}, \ldots, C_{ik}$, but do not know $C_{i,k+1}$. The same notation is also used for variances since they are specific expectations. The reader who is not familiar with conditional expectations should not refrain from further reading because this terminology is easily understandable and the usual rules for the calculation with expected values also apply to conditional expected values. Any special rule will be indicated wherever it is used.

We want to point out again that the equations (3) constitute an assumption which is not imposed by us but rather implicitly underlyies the chain ladder method. This is based on two aspects of the basic chain ladder equation (1): One is the fact that (1) uses the same age-to-age factor $\mathbf{f}_k$ for different accident years $i = I+1-k, \ldots, I$. Therefore equations (3) also postulate age-to-age parameters f_k which are the same for all accident years. The other is the fact that (1) uses only the most recent observed value $C_{i,I+1-i}$ as basis for the projection to ultimate ignoring on the one hand all amounts $C_{i1}, \ldots, C_{i,I-i}$ observed earlier and on the other hand the fact that $C_{i,I+1-i}$ could substantially deviate from its expected value. Note that it would easily be possible to also project to ultimate the amounts $C_{i1}, \ldots, C_{i,I-i}$ of the earlier development years with the help of the age-to-age factors $\mathbf{f}_1, \ldots, \mathbf{f}_{I-1}$ and to combine all these projected amounts together with $C_{i,I+1-i}\mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{I-1}$ into a common estimator for C_{iI}. Moreover, it would also easily be possible to use the values $C_{j,I+1-i}$ of the earlier accident years $j < i$ as additional estimators for $E(C_{i,I+1-i})$ by translating them into accident year i with the help of a measure of volume for each accident year. These possibilities are all ignored by the chain ladder method which uses $C_{i,I+1-i}$ as the only basis for the projection to ultimate. This means that the chain ladder method implicitly must use an assumption which states that the information contained in $C_{i,I+1-i}$ cannot be augmented by additionally using $C_{i1}, \ldots, C_{i,I-i}$ or $C_{1,I+1-i}, \ldots, C_{i-1,I+1-i}$. This is very well reflected by the equations (3).

Having now formulated this first assumption underlying the chain ladder method we want to emphasize that this is a rather strong assumption which has important consequences and which cannot be taken as met for every run-off triangle. Thus the widespread impression the chain ladder method would work with almost no assumptions is not justified. In Chapter 5 we will elaborate on the linearity constraint contained in assumption (3). But here we want to point out another consequence of formula (3). We can rewrite (3) into the form

$$E(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik}) = f_k$$

because C_{ik} is a scalar under the condition that we know C_{i1}, ..., C_{ik}. This form of (3) shows that the expected value of the individual development factor $C_{i,k+1}/C_{ik}$ equals f_k irrespective of the prior development C_{i1}, ..., C_{ik} and especially of the foregoing development factor $C_{ik}/C_{i,k-1}$. As is shown in Appendix G, this implies that subsequent development factors $C_{ik}/C_{i,k-1}$ and $C_{i,k+1}/C_{ik}$ are uncorrelated. This means that after a rather high value of $C_{ik}/C_{i,k-1}$ the expected size of the next development factor $C_{i,k+1}/C_{ik}$ is the same as after a rather low value of $C_{ik}/C_{i,k-1}$. We therefore should not apply the chain ladder method to a business where we usually observe a rather small increase $C_{i,k+1}/C_{ik}$ if $C_{ik}/C_{i,k-1}$ is higher than in most other accident years, and vice versa. Appendix G also contains a test procedure to check this for a given run-off triangle.

3. Analysis of the Age-to-Age Factor Formula: the Key to Measuring the Variability

Because of the randomness of all realizations C_{ik} we can not infer the true values of the increase factors $f_1, \ldots, f_{I-1}$ from the data. They only can be estimated and the chain ladder method calculates estimators $\mathbf{f}_1, \ldots, \mathbf{f}_{I-1}$ according to formula (2). Among the properties which a good estimator should have, one prominent property is that the estimator should be unbiased, i.e. that its expected value $E(\mathbf{f}_k)$ (under the assumption that the whole run-off triangle is not yet known) is equal to the true value f_k, i.e. that $E(\mathbf{f}_k) = f_k$. Indeed, this is the case here as is shown in Appendix A under the additional assumption that

(4) the variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \neq j$ are independent.

Because the chain ladder method neither in (1) nor in (2) takes into account any dependency between the accident years we can conclude that the independence of the accident years is also an implicit assumption of the chain ladder method. We will therefore assume (4) for all further calculations. Assumption (4), too, cannot be taken as being met for every run-off triangle because certain calendar year effects (such as a major change in claims handling or in case reserving or greater changes in the inflation rate) can affect several accident years in the same way and can thus distort the independence. How such a situation can be recognized is shown in Appendix H.

A closer look at formula (2) reveals that

$$\mathbf{f}_k = \frac{\sum_{j=1}^{I-k} C_{j,k+1}}{\sum_{j=1}^{I-k} C_{jk}} = \sum_{j=1}^{I-k} \frac{C_{jk}}{\sum_{j=1}^{I-k} C_{jk}} \cdot \frac{C_{j,k+1}}{C_{jk}}$$

is a weighted average of the observed individual development factors $C_{j,k+1}/C_{jk}$, $1 \le j \le I-k$, where the weights are proportional to C_{jk}. Like $\mathbf{f}_k$ every individual development factor $C_{j,k+1}/C_{jk}$, $1 \le j \le I-k$, is also an unbiased estimator of f_k because

$$\begin{aligned}
E(C_{j,k+1}/C_{jk}) &= E(E(C_{j,k+1}/C_{jk} | C_{j1},\ldots,C_{jk})) && \text{(a)}\\
&= E(E(C_{j,k+1} | C_{j1},\ldots,C_{jk})/C_{jk}) && \text{(b)}\\
&= E(C_{jk}f_k/C_{jk}) && \text{(c)}\\
&= E(f_k) && \\
&= f_k \ . && \text{(d)}
\end{aligned}$$

Here equality (a) holds due to the iterative rule $E(X) = E(E(X|Y))$ for expectations, (b) holds because, given C_{j1} to C_{jk}, C_{jk} is a scalar, (c) holds due to assumption (3) and (d) holds because f_k is a scalar. (When applying expectations iteratively, e.g. $E(E(X|Y))$, one first takes the conditional expectation $E(X|Y)$ assuming Y being known and then averages over all possible realizations of Y.)

Therefore the question arises as to why the chain ladder method uses just $\mathbf{f_k}$ as estimator for f_k and not the simple average

$$\frac{1}{I-k} \sum_{j=1}^{I-k} C_{j,k+1}/C_{jk}$$

of the observed development factors which also would be an unbiased estimator as is the case with any weighted average

$$g_k = \sum_{j=1}^{I-k} w_{jk}\, C_{j,k+1}/C_{jk} \qquad \text{with} \qquad \sum_{j=1}^{I-k} w_{jk} = 1$$

of the observed development factors. (Here, w_{jk} must be a scalar if $C_{j1}, \ldots, C_{jk}$ are known.)

Here we recall one of the principles of the theory of point estimation which states that among several unbiased estimators preference should be given to the one with the smallest variance, a principle which is easy to understand. We therefore should choose the weights w_{jk} in such a way that the variance of g_k is minimal. In Appendix B it is shown that this is the case if and only if (for fixed k and all j)

w_{jk} is inversely proportional to $\text{Var}(C_{j,k+1}/C_{jk}|C_{j1},\ldots,C_{jk})$.

The fact that the chain ladder estimator $\mathbf{f_k}$ uses weights which are proportional to C_{jk} therefore means that C_{jk} is assumed to be inversely proportional to $\text{Var}(C_{j,k+1}/C_{jk}|C_{j1},\ldots,C_{jk})$, or stated the other way around, that

$$\text{Var}(C_{j,k+1}/C_{jk}|C_{j1},\ldots,C_{jk}) = \alpha_k^2/C_{jk}$$

with a proportionality constant α_k^2 which may depend on k but not on j and which must be non-negative because variances are always non-negative. Since here C_{jk} is a scalar and because generally $Var(X/c) = Var(X)/c^2$ for any scalar c, we can state the above proportionality condition also in the form

(5) $\quad Var(C_{j,k+1}|C_{j1},\ldots,C_{jk}) = C_{jk}\alpha_k^2, \quad 1 \le j \le I, \ 1 \le k \le I-1,$

with unknown proportionality constants α_k^2, $1 \le k \le I-1$.

As it was the case with assumptions (3) and (4), assumption (5) also has to be considered a basic condition implicitly underlying the chain ladder method. Again, condition (5) cannot a priori be assumed to be met for every run-off triangle. In Chapter 5 we will show how to check a given triangle to see whether (5) can be considered met or not. But before we turn to the most important consequence of (5): Together with (3) and (4) it namely enables us to quantify the uncertainty in the estimation of C_{iI} by $\mathbf{C_{iI}}$.

4. Quantifying the Variability of the Ultimate Claims Amount

The aim of the chain ladder method and of every claims reserving method is the estimation of the ultimate claims amount C_{iI} for the accident years i = 2, ..., I. The chain ladder method does this by formula (1), i.e. by

$$\mathbf{C_{iI}} = C_{i,I+1-i}\cdot\mathbf{f_{I+1-i}}\cdot\ldots\cdot\mathbf{f_{I-1}} \; .$$

This formula yields only a point estimate for C_{iI} which will normally turn out to be more or less wrong, i.e. there is only a very small probability for C_{iI} being equal to $\mathbf{C}_{iI}$. This probability is even zero if C_{iI} is considered to be a continuous variable. We therefore want to know in addition if the estimator $\mathbf{C}_{iI}$ is at least on average equal to the mean of C_{iI} and how large on average the error is. Precisely speaking we first would like to have the expected values $E(\mathbf{C}_{iI})$ and $E(C_{iI})$, $2 \leq i \leq I$, being equal. In Appendix C it is shown that this is indeed the case as a consequence of assumptions (3) and (4).

The second thing we want to know is the average distance between the forecast $\mathbf{C}_{iI}$ and the future realization C_{iI}. In Mathematical Statistics it is common to measure such distances by the square of the ordinary Euclidean distance ('quadratic loss function'). This means that one is interested in the size of the so-called mean squared error

$$\mathrm{mse}(\mathbf{C}_{iI}) = E((C_{iI} - \mathbf{C}_{iI})^2 | D)$$

where $D = \{ C_{ik} \mid i+k \leq I+1 \}$ is the set of all data observed so far. It is important to realize that we have to calculate the mean squared error on the condition of knowing all data observed so far because we want to know the error due to <u>future</u> randomness only. If we calculated the unconditional error $E(C_{iI}-\mathbf{C}_{iI})^2$, which due to the iterative rule for expectations is equal to the mean value $E(E((C_{iI} - \mathbf{C}_{iI})^2|D))$ of the conditional mse over all possible data sets D, we also would include all deviations from the data observed so far which obviously makes no sense if we want to establish a confidence interval for C_{iI} on the basis of the given particular run-off triangle D.

The mean squared error is exactly the same concept which also underlyies the notion of the variance

$$\mathrm{Var}(X) = E(X - E(X))^2$$

of any random variable X. Var(X) measures the average distance of X from its mean value E(X).

Due to the general rule $E(X-c)^2 = \mathrm{Var}(X) + (E(X)-c)^2$ for any scalar c we have

$$\mathrm{mse}(\mathbf{C}_{iI}) = \mathrm{Var}(C_{iI}|D) + (E(C_{iI}|D) - \mathbf{C}_{iI})^2$$

because $\mathbf{C}_{iI}$ is a scalar under the condition that all data D are known. This equation shows that the mse is the sum of the pure future random error $\mathrm{Var}(C_{iI}|D)$ and of the estimation error which is measured by the squared deviation of the estimate $\mathbf{C}_{iI}$ from its target $E(C_{iI}|D)$. On the other hand, the mse does not take into account any future changes in the underlying model, i.e. future deviations from the assumptions (3), (4) and (5), an extreme example of which was the emergence of asbestos. Modelling such deviations is beyond the scope of this paper.

As is to be expected and can be seen in Appendix D, $\mathrm{mse}(\mathbf{C}_{iI})$ depends on the unknown model parameters f_k and α_k^2. We therefore must develop an estimator for $\mathrm{mse}(\mathbf{C}_{iI})$ which can be calculated from the known data D only. The square root of such an estimator is usually called '<u>standard error</u>' because it is an estimate of the standard deviation of C_{iI} in cases in which we have to estimate the mean value, too. The standard error $\mathrm{s.e.}(\mathbf{C}_{iI})$ of

$\mathbf{C}_{iI}$ is at the same time the standard error s.e.$(\mathbf{R}_i)$ of the reserve estimate

$$\mathbf{R}_i = \mathbf{C}_{iI} - C_{i,I+1-i}$$

of the outstanding claims reserve

$$R_i = C_{iI} - C_{i,I+1-i}$$

because

$$\text{mse}(\mathbf{R}_i) = E((\mathbf{R}_i - R_i)^2|D) = E((\mathbf{C}_{iI} - C_{iI})^2|D) =$$
$$= \text{mse}(\mathbf{C}_{iI})$$

and because the equality of the mean squared errors also implies the equality of the standard errors. This means that

$$(6) \qquad \text{s.e.}(\mathbf{R}_i) = \text{s.e.}(\mathbf{C}_{iI}) \ .$$

The derivation of a formula for the standard error s.e.$(\mathbf{C}_{iI})$ of $\mathbf{C}_{iI}$ turns out to be the most difficult part of this paper; it is done in Appendix D. Fortunately, the resulting formula is simple:

$$(7) \qquad (\text{s.e.}(\mathbf{C}_{iI}))^2 = \mathbf{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha}_k^2}{\mathbf{f}_k^2} \left(\frac{1}{\mathbf{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

where

$$(8) \qquad \boldsymbol{\alpha}_k^2 = \frac{1}{I-k-1} \sum_{j=1}^{I-k} C_{jk} \left(\frac{C_{j,k+1}}{C_{jk}} - \mathbf{f}_k \right)^2 , \qquad 1 \le k \le I-2.$$

is an unbiased estimator of α_k^2 (the unbiasedness being shown in Appendix E) and

$$\mathbf{C}_{ik} = C_{i,I+1-i} \cdot \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{k-1} \ , \quad k > I+1-i,$$

are the amounts which are automatically obtained if the run-off triangle is completed step by step according to the chain ladder method. In (7), for notational convenience we have also set

$$c_{i,I+1-i} = C_{i,I+1-i}.$$

Formula (8) does not yield an estimator for α_{I-1} because it is not possible to estimate the two parameters f_{I-1} and α_{I-1} from the single observation $C_{1,I}/C_{1,I-1}$ between development years I-1 and I. If $f_{I-1} = 1$ and if the claims development is believed to be finished after I-1 years we can put $\alpha_{I-1} = 0$. If not, we extrapolate the usually decreasing series $\alpha_1, \alpha_2, \ldots, \alpha_{I-3}, \alpha_{I-2}$ by one additional member, for instance by means of loglinear regression (cf. the example in Chapter 6) or more simply by requiring that

$$\alpha_{I-3} / \alpha_{I-2} = \alpha_{I-2} / \alpha_{I-1}$$

holds at least as long as $\alpha_{I-3} > \alpha_{I-2}$. This last possibility leads to

$$(9) \qquad \alpha_{I-1}^2 = \min\left(\alpha_{I-2}^4/\alpha_{I-3}^2, \min(\alpha_{I-3}^2, \alpha_{I-2}^2) \right).$$

We now want to establish a confidence interval for our target variables C_{iI} and R_i. Because of the equation

$$C_{iI} = C_{i,I+1-i} + R_i$$

the ultimate claims amount C_{iI} consists of a known part $C_{i,I+1-i}$ and an unknown part R_i. This means that the probability distribution function of C_{iI} (given the observations D which include $C_{i,I+1-i}$) is completely determined by that of R_i. We therefore need to establish a confidence interval for R_i only and can then simply shift it to a confidence interval for C_{iI}.

For this purpose we need to know the distribution function of R_i. Up to now we only have estimates $\mathbf{R}_i$ and s.e.($\mathbf{R}_i$) for the mean and the standard deviation of this distribution. If the volume of the outstanding claims is large enough we can, due to the central limit theorem, assume that this distribution function is a Normal distribution with an expected value equal to the point estimate given by $\mathbf{R}_i$ and a standard deviation equal to the standard error s.e.($\mathbf{R}_i$). A symmetric 95%-confidence interval for R_i is then given by

$$(\mathbf{R}_i - 2\cdot \text{s.e.}(\mathbf{R}_i) \ , \ \mathbf{R}_i + 2\cdot \text{s.e.}(\mathbf{R}_i)).$$

But the symmetric Normal distribution may not be a good approximation to the true distribution of R_i if this latter distribution is rather skewed. This will especially be the case if s.e.($\mathbf{R}_i$) is greater than 50 % of $\mathbf{R}_i$. This can also be seen at the above Normal distribution confidence interval whose lower limit then becomes negative even if a negative reserve is not possible.

In this case it is recommended to use an approach based on the Lognormal distribution. For this purpose we approximate the unknown distribution of R_i by a Lognormal distribution with parameters μ_i and σ_i^2 such that mean values as well as variances of both distributions are equal, i.e. such that

$$\exp(\mu_i + \sigma_i^2/2) = \mathbf{R}_i \ ,$$

$$\exp(2\mu_i + \sigma_i^2)(\exp(\sigma_i^2)-1) = (\text{s.e.}(\mathbf{R}_i))^2 \ .$$

This leads to

$$(10)\qquad \begin{aligned} \sigma_i^2 &= \ln(1 + (\text{s.e.}(\mathbf{R_i}))^2/\mathbf{R_i}^2) \ , \\ \mu_i &= \ln(\mathbf{R_i}) - \sigma_i^2/2 \ . \end{aligned}$$

Now, if we want to estimate the 90th percentile of R_i, for example, we proceed as follows. First we take the 90th percentile of the Standard Normal distribution which is 1.28. Then $\exp(\mu_i+1.28\sigma_i)$ with μ_i and σ_i^2 according to (10) is the 90th percentile of the Lognormal distribution and therefore also approximately of the distribution of R_i. For instance, if $\text{s.e.}(\mathbf{R_i})/\mathbf{R_i} = 1$, then $\sigma_i^2 = \ln(2)$ and the 90th percentile is $\exp(\mu_i + 1.28\sigma_i) = \mathbf{R_i}\exp(1.28\sigma_i - \sigma_i^2/2) = \mathbf{R_i}\exp(.719) = 2.05\cdot\mathbf{R_i}$. If we had assumed that R_i has approximately a Normal distribution, we would have obtained in this case $\mathbf{R_i} + 1.28\cdot\text{s.e.}(\mathbf{R_i}) = 2.28\cdot\mathbf{R_i}$ as 90th percentile.

This may come as a surprise since we might have expected that the 90th percentile of a Lognormal distribution always must be higher than that of a Normal distribution with same mean and variance. But there is no general rule, it depends on the percentile chosen and on the size of the ratio $\text{s.e.}(\mathbf{R_i})/\mathbf{R_i}$. The Lognormal approximation only prevents a negative lower confidence limit. In order to set a specific lower confidence limit we choose a suitable percentile, for instance 10%, and proceed analogously as with the 90% before. The question of which confidence probability to choose has to be decided from a business policy point of view. The value of 80% = 90% - 10% taken here must be regarded merely as an example.

We have now shown how to establish confidence limits for every R_i and therefore also for every $C_{iI} = C_{i,I+1-i} + R_i$. We may also be interested in having confidence limits for the overall reserve

$$R = R_2 + \ldots + R_I \ ,$$

and the question is whether, in order to estimate the variance of R, we can simply add the squares $(\text{s.e.}(\mathbf{R}_i))^2$ of the individual standard errors as would be the case with standard deviations of independent variables. But unfortunately, whereas the R_i's itself are independent, the estimators $\mathbf{R}_i$ are not because they are all influenced by the same age-to-age factors $\mathbf{f}_k$, i.e. the $\mathbf{R}_i$'s are positively correlated. In Appendix F it is shown that the square of the standard error of the overall reserve estimator

$$\mathbf{R} = \mathbf{R}_2 + \ldots + \mathbf{R}_I$$

is given by

(11) $$(\text{s.e.}(\mathbf{R}))^2 =$$

$$= \sum_{i=2}^{I} \left\{ (\text{s.e.}(\mathbf{R}_i))^2 + \mathbf{C}_{iI} \Big(\sum_{j=i+1}^{I} \mathbf{C}_{jI} \Big) \sum_{k=I+1-i}^{I-1} \frac{2\boldsymbol{\alpha}_k^2 / \mathbf{f}_k^2}{\sum_{n=1}^{I-k} C_{nk}} \right\}$$

Formula (11) can be used to establish a confidence interval for the overall reserve amount R in quite the same way as it was done before for R_i. Before giving a full example of the calculation of the standard error, we will deal in the next chapter with the problem of how to decide for a given run-off triangle whether the chain ladder assumptions (3) and (5) are met or not.

5. Checking the Chain Ladder Assumptions Against the Data

As has been pointed out before, the three basic implicit chain ladder assumptions

(3) $E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k$,

(4) Independence of accident years ,

(5) $\mathrm{Var}(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}\alpha_k^2$,

are not met in every case. In this chapter we will indicate how these assumptions can be checked for a given run-off triangle. We have already mentioned in Chapter 3 that Appendix H develops a test for calendar year influences which may violate (4). We therefore can concentrate in the following on assumptions (3) and (5).

First, we look at the equations (3) for an arbitrary but fixed k and for i = 1, ..., I. There, the values of C_{ik}, $1 \le i \le I$, are to be considered as given non-random values and equations (3) can be interpreted as an ordinary regression model of the type

$$Y_i = c + x_i b + \epsilon_i \ , \qquad 1 \le i \le I,$$

where c and b are the regression coefficients and ϵ_i the error term with $E(\epsilon_i) = 0$, i.e. $E(Y_i) = c + x_i b$. In our special case, we have $c = 0$, $b = f_k$ and we have observations of the independent variable $Y_i = C_{i,k+1}$ at the points $x_i = C_{ik}$ for i =

1, ..., I-k. Therefore, we can estimate the regression coefficient $b = f_k$ by the usual least squares method

$$\sum_{i=1}^{I-k} (C_{i,k+1} - C_{ik}f_k)^2 \overset{!}{=} \text{minimum} .$$

If the derivative of the left hand side with respect to f_k is set to 0 we obtain for the minimizing parameter f_k the solution

$$(12) \qquad f_{k0} = \sum_{i=1}^{I-k} C_{ik}C_{i,k+1} \Big/ \sum_{i=1}^{I-k} C_{ik}^2 .$$

This is not the same estimator for f_k as according to the chain ladder formula (2). We therefore have used an additional index '0' at this new estimator for f_k. We can rewrite f_{k0} as

$$f_{k0} = \sum_{i=1}^{I-k} \frac{C_{ik}^2}{\sum_{i=1}^{I-k} C_{ik}^2} \cdot \frac{C_{i,k+1}}{C_{ik}}$$

which shows that f_{k0} is the C_{ik}^2-weighted average of the individual development factors $C_{i,k+1}/C_{ik}$, whereas the chain ladder estimator $\mathbf{f_k}$ is the C_{ik}-weighted average. In Chapter 3 we saw that these weights are inversely proportional to the underlying variances $\mathrm{Var}(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik})$. Correspondingly, the estimator f_{k0} assumes

$\mathrm{Var}(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik})$ being proportional to $1/C_{ik}^2$,

or equivalently

$\mathrm{Var}(C_{i,k+1}|C_{i1},\ldots,C_{ik})$ being proportional to 1

which means that $\mathrm{Var}(C_{i,k+1}|C_{i1},\ldots,C_{ik})$ is the same for all observations i = 1, ..., I-k. This is not in agreement with the chain ladder assumption (5).

Here we remember that indeed the least squares method implicitly assumes equal variances $\mathrm{Var}(Y_i) = \mathrm{Var}(\epsilon_i) = \sigma^2$ for all i. If this assumption is not met, i.e. if the variances $\mathrm{Var}(Y_i) = \mathrm{Var}(\epsilon_i)$ depend on i, one should use a weighted least squares approach which consists of minimizing the weighted sum of squares

$$\sum_{i=1}^{I} w_i (Y_i - c - x_i b)^2$$

where the weights w_i are in inverse proportion to $\mathrm{Var}(Y_i)$.

Therefore, in order to be in agreement with the chain ladder variance assumption (5), we should use regression weights w_i which are proportional to $1/C_{ik}$ (more precisely to $1/(C_{ik}\alpha_k^2)$, but α_k^2 can be amalgamated with the proportionality constant because k is fixed). Then minimizing

$$\sum_{i=1}^{I-k} (C_{i,k+1} - C_{ik} f_k)^2 \,/\, C_{ik}$$

with respect to f_k yields indeed

$$f_{k1} = \sum_{i=1}^{I-k} C_{i,k+1} \,/\, \sum_{i=1}^{I-k} C_{ik}$$

which is identical to the usual chain ladder age-to-age factor $\mathbf{f_k}$.

It is tempting to try another set of weights, namely $1/C_{ik}^2$ because then the weighted sum of squares becomes

$$\sum_{i=1}^{I-k} (C_{i,k+1} - C_{ik}f_k)^2 / C_{ik}^2 = \sum_{i=1}^{I-k} \left(\frac{C_{i,k+1}}{C_{ik}} - f_k \right)^2 .$$

Here the minimizing procedure yields

$$(13) \qquad f_{k2} = \frac{1}{I-k} \sum_{i=1}^{I-k} \frac{C_{i,k+1}}{C_{ik}} ,$$

which is the ordinary unweighted average of the development factors. The variance assumption corresponding to the weights used is

$$\mathrm{Var}(C_{i,k+1}|C_{i1},\ldots,C_{ik}) \quad \text{being proportional to } C_{ik}^2$$

or equivalently

$$\mathrm{Var}(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik}) \quad \text{being proportional to 1.}$$

The benefit of transforming the estimation of the age-to-age factors into the regression framework is the fact that the usual regression analysis instruments are now available to check the underlying assumptions, especially the linearity and the variance assumption. This check is usually done by carefully inspecting plots of the data and of the residuals:

First, we plot $C_{i,k+1}$ against C_{ik}, $i = 1, \ldots, I-k$, in order to see if we really have an approximately linear relationship around a straight line through the origin with slope $\mathbf{f_k} = f_{k1}$. Second, if linearity seems acceptable, we plot the weighted residuals

$$(C_{i,k+1} - C_{ik}\mathbf{f_k}) / \sqrt{C_{ik}} , \quad 1 \le i \le I-k,$$

(whose squares have been minimized) against C_{ik} in order to see if the employed variance assumption really leads to a plot in which the residuals do not show any specific trend but appear purely random. It is recommended to compare all three residual plots (for $i = 1, \ldots, I-k$)

Plot 0: $C_{i,k+1} - C_{ik}f_{k0}$ against C_{ik},

Plot 1: $(C_{i,k+1} - C_{ik}f_{k1})/\sqrt{C_{ik}}$ against C_{ik},

Plot 2: $(C_{i,k+1} - C_{ik}f_{k2})/C_{ik}$ against C_{ik},

and to find out which one shows the most random behaviour. All this should be done for every development year k for which we have sufficient data points, say at least 6, i.e. for $k \leq I-6$.

Some experience with least squares residual plots is useful, especially because in our case we have only very few data points. Consequently, it is not always easy to decide whether a pattern in the residuals is systematic or random. However, if Plot 1 exhibits a nonrandom pattern, and either Plot 0 or Plot 2 does not, and if this holds true for several values of k, we should seriously consider replacing the chain ladder age-to-age factors $f_{k1} = \mathbf{f}_k$ with f_{k0} or f_{k2} respectively. The following numerical example will clarify the situation a bit more.

6. Numerical Example

The data for the following example are taken from the 'Historical Loss Development Study', 1991 Edition, published by the Reinsurance Association of America (RAA). There, we find on page 96 the following run-off triangle of Automatic Facultative business in General Liability (excluding Asbestos & Environmental):

	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i6}	c_{i7}	c_{i8}	c_{i9}	c_{i10}
i=1	5012	8269	10907	11805	13539	16181	18009	18608	18662	18834
i=2	106	4285	5396	10666	13782	15599	15496	16169	16704	
i=3	3410	8992	13873	16141	18735	22214	22863	23466		
i=4	5655	11555	15766	21266	23425	26083	27067			
i=5	1092	9565	15836	22169	25955	26180				
i=6	1513	6445	11702	12935	15852					
i=7	557	4020	10946	12314						
i=8	1351	6947	13112							
i=9	3133	5395								
i=10	2063									

The above figures are cumulative incurred case losses in \$ 1000. We have taken the accident years from 1981 (i=1) to 1990 (i=10) which is enough for the sake of example but does not mean that we believe to have reached the ultimate claims amount after 10 years of development.

We first calculate the age-to-age factors $\mathbf{f_k} = f_{k,1}$ according to formula (2). The result is shown in the following table together with the alternative factors f_{k0} according to (12) and f_{k2} according to (13):

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
f_{k0}	2.217	1.569	1.261	1.162	1.100	1.041	1.032	1.016	1.009
f_{k1}	2.999	1.624	1.271	1.172	1.113	1.042	1.033	1.017	1.009
f_{k2}	8.206	1.696	1.315	1.183	1.127	1.043	1.034	1.018	1.009

If one has the run-off triangle on a personal computer it is very easy to produce the plots recommended in Chapter 5 because most spreadsheet programs have the facility of plotting X-Y graphs. For every $k = 1, \ldots, 8$ we make a plot of the amounts $C_{i,k+1}$ (y-axis) of development year k+1 against the amounts C_{ik} (x-axis) of development year k for $i = 1, \ldots, 10-k$, and draw a straight line through the origin with slope f_{k1}. The plots for k = 1 to 8 are shown in the upper graphs of Figures 1 to 8, respectively. (All figures are to be found at the end of the paper after the appendices.) The number above each point mark indicates the corresponding accident year. (Note that the point mark at the upper or right hand border line of each graph does not belong to the plotted points $(C_{ik}, C_{i,k+1})$, it has only been used to draw the regression line.) In the lower graph of each of the Figures 1 to 8 the corresponding weighted residuals $(C_{i,k+1} - C_{ik})/\sqrt{C_{ik}}$ are plotted against C_{ik} for $i = 1, \ldots, 10-k$.

The two plots for k = 1 (Figure 1) clearly show that the regression line does not capture the direction of the data points very well. The line should preferably have a positive intercept on the y-axis and a flatter slope. However, even then we would have a high dispersion. Using the line through the origin we will probably underestimate any future C_{i2} if C_{i1} is less than 2000 and will overestimate it if C_{i1} is more than 4000. Fortunately, in the one relevant case i = 10 we have $C_{10,1} = 2063$ which means that the resulting forecast $C_{10,2} = C_{10,1}\mathbf{f_2} =$

$2063 \cdot 2.999 = 6187$ is within the bulk of the data points plotted. In any case, Figure 1 shows that any forecast of $C_{10,2}$ is associated with a high uncertainty of about ± 3000 or almost $\pm 50\%$ of an average-sized $C_{i,2}$ which subsequently is even enlarged when extrapolating to ultimate. If in a future accident year we have a value C_{i1} outside the interval (2000, 4000) it is reasonable to introduce an additional parameter by fitting a regression line with positive intercept to the data and using it for the projection to C_{i2}. Such a procedure of employing an additional parameter is acceptable between the first two development years in which we have the highest number of data points of all years.

The two plots for $k = 2$ (Figure 2) are more satisfactory. The data show a clear trend along the regression line and quite random residuals. The same holds for the two plots for $k = 4$ (Figure 4). In addition, for both $k = 2$ and $k = 4$ a weighted linear regression including a parameter for intercept would yield a value of the intercept which is not significantly different from zero. The plots for $k = 3$ (Figure 3) seem to show a curvature to the left but because of the few data points we can hope that this is incidental. Moreover, the plots for $k = 5$ have a certain curvature to the right such that we can hope that the two curvatures offset each other. The plots for $k = 6$, 7 and 8 are quite satisfactory. The trends in the residuals for $k = 7$ and 8 have no significance in view of the very few data points.

We need not to look at the regression lines with slopes f_{k0} or f_{k2} as these slopes are very close to $\mathbf{f_k}$ (except for k=1). But we should look at the corresponding plots of weighted residuals in order to see whether they appear more satisfactory than the previous ones. (Note that due to the different weights the residuals will be different even if the slopes are equal.) The residual plots for f_{k0} and k = 1 to 4 are shown in Figures 9 and 10. Those for f_{k2} and k = 1 to 4 are shown in Figures 11 and 12. In the residual plot for $f_{1,0}$ (Figure 9, upper graph) the point furthest to the left is not an outlier as it is in the plots for $f_{1,1} = \mathbf{f_1}$ (Figur 1, lower graph) and $f_{1,2}$ (Figure 11, upper graph). But with all three residual plots for k=1 the main problem is the missing intercept of the regression line which leads to a decreasing trend in the residuals. Therefore the improvement of the outlier is of secondary importance. For k = 2 the three residuals plots do not show any major differences between each other. The same holds for k = 3 and 4. The residual plots for k = 5 to 8 are not important because of the small number of data points. Altogether, we decide to keep the usual chain ladder method, i.e. the age-to-age factors $\mathbf{f_k} = f_{k,1}$, because the alternatives $f_{k,0}$ or $f_{k,2}$ do not lead to a clear improvement.

Next, we can carry through the tests for calendar year influences (see Appendix H) and for correlations between subsequent development factors (see Appendix G). For our example neither test leads to a rejection of the underlying assumption as is shown in the appendices mentioned.

Having now finished all preliminary analyses we calculate the estimated ultimate claims amounts C_{iI} according to formula (1), the reserves $R_i = C_{iI} - C_{i,I+1-i}$ and its standard errors (7). For the standard errors we need the estimated values of α_k^2 which according to formula (8) are given by

k	1	2	3	4	5	6	7	8	9
α_k^2	27883	1109	691	61.2	119	40.8	1.34	7.88	

A plot of $\ln(\alpha_k^2)$ against k is given in Figure 13 and shows that there indeed seems to be a linear relationship which can be used to extrapolate $\ln(\alpha_9^2)$. This yields $\alpha_9^2 = \exp(-.44) = .64$. But we use formula (9) which is more easily programmable and in the present case is a bit more on the safe side: it leads to $\alpha_9^2 =$ 1.34. Using formula (11) for s.e.(R) as well we finally obtain

	$C_{i,10}$	R_i	$s.e(C_{i,10}) = s.e.(R_i)$	$s.e.(R_i)/R_i$
i=2	16858	154	206	134 %
i=3	24083	617	623	101 %
i=4	28703	1636	747	46 %
i=5	28927	2747	1469	53 %
i=6	19501	3649	2002	55 %
i=7	17749	5435	2209	41 %
i=8	24019	10907	5358	49 %
i=9	16045	10650	6333	59 %
i=10	18402	16339	24566	150 %
Overall		52135	26909	52 %

(The numbers in the 'Overall'-row are **R**, s.e.(**R**) and s.e.(**R**)/**R**.) For i = 2, 3 and 10 the percentage standard error (last column) is more than 100% of the estimated reserve R_i. For i = 2 and 3 this is due to the small amount of the corresponding reserve and is not important because the absolute amounts of the standard errors are rather small. But the standard error of 150 % for the most recent accident year i = 10 might lead to some concern in practice. The main reason for this high standard error is the high uncertainty of forecasting next year's value $C_{10,2}$ as was seen when examining the plot of C_{i2} against C_{i1}. Thus, one year later we will very likely be able to give a much more precise forecast of $C_{10,10}$.

Because all standard errors are close to or above 50 % we use the Lognormal distribution in all years for the calculation of confidence intervals. We first calculate the upper 90%-confidence limit (or with any other chosen percentage) for the overall outstanding claims reserve R. Denoting by μ and σ^2 the parameters of the Lognormal distribution approximating the distribution of R and using s.e.(**R**)/**R** = .52 we have σ^2 = .236 (cf. (10)) and, in the same way as in Chapter 4, the 90th percentile is $\exp(\mu + 1.28\sigma) = \mathbf{R}\cdot\exp(1.28\sigma-\sigma^2/2) = 1.655\cdot\mathbf{R}$ = 86298. Now we allocate this overall amount to the accident years i = 2,..., 10 in such a way that we reach the same level of confidence for every accident year. Each level of confidence corresponds to a certain percentile t of the Standard Normal distribution and - according to Chapter 4 - the corresponding percentile of the distribution of R_i is $\mathbf{R_i}\exp(t\sigma_i - \sigma_i^2/2)$ with $\sigma_i^2 = \ln(1 + (\text{s.e.}(\mathbf{R_i}))^2/\mathbf{R_i}^2)$. We therefore only have to choose t in such a way that

$$\sum_{i=2}^{I} \mathbf{R_i}\cdot\exp(t\sigma_i - \sigma_i^2/2) = 86298 .$$

This can easily be solved with the help of spreadsheet software (e.g. by trial and error) and yields t = 1.13208 which corresponds to the 87th percentile per accident year and leads to the following distribution of the overall amount 86298:

	$\mathbf{R_i}$	s.e. $(\mathbf{R_i})/\mathbf{R_i}$	σ_i^2	upper confidence limit $\mathbf{R_i}\exp(t\sigma_i-\sigma_i^2/2)$
i=2	154	1.34	1.028	290
i=3	617	1.01	.703	1122
i=4	1636	.46	.189	2436
i=5	2747	.53	.252	4274
i=6	3649	.55	.263	5718
i=7	5435	.41	.153	7839
i=8	10907	.49	.216	16571
i=9	10650	.59	.303	17066
i=10	16339	1.50	1.182	30981
Total	52135			86298

In order to arrive at the lower confidence limits we proceed completely analogously. The 10th percentile, for instance, of the total outstanding claims amount is $\mathbf{R}\cdot\exp(-1.28\sigma-\sigma^2/2) = .477\cdot\mathbf{R} = 24871$. The distribution of this amount over the individual accident years is made as before and leads to a value of $t = -.8211$ which corresponds to the 21st percentile. This means that a 87% - 21% = 66% confidence interval for each accident year leads to a 90% - 10% = 80% confidence interval for the overall reserve amount. In the following table, the confidence intervals thus obtained for R_i are already shifted (by adding $C_{i,I+1-i}$) to confidence intervals for the ultimate claims amounts C_{iI} (for instance, the upper limit 16994 for i=2 has been obtained by adding $C_{2,9}$ = 16704 and 290 from the preceding table):

	$C_{i,10}$	confidence intervals for 80% prob. overall	empirical limits
i=2	16858	(16744 , 16994)	(16858 , 16858)
i=3	24083	(23684 , 24588)	(23751 , 24466)
i=4	28703	(28108 , 29503)	(28118 , 29446)
i=5	28927	(27784 , 30454)	(27017 , 31699)
i=6	19501	(17952 , 21570)	(16501 , 22939)
i=7	17749	(15966 , 20153)	(14119 , 23025)
i=8	24019	(19795 , 29683)	(16272 , 48462)
i=9	16045	(11221 , 22461)	(8431 , 54294)
i=10	18402	(5769 , 33044)	(5319 , 839271)

The column "empirical limits" contains the minimum and maximum size of the ultimate claims amount resulting if in formula (1) each age-to-age factor f_k is replaced with the minimum (or maximum) individual development factor observed so far. These factors are defined by

$$f_{k,min} = \min \{ C_{i,k+1}/C_{ik} \mid 1 \le i \le I-k \}$$

$$f_{k,max} = \max \{ C_{i,k+1}/C_{ik} \mid 1 \le i \le I-k \}$$

and can be taken from the table of all development factors which can be found in Appendices G and H. They are

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
$f_{k,min}$	1.650	1.259	1.082	1.102	1.009	.993	1.026	1.003	1.009
$f_{k,max}$	40.425	2.723	1.977	1.292	1.195	1.113	1.043	1.033	1.009

In comparison with the confidence intervals, these empirical limits are narrower in the earlier accident years $i \le 4$ and wider in the more recent accident years $i \ge 5$. This was to be expected because the small number of development factors observed between the late development years only leads to a rather small variation between the minimum and maximum factors. Therefore these empirical limits correspond to a confidence probability which is rather small in the early accident years and becomes larger and larger towards the recent accident years. Thus, this empirical approach to establishing confidence limits does not seem to be reasonable.

If we used the Normal distribution instead of the Lognormal we had obtained a 90th percentile of $\mathbf{R} + 1.28 \cdot \mathbf{R} \cdot (\text{s.e.}(\mathbf{R})/\mathbf{R}) = 1.661 \cdot \mathbf{R}$ (which is almost the same as the $1.655 \cdot \mathbf{R}$ with the Lognormal) and a 10th percentile of $\mathbf{R} - 1.28 \cdot \mathbf{R} \cdot (\text{s.e.}(\mathbf{R})/\mathbf{R}) = .34 \cdot \mathbf{R}$ (which is lower than the $.477 \cdot \mathbf{R}$ with the Lognormal). Also, the allocation to the accident years would be different.

Finally, we compare the standard errors obtained to the output of the claims reserving software package ICRFS by Ben Zehnwirth.

This package is a modelling framework in which the user can specify his own model within a large class of models. But it also contains some predefined models, inter alia also a 'chain ladder model'. But this is not the usual chain ladder method, instead, it is a loglinearized approximation of it. Therefore, the estimates of the oustanding claims amounts differ from those obtained here with the usual chain ladder method. Moreover, it works with the logarithms of the incremental amounts $C_{i,k+1}-C_{ik}$ and one must therefore eliminate the negative increment $C_{2,7}-C_{2,6}$. In addition, $C_{2,1}$ was identified as an outlier and was eliminated. Then the ICRFS results were quite similar to the chain ladder results as can be seen in the following table:

	est. outst. claims amount R_i		standard error	
	chain ladder	ICRFS	chain ladder	ICRFS
i=2	154	394	206	572
i=3	617	825	623	786
i=4	1636	2211	747	1523
i=5	2747	2743	1469	1724
i=6	3649	4092	2002	2383
i=7	5435	5071	2209	2972
i=8	10907	11899	5358	6892
i=9	10650	14569	6333	9689
i=10	16339	25424	24566	23160
Overall	52135	67228	26909	28414

Even though the reserves R_i for i=9 and i=10 as well as the overall reserve R differ considerably they are all within one standard error and therefore not significantly different. But it should be remarked that this manner of using ICRFS is not intended by Zehnwirth because any initial model should be further adjusted according to the indications and plots given by the program. In this particular case there were strong indications for developing the model further but then one would have to give up the 'chain ladder model'.

7. Final Remark

This paper develops a rather complete methodology of how to attack the claims reserving task in a statistically sound manner on the basis of the well-known and simple chain ladder method. However, the well-known weak points of the chain ladder method should not be concealed: These are the fact that the estimators of the last two or three factors f_I, f_{I-1}, f_{I-2} rely on very few observations and the fact that the known claims amount C_{I1} of the last accident year (sometimes $C_{I-1,2}$, too) forms a very uncertain basis for the projection to ultimate. This is most clearly seen if C_{I1} happens to be 0: Then we have $\mathbf{C}_{iI} = 0$, $\mathbf{R}_I = 0$ and s.e.$(\mathbf{R}_I) = 0$ which obiously makes no sense. (Note that this weakness often can be overcome by translating and mixing the amounts C_{i1} of earlier accident years $i < I$ into accident year I with the help of a measure of volume for each accident year.)

Thus, even if the statistical instruments developed do not reject the applicability of the chain ladder method, the result must be judged by an actuary and/or underwriter who knows the business under consideration. Even then, unexpected future changes can make all estimations obsolete. But for the many normal cases it is good to have a sound and simple method. Simple methods have the disadvantage of not capturing all aspects of reality but have the advantage that the user is in a position to know exactly how the method works and where its weaknesses are. Moreover, a simple method can be explained to non-actuaries in more detail. These are invaluable advantages of simple models over more sophisticated ones.

Appendix A: Unbiasedness of Age-to-Age Factors

Proposition: Under the assumptions

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$$

(4) The variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \neq j$ are independent.

the age-to-age factors $\mathbf{f_1}, \ldots, \mathbf{f_{I-1}}$ defined by

$$(2) \quad \mathbf{f_k} = \sum_{j=1}^{I-k} C_{j,k+1} \Big/ \sum_{j=1}^{I-k} C_{jk}, \qquad 1 \le k \le I-1,$$

are unbiased, i.e. we have $E(\mathbf{f_k}) = f_k$, $1 \le k \le I-1$.

Proof: Because of the iterative rule for expectations we have

$$(A1) \qquad E(\mathbf{f_k}) = E(E(\mathbf{f_k}|B_k))$$

for any set B_k of variables C_{ij} assumed to be known. We take

$$B_k = \{ C_{ij} \mid i+j \le I+1, j \le k \}, \qquad 1 \le k \le I-1.$$

According to the definition (2) of $\mathbf{f_k}$ and because C_{jk}, $1 \le j \le I-k$, is contained in B_k and therefore has to be treated as scalar, we have

$$(A2) \qquad E(\mathbf{f_k}|B_k) = \sum_{j=1}^{I-k} E(C_{j,k+1}|B_k) \Big/ \sum_{j=1}^{I-k} C_{jk}.$$

Because of the independence assumption (4) conditions relating to accident years other than that of $C_{j,k+1}$ can be omitted, i.e. we get

$$(A3) \qquad E(C_{j,k+1}|B_k) = E(C_{j,k+1}|C_{j1},\ldots,C_{jk}) = C_{jk}f_k$$

using assumption (3) as well. Inserting (A3) into (A2) yields

$$\text{(A4)} \qquad E(\mathbf{f}_k | B_k) = \sum_{j=1}^{I-k} C_{jk} f_k \Big/ \sum_{j=1}^{I-k} C_{jk} = f_k \ .$$

Finally, (A1) and (A4) yield

$$E(\mathbf{f}_k) = E(f_k) = f_k$$

because f_k is a scalar.

Appendix B: Minimizing the Variance of Independent Estimators

Proposition: Let $T_1, \ldots, T_I$ be independent unbiased estimators of a parameter t, i.e. with

$$E(T_i) = t \ , \quad 1 \le i \le I,$$

then the variance of a linear combination

$$T = \sum_{i=1}^{I} w_i T_i$$

under the constraint

$$\text{(B1)} \qquad \sum_{i=1}^{I} w_i = 1$$

(which guarantees $E(T) = t$) is minimal iff the coefficients w_i are inversely proportional to $Var(T_i)$, i.e. iff

$$w_i = c/Var(T_i) \ , \quad 1 \le i \le I.$$

Proof: We have to minimize

$$Var(T) = \sum_{i=1}^{I} w_i^2 Var(T_i)$$

(due to the independence of $T_1, \ldots, T_I$) with respect to w_i under the constraint (B1). A necessary condition for an extremum is that the derivatives of the Lagrangian are zero, i.e. that

$$\text{(B2)} \qquad \frac{\partial}{\partial w_i} \left(\sum_{i=1}^{I} w_i^2 Var(T_i) + \lambda(1 - \sum_{i=1}^{I} w_i) \right) = 0 \ , \quad 1 \le i \le I,$$

with a constant multiplier λ whose value can be determined by additionally using (B1). (B2) yields

$$2w_i Var(T_i) - \lambda = 0$$

or

$$w_i = \lambda \,/\, (2 \cdot \mathrm{Var}(T_i)) \;.$$

These weights w_i indeed lead to a minimum as can be seen by calculating the extremal value of Var(T) and applying Schwarz's inequality.

Corrollary: In the chain ladder case we have estimators $T_i = C_{i,k+1}/C_{ik}$, $1 \le i \le I-k$, for f_k where the variables of the set

$$A_k = \bigcup_{i=1}^{I-k} \{ C_{i1}, \ldots, C_{ik} \}$$

of the corresponding accident years $i = 1, \ldots, I-k$ up to development year k are considered to be given. We therefore want to minimize the conditional variance

$$\mathrm{Var}(\sum_{i=1}^{I-k} w_i T_i | A_k) \;.$$

From the above proof it is clear that the minimizing weights should be inversely proportional to $\mathrm{Var}(T_i|A_k)$. Because of the independence (4) of the accident years, conditions relating to accident years other than that of $T_i = C_{i,k+1}/C_{ik}$ can be omitted. We therefore have

$$\mathrm{Var}(T_i|A_k) = \mathrm{Var}(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik})$$

and arrive at the result that

the minimizing weights should be
inversely proportional to $\mathrm{Var}(C_{i,k+1}/C_{ik}|C_{i1},\ldots,C_{ik})$.

Appendix C: Unbiasedness of the Estimated Ultimate Claims Amount

Proposition: Under the assumptions

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$E(C_{i,k+1} | C_{i1}, \ldots, C_{ik}) = C_{ik} f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

(4) The variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \neq j$ are independent.

the expected values of the estimator

(1) $\mathbf{C}_{iI} = C_{i,I+1-i} \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{I-1}$

for the ultimate claims amount and of the true ultimate claims amount C_{iI} are equal, i.e. we have $E(\mathbf{C}_{iI}) = E(C_{iI})$, $2 \le i \le I$.

Proof: We first show that the age-to-age factors $\mathbf{f}_k$ are uncorrelated. With the same set

$$B_k = \{ C_{ij} \mid i+j \le I+1,\ j \le k \}, \quad 1 \le k \le I-1,$$

of variables assumed to be known as in Appendix A we have for $j < k$

$$\begin{aligned}
E(\mathbf{f}_j \mathbf{f}_k) &= E(E(\mathbf{f}_j \mathbf{f}_k | B_k)) && \text{(a)}\\
&= E(\mathbf{f}_j E(\mathbf{f}_k | B_k)) && \text{(b)}\\
&= E(\mathbf{f}_j f_k) && \text{(c)}\\
&= E(\mathbf{f}_j) f_k && \text{(d)}\\
&= f_j f_k\ . && \text{(e)}
\end{aligned}$$

Here (a) holds because of the iterative rule for expectations, (b) holds because $\mathbf{f}_j$ is a scalar for B_k given and for $j < k$, (c) holds due to (A4), (d) holds because f_k is a scalar and (e) was shown in Appendix A.

This result can easily be extended to arbitrary products of different f_k's, i.e. we have

(C1) $E(f_{I+1-i}\cdot\ldots\cdot f_{I-1}) = f_{i+1-i}\cdot\ldots\cdot f_{I-1}$.

This yields

$$\begin{aligned}
E(C_{iI}) &= E(E(C_{iI}|C_{i1},\ldots,C_{i,I+1-i})) && \text{(a)}\\
&= E(E(C_{i,I+1-i}f_{I+1-i}\cdot\ldots\cdot f_{I-1}|C_{i1},\ldots,C_{i,I+1-i})) && \text{(b)}\\
&= E(C_{i,I+1-i}E(f_{I+1-i}\cdot\ldots\cdot f_{I-1}|C_{i1},\ldots,C_{i,I+1-i})) && \text{(c)}\\
&= E(C_{i,I+1-i}E(f_{I+1-i}\cdot\ldots\cdot f_{I-1})) && \text{(d)}\\
&= E(C_{i,I+1-i})\cdot E(f_{I+1-i}\cdot\ldots\cdot f_{I-1}) && \text{(e)}\\
&= E(C_{i,I+1-i})\cdot f_{I+1-i}\cdot\ldots\cdot f_{I-1}\ . && \text{(f)}
\end{aligned}$$

Here (a) holds because of the iterative rule for expectations, (b) holds because of the definition (1) of C_{iI}, (c) holds because $C_{i,I+1-i}$ is a scalar under the stated condition, (d) holds because conditions which are independent from the conditioned variable $f_{I+1-i}\cdot\ldots\cdot f_{I-1}$ can be omitted (observe assumption (4) and the fact that f_{I+1-i}, ..., f_{I-1} only depend on variables of accident years < i), (e) holds because $E(f_{I+1-i}\cdot\ldots\cdot f_{I-1})$ is a scalar and (f) holds because of (C1).

Finally, repeated application of the iterative rule for expectations and of assumption (3) yields for the expected value of the true reserve C_{iI}

$$\begin{aligned}
E(C_{iI}) &= E(E(C_{iI}|C_{i1},\ldots,C_{i,I-1}))\\
&= E(C_{i,I-1}f_{I-1})\\
&= E(C_{i,I-1})f_{I-1}\\
&= E(E(C_{i,I-1}|C_{i1},\ldots,C_{I-2}))f_{I-1}
\end{aligned}$$

$$= E(C_{i,I-2} f_{I-2}) f_{I-1}$$

$$= E(C_{i,I-2}) f_{I-2} f_{I-1}$$

$$= \text{etc.}$$

$$= E(C_{i,I+1-i}) f_{I+1-i} \cdot \ldots \cdot f_{I-1}$$

$$= E(C_{iI}) \ .$$

Appendix D: Calculation of the Standard Error of $\hat{C}_{iI}$

Proposition: Under the assumptions

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

(4) The variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \ne j$ are independent.

(5) There are unknown constants $\alpha_1, \ldots, \alpha_{I-1}$ with

$Var(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}\alpha_k^2, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

the standard error s.e.($\hat{C}_{iI}$) of the estimated ultimate claims amount $\hat{C}_{iI} = C_{i,I+1-i}\hat{f}_{I+1-i}\cdot\ldots\cdot\hat{f}_{I-1}$ is given by the formula

$$(s.e.(\hat{C}_{iI}))^2 = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\alpha}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

where $\hat{C}_{ik} = C_{i,I+1-i}\hat{f}_{I+1-i}\cdots\hat{f}_{k-1}$, $k > I+1-i$, are the estimated values of the future C_{ik} and $\hat{C}_{i,I+1-i} = C_{i,I+1-i}$.

Proof: As stated in Chapter 4, the standard error is the square root of an estimator of mse($\hat{C}_{iI}$) and we have also seen that

(D1) $\quad mse(\hat{C}_{iI}) = Var(C_{iI}|D) + (E(C_{iI}|D) - \hat{C}_{iI})^2$.

In the following, we use the abbreviations

$E_i(X) = E(X|C_{i1}, \ldots, C_{i,I+1-i})$,

$Var_i(X) = Var(X|C_{i1}, \ldots, C_{i,I+1-i})$.

Because of the independence of the accident years we can omit in (D1) that part of the condition $D = \{ C_{ik} \mid i+k \le I+1 \}$ which is independent from C_{iI}, i.e. we can write

(D2) $\quad mse(\hat{C}_{iI}) = Var_i(C_{iI}) + (E_i(C_{iI}) - \hat{C}_{iI})^2$.

We first consider $\mathrm{Var}_i(C_{iI})$. Because of the general rule $\mathrm{Var}(X) = E(X^2) - (E(X))^2$ we have

(D3) $\quad \mathrm{Var}_i(C_{iI}) = E_i(C_{iI}^2) - (E_i(C_{iI}))^2$.

For the calculation of $E_i(C_{iI})$ we use the fact that for $k \geq I+1-i$

$$\text{(D4)} \quad \begin{aligned} E_i(C_{i,k+1}) &= E_i(E(C_{i,k+1}|C_{i1}, \ldots, C_{ik})) \\ &= E_i(C_{ik}f_k) \\ &= E_i(C_{ik})f_k \ . \end{aligned}$$

Here, we have used the iterative rule for expectations in its general form $E(X|Z) = E(E(X|Y)|Z)$ for $\{Y\} \supset \{Z\}$ (mostly we have $\{Z\} = \varnothing$). By successively applying (D4) we obtain for $k \geq I+1-i$

$$\text{(D5)} \quad \begin{aligned} E_i(C_{i,k+1}) &= E_i(C_{i,I+1-i})f_{I+1-i}\cdot\ldots\cdot f_k \\ &= C_{i,I+1-i}f_{I+1-i}\cdot\ldots\cdot f_k \end{aligned}$$

because $C_{i,I+1-i}$ is a scalar under the condition 'i'.

For the calculation of the first term $E_i(C_{iI}^2)$ of (D3) we use the fact that for $k \geq I+1-i$

$$\text{(D6)} \quad \begin{aligned} E_i(C_{i,k+1}^2) &= E_i(E(C_{i,k+1}^2|C_{i1}, \ldots, C_{ik}) && \text{(a)} \\ &= E_i(\ \mathrm{Var}(C_{i,k+1}|C_{i1}, \ldots, C_{ik}) + && \text{(b)} \\ &\qquad + (E(C_{i,k+1}|C_{i1}, \ldots, C_{ik}))^2\ ) \\ &= E_i(\ C_{ik}\alpha_k^2 + (C_{ik}f_k)^2\ ) && \text{(c)} \\ &= E_i(C_{ik})\alpha_k^2 + E_i(C_{ik}^2)f_k^2\ . \end{aligned}$$

Here, (a) holds due to the iterative rule for expectations, (b) due to the rule $E(X^2) = \mathrm{Var}(X) + (E(X))^2$ and (c) holds due to (3) and (5).

Now, we apply (D6) and (D5) successively to get

$$(D7) \quad E_i(C_{iI}^2) = E_i(C_{i,I-1})\alpha_{I-1}^2 + E_i(C_{i,I-1}^2)f_{I-1}^2 \qquad (D6)$$

$$= C_{i,I+1-1}f_{I+1-1}\cdots f_{I-2}\alpha_{I-1}^2 + \qquad (D5)$$

$$+ E_i(C_{i,I-2})\alpha_{I-2}^2 f_{I-1}^2 + \qquad (D6)$$

$$+ E_i(C_{i,I-2}^2)f_{I-2}^2 f_{I-1}^2$$

$$= C_{i,I+1-1}f_{I+1-1}\cdots f_{I-2}\alpha_{I-1}^2 +$$

$$+ C_{i,I+1-1}f_{I+1-1}\cdots f_{I-3}\alpha_{I-2}^2 f_{I-1}^2 + \qquad (D5)$$

$$+ E_i(C_{i,I-3})\alpha_{I-3}^2 f_{I-2}^2 f_{I-1}^2 + \qquad (D6)$$

$$+ E_i(C_{i,I-3}^2)f_{I-3}^2 f_{I-2}^2 f_{I-1}^2$$

$$= \text{etc.}$$

$$= C_{i,I+1-i} \sum_{k=I+1-i}^{I-1} f_{I+1-i}\cdots f_{k-1}\alpha_k^2 f_{k+1}^2\cdots f_{I-1}^2$$

$$+ C_{i,I+1-i}^2 f_{I+1-i}^2\cdot\ldots\cdot f_{I-1}^2$$

where in the last step we have used $E_i(C_{i,I+1-i}) = C_{i,I+1-i}$ and $E_i(C_{i,I+1-i}^2) = C_{i,I+1-i}^2$ because under the condition 'i' $C_{i,I+1-i}$ is a scalar.

Due to (D5) we have

$$(D8) \quad (E_i(C_{iI}))^2 = C_{i,I+1-i}^2 f_{I+1-i}^2\cdot\ldots\cdot f_{I-1}^2 .$$

Inserting (D7) and (D8) into (D3) yields

$$(D9) \quad Var_i(C_{iI}) = C_{i,I+1-i} \sum_{k=I+1-i}^{I-1} f_{I+1-i}\cdots f_{k-1}\alpha_k^2 f_{k+1}^2\cdots f_{I-1}^2$$

We estimate this first summand of $mse(\mathbf{C_{iI}})$ by replacing the unknown parameters f_k, α_k^2 with their unbiased estimators $\mathbf{f_k}$ and $\boldsymbol{\alpha_k}^2$, i.e. by

$$(D10) \quad C_{i,I+1-i} \sum_{k=I+1-i}^{I-1} \mathbf{f_{I+1-i}}\cdots\mathbf{f_{k-1}}\cdot\boldsymbol{\alpha}_{\mathbf{k}}^2\cdot\mathbf{f}_{\mathbf{k+1}}^2\cdots\mathbf{f}_{\mathbf{I-1}}^2 =$$

$$= C_{i,I+1-i}^2 \mathbf{f}_{I+1-i}^2 \cdots \mathbf{f}_{I-1}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha}_k^2/\mathbf{f}_k^2}{C_{i,I+1-i}\mathbf{f}_{I+1-i}\cdots\mathbf{f}_{k-1}}$$

$$= \mathbf{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha}_k^2/\mathbf{f}_k^2}{\mathbf{C}_{ik}}$$

where we have used the notation $\mathbf{C}_{ik}$ introduced in the proposition for the estimated amounts of the future C_{ik}, $k > I+1-i$, including $\mathbf{C}_{i,I+1-i} = C_{i,I+1-i}$.

We now turn to the second summand of the expression (D2) for $\mathrm{mse}(\mathbf{C}_{iI})$. Because of (D5) we have

$$E_i(C_{iI}) = C_{i,I+1-i} f_{I+1-i} \cdot \ldots \cdot f_{I-1}$$

and therefore

(D11) $(E_i(C_{iI}) - \mathbf{C}_{iI})^2 =$

$$= C_{i,I+1-i}^2 (f_{I+1-i} \cdot \ldots \cdot f_{I-1} - \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{I-1})^2 .$$

This expression cannot simply be estimated by replacing f_k with $\mathbf{f}_k$ because this would yield 0 which is not a good estimator because $\mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{I-1}$ generally will be different from $f_{I+1-i} \cdot \ldots \cdot f_{I-1}$ and therefore the squared difference will be positive. We therefore must take a different approach. We use the algebraic identity

$$F = f_{I+1-i} \cdot \ldots \cdot f_{I-1} - \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{I-1}$$
$$= S_{I+1-i} + \ldots + S_{I-1}$$

with

$$S_k = \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{k-1} f_k f_{k+1} \cdot \ldots \cdot f_{I-1} -$$
$$- \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{k-1} \mathbf{f}_k f_{k+1} \cdot \ldots \cdot f_{I-1}$$
$$= \mathbf{f}_{I+1-i} \cdot \ldots \cdot \mathbf{f}_{k-1} (f_k - \mathbf{f}_k) f_{k+1} \cdot \ldots \cdot f_{I-1} .$$

This yields

$$F^2 = (S_{I+1-i} + \dots + S_{I-1})^2$$

$$= \sum_{k=I+1-i}^{I-1} S_k^2 + 2 \sum_{j<k} S_j S_k \, .$$

where in the last summation j and k run from $I+1-i$ to $I-1$. Now we replace S_k^2 with $E(S_k^2|B_k)$ and S_jS_k, $j < k$, with $E(S_jS_k|B_k)$. This means that we approximate S_k^2 and S_jS_k by varying and averaging as little data as possible so that as many values C_{ik} as possible from data observed are kept fixed. Due to (A4) we have $E(f_k-\mathbf{f}_k|B_k) = 0$ and therefore $E(S_jS_k|B_k) = 0$ for $j < k$ because all $\mathbf{f}_r$, $r < k$, are scalars under B_k. Because of

(D12) $$E((f_k-\mathbf{f}_k)^2|B_k) = Var(\mathbf{f}_k|B_k)$$

$$= \sum_{j=1}^{I-k} Var(C_{j,k+1}|B_k)/(\sum_{j=1}^{I-k} C_{jk})^2$$

$$= \sum_{j=1}^{I-k} Var(C_{j,k+1}|C_{j1},\dots,C_{jk})/(\sum_{j=1}^{I-k} C_{jk})^2$$

$$= \sum_{j=1}^{I-k} C_{jk}\alpha_k^2 \ / \ (\sum_{j=1}^{I-k} C_{jk})^2$$

$$= \alpha_k^2 \ / \ \sum_{j=1}^{I-k} C_{jk}$$

we obtain

$$E(S_k^2|B_k) = \mathbf{f}_{I+1-i}^2 \cdots \mathbf{f}_{k-1}^2 \alpha_k^2 f_{k+1}^2 \cdots f_{I-1}^2 \ / \ \sum_{j=1}^{I-k} C_{jk} \, .$$

Taken together, we have replaced $F^2 = (\Sigma S_k)^2$ with $\Sigma_k E(S_k^2|B_k)$ and because all terms of this sum are positive we can replace all unknown parameters f_k, α_k^2 with their unbiased estimators

f_k, α_k^2. Altogether, we estimate $F^2 = (f_{I+1-i}\cdot\ldots\cdot f_{I-1} - \hat{f}_{I+1-i}\cdot\ldots\cdot\hat{f}_{I-1})^2$ by

$$\sum_{k=I+1-i}^{I-1} \left(\hat{f}^2_{I+1-i}\cdots\hat{f}^2_{k-1}\cdot\hat{\alpha}^2_k\cdot\hat{f}^2_{k+1}\cdots\hat{f}^2_{I-1} \Big/ \sum_{j=1}^{I-k} C_{jk} \right) =$$

$$= \hat{f}^2_{I+1-i}\cdot\ldots\cdot\hat{f}^2_{I-1} \sum_{k=I+1-i}^{I-1} \frac{\hat{\alpha}_k^2/\hat{f}_k^2}{\sum_{j=1}^{I-k} C_{jk}} .$$

Using (D11), this means that we estimate $(E_i(C_{iI}) - \hat{C}_{iI})^2$ by

$$\text{(D13)} \quad C^2_{i,I+1-i}\hat{f}^2_{I+1-i}\cdot\ldots\cdot\hat{f}^2_{I-1} \sum_{k=I+1-i}^{I-1} \frac{\hat{\alpha}_k^2/\hat{f}_k^2}{\sum_{j=1}^{I-k} C_{jk}} =$$

$$= \hat{C}^2_{iI} \sum_{k=I+1-i}^{I-1} \frac{\hat{\alpha}_k^2/\hat{f}_k^2}{\sum_{j=1}^{I-k} C_{jk}} .$$

From (D2), (D10) and (D13) we finally obtain the estimator $(\mathrm{s.e.}(\hat{C}_{iI}))^2$ for $\mathrm{mse}(\hat{C}_{iI})$ as stated in the proposition.

Appendix E: Unbiasedness of the Estimator α_k^2

Proposition: Under the assumptions

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

(4) The variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \neq j$ are independent.

(5) There are unknown constants $\alpha_1, \ldots, \alpha_{I-1}$ with

$Var(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}\alpha_k^2, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

the estimators

$$\alpha_k^2 = \frac{1}{I-k-1} \sum_{j=1}^{I-k} C_{jk} \left(\frac{C_{j,k+1}}{C_{jk}} - f_k \right)^2, \quad 1 \le k \le I-2,$$

of α_k^2 are unbiased, i.e. we have

$$E(\alpha_k^2) = \alpha_k^2, \quad 1 \le k \le I-2.$$

Proof: In this proof all summations are over the index j from j=1 to j=I-k. The definition of α_k^2 can be rewritten as

(E1) $$(I-k-1)\alpha_k^2 = \Sigma \left(C_{j,k+1}^2/C_{jk} - 2\cdot C_{j,k+1}f_k + C_{jk}f_k^2 \right) = \Sigma \left(C_{j,k+1}^2/C_{jk} \right) - \Sigma \left(C_{jk}f_k^2 \right)$$

using $\Sigma C_{j,k+1} = f_k \Sigma C_{jk}$ according to the definition of f_k. Using again the set

$$B_k = \{ C_{ij} \mid i+j \le I+1, \ j \le k \}$$

of variables C_{ij} assumed to be known, (E1) yields

(E2) $$E((I-k-1)\alpha_k^2|B_k) = \Sigma E(C_{j,k+1}^2|B_k)/C_{jk} - \Sigma C_{jk}E(f_k^2|B_k)$$

because C_{jk} is a scalar under the condition of B_k being known. Due to the independence (4) of the accident years, conditions which are independent from the conditioned variable can be omitted in $E(C_{j,k+1}^2|B_k)$, i.e.

$$\text{(E3)}\quad E(C_{j,k+1}^2|B_k) = E(C_{j,k+1}^2|C_{j1},\dots,C_{jk})$$
$$= \mathrm{Var}(C_{j,k+1}|C_{j1},\dots,C_{jk}) + (E(C_{j,k+1}|C_{j1},\dots,C_{jk}))^2$$
$$= C_{jk}\alpha_k^2 + (C_{jk}f_k)^2$$

where the rule $E(X^2) = \mathrm{Var}(X) + (E(X))^2$ and the assumptions (5) and (3) have also been used.

From (D12) and (A4) we gather

$$\text{(E4)}\quad E(\mathbf{f_k}^2|B_k) = \mathrm{Var}(\mathbf{f_k}|B_k) + (E(\mathbf{f_k}|B_k))^2$$
$$= \alpha_k^2 \,/\, \Sigma C_{jk} + f_k^2 \,.$$

Inserting (E3) and (E4) into (E2) we obtain

$$E((I-k-1)\boldsymbol{\alpha_k}^2|B_k) =$$
$$= \sum_{j=1}^{I-k} (\alpha_k^2 + C_{jk}f_k^2) - \sum_{j=1}^{I-k} (C_{jk}\alpha_k^2/\sum_{j=1}^{I-k} C_{jk} + C_{jk}f_k^2)$$
$$= (I-k)\alpha_k^2 - \alpha_k^2$$
$$= (I-k-1)\alpha_k^2 \,.$$

From this we immediately obtain $E(\boldsymbol{\alpha_k}^2|B_k) = \alpha_k^2$.

Finally, the iterative rule for expectations yields

$$E(\boldsymbol{\alpha_k}^2) = E(E(\boldsymbol{\alpha_k}^2|B_k)) = E(\alpha_k^2) = \alpha_k^2 \,.$$

Appendix F: The Standard Error of the Overall Reserve Estimate

Proposition: Under the assumptions

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

(4) The variables $\{C_{i1}, \ldots, C_{iI}\}$ and $\{C_{j1}, \ldots, C_{jI}\}$ of different accident years $i \ne j$ are independent.

(5) There are unknown constants $\alpha_1, \ldots, \alpha_{I-1}$ with

$Var(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}\alpha_k^2, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$

the standard error s.e.(**R**) of the overall reserve estimate

$$\mathbf{R} = \mathbf{R}_2 + \ldots + \mathbf{R}_I$$

is given by

$$(s.e.(\mathbf{R}))^2 = \sum_{i=2}^{I} \left\{ (s.e.(\mathbf{R}_i))^2 + \mathbf{C}_{iI}\left(\sum_{j=i+1}^{I} \mathbf{C}_{jI}\right) \sum_{k=I+1-i}^{I-1} \frac{2\alpha_k^2/\mathbf{f}_k^2}{\sum_{n=1}^{I-k} C_{nk}} \right\}$$

Proof: This proof is analogous to that in Appendix D. The comments will therefore be brief.

We first must determine the mean squared error mse(**R**) of **R**.

Using again $D = \{ C_{ik} \mid i+k \le I+1 \}$ we have

$$\text{(F1)} \quad mse\left(\sum_{i=2}^{I} \mathbf{R}_i\right) = E\left(\left(\sum_{i=2}^{I} \mathbf{R}_i - \sum_{i=2}^{I} R_i\right)^2 \middle| D\right)$$

$$= E\left(\left(\sum_{i=2}^{I} \mathbf{C}_{iI} - \sum_{i=2}^{I} C_{iI}\right)^2 \middle| D\right)$$

$$= Var\left(\sum_{i=2}^{I} C_{iI} \middle| D\right) + \left(E\left(\sum_{i=2}^{I} C_{iI} \middle| D\right) - \sum_{i=2}^{I} \mathbf{C}_{iI} \right)^2 .$$

The independence of the accident years yields

$$\text{(F2)}\qquad \mathrm{Var}\Big(\sum_{i=2}^{I} C_{iI}\,\Big|\,D\Big) = \sum_{i=2}^{I} \mathrm{Var}(C_{iI}\,|\,C_{i1},\ \ldots,\ C_{i,I+1-i})\ ,$$

whose summands have been calculated in Appendix D, see (D9).

Furthermore

$$\text{(F3)}\qquad \Big(\ E\Big(\sum_{i=2}^{I} C_{iI}\,\Big|\,D\Big) - \sum_{i=2}^{I} \mathbf{C}_{iI}\ \Big)^2 = \Big(\sum_{i=2}^{I}\Big(\ E(C_{iI}|D) - \mathbf{C}_{iI}\ \Big)\ \Big)^2 =$$

$$= \sum_{2\le i,j\le I} (E(C_{iI}|D) - \mathbf{C}_{iI})\cdot(E(C_{jI}|D) - \mathbf{C}_{jI})$$

$$= \sum_{2\le i,j\le I} C_{i,I+1-i}C_{j,I+1-j}F_iF_j$$

$$= \sum_{i=2}^{I} (C_{i,I+1-i}F_i)^2 + 2\sum_{i<j} C_{i,I+1-i}C_{j,I+1-j}F_iF_j$$

with (like in (D11))

$$F_i = f_{I+1-i}\cdots f_{I-1} - \mathbf{f}_{I+1-i}\cdots\mathbf{f}_{I-1}$$

which is identical to F of Appendix D but here we have to carry the index i, too. In Appendix D we have shown (cf. (D2) and (D11)) that

$$\mathrm{mse}(\mathbf{R}_i) = \mathrm{Var}(C_{iI}|C_{i1},\ldots,C_{i,I+1-i}) + (C_{i,I+1-i}F_i)^2\ .$$

Comparing this with (F1), (F2) and (F3) we see that

$$\text{(F4)}\qquad \mathrm{mse}\Big(\sum_{i=2}^{I} \mathbf{R}_i\Big) = \sum_{i=2}^{I} \mathrm{mse}(\mathbf{R}_i) + \sum_{2\le i<j\le I} 2\cdot C_{i,I+1-i}C_{j,I+1-j}F_iF_j.$$

We therefore need only develop an estimator for F_iF_j. A procedure completely analogous to that for F^2 in the proof of Appendix D yields for F_iF_j, i<j, the estimator

$$\sum_{k=I+1-i}^{I-1} \mathbf{f}_{I+1-j}\cdots\mathbf{f}_{I-i}\mathbf{f}^2_{I+1-i}\cdots\mathbf{f}^2_{k-1}\boldsymbol{\alpha}^2_k\mathbf{f}^2_{k+1}\cdots\mathbf{f}^2_{I-1}\Big/\sum_{n=1}^{I-k} C_{nk}\ ,$$

which immediately leads to the result stated in the proposition.

Appendix G: Testing for Correlations between Subsequent Development Factors

In this appendix we first prove that the basic assumption (3) of the chain ladder method implies that subsequent development factors $C_{ik}/C_{i,k-1}$ and $C_{i,k+1}/C_{ik}$ are not correlated. Then we show how we can test if this uncorrelatedness is met for a given run-off triangle. Finally, we apply this test procedure to the numerical example of Chapter 6.

Proposition: Under the assumption

(3) There are unknown constants $f_1, \ldots, f_{I-1}$ with

$$E(C_{i,k+1}|C_{i1},\ldots,C_{ik}) = C_{ik}f_k, \quad 1 \le i \le I, \quad 1 \le k \le I-1.$$

subsequent development factors $C_{ik}/C_{i,k-1}$ and $C_{i,k+1}/C_{ik}$ are uncorrelated, i.e. we have (for $1 \le i \le I$, $2 \le k \le I-1$)

$$E\left(\frac{C_{ik}}{C_{i,k-1}}\cdot\frac{C_{i,k+1}}{C_{ik}}\right) = E\left(\frac{C_{ik}}{C_{i,k-1}}\right)\cdot E\left(\frac{C_{i,k+1}}{C_{ik}}\right).$$

Proof: For $j \le k$ we have

$$\text{(G1)} \quad E(C_{i,k+1}/C_{ij}) = E(E(C_{i,k+1}/C_{ij}|C_{i1},\ldots,C_{ik})) \quad \text{(a)}$$

$$= E(E(C_{i,k+1}|C_{i1},\ldots,C_{ik})/C_{ij}) \quad \text{(b)}$$

$$= E(C_{ik}f_k/C_{ij}) \quad \text{(c)}$$

$$= E(C_{ik}/C_{ij})f_k\,. \quad \text{(d)}$$

Here equation (a) holds due to the iterative rule $E(X) = E(E(X|Y))$ for expectations, (b) holds because, given $C_{i1}, \ldots, C_{ik}$, C_{ij} is a scalar for $j \le k$, (c) holds due to (3) and (d) holds because f_k is a scalar.

From (G1) we obtain through the specialization $j = k$

(G2) $\quad E(C_{i,k+1}/C_{ik}) = E(C_{ik}/C_{ik})f_k = f_k$

and through $j = k-1$

(G3) $$E\left(\frac{C_{ik}}{C_{i,k-1}}\cdot\frac{C_{i,k+1}}{C_{ik}}\right) = E\left(\frac{C_{i,k+1}}{C_{i,k-1}}\right) \overset{(G1)}{=} E\left(\frac{C_{ik}}{C_{i,k-1}}\right)f_k .$$

Inserting (G2) into (G3) completes the proof.

Designing the test procedure:

The usual test for uncorrelatedness requires that we have identically distributed pairs of observations which come from a Normal distribution. Both conditions are usually not fulfilled for adjacent columns of development factors. (Note that due to (G2) the development factors $C_{i,k+1}/C_{ik}$, $1 \le i \le I-k$, have the same expectation but assumption (5) implies that they have different variances.) We therefore use the test with Spearman's rank correlation coefficient because this test is distribution-free and because by using ranks the differences in the variances of $C_{i,k+1}/C_{ik}$, $1 \le i \le I-k$, become less important. Even if these differences are negligeable the test will only be of an approximate nature because, strictly speaking, it is a test for independence rather than for uncorrelatedness. But we will take this into account when fixing the critical value of the test statistic.

For the application of Spearman's test we consider a fixed development year k and rank the development factors $C_{i,k+1}/C_{ik}$ observed so far according to their size starting with the smallest one on rank one and so on. Let r_{ik}, $1 \le i \le I-k$, denote the rank of $C_{i,k+1}/C_{ik}$ obtained in this way, $1 \le r_{ik} \le I-k$. Then we do the same with the preceding development factors $C_{ik}/C_{i,k-1}$, $1 \le i \le I-k$, leaving out $C_{I+1-k,k}/C_{I+1-k,k-1}$ for which the subsequent development factor has not yet been observed. Let s_{ik}, $1 \le i \le I-k$, be the ranks obtained in this way, $1 \le s_{ik} \le I-k$. Now, Spearman's rank correlation coefficient T_k is defined to be

$$\text{(G4)} \qquad T_k = 1 - 6 \sum_{i=1}^{I-k} (r_{ik} - s_{ik})^2 \;/\; ((I-k)^3 - I + k) \;.$$

From a textbook of Mathematical Statistics it can be seen that

$$-1 \le T_k \le +1 \;,$$

and, under the null-hypothesis,

$$E(T_k) = 0 \;,$$

$$\mathrm{Var}(T_k) = 1/(I-k-1) \;.$$

A value of T_k close to 0 indicates that the development factors between development years k-1 and k and those between years k and k+1 are not correlated. Any other value of T_k indicates that the factors are (positively or negatively) correlated.

For a formal test we do not want to consider every pair of columns of adjacent development years separately in order to avoid an accumulation of the error probabilities. We therefore consider the triangle as a whole. This also is preferable from a practical point of view because it is more important to know whether correlations globally prevail than to find a small part of the triangle with correlations. We therefore combine all values T_2, T_3, ..., T_{I-2} obtained in the same way like T_k. (There is no T_1 because there are no development factors before development year k=1 and similarly there is also no T_I; even T_{I-1} is not included because there is only one rank and therefore no randomness.) According to Appendix B we should not form an unweighted average of T_2, ..., T_{I-2} but rather use weights which are inversely proportional to $Var(T_k) = 1/(I-k-1)$. This leads to weights which are just equal to one less than the number of pairs (r_{ik}, s_{ik}) taken into account by T_k which seems very reasonable.

We thus calculate

$$\text{(G5)} \qquad T = \sum_{k=2}^{I-2} (I-k-1) T_k \Big/ \sum_{k=2}^{I-2} (I-k-1)$$

$$= \sum_{k=2}^{I-2} \frac{I-k-1}{(I-2)(I-3)/2} T_k \ ,$$

$$E(T) = \sum_{k=2}^{I-2} E(T_k) = 0 \ ,$$

$$\text{(G6)} \qquad Var(T) = \sum_{k=2}^{I-2} (I-k-1)^2 \, Var(T_k) \Big/ \left(\sum_{k=2}^{I-2} (I-k-1) \right)^2$$

$$= \sum_{k=2}^{I-2} (I-k-1) \Big/ \left(\sum_{k=2}^{I-2} (I-k-1) \right)^2$$

$$= \frac{1}{(I-2)(I-3)/2}$$

where for the calculation of Var(T) we used the fact that under the null-hypothesis subsequent development factors and therefore also different T_k's are uncorrelated.

Because the distribution of a single T_k with $I-k \geq 10$ is Normal in good approximation and because T is the aggregation of several uncorrelated T_k's (which all are symmetrically distributed around their mean 0) we can assume that T has approximately a Normal distribution and use this to design a significance test. Usually, when applying a significance test one rejects the null-hypothesis if it is very unlikely to hold, e.g. if the value of the test statistic is outside its 95% confidence interval. But in our case we propose to use only a 50% confidence interval because the test is only of an approximate nature and because we want to detect correlations already in a substantial part of the run-off triangle. Therefore, as the probability for a Standard Normal variate lying in the interval (-.67, .67) is 50% we do not reject the null-hypothesis of having uncorrelated development factors if

$$- \frac{.67}{\sqrt{((I-2)(I-3)/2)}} \leq T \leq + \frac{.67}{\sqrt{((I-2)(I-3)/2)}} .$$

If T is outside this interval we should be reluctant with the application of the chain ladder method and analyze the correlations in more detail.

<u>Application to the example of Chapter 6</u>:

We start with the table of all development factors:

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
i=1	1.6	1.32	1.08	1.15	1.20	1.11	1.033	1.00	1.01
i=2	40.4	1.26	1.98	1.29	1.13	0.99	1.043	1.03	
i=3	2.6	1.54	1.16	1.16	1.19	1.03	1.026		
i=4	2.0	1.36	1.35	1.10	1.11	1.04			
i=5	8.8	1.66	1.40	1.17	1.01				
i=6	4.3	1.82	1.11	1.23					
i=7	7.2	2.72	1.12						
i=8	5.1	1.89							
i=9	1.7								

As described above we first rank column F_1 according to the size of the factors, then leave out the last element and rank the column again. Then we do the same with columns F_2 to F_8. This yields the following table:

r_{i1}	s_{i2}	r_{i2}	s_{i3}	r_{i3}	s_{i4}	r_{i4}	s_{i5}	r_{i5}	s_{i6}	r_{i6}	s_{i7}	r_{i7}	s_{i8}	r_{i8}
1	1	2	2	1	1	2	2	5	4	4	3	2	1	1
9	8	1	1	7	6	6	5	3	2	1	1	3	2	2
4	3	4	4	4	3	3	3	4	3	2	2	1		
3	2	3	3	5	4	1	1	2	1	3				
8	7	5	5	6	5	4	4	1						
5	4	6	6	2	2	5								
7	6	8	7	3										
6	5	7												
2														

We now add the squared differences between adjacent rank columns of equal length, i.e. we add $(s_{ik} - r_{ik})^2$ over i for every k, $2 \le k \le 8$. This yields 68, 74, 20, 24, 6, 6 and 0. (Remember that we have to leave out k = 1 because there is no s_{i1}, and k = 9 because there is only one pair of ranks and therefore no randomness.) From these figures we obtain Spearman's rank correlation coefficients T_k according to formula (G4):

k	2	3	4	5	6	7	8
T_k	4/21	-9/28	3/7	-1/5	2/5	-1/2	1
I-k-1	7	6	5	4	3	2	1

The (I-k-1)-weighted average of the T_k's is T = .070 (see formula (G5)). Because of Var(T) = 1/28 (see (G6)) the 50% confidence limits for T are $\pm.67/\sqrt{28} = \pm.127$. Thus, T is within its 50%-interval and the hypothesis of having uncorrelated development factors is not rejected.

Appendix H: Testing for Calendar Year Effects

One of the three basic assumptions underlying the chain ladder method was seen to be assumption (4) of the independence of the accident years. The main reason why this independence can be violated in practice is the fact that we can have certain calendar year effects such as major changes in claims handling or in case reserving or external influences such as substantial changes in court decisions or inflation. Note that a constant rate of inflation which has not been removed from the data is extrapolated into the future by the chain ladder method. In the following, we first generally describe a procedure to test for such calendar year influences and then apply it to our example.

Designing the test procedure:

A calendar year influence affects one of the diagonals

$$D_j = \{ C_{j1}, C_{j-1,2}, \dots, C_{2,j-1}, C_{1j} \}, \quad 1 \le j \le I,$$

and therefore also influences the adjacent development factors

$$A_j = \{ C_{j2}/C_{j1}, C_{j-1,3}/C_{j-1,2}, \dots, C_{1,j+1}/C_{1j} \}$$

and

$$A_{j-1} = \{ C_{j-1,2}/C_{j-1,1}, C_{j-2,3}/C_{j-2,2}, \dots, C_{1j}/C_{1,j-1} \}$$

where the elements of D_j form either the denominator or the numerator. Thus, if due to a calendar year influence the elements of D_j are larger (smaller) than usual, then the elements of A_{j-1} are also larger (smaller) than usual and the elements of A_j are smaller (larger) than usual.

Therefore, in order to check for such calendar year influences we only have to subdivide all development factors into 'smaller' and 'larger' ones and then to examine whether there are diagonals where the small development factors or the large ones clearly prevail. For this purpose, we order for every k, $1 \leq k \leq I-1$, the elements of the set

$$F_k = \{ C_{i,k+1}/C_{ik} \mid 1 \leq i \leq I-k \} ,$$

i.e. of the column of all development factors observed between development years k and k+1, according to their size and subdivide them into one part LF_k of larger factors being greater than the median of F_k and into a second part SF_k of smaller factors below the median of F_k. (The median of a set of real numbers is defined to be a number which divides the set into two parts with the same number of elements.) If the number I-k of elements of F_k is odd there is one element of F_k which is equal to the median and therefore assigned to neither of the sets LF_k and SF_k; this element is eliminated from all further considerations.

Having done this procedure for each set F_k, $1 \leq k \leq I-1$, every development factor observed is

- either eliminated (like e.g. the only element of F_{I-1})
- or assigned to the set $L = LF_1 + \ldots + LF_{I-2}$ of larger factors
- or assigned to the set $S = SF_1 + \ldots + SF_{I-2}$ of smaller factors. In this way, every development factor which is not eliminated has a 50% chance of belonging to either L or S.

Now we count for every diagonal A_j, $1 \le j \le I-1$, of development factors the number L_j of large factors, i.e. elements of L, and the number S_j of small factors, i.e. elements of S. Intuitively, if there is no specific change from calendar year j to calendar year j+1, A_j should have about the same number of small factors as of large factors, i.e. L_j and S_j should be of approximately the same size apart from pure random fluctuations. But if L_j is significantly larger or smaller than S_j or, equivalently, if

$$Z_j = \min(L_j, S_j) ,$$

i.e. the smaller of the two figures, is significantly smaller than $(L_j+S_j)/2$, then there is some reason for a specific calendar year influence.

In order to design a formal test we need the first two moments of the probability distribution of Z_j under the hypothesis that each development factor has a 50 % probability of belonging to either L or S. This distribution can easily be established. We give an example for the case where $L_j+S_j = 5$, i.e. where the set A_j contains 5 development factors without counting any eliminated factor. Then the number L_j has a Binomial distribution with n = 5 and p = .5, i.e.

$$\text{prob}(L_j = m) = \binom{n}{m} \frac{1}{2^n} = \binom{5}{m} \frac{1}{2^5} , \quad m = 0, 1, \ldots, 5.$$

Therefore

$$\text{prob}(S_j = 5) = \text{prob}(L_j = 0) = 1/32 ,$$

$$\text{prob}(S_j = 4) = \text{prob}(L_j = 1) = 5/32 ,$$

$$\text{prob}(S_j = 3) = \text{prob}(L_j = 2) = 10/32 \ ,$$

$$\text{prob}(S_j = 2) = \text{prob}(L_j = 3) = 10/32 \ ,$$

$$\text{prob}(S_j = 1) = \text{prob}(L_j = 4) = 5/32 \ ,$$

$$\text{prob}(S_j = 0) = \text{prob}(L_j = 5) = 1/32 \ .$$

This yields

$$\text{prob}(Z_j = 0) = \text{prob}(L_j = 0) + \text{prob}(S_j = 0) = 2/32 \ ,$$

$$\text{prob}(Z_j = 1) = \text{prob}(L_j = 1) + \text{prob}(S_j = 1) = 10/32 \ ,$$

$$\text{prob}(Z_j = 2) = \text{prob}(L_j = 2) + \text{prob}(S_j = 2) = 20/32 \ ,$$

$$E(Z_j) = (0 \cdot 2 + 1 \cdot 10 + 2 \cdot 20)/32 = 50/32 \ ,$$

$$E(Z_j^2) = (0 \cdot 2 + 1 \cdot 10 + 4 \cdot 20)/32 = 90/32 \ ,$$

$$\text{Var}(Z_j) = E(Z_j^2) - (E(Z_j))^2 = 95/256 \ .$$

The derivation of the general formula is straightforward but tedious. We therefore give only its result. If $n = L_j+S_j$ and $m = [(n-1)/2]$ denotes the largest integer $\leq (n-1)/2$ then

$$\text{(H1)} \qquad E(Z_j) = \frac{n}{2} - \binom{n-1}{m} \frac{n}{2^n} \ ,$$

$$\text{(H2)} \qquad \text{Var}(Z_j) = \frac{n(n-1)}{4} - \binom{n-1}{m} \frac{n(n-1)}{2^n} + E(Z_j) - (E(Z_j))^2 \ .$$

It is not advisable to test each Z_j separately in order to avoid an accumulation of the error probabilities. Instead, we consider

$$Z = Z_2 + \ldots + Z_{I-1}$$

where we have left out Z_1 because A_1 contains at most one element which is not eliminated and therefore Z_1 is not a random variable but always $= 0$. Similarly, we have to leave out any other Z_j if $L_j+S_j \leq 1$. Because under the null-hypothesis different Z_j's are (almost) uncorrelated we have

$$E(Z) = E(Z_2) + \ldots + E(Z_{I-1}) ,$$

$$Var(Z) = Var(Z_2) + \ldots + Var(Z_{I-1})$$

and we can assume that Z approximately has a Normal distribution. This means that we reject (with an error probability of 5 %) the hypothesis of having no significant calendar year effects only if not

$$E(Z) - 2\cdot\sqrt{Var(Z)} \leq Z \leq E(Z) + 2\cdot\sqrt{Var(Z)} .$$

<u>Application to the example of Chapter 6</u>:

We start with the triangle of all development factors observed:

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
i=1	1.6	1.32	1.08	1.15	1.20	1.11	1.033	1.00	1.01
i=2	40.4	1.26	1.98	1.29	1.13	0.99	1.043	1.03	
i=3	2.6	1.54	1.16	1.16	1.19	1.03	1.026		
i=4	2.0	1.36	1.35	1.10	1.11	1.04			
i=5	8.8	1.66	1.40	1.17	1.01				
i=6	4.3	1.82	1.11	1.23					
i=7	7.2	2.72	1.12						
i=8	5.1	1.89							
i=9	1.7								

We have to subdivide each column F_k into the subset SF_k of 'smaller' factors below the median of F_k and into the subset LF_k of 'larger' factors above the median. This can be done very easily with the help of the rank columns r_{ik} established in Appendix G: The half of factors with small ranks belongs to SF_k, those with large ranks to LF_k and if the total number is odd we have to eliminate the mean rank. Replacing a small rank with

'S', a large rank with 'L' and a mean rank with '*' we obtain the following picture:

	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8	j=9
j=1	S	S	S	S	L	L	*	S	*
j=2	L	S	L	L	*	S	L	L	
j=3	S	S	*	S	L	S	S		
j=4	S	S	L	S	S	L			
j=5	L	L	L	L	S				
j=6	*	L	S	L					
j=7	L	L	S						
j=8	L	L							
j=9	S								

We now count for every diagonal A_j, $2 \le j \le 9$, the number L_j of L's and the number S_j of S's. With the notations $Z_j = \min(L_j, S_j)$, $n = S_j + L_j$, $m = [(n-1)/2]$ as above and using the formulae (H1), (H2) for $E(Z_j)$ and $Var(Z_j)$ we obtain the following table:

j	S_j	L_j	Z_j	n	m	$E(Z_j)$	$Var(Z_j)$
2	1	1	1	2	0	.5	.25
3	3	0	0	3	1	.75	.1875
4	3	1	1	4	1	1.25	.4375
5	1	3	1	4	1	1.25	.4375
6	1	3	1	4	1	1.25	.4375
7	2	4	2	6	2	2.0625	.6211
8	4	4	4	8	3	2.90625	.8037
9	4	4	4	8	3	2.90625	.8037
Total			14			12.875	$3.9785 = (1.9946)^2$

The test statistic $Z = \Sigma Z_j = 14$ is not outside its 95%-range $(12.875 - 2\cdot 1.9946,\ 12.875 + 2\cdot 1.9946) = (8.886,\ 16.864)$ and therefore the null-hypothesis of not having significant calendar year influences is not rejected so that we can continue to apply the chain ladder method.

Figure 1: Regression and Residuals
Ci2 against Ci1

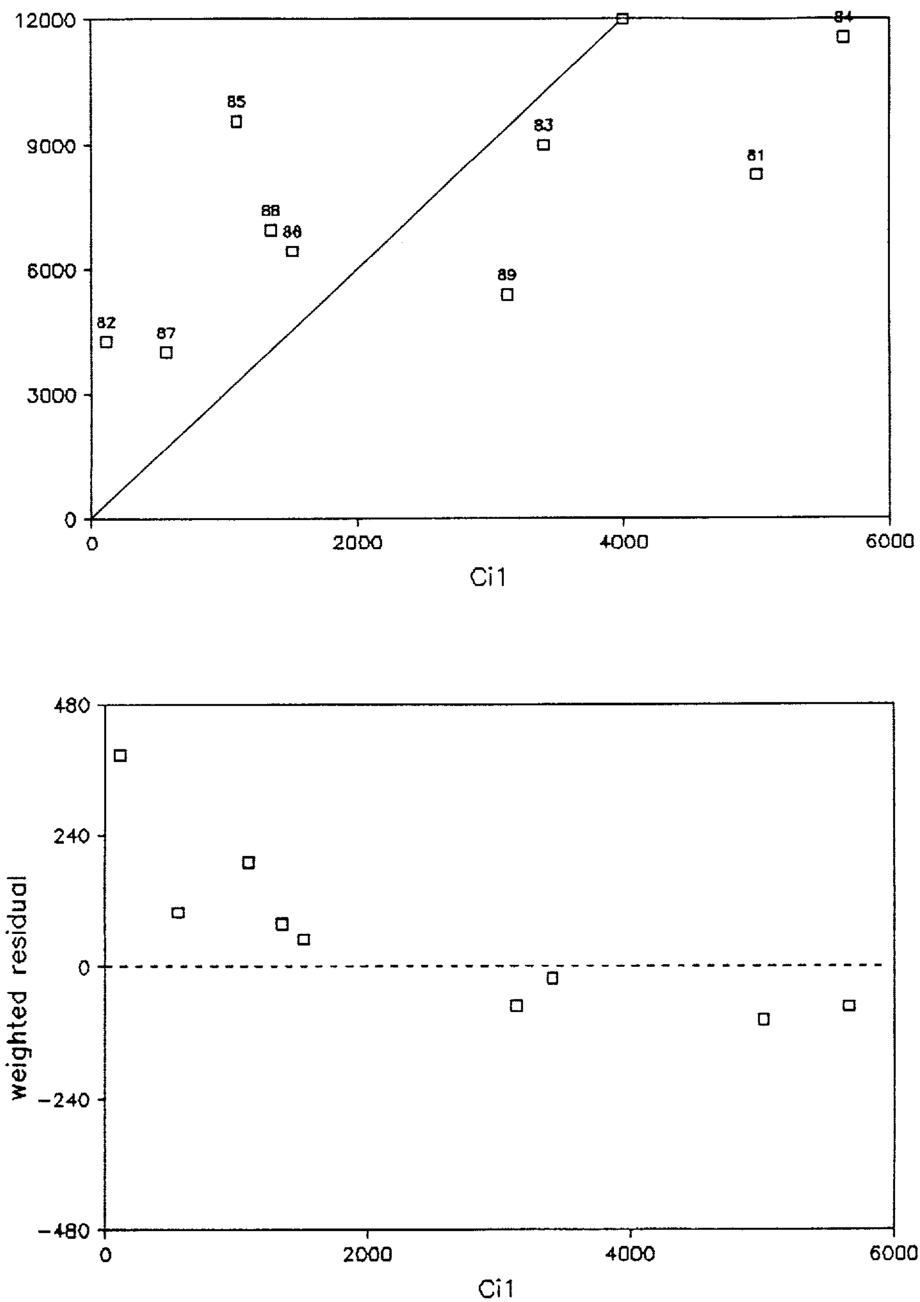


Figure 2: Regression and Residuals

Ci3 against Ci2

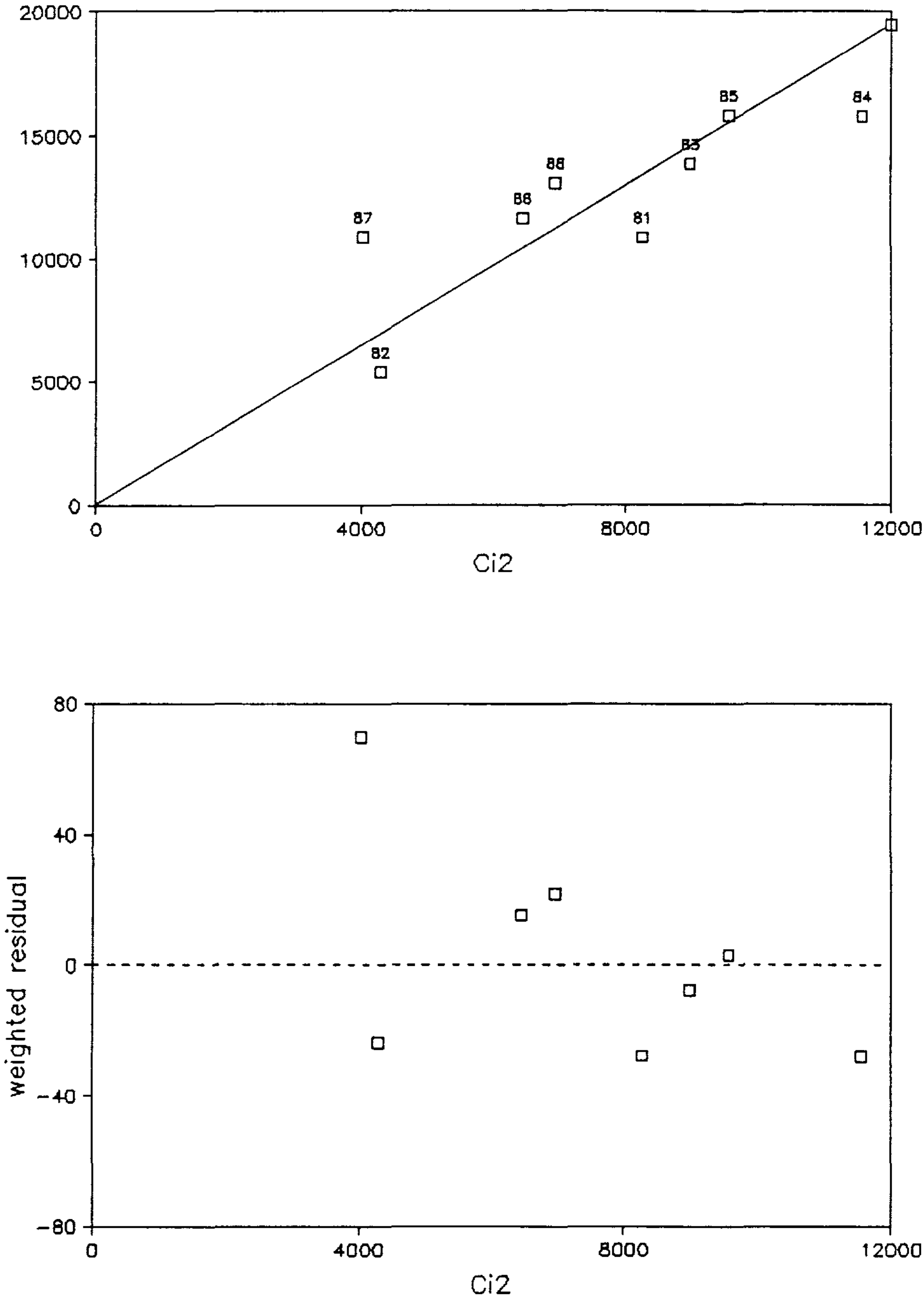
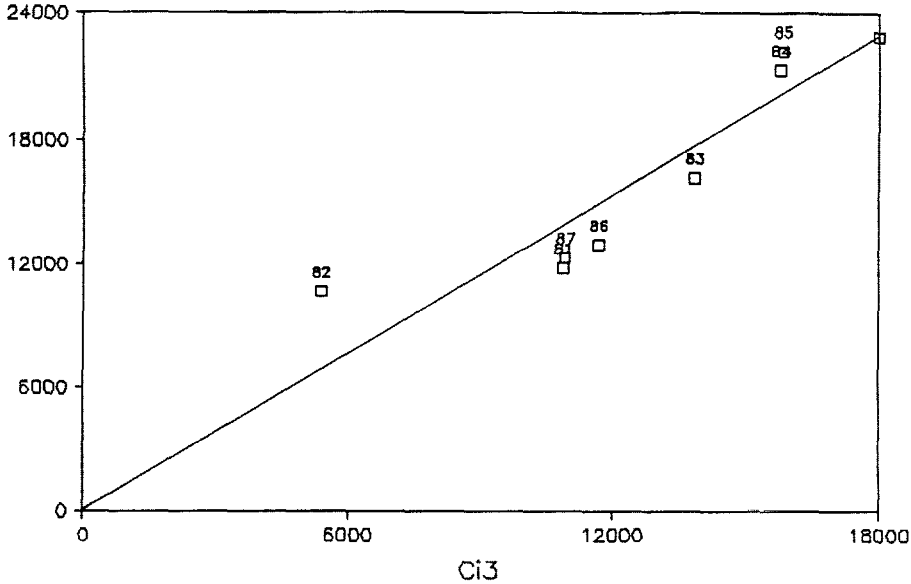


Figure 3: Regression and Residuals

Ci4 against Ci3



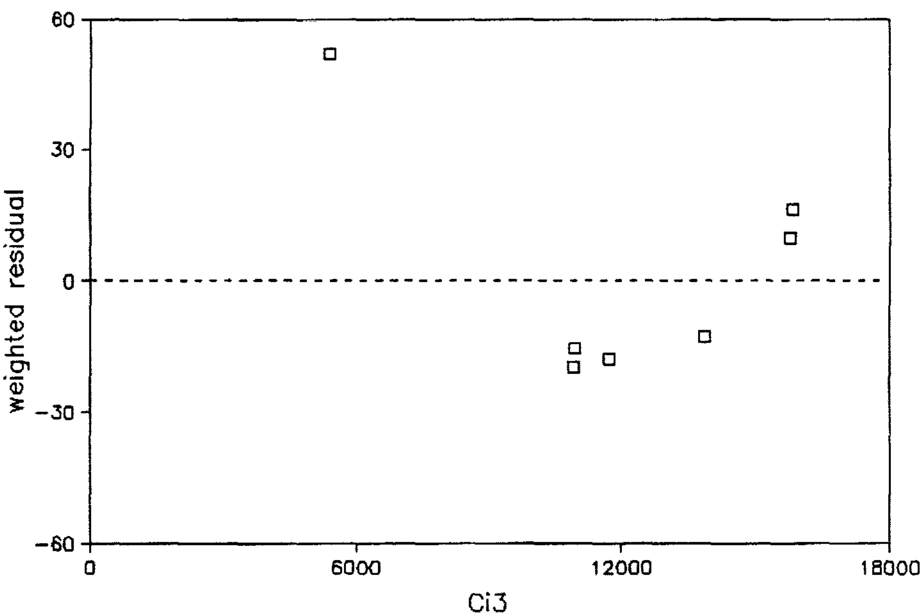


Figure 3: Regression and Residuals

Ci4 against Ci3

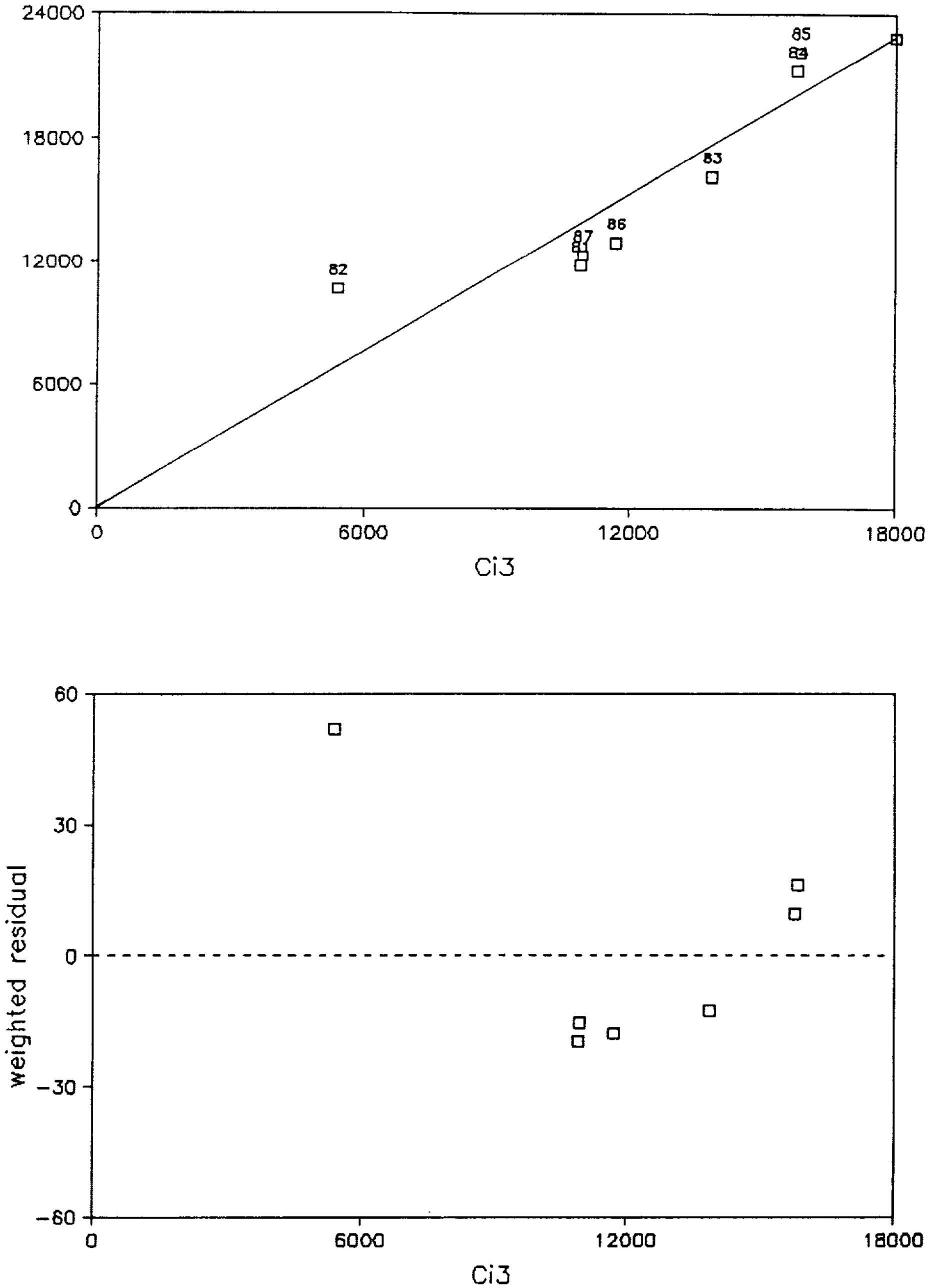


Figure 4: Regression and Residuals

Ci5 against Ci4

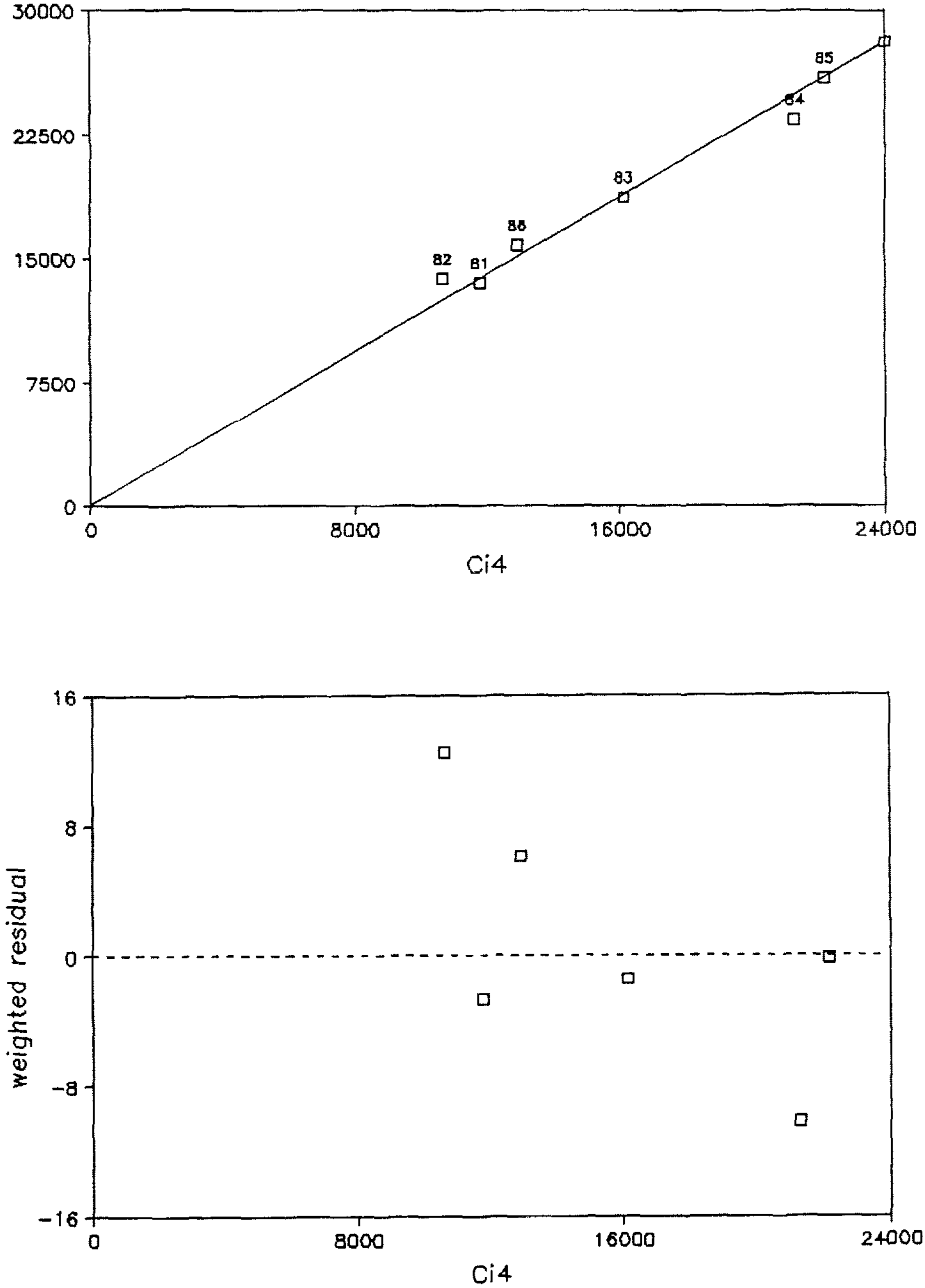


Figure 5: Regression and Residuals

Ci6 against Ci5

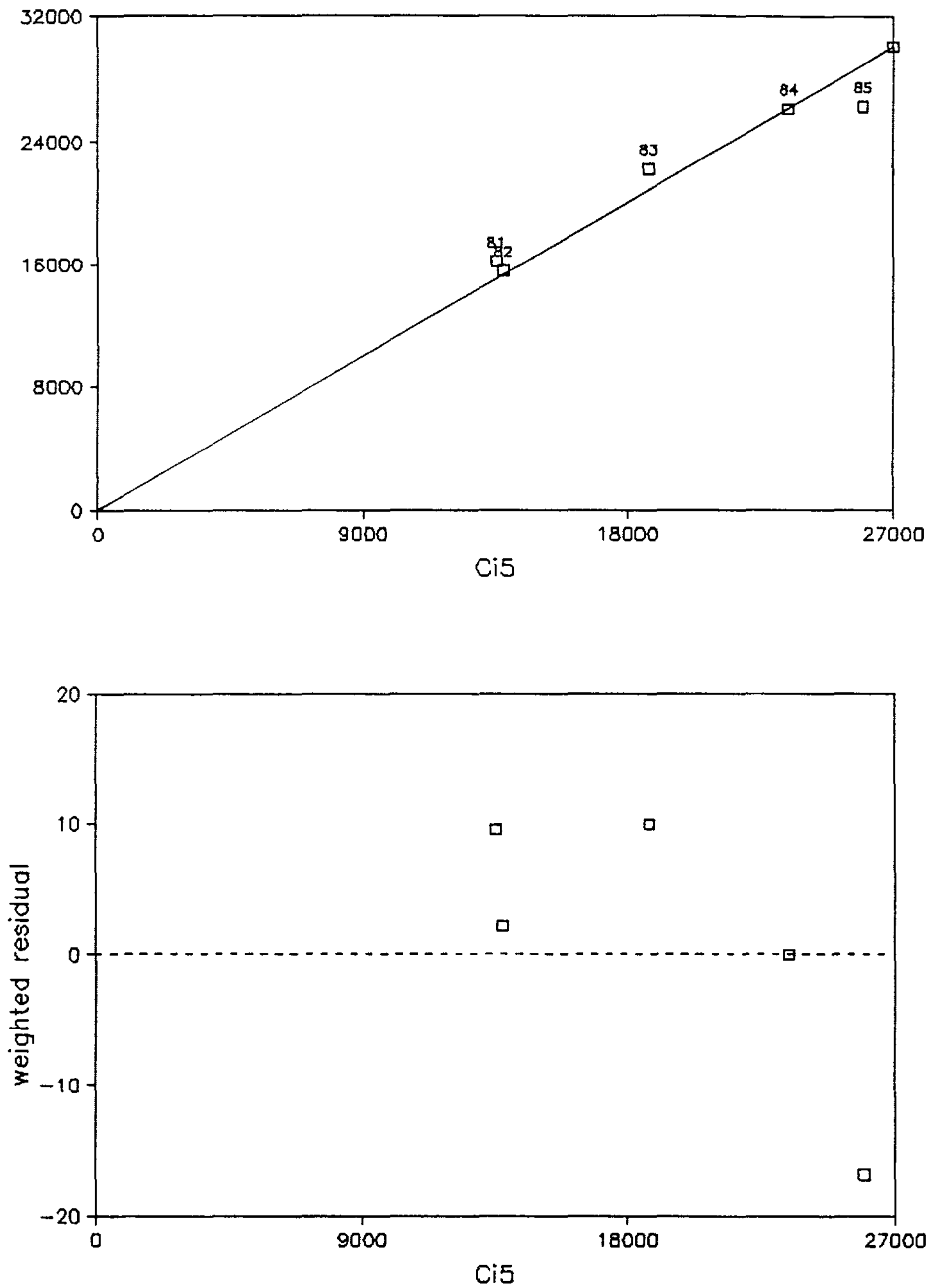


Figure 6: Regression and Residuals
Ci7 against Ci6

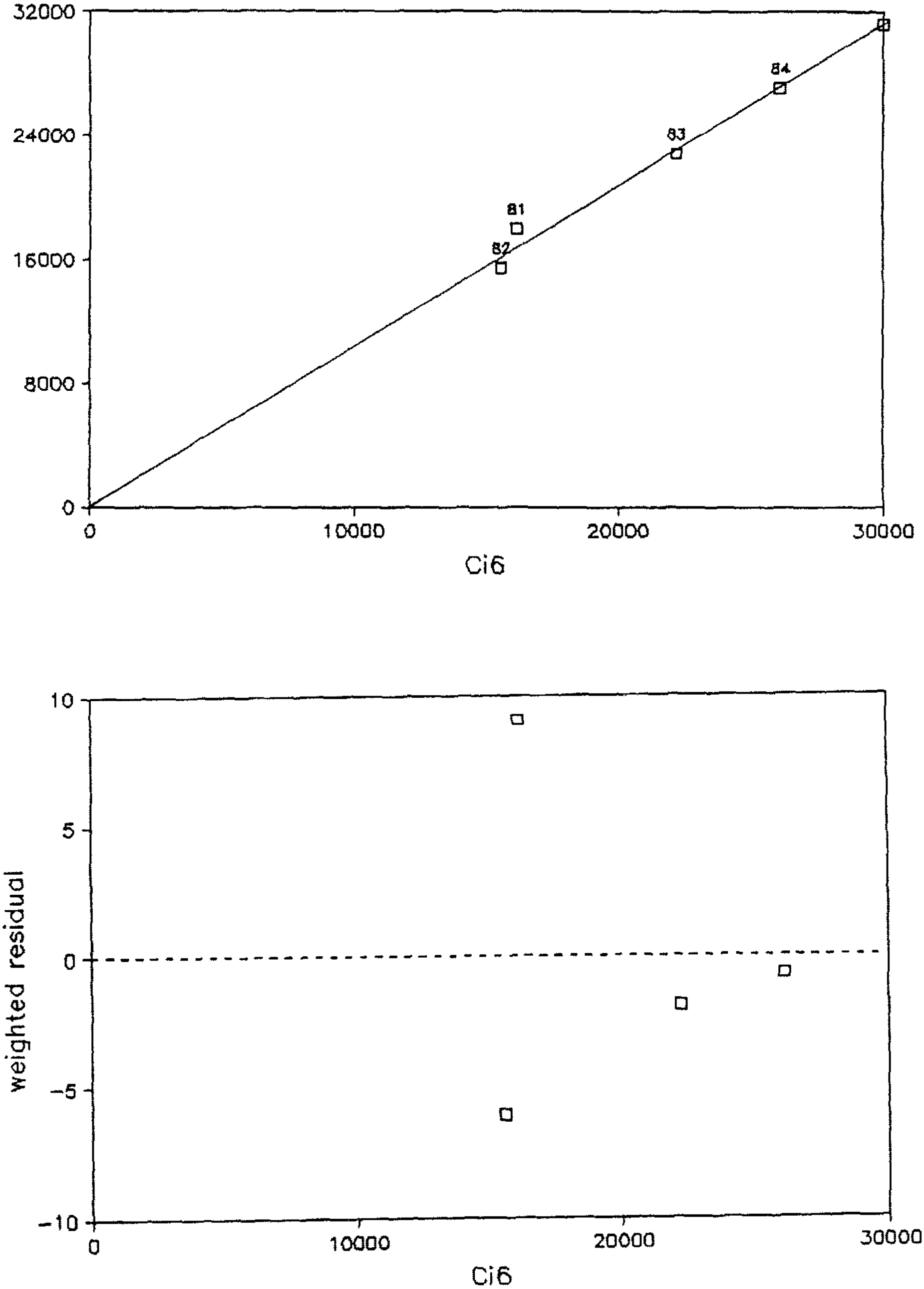
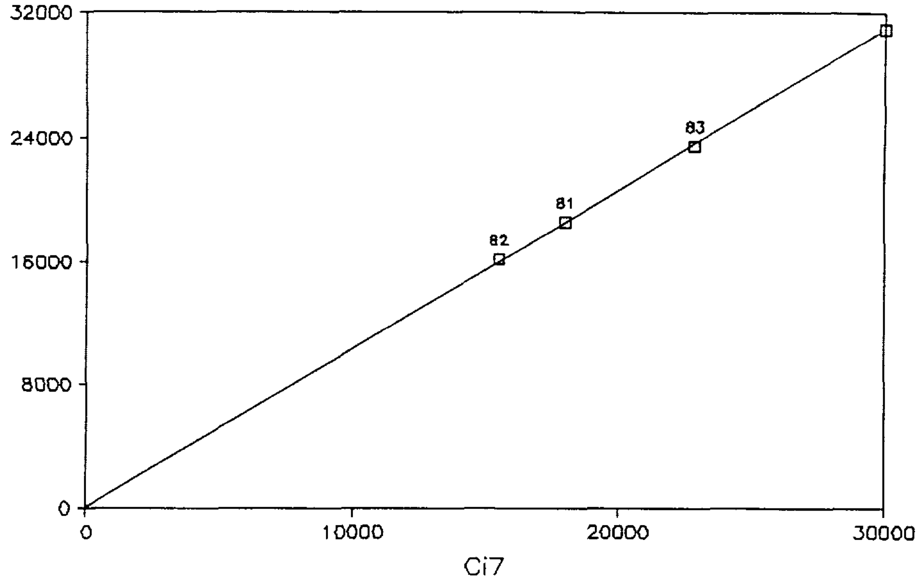


Figure 7: Regression and Residuals

Ci8 against Ci7



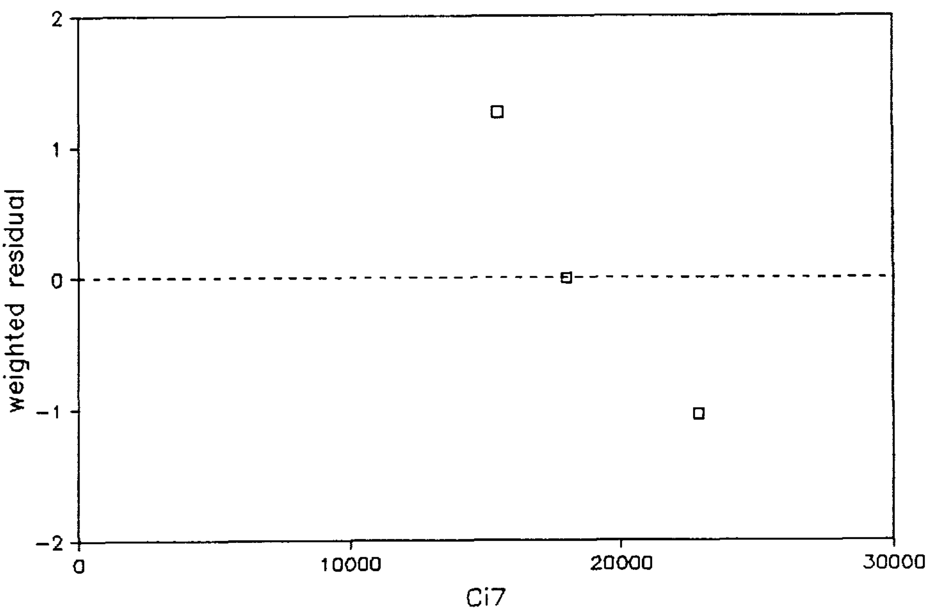


Figure 8: Regression and Residuals

Ci9 against Ci8

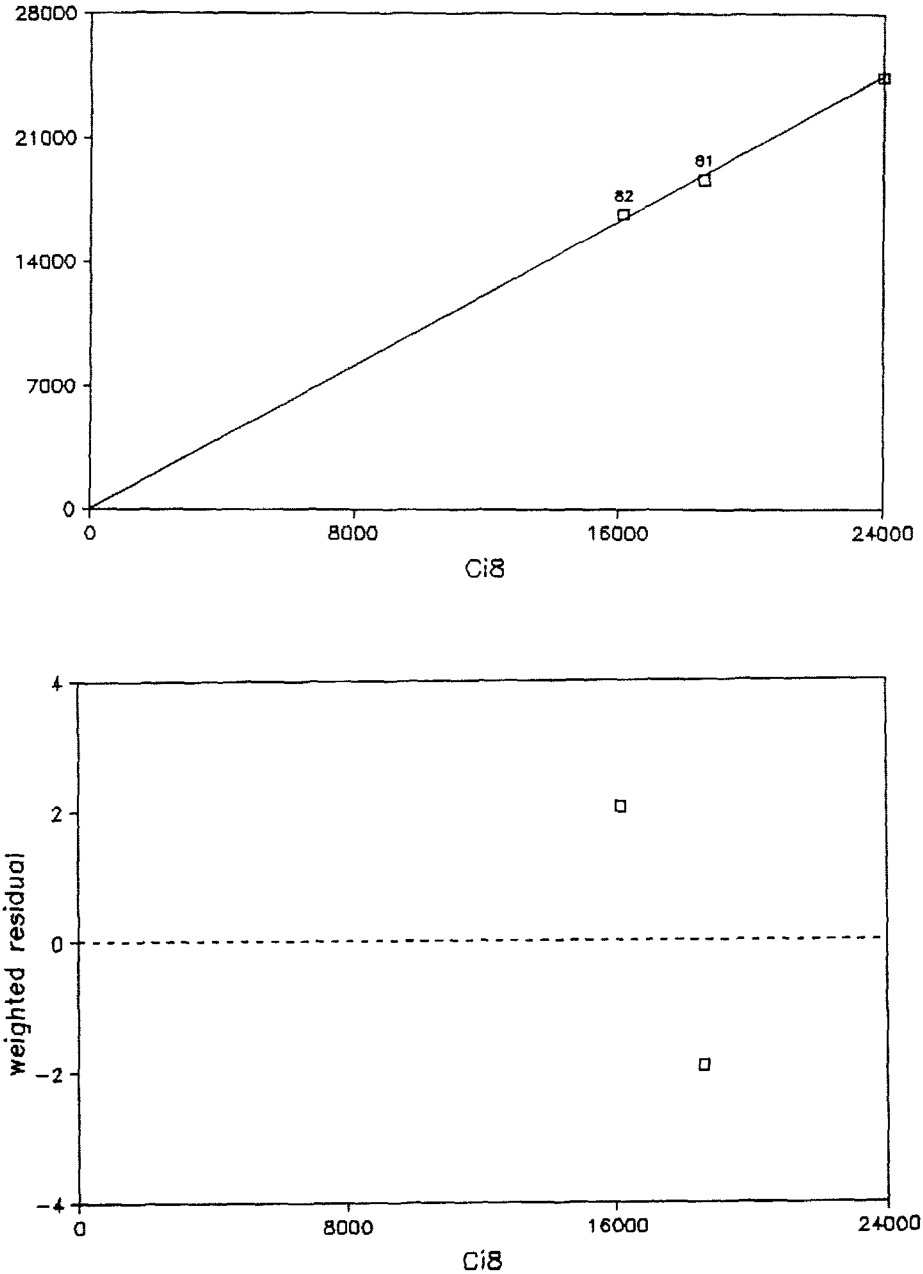
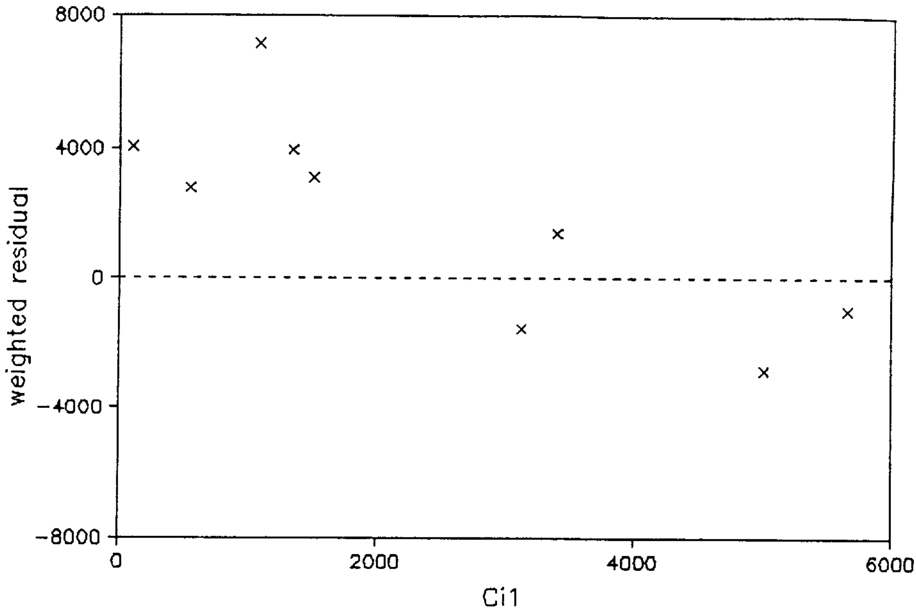


Figure 9: Residual Plots for fk0



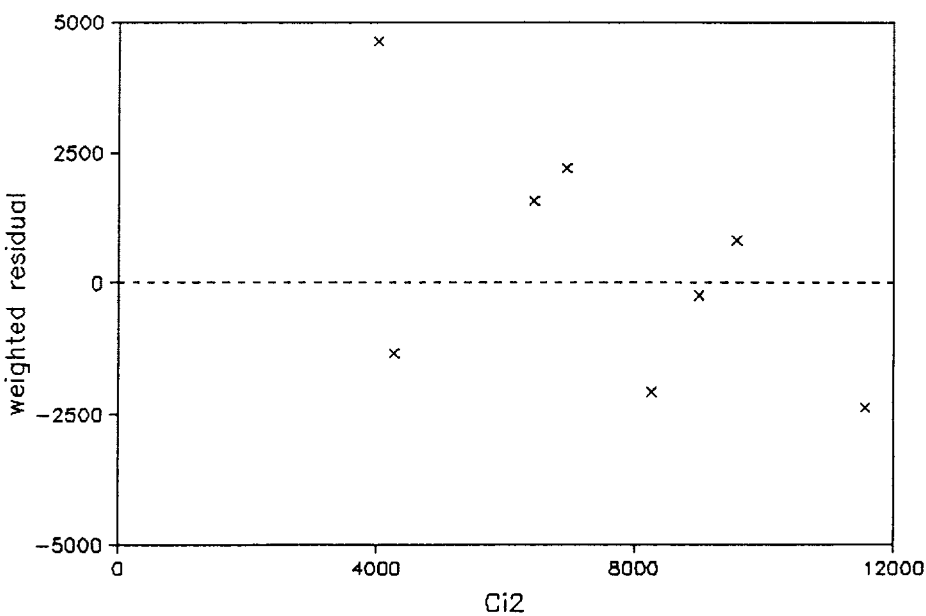


Figure 10: Residual Plots for fk0

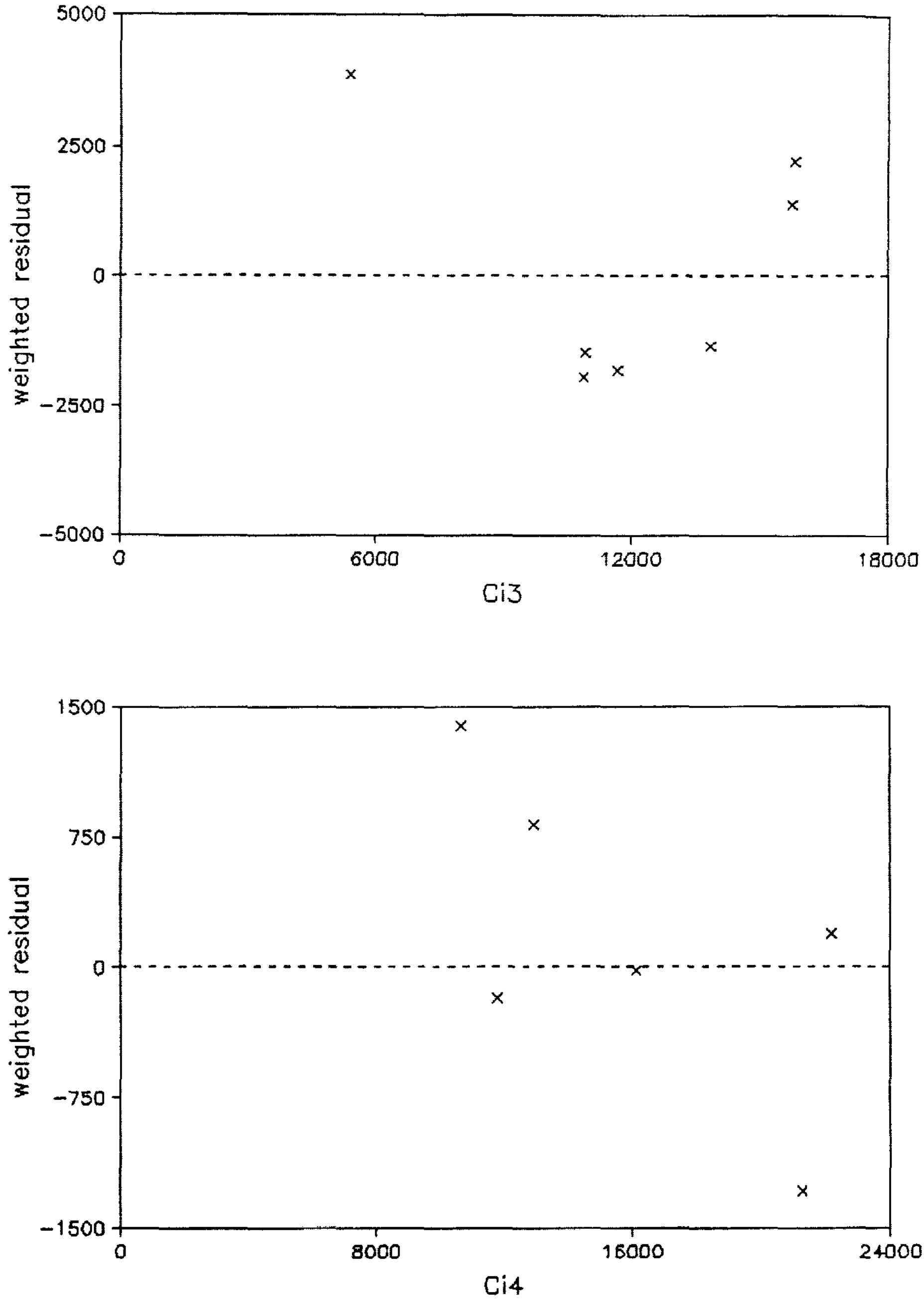
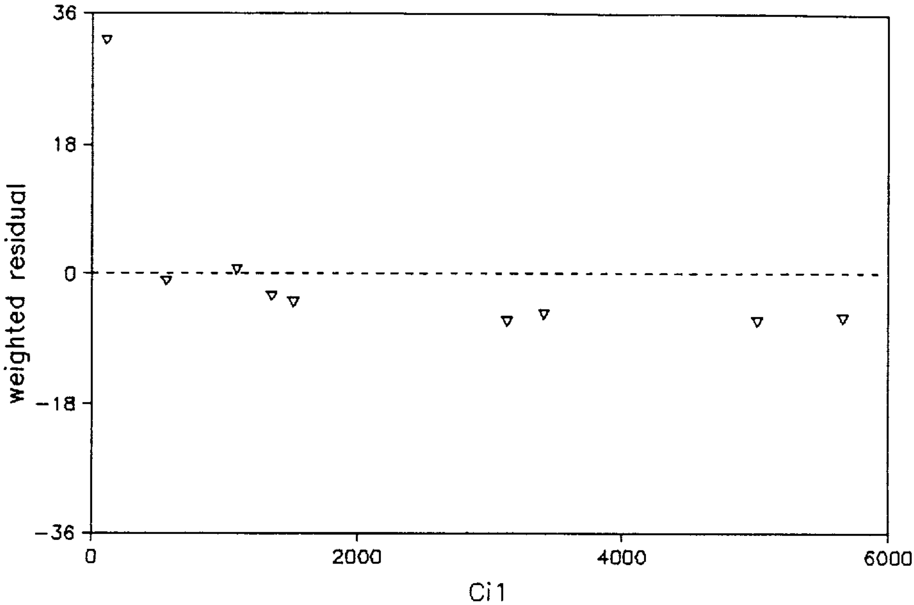


Figure 11: Residual Plots for fk2



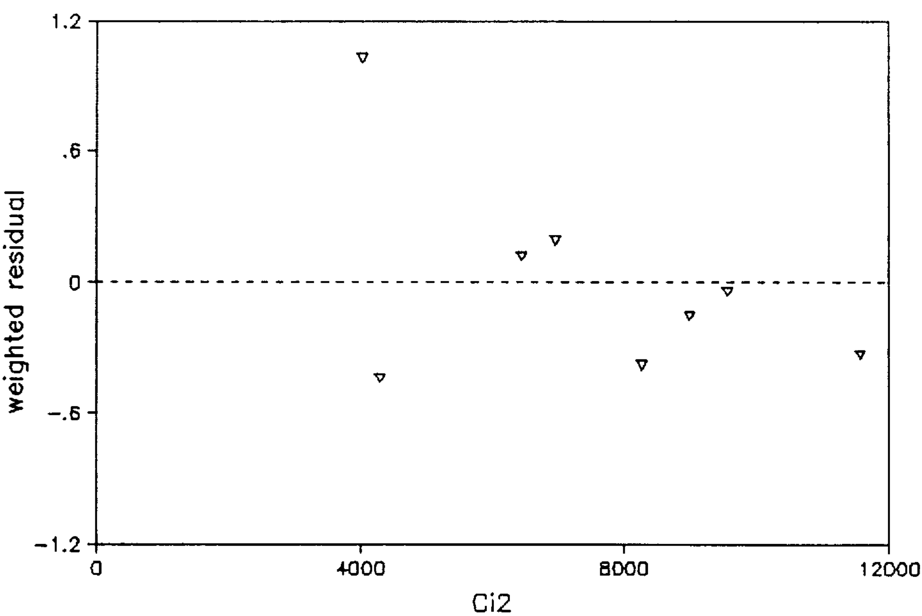
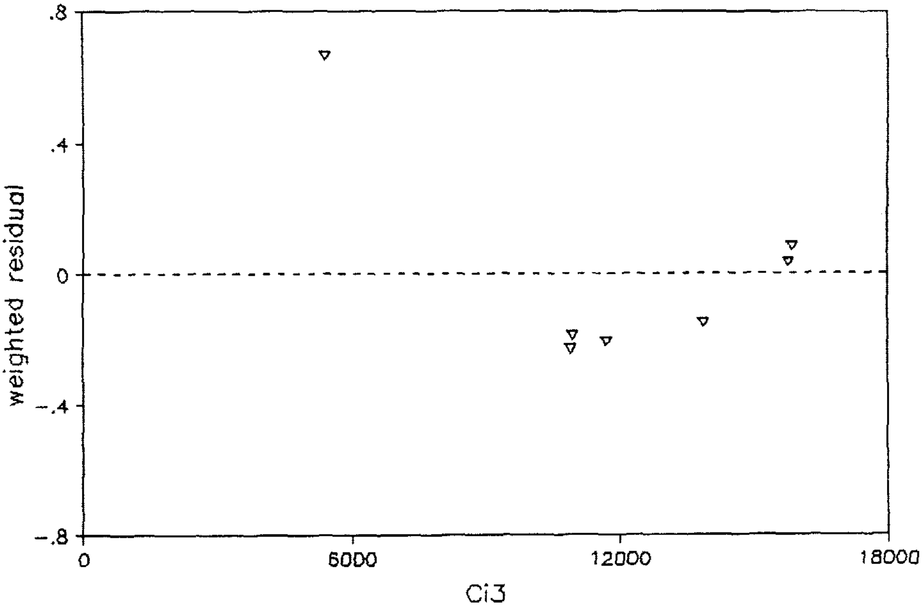


Figure 12: Residual Plots for fk2



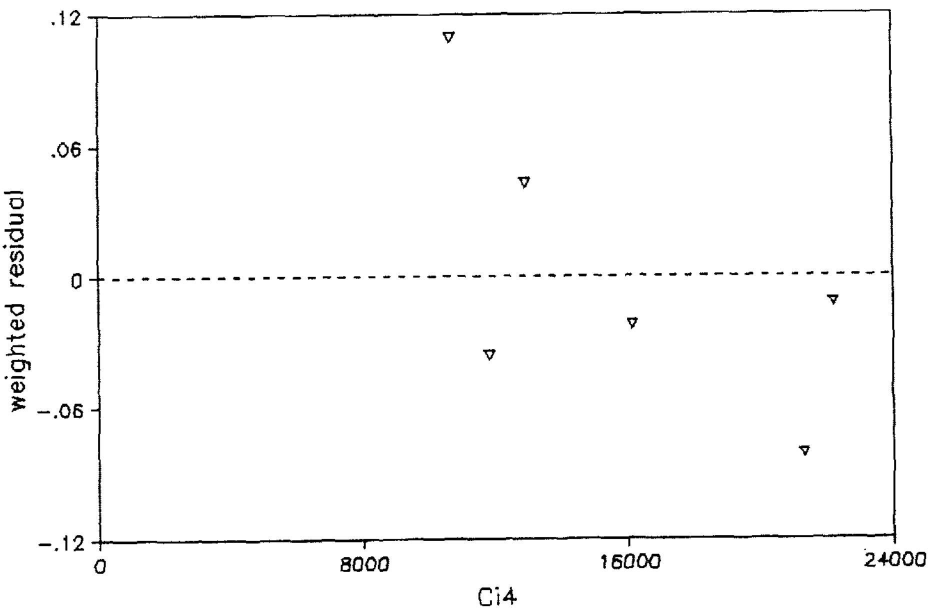
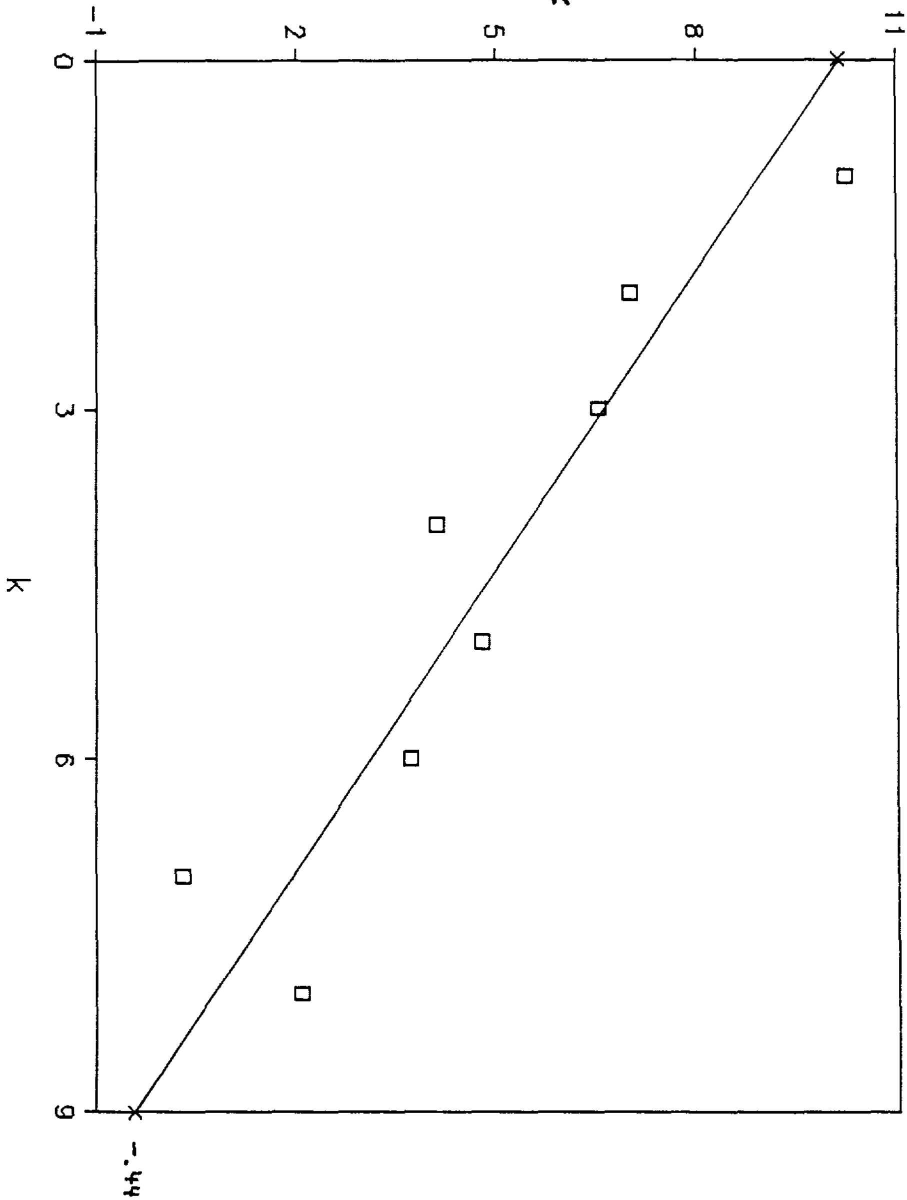


Figure 13: Plot of $\ln(\alpha_k^2)$ against k



Unbiased Loss Development Factors

by Daniel M. Murphy

UNBIASED LOSS DEVELOPMENT FACTORS

By Daniel M. Murphy, FCAS, MAAA

UNBIASED LOSS DEVELOPMENT FACTORS

Abstract

Casualty Actuarial Society literature is inconclusive regarding whether the loss development technique is biased or unbiased, or which of the traditional methods of estimating link ratios is best. This paper presents a mathematical framework to answer those questions for the class of linear link ratio estimators used in practice. A more accurate method of calculating link ratios is derived based on classical regression theory. The circumstances under which the traditional methods could be considered optimal are discussed. It is shown that two traditional estimators may in fact be least squares estimators depending on the set of assumptions one believes governs the process of loss development. Formulas for variances of, and confidence intervals around, point estimates of ultimate loss and loss reserves are derived. A triangle of incurred loss dollars is analyzed to demonstrate the concepts and techniques. A summary of a simulation study is presented and suggests that the performance of the incurred loss development technique based on the more general least squares estimator may approach that of the Bornhuetter-Ferguson and Stanard-Buhlmann techniques in some situations. The requisite mathematics is within the reach of the actuarial student equipped with the first three exams.

1. INTRODUCTION

Three standard methods of estimating link ratios in practice are the Simple Average Development (SAD) method — the arithmetic average of the link ratios; the Weighted Average Development (WAD) method — the sum of losses at the end of the development period divided by the sum of the losses at the beginning; and the Geometric Average Development (GAD) method — the n^{th} root of the product of n link ratios. Casualty Actuarial literature is inconclusive regarding which method is "best" or even whether the methods are biased or unbiased.[1] The purpose of this paper is to present a mathematical framework for evaluating the accuracy of these methods, to suggest alternatives, and to unearth valuable information about the variance of the estimates of developed ultimate loss. It is assumed that the actuary has exhausted all leads to discover systematic or operational reasons why a development triangle may appear as it does, and the only concern now is how to deal with the remaining noise.

Proofs of the technical theorems are relegated to the Appendix. The mathematics within the body of the paper is intended to motivate discussion and application.

An example will help motivate the exposition, so consider the accident year incurred loss development triangle and its triangle of link ratios in Figures 1A and 1B. The specific content of the example triangle is incidental to the purpose of this paper. It is hoped that the data is sufficiently realistic to exemplify adequately the application of these results. The extension of the results to other kinds of triangles should be self-evident.

Denote the link ratio as $\mathbf{b}$, and the SAD, WAD, and GAD estimates of $\mathbf{b}$ as $\mathbf{b}_{SAD}$, $\mathbf{b}_{WAD}$, and $\mathbf{b}_{GAD}$ respectively. For 12-24 months of development in the example triangle, these statistics evaluate to $\mathbf{b}_{SAD}=3.953$, $\mathbf{b}_{WAD}=2.480$, and $\mathbf{b}_{GAD}=3.129$. To determine which estimate is best, we must first unveil the hidden assumptions implicit in the actuarial technique called loss development.

[1] See, for example, James N. Stanard, "A Simulation Test of Prediction Errors of Loss Reserve Estimation Techniques," and John P. Robertson's discussion in the *Proceedings of the Casualty Actuarial Society*, LXXII, 1985.

2. POINT ESTIMATES

When we say that we expect the value of incurred losses as of, say, 24 months to equal the incurred value as of 12 months multiplied by a link ratio, it is possible that what we really mean is this: the value of incurred losses as of 24 months is a random variable whose expected value is conditional on the 12 month incurred value, and equals that 12 month value multiplied by an unknown constant. Symbolically,

$$y = \mathbf{b}\,x + e$$

where x and y are the current and next evaluations respectively, $\mathbf{b}$ is the unknown constant development factor, called the age-to-age factor or link ratio, and e represents random variation. The first step in developing losses is estimating the link ratios.

Expected Value of the Link Ratio

Let us generalize and suppose that the relationship between x and y is truly linear rather than strictly multiplicative. The more general model is

Model I $y = \mathbf{a} + \mathbf{b}\,x + e$
$E(e)=0$, $\mathrm{Var}(e)$ is constant across accident years, the e's are uncorrelated between accident years and are independent of x.

This model is clearly a regression of 24-month losses y on 12-month losses x. Although x is *a priori* a random variable, once an evaluation is made it is treated as a constant for the purpose of loss development. More precisely, the model says that the expected value of the random variable y conditional on the random variable x is linear in x: $E(y \mid x)=\mathbf{a}+\mathbf{b}x$. With this understanding of the relationship between x and y, all classical results of least squares regression may be brought to bear on the theory of loss development.[2] For the remainder of this paper all expectations are conditional on the current evaluation.

It is a well known theorem, the Gauss-Markoff Theorem, that the "best estimates" of $\mathbf{a}$ and $\mathbf{b}$ are the least squares estimates, denoted $\hat{a}$ and $\hat{b}$:

$$\hat{b} = \frac{\Sigma(x-\bar{x})y}{\Sigma(x-\bar{x})^2} \quad \text{and} \quad \hat{a} = \bar{y} - \hat{b}\bar{x} \,.$$

For example, the least squares estimates $\hat{a}$ and $\hat{b}$ for the 12-24 month development period in the triangle of Figure 1 are $\hat{a}$ = \$373.63 (all amounts will be given in thousands of dollars) and $\hat{b}$ = 2.027. These estimates were calculated using a popular spreadsheet software package.

[2] See, for example, Henry Scheffé, *The Analysis of Variance*, Wiley, 1956, p. 195.

The indicated regression line is shown in Figure 2A.

The method of estimating link ratios[3] by least squares under the assumptions of Model I will be called the Least Squares Linear (LSL) method. The least squares estimators of the line's parameters will be denoted $\mathbf{a}_{LSL}$ and $\mathbf{b}_{LSL}$.

Five properties of the least squares estimates are particularly appealing.[4]

1. The least squares estimates are linear functions of the variables $y \mid x$.
2. They are unbiased; i.e., $E(\mathbf{a}_{LSL})=\mathbf{a}$ and $E(\mathbf{b}_{LSL})=\mathbf{b}$.
3. Within the class of all linear unbiased estimates of **a** and **b**, the least squares estimates have the smallest variance. Least squares estimators are therefore called B.L.U.E.: Best Linear Unbiased Estimators.
4. The vertical deviations of the (x,y) observations from the regression line sum to zero; in other words, the average residual is zero.
5. The least squares regression line passes through the sample average $(\bar{x},\bar{y})$.

Before continuing, glance again at Figure 2A. By visual inspection one might say that the y-intercept is close enough to zero that it could reasonably be ignored in the predicted relationship between y and x.[5] If one believes the y-intercept should truly be zero in the first place, perhaps the model to use is

Model II $\quad y = \mathbf{b}\,x + e$
$E(e)=0$, $\mathrm{Var}(e)$ is constant across accident years, the e's are uncorrelated between accident years and are independent of x.

This model would be inappropriate if there were a significant probability that $x = 0$.

The BLUE estimator for **b** under Model II is

$$\hat{b}=\frac{\Sigma xy}{\Sigma x^2} \quad .$$

The method of estimating link ratios by least squares under this strictly multiplicative development model will be called the Least Squares Multiplicative (LSM) method. The least squares estimator of the line's parameter will be denoted $\mathbf{b}_{LSM}$.

---

[3] The estimate $\hat{a}$ of the constant term can be considered a "link ratio" if the link ratio function is viewed as being vector valued $(\hat{a},\hat{b})$.

[4] These results can be found in many introductory texts on statistical regression. Property 3 is the Gauss-Markoff Theorem.

[5] Although it will be demonstrated that the y-intercept is significantly different from zero.

In the example triangle the 12-24 month LSM link ratio is $\mathbf{b}_{LSM} = 2.204$. Figure 2B illustrates the difference between the LSL and LSM indicated regression lines.

Does $\mathbf{b}_{LSM}$ satisfy the five properties of the LSL estimator above? Obviously, $\mathbf{b}_{LSM}$ is a linear function of the y's (again, conditional on the known x values). The fact that it is unbiased is easy to prove. It has minimum variance within the class of linear unbiased estimators by virtue of the Gauss-Markoff Theorem because it is the least squares estimator. But $\mathbf{b}_{LSM}$ does not necessarily satisfy Properties 4 and 5. At first, the fact that $\mathbf{b}_{LSM}$ does not zero out the sum of the residuals nor determine a regression line passing through $(\bar{x},\bar{y})$ may seem to be a drawback. But on second thought, it must be inevitable. Indeed, a least squares regression line is required to satisfy two conditions: it must be close to the data and it must zero out the residuals. A two parameter line is free to satisfy two conditions. But a one-parameter line has the ability to satisfy only one condition. LSM satisfies the first, so it cannot be expected to satisfy the second as well.

If one were to define a "good" linear unbiased estimator as one which satisfies Properties 4 and 5, but not necessarily Property 3, then $\mathbf{b}_{WAD}$ would be best (Theorem 1). However, the price of adopting $\mathbf{b}_{WAD}$ rather than $\mathbf{b}_{LSM}$ is an increase in the probability that the prediction of losses as of the next evaluation would be off the mark because the variance of $\mathbf{b}_{WAD}$ is greater than the variance of $\mathbf{b}_{LSM}$.[6] Such are the standards by which $\mathbf{b}_{WAD}$ may be considered "optimal."

In the example, with $\mathbf{b}_{LSM} = 2.204$ for 12-24 months of development, the average residual is \$227.9 and the standard deviation of the residuals is \$876.5. With $\mathbf{b}_{WAD} = 2.480$, the average residual is \$0.4 and the standard deviation of the residuals is \$953.1.

Let us continue now to attack the assumptions of LSL and LSM to discover what we can about $\mathbf{b}_{SAD}$ and $\mathbf{b}_{GAD}$. Take the constant variance assumption for example. The impact of trend would imply that the variance of e is not constant across accident years. On-leveling the loss triangle may adjust for such heteroskedasticity but in addition may introduce unwelcome side effects. A model that speaks directly to the issue of non-constant variances is

Model III $\quad y = \mathbf{b}\, x + x\, e$

$E(e)=0$, $Var(e)$ is constant across accident years, the e's are uncorrelated between accident years and are independent of x.

This model differs from Model II in that it explicitly postulates a dependent relationship between the current evaluation and the error term, xe. By dividing both sides of this equation by x we see that this model also says that the ratio of consecutive evaluations is constant across accident years. In other words, it is the development percent, not the development dollars, and the random deviation in that percent that behave consistently from one accident year to the next. This model's BLUE for $\mathbf{b}$ is $\mathbf{b}_{SAD}$ (Theorem 3). The technique of estimating link ratios under the assumptions of Model III will be called the SAD method.

[6] Again, the Gauss-Markoff Theorem. This fact is proved directly for this actuarial problem as Theorem 2. Intuitively, $Var(\mathbf{b}_{LSM}) \le Var(\mathbf{b}_{WAD})$ because $\mathbf{b}_{LSM}$ gives more weight to the larger values of x.

Another model that can adjust for trend is

Model IV $y = \mathbf{b}xe$
$E(e)=1$, $Var(e)$ is constant across accident years, the e's are uncorrelated between accident years and are independent of x.

This model says that random noise shocks the development process multiplicatively, and may be appropriate in those situations in which the random error in the percentage development is itself expected to be skewed. The BLUE for **b** under Model IV is the geometric average of the link ratios, $\mathbf{b}_{GAD}$ (Theorem 4). The technique of estimating link ratios under the assumptions of Model IV will be called the GAD method.

For the remainder of the paper, results will be stated in terms of the LSL and LSM methods. Results for SAD and GAD, which are left to the reader, can be derived directly or by applying the results below to the transformed SAD and GAD models on which Theorems 3 and 4, respectively, depend.

Estimate of the Next Evaluation

The point estimate of the expected value of incurred losses as of the next evaluation given the current evaluation is

LSL	LSM
$y_{LSL} = \mathbf{a}_{LSL} + \mathbf{b}_{LSL}\, x$	$y_{LSM} = \mathbf{b}_{LSM}\, x$

The estimates are unbiased under the assumptions of their respective models (Theorem 5). For the example triangle the LSL and LSM estimates of the 24-month evaluation of accident year 1991 are, respectively, \$2983 = \$374 + 2.027 x \$1287 and \$2837 = 2.204 x \$1287.

Estimated Ultimate Loss: A Single Accident Year

The Chain Ladder Method states that if $\mathbf{b}_1$ is a link ratio from 12 to 24 months, $\mathbf{b}_2$ is a link ratio from 24 to 36 months, etc., and if U is the number of links required to reach ultimate, then $\mathbf{B}_U=\mathbf{b}_1\mathbf{b}_2\cdots\mathbf{b}_U$ is the (to ultimate) loss development factor (LDF). The implicit assumption is that future development is independent of prior development. This assumption implies a type of "transitive" property of loss development: if the conditional expectation of y given x is $\mathbf{b}_1x$ and the conditional expectation of z given y is $\mathbf{b}_2y$ then the conditional expectation of z given x is $\mathbf{b}_1\mathbf{b}_2x$.[7]

This all-important Chain Ladder Independence Assumption (CLIA) says that the relationship between consecutive evaluations does not depend on the relationship between any other pair of consecutive evaluations. In mathematical terms, the random variable corresponding to losses evaluated at one point in time *conditional on the previous evaluation* is independent of any other evaluation *conditional on its previous evaluation*. A direct result of this assumption is the fact

[7] See Lemma 1 in Appendix A. This assumption may not hold in practice, for example, when a claims department issues orders to "strengthen reserves" after having operated for some time under a less conservative strategy.

that an unbiased estimate of a to-ultimate loss development factor is the product of the unbiased link ratio estimates; i.e., $\hat{B}_U = \hat{b}_1 \hat{b}_2 \cdots \hat{b}_U$.

The very simplicity of the closed form LDF is one of the beauties of the multiplicative development approach. A closed form expression for the intercept term of the more general LSL approach is not nearly as simple, but this should not be considered a deterrent because a closed form, to-ultimate expression is unnecessary. Instead, this paper proposes the use of a recursive formula. A recursive estimate of developing ultimate loss illuminates the missing portion of the triangle (clarifying the communication of the analysis to management and clients), enables the actuary to switch models mid-chain, and is easy to program, even in a spreadsheet. Perhaps the most compelling reason, however, is that a recursive estimate is invaluable for calculating variances of predicted losses (Section 3), so the point estimates may as well be calculated in the same step.

The mathematical theory for developing recursive estimates of ultimate loss conditional on the current evaluation proceeds as follows. Consider a single fixed accident year. Let x_0 denote the (known) current evaluation and let $x_n \mid x_0$ denote the random variable corresponding to the n^{th} subsequent (unknown) evaluation conditional on the current evaluation. The goal is to find an unbiased estimator for $x_n \mid x_0$. By definition, an unbiased estimate of $x_n \mid x_0$ is one which estimates $\mu_n = E(x_n \mid x_0)$. The unbiased chain estimate is built from the individual links $x_n \mid x_{n-1}$ of losses as of one age conditional on losses at the previous age.

Under the more general LSL model, it is assumed that for each n there exist constants $\mathbf{a}_n$ and $\mathbf{b}_n$ such that the random variable x_n conditional on x_{n-1} can be expressed as

$$x_n \mid x_{n-1} = \mathbf{a}_n + \mathbf{b}_n x_{n-1} + e_n \ .$$

It is also assumed that $E(e_n)=0$, that $Var(e_n)=\sigma^2_n$, and that the e_n's are independent of all the x's and, by the CLIA, of each other. Theorem 6 proves that the following recursive formulas yield unbiased estimates of future evaluation.

LSL	LSM
$\hat{\mu}_1 = \hat{a}_1 + \hat{b}_1 x_0$	$\hat{\mu}_1 = \hat{b}_1 x_0$
$\hat{\mu}_n = \hat{a}_n + \hat{b}_n \hat{\mu}_{n-1}$	$\hat{\mu}_n = \hat{b}_n \hat{\mu}_{n-1}$

An unbiased estimate of ultimate loss conditional on the current evaluation is therefore $\hat{\mu}_U$.

For the example, the LSM estimate for 24-36 months of development is $\mathbf{b}_{LSM}=1.133$. Therefore, the prediction of accident year 1991 losses evaluated as of 36 months would be \$3380 = \$2983 x 1.133 if LSL had been used for the 12-24 development period; if LSM had been used, the estimate would be \$3214 = \$2837 x 1.133. The LSM prediction of accident year 1990 losses as of 36 months would be \$3167 = \$2795 x 1.133.

Estimated Total Ultimate Loss: Multiple Accident Years

It should be obvious that an estimate of total ultimate loss for more than one accident year combined could be obtained by adding up the separate accident year $\hat{\mu}_U$'s. However, for the purpose of calculating variances, a recursive expression is preferred because development estimates of ultimate loss for different accident years are not independent.

The idea behind the recursive estimate for multiple accident years is this. Starting at the bottom left corner of the triangle, add up columns of estimated future evaluations. Find a recursive unbiased estimate of those column sums. Then an unbiased estimate of total losses at ultimate will be the final sum.

The formulas are developed as follows. To keep the notation from becoming too convoluted, index the rows of the triangle in reverse order so that the youngest accident year is the zeroth row, the next youngest is row 1, and so on. Next, index the columns so that the 12 month column is the zeroth column, the 24 month column is column 1, etc. A full triangle of N+1 accident years appears as in Figure 3. If

$$S_n = \sum_{i=0}^{n-1} x_{i,n} | x_{i,i}$$

denotes the sum of the accident years' future evaluations conditional on the accident years' current evaluations, then an unbiased estimate of the future evaluation of multiple accident years is an estimate of $E(S_n)$. Let M_n denote this expectation. Recursive formulas for estimates of M_n are:

LSL	LSM
$\hat{M}_1 = \hat{a}_1 + \hat{b}_1 x_{0,0}$	$\hat{M}_1 = \hat{b}_1 x_{0,0}$
$\hat{M}_n = n\hat{a}_n + \hat{b}_n (\hat{M}_{n-1} + x_{n-1,n-1})$	$\hat{M}_n = \hat{b}_n (\hat{M}_{n-1} + x_{n-1,n-1})$

Stop when n=U, the age at which all accident years are assumed to have reached ultimate. These estimates are unbiased under the assumptions of their respective models. See Theorem 10.

The completed triangle of Figure 1A is shown in Figure 4 where it was assumed that LSL is appropriate through 84 months of development, LSM thereafter, and that losses are fully mature (i.e., case reserves are adequate, on average) after 108 months. Then, for example, $\hat{M}_1$ =\$2,982 because the 1991 accident year is the only one for which 24 months is a future development point. Accident years 1991 and 1990 are the only years which have yet to reach the age of 36 months, so $\hat{M}_2$ = \$3,268 + \$3,470 = \$6,738. And so on. Accident years 1984 through 1991 have yet to reach ultimate (108 months) so $\hat{M}_8$ = \$47,554.

Estimated Reserves for Outstanding Losses

Unbiased estimates of outstanding losses are

$$\hat{\mu}_U - \text{paid to date}$$

for a single accident year and

$$\hat{M}_U - \text{Total Paid To Date}$$

for multiple accident years.

Estimated Pure Premiums and Loss Ratios

Assuming exposures and earned premiums are static variables,[8] unbiased estimates of the pure premium rate and of the loss ratio for a single accident year are

$$\frac{\hat{\mu}_U}{\text{exposure}} \quad \text{and} \quad \frac{\hat{\mu}_U}{\text{earned premium}},$$

respectively. For multiple accident years, the estimates are

$$\frac{\hat{M}_U}{\text{Total Exposure}} \quad \text{and} \quad \frac{\hat{M}_U}{\text{Total Earned Premium}}$$

Of course, the latter statistics are most useful when all quantities are brought onlevel.

[8] Audit and reinsurance exposures and premiums may be random variables.

3. VARIANCE

The least squares point estimators of Section 2 are functions of random variables. As such, they are themselves random variables with their own inherent variances. Exact formulas for, and estimates of, these variances will be addressed in turn.

Variance of the Link Ratio Estimates

It is well known[9] that the exact variances of the link ratio estimators are

LSL	LSM
$\mathrm{Var}(\hat{a}) = \frac{\sum x^2}{n\sum(x-\bar{x})^2}\sigma^2$	
$\mathrm{Var}(\hat{b}) = \frac{\sigma^2}{\sum(x-\bar{x})^2}$	$\mathrm{Var}(\hat{b}) = \frac{\sigma^2}{\sum x^2}$ (1)

where

$$\bar{x} = \frac{1}{I}\sum x_i$$

and I is the number of accident years used in the estimate of the link ratio. Unbiased estimates of these variances are obtained by plugging in the unbiased estimate s^2 of σ^2 where s^2 is the Mean Square for Error (MSE) of the link ratio regression. The MSE or its square root s (the standard error of the estimate) is a standard statistic produced in the output of regression software. Most regression software will calculate an estimate of the square root of the variance in equation (1), sometimes called the standard error of the coefficient.

For 12-24 months of development in the example triangle $s^2_{LSL}=848.8^2$. Estimates of the standard deviations of the 12-24 month LSL intercept and slope factor are 77.35 and 0.194, respectively. For the LSM model the MSE is 876.5^2 and the standard error of the coefficient is 0.157. The spreadsheet software used to calculate these statistics automatically generates s_{LSL} and the standard error of the coefficient. The average x^2 value had to be calculated "by hand" to derive the estimate of $\mathrm{Var}(\hat{a})$.

Variance of Estimated Ultimate Loss: A Single Accident Year

Before continuing, it is time to make an important distinction. The point estimate of ultimate loss $\hat{\mu}_U$ calculated recursively above is an estimate of the expected value of the (conditional on

[9] See for example Robert B. Miller and Dean W. Wichern, *Intermediate Business Statistics*, Holt, Rinehart and Winston, 1977.

x_0) ultimate loss x_U.[10] Actual ultimate loss will vary from its expected value in accordance with its inherent variation about its developed mean μ_U. As a result, the risk that actual ultimate loss will differ from the estimate $\hat{\mu}_U$ is comprised of two components: the variance in the estimate of the expected value of $x_U \mid x_0$ — Parameter Risk — and the inherent variability of ultimate loss about its mean μ_U — Process Risk.[11] Symbolically, if (conditional on x_0) ultimate loss for a given accident year is expressed as the sum of its (conditional) mean plus a random error term ϵ_U

$$x_U \mid x_0 = \mu_U + \epsilon_U$$

then the variance in the prediction $pred_U$ of ultimate loss is

$$\begin{aligned} \mathrm{Var}(pred_U) &= \mathrm{Var}(\hat{\mu}_U) + \mathrm{Var}(\epsilon_U) \\ &= \text{Parameter Risk} + \text{Process Risk} \\ &= \text{Total Risk}\ . \end{aligned}$$

The following recursive formulas for exact values of these two variance components are derived in Theorems 8 and 9.

Parameter Risk

	LSL	LSM
For n=1:	$\mathrm{Var}\,\hat{\mu}_1 = \frac{\sigma_1^2}{I_1} + (x_0-\bar{x}_0)^2\,\mathrm{Var}\,\hat{b}_1$	$\mathrm{Var}\,\hat{\mu}_1 = x_0^2\,\mathrm{Var}\,\hat{b}_1$
For n>1:	$\mathrm{Var}\,\hat{\mu}_n = \frac{\sigma_n^2}{I_n} + (\mu_{n-1}-\bar{x}_{n-1})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\hat{\mu}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{\mu}_{n-1}$	$\mathrm{Var}\,\hat{\mu}_n = \mu_{n-1}^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\hat{\mu}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{\mu}_{n-1}$

[10] For better or for worse, it is usually the expected value of an unknown quantity – e.g., rates or reserves – that actuaries are called upon to produce. The "Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves" is rather vague on that issue, but "The "Statement of Principles Regarding Property and Casualty Insurance Ratemaking" (Principle 1) and, for example, "Actuarial Standard of Practice No. 7: Performing Cash Flow Testing for Insurers" (section 5.5) are quite explicit.

[11] This process risk is the conditional variance of developing losses about the conditional mean. As pertaining to triangles of incurred loss dollars, it includes, but is not limited to, the unconditional *a priori* process risk of the loss distribution (mitigated by the knowledge of losses emerged to date), the random variation of the claims occurrence and reporting patterns, and the random variation within case reserves.

Process Risk

$$\text{Var}(x_1 \mid x_0) = \sigma_1^2$$

and

$$\text{Var}(x_n \mid x_0) = \sigma_n^2 + b_n^2 \, \text{Var}(x_{n-1} \mid x_0)$$

The equation for Process Risk is the same under both models. Unbiased estimates of these variances are obtained by plugging in unbiased estimates s_n^2 for σ_n^2, $\hat{\mathbf{b}}_n$ for $\mathbf{b}_n$, and $\hat{\mu}_n$ for μ_n.

Parameter Risk and Total Risk are illustrated in the familiar graphs of Figures 5A and 5B where ± 2 standard deviation prediction bands are drawn around the LSL and LSM estimates, respectively, of 12-24 months of development from the example triangle. First, Parameter Risk is represented by two curves ± 2 standard deviations (the square root of the estimated Parameter Risk) away from the least squares line. Total prediction risk is represented by two curves ± 2 standard deviations (the square root of the sum of estimated Parameter Risk plus estimated Process Risk) away from the least squares line. The actuary may represent Process Risk to the layman as the distance between the Total Risk and Parameter Risk bands; of course, this is technically incorrect.

Notice that in Figure 5A the Parameter Risk bands widen in both directions as x moves away from its average value of $824 and that in Figure 5B the bands widen as x moves away from zero. This occurs because the equation for parameter risk is a function of distance of x_0 from the average value of x for the LSL model and a function of the absolute value of x_0 for LSM.

There is a subtle difference between a "prediction band" which measures the error one would expect in a prediction based on the regression, and the more common "confidence band" which measures the fit of the regression relative to the sample data. The concept of the confidence band is illustrated in Figure 5C where, for example, a one-standard-deviation confidence band is drawn around the LSL regression of 12-24 months of development. The radius of the confidence band is the square root of the MSE, 848.8. Using the techniques of the next section, it can be shown that one should expect about 34% of the data points to fall outside the confidence band. In other words, one should expect about six outliers. In this case, there are only four. The identification of outliers can provide the actuary with useful information before he or she enters into fact-finding interviews with the claims and underwriting departments. The identification of outliers provides information of a more technical nature as well. Indeed, note that the outliers in Figure 5C occur at the higher values of x. This suggests that the variance of y is not independent of x. The assumptions of the SAD or GAD methods, or a variant, may more appropriately describe the random processes underlying these particular data.

As a final note, ultimate loss is not ultimate until the final claim is closed. Suppose it takes C development periods, $C \geq U$, to close out the accident year. Then the estimate of ultimate loss is not of $x_U \mid x_0$ but of $x_C \mid x_0$. Although estimated ultimate loss through U development periods may be the same as estimated ultimate loss through C development periods, the variances of the two estimates are not the same. Even if it is true that $\mathbf{b}_n = 1$ for $n > U$, whereby parameter risk halts at $n = U$, process risk continues to add up, so recursive estimates of $\mathrm{Var}(x_n \mid x_0)$ should be carried out beyond $n = U$.

In the example, it was assumed that an accident year will be closed after 144 months based on a visual inspection of Figure 1B (accident year 1974 was considered a data anomaly). The recursive projection of ultimate accident year 1991 loss was already displayed in Figure 4. The detailed calculation of the variance (Total Risk) is shown in Figure 6A.

Variance of Estimated Ultimate Loss: Multiple Accident Years

Actual total ultimate loss S_U for multiple (open) accident years will vary from the estimate $\hat{M}_U$ as a result of two sources of uncertainty: PARAMETER RISK — the variance in the estimate of M_U — and PROCESS RISK — the inherent variance of S_U about its developed mean M_U. Symbolically, if we express total ultimate loss for multiple accident years (conditional on the current evaluation of all accident years) as the sum of its mean M_U plus a random error term E_U

$$S_U = M_U + E_U$$

then for a given accident year the variance in the prediction $PRED_U$ of ultimate loss is

$$\begin{aligned}\mathrm{Var}(PRED_U) &= \mathrm{Var}(\hat{M}_U) + \mathrm{Var}(E_U) \\ &= \text{PARAMETER RISK} + \text{PROCESS RISK} \\ &= \text{TOTAL RISK}\ .\end{aligned}$$

In Theorems 10 and 11 are derived the following recursive formulas for exact values of these two variance components.

PARAMETER RISK

	LSL	LSM
For n=1:	$$\mathrm{Var}\,\hat{M}_1 = \frac{\sigma_1^2}{I_1} + (x_{0,0}-\bar{x}_0)^2\,\mathrm{Var}\,\hat{b}_1$$	$$\mathrm{Var}\,\hat{M}_1 = x_{0,0}^2\,\mathrm{Var}\,\hat{b}_1$$
For n>1:	$$\mathrm{Var}\,\hat{M}_n = n^2\frac{\sigma_n^2}{I_n} + (M_{n-1}+x_{n-1,n-1}-n\bar{x}_{n-1})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{M}_{n-1}$$	$$\mathrm{Var}\,\hat{M}_n = (M_{n-1}+x_{n,n})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{M}_{n-1}$$

where

$$\bar{x}_{n-1} = \frac{1}{I_n}\sum_{i=n}^{N} x_{i,n-1}$$

is the average "x value," and I_n=N-n+1 (assuming a full column in the triangle) is the number of data points, in the regression estimate of the n^{th} link ratio.

PROCESS RISK

$$\mathrm{Var}(E_1) = \sigma_1^2$$

and

$$\mathrm{Var}(E_n) = n\sigma_n^2 + b_n^2\,\mathrm{Var}(E_{n-1}) \quad .$$

The equation for Process Risk is the same under both models. Unbiased estimates of these variances are obtained by plugging in unbiased estimates s_n^2 for σ_n^2, $\hat{b}_n$ for $\mathbf{b}_n$, and $\hat{M}_n$ for M_n.

For the example, Figure 6B shows the calculation of the estimate of the variance of the estimate of total ultimate loss for accident years 1984 through 1991 combined. Most of the basic statistics are the same those appearing in Figure 6A.

Variance of Estimated Outstanding Losses: Single or Multiple Accident Years

Assuming paid losses are constant at any given evaluation,[12] it is obvious that the variance of a reserve equals the variance of total ultimate losses:

$$\begin{aligned}\text{Var}(\textit{Estimated Reserve}) &= \text{Var}(\textit{Estimated Ultimate Loss} - \text{Paid Loss}) \\ &= \text{Var}(\textit{Estimated Ultimate Loss}) \quad .\end{aligned}$$

This equality holds for estimated reserves for a single accident year and for multiple accident years.

Variance of Estimated Pure Premiums and Loss Ratios: Single or Multiple Accident Years

Assuming static exposures and pure premiums, the variances of the estimated pure premium rate and of the estimated loss ratio are

$$\frac{\text{Var}(\textit{Estimated Ultimate Loss})}{\text{exposures}^2} \quad \text{and} \quad \frac{\text{Var}(\textit{Estimated Ultimate Loss})}{\text{earned premiums}^2} \quad .$$

Again, these formulas hold for single or multiple accident years.

One final note before leaving this section. Aggregate losses are often expressed as the compound product of a frequency distribution (e.g., Poisson or negative binomial) and a severity distribution (e.g., lognormal or Pareto). In practice, parameters for those distributions are estimates, the result being that the variance of the aggregate loss distribution depends not only on the inherent variance of the postulated frequency and severity distributions but on the variance of the parameter estimates. The parameter error of the frequency distribution could be estimated by applying the above techniques to the frequency triangle, defined to be the triangle of claim counts per exposure. The parameter error of the severity distribution could be estimated by applying the above techniques to the incurred (or paid) severity triangle, defined as the triangle of cumulative incurred (paid) dollars divided by cumulative incurred (paid) claims. Furthermore, since it is the mean of the distributions that are usually sought, only the Parameter Risk above need be considered.

[12] Salvage and subrogation could be handled as a separate category.

4. CONFIDENCE INTERVALS

Confidence intervals necessarily are phrased in terms of a probability measure. As a result, this discussion can no longer avoid making assumptions about the probability distribution of the error terms, e_n. The traditional assumption is that they are normally distributed (lognormally distributed under GAD which may be a bit more believable).

Confidence Intervals Around the Link Ratios

Let α be the probability measurement of the width of the confidence interval. Then $100\alpha\%$ confidence intervals around the true LSL link ratios $(\mathbf{a}_n, \mathbf{b}_n)$ are:

$$\hat{a}_n \pm t_{\frac{\alpha}{2}}(I_n-2)\sqrt{\hat{Var}(\hat{a}_n)}$$

and

$$\hat{b}_n \pm t_{\frac{\alpha}{2}}(I_n-2)\sqrt{\hat{Var}(\hat{b}_n)}$$

where $t_\alpha(I_n-2)$ denotes the two-tailed α point (the "t-value")[13] of Student's t distribution with I_n-2 degrees of freedom and where I_n is the number of accident years used in the estimate of the n^{th} link ratio. The degrees of freedom under LSL is I_n-2 because two parameters are estimated under that model. These formulas may be used for the LSM model as well; in that case the degrees of freedom are I_n-1.

To demonstrate how these formulas can be used, suppose we want to test the hypothesis that the 12-24 month LSL constant term is not significantly greater than zero. Recall that this constant term was estimated to be \$373.63. Refer to Figure 7. There are 18 data points and two parameters, so the degrees of freedom equals 16. At the 99% confidence level, the one-tailed t-value is 2.62. It was shown above that the estimated variance of the constant term is 77.35^2. Then, if the constant term were truly zero, there would be a 99% chance that the estimated intercept would be less than or equal to 202.66 = 77.35 x 2.62. Since the estimated value of the intercept falls outside the confidence interval, it appears that LSL is an appropriate model for this young stage of development. In fact, it appears that LSL is appropriate for the youngest six stages of development. The confidence of that statement is $94\% = .99^6$.

As another example, the decision to assume that case reserves are reasonably adequate by 108 months is based on the apparent random nature of the link ratios thereafter. Notice in Figure 7 that the LSM link ratios are either at, or well within, one standard deviation ("Std(b)") of unity for 120-132 months and beyond, but the 108-120 link ratio (.992) is more than one

[13] This assumes that the available t-table is presented in terms of a one-tailed test, or, if not, that the actuary is able to look up the appropriate value accordingly.

standard deviation away from unity. Somewhat subjectively, it was deemed appropriate to ignore this significant average negative development, as well as the relatively insignificant positive development thereafter. If the actuary were to set the 108-120 factor to, say, an interpolated value between the 96-108 and 120-132 factors, it may generally be considered a not unreasonable application of actuarial judgment and may just so happen to reflect an amount of conservativism consistent with the risk posture of the owners of the enterprise. However, in the end, the ability of that actuary to convince management that this judgment is appropriate depends on the level of trust established between the parties.

It is clear that near the tail of the triangle the degrees of freedom drop prohibitively. Inferences about the link ratios become less precise. If it can be assumed that the variances of the residuals in the development model are the same for all development periods (i.e., $\sigma_i = \sigma_j$ for all i and j), then a single estimate of the MSE can be obtained by solving for all link ratios simultaneously. The result is that the t-value should become reasonably small[14] and can make for "tighter" inferences for all development periods.

Confidence Intervals Around Estimated Ultimate Loss

This section will begin with the GAD model because all results are exact.[15] Under the transformed GAD model

$$\ln(x_n) = \ln(\mathbf{b}_n) + \ln(x_{n-1}) + \ln(e_n)$$

or

$$x_n' = \mathrm{b}_n' + x_{n-1}' + e_n'$$

the point estimate of ultimate transformed loss is

$$pred' = \beta'_C = \beta'_U = x_0' + \sum_{j=1}^{U} \hat{b}_j'$$

and the estimate of the variance of the prediction is

[14] For an NxN triangle, df=(N-1)(N-2)/2 under LSM if no data points are discarded. For example, with a moderately-sized 5x5 triangle the two-tailed 90%-ile t-value is only 18% greater than the smallest possible 90%-ile t-value, namely the 90%-ile point on the standard normal curve. This can be especially important for the small triangles that consultants or companies underwriting new products are wont to see.

[15] Commonly used probability distributions are location oriented, so additive models such as the transformed GAD model are quite tractable. The use of scale-oriented probability distributions may yield results more directly applicable to the multiplicative models actuaries favor.

$$\hat{Var}(pred') = (C + \sum_{j=1}^{u} \frac{1}{I_j}) s'^2 \qquad \text{(Theorem 13)}$$

where we assume under transformed GAD that all σ_n's are equal. It is well known that in this case the MSE is proportional to a chi-square random variable with degrees of freedom equal to the number of data points less the number of estimated parameters. Therefore, a one-sided $100\alpha\%$ confidence interval[16] for ultimate transformed losses x_C' given the current transformed evaluation x_0' is exactly equal to

$$\hat{\mu}_C' \pm t_\alpha(df)\sqrt{\hat{Var}(pred')} \quad .$$

The corresponding $100\alpha\%$ confidence interval around the "untransformed" prediction of ultimate loss x_C given x_0 is

$$\exp\left(\hat{\mu}_C' \pm t_\alpha(df)\sqrt{\hat{Var}(pred')}\right) \quad .$$

If df is large enough, $t_\alpha(df)$ may be replaced by z_α, the standard normal point, without significant loss in accuracy.[17]

With this justification, an *approximate* $100\alpha\%$ confidence interval around a prediction under any of the models is

$$pred \pm t_{\frac{\alpha}{2}}(df)\sqrt{\hat{Var}(pred)} \quad .$$

Figures 6A and 6B show how this approach is used to derive estimates of ultimate loss at the 80% confidence level.

<u>Confidence Intervals around Reserves</u>

Confidence intervals around reserves are obtained by subtracting paid dollars from the endpoints of the confidence intervals around ultimate loss. This is simply due to the fact that if

[16] At the risk of pedantry, "prediction interval" is more correct.

[17] This is often done in practice, particularly in time series analysis, even when df is not large. The t distribution is preferred, however, because the thinner tails of the standard normal will understate the radius of the confidence interval. For another perspective on this subject, see Everette S. Gardner Jr., "A Simple Method of Computing Prediction Intervals for Time Series Forecasts," *Management Science*, Vol. 34, No. 4, April 1988, p. 541-546.

$$\alpha = P(\text{lower bound} \leq \text{ultimate loss} \leq \text{upper bound})$$

then it is also true that

$$\alpha = P(\text{lower bound - paid} \leq \text{outstanding loss} \leq \text{upper bound - paid}).$$

Figures 6A and 6B demonstrate the application of this concept as well. The 80% risk load for all outstanding losses for accident years 1984 through 1991 is about 27% of the expected value. It would be interesting to see how much this load is reduced for the same level of confidence when an analysis of paid dollar triangles is also conducted. Incurred and paid estimates should be negatively correlated, therefore the variance of their average should be reduced even more than if independence were simply assumed.

Confidence Intervals around Pure Premiums and Loss Ratios

Confidence intervals around pure premiums and loss ratios are obtained by dividing the endpoints of the confidence intervals around ultimate loss by exposures or premiums, respectively. This scale shift is akin to the location shift for the confidence intervals around reserves.

5. AN ARGUMENT IN SUPPORT OF A NON-ZERO CONSTANT TERM

When the current evaluation is zero, the practice is to abandon the multiplicative loss development methods and adopt an alternative, e.g., Bornhuetter-Ferguson, Stanard-Buhlmann, or a variation on frequency-severity. LSL may be a fourth possibility.

To elaborate, consider the development of reported claim counts. Let N be the true ultimate number of claims for a given accident year. Let r_i be the random report year of the i^{th} claim. Assume that the r_i are independent and identically distributed for all claims so that if p_n is the probability that a claim is reported before the end of the n^{th} year, then p_n is independent of i. Based on these assumptions it is not difficult to show that if x_n is the number of reported claims at the n^{th} evaluation then

$$E(x_n | x_{n-1}) = N\frac{p_n - p_{n-1}}{1 - p_{n-1}} + \frac{1 - p_n}{1 - p_{n-1}} x_{n-1} \qquad (2)$$

which is of the form $\mathbf{a}_n + \mathbf{b}_n x_{n-1}$. Clearly the constant term $\mathbf{a}_n$ is non-zero until all claims are reported.

Figure 8A shows the true development line for evaluation 1 to evaluation 2 when N=40 and the p_n's are 1/2, 3/4, 7/8, ... , $1-½^n$, along with a scattering of ten random data points.

Equation (2) becomes even more interesting when the reporting pattern is exponential, as might be expected from a Poisson frequency process. In that case it is straightforward to prove that the LSL coefficients $(\mathbf{a}_n,\mathbf{b}_n)$ are identical for every age n. This somewhat surprising result can be put to good use when the triangle is too small to give stable LSL estimates of individual link ratios, as will be demonstrated in the following section.

From Equation (2) one can see that the slope factor $\mathbf{b}_n$ does not depend on the exposure (N) but only on the reporting pattern, and that the constant term $\mathbf{a}_n$ is proportional to the exposure. An increase in exposure from one accident year to the next will cause an upward, parallel shift in the development regression line. Equation (2) may also be used as a paradigm for loss dollars, although the bias of case reserves complicates the analysis, and systematic factors such as trend can change expected ultimate loss dollars from one accident year to the next. Development triangles, therefore, can be expected to display data samples randomly distributed about not a single regression line but about multiple parallel regression lines as claim frequency increases, as the volume of business expands, or simply through the impact of trend. This is pictured in Figure 8B where a random sample is displayed about the regression line of Figure 8A and about a parallel line determined by N=80. The estimated regression line based on all the points combined will indicate a less significant constant term.

6. COMPARING THE MODELS USING SIMULATION

In the 1985 *Proceedings* Mr. James N. Stanard published the results of a simulation study of the accuracy of four simple methods of estimating ultimate losses using a 5x5 incurred loss triangle. For the exposure tested[18] it was demonstrated that WAD loss development was clearly inferior to three additive methods — Bornhuetter-Ferguson (BF), Stanard-Buhlmann[19] (SB), and a little-used method called the Additive Model (ADD) — because it had greater average bias and a larger variance. The additive methods differ from the multiplicative methods in that they adjust incurred losses to date by an estimated dollar increase to reach ultimate, whereas the multiplicative methods adjust by an estimated percentage increase. ADD's estimated increase is a straightforward calculation of differences in column means, $\bar{y}-\bar{x}$. BF and SB estimated increases are based on inverted LDFs and are therefore nonlinear functions of the y's.

Stanard's simulation was replicated here to test additionally the accuracy of LSM, LSL, SAD and GAD.[20] The model does not attempt to predict "beyond the triangle," which is to say that the methods project incurred losses to the most mature age available in the triangle, namely the age of the first accident year. In the discussion below, by "ultimate loss" is meant case incurred loss as of the most mature available age.

The LSL method was modified to use LSM in those instances when the development factors were "obviously wrong," defined to be when either the slope or the constant term was negative. In real-life situations, this rudimentary adjustment for outliers can be expected to be improved upon with more discerning application of actuarial judgment. The reason this modification was necessary is due to the fact that a model that fits data well does not necessarily predict very well. As an extreme example, LSL provides an exact fit to the sample data for the penultimate link ratio (two equations, two unknowns), but the coefficients so determined reveal nothing about the random processes that might cause another accident year to behave differently. It is not possible to identify every conceivable factor that could explain the otherwise "unexplained" variance of a model. Such unidentified variables are reflected through the averaging process of statistical analysis: as the number of data points minus the number of parameters (the definition of degrees of freedom) increases, the model captures more of the unexplained factors and

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[18] Normally distributed frequency with mean = 40 and standard deviation = $\sqrt{40}$ claims per year, uniform occurrence date during the year, lognormal severity with mean = $10,400 and standard deviation = $34,800, exponential report lag with mean = 18 months, exponential payment lag with mean = 12 months, and case reserve error proportional to a random factor equal to a lognormal random variable with mean = 1 and variance = 2, and to a systematic factor equal to the impact of trend between the date the reserve is set and the date the claim is paid.

[19] Which Mr. Stanard called the "Adjustment to Total Known Losses" method, a.k.a. the "Cape Cod Method."

[20] For the details behind the computer model, the reader is referred to Mr. Stanard's published results. The simulation was reproduced in C on an IBM PS/2 Model 70 with a math coprocessor. The most complicated scenarios requiring 15000 iterations took about an hour and a half to process.

becomes a better predictor.

In Exhibits I through IV, the average bias and standard deviation of the first accident year are zero because the simulation defines ultimate to be the current age of that accident year.

Exhibit I: Claim Counts Only

In this case, 5000 claim count triangles were simulated, the "actual ultimate" as of the last column was simulated as well, accident year ultimates were estimated using the various methods, and averages and standard deviations of the prediction errors were calculated.

Of the multiplicative estimators, LSM has the smallest bias and the smallest variance for every accident year. As can be expected, WAD is close behind. The remaining methods could perhaps be ordered BF, SB, ADD, and LSL, in increasing order of accuracy as measured by the standard deviation of the accident-year-total projection.

Consider first the average bias. In Figure 9A is graphed the relationship between incurred counts at 12 months, x, with incurred losses at 24 months, y, which we know from the previous section must be a linear relationship with a positive constant term. The ADD and WAD estimates are also shown. All relationships are shown in their idealized states where LSL is collinear with the true relationship and where the point $(\bar{x},\bar{y})$ coincides with its expectation $(E(x),E(y))$. Note that the ADD model is parallel to the line $y=x$ because it adds the same amount for every value of x. The conditional (on x) bias is the signed, vertical distance from the estimated relationship to the true relationship. As is clear from Figure 9A, WAD and ADD can be expected to overstate y for $x > E(x)$ and understate y for $x < E(x)$. The weighted average of the conditional bias across all values of x, weighted by the probability density $f(x)$, is simulated by the average bias that appears in Exhibit I.

Ideally, this weighted average of the bias across all values of x should be expected to be zero, which it is for the Additive Model. ADD estimates $E(y)$-$E(x)$ via $\bar{y}-\bar{x}$ calculated from prior accident years. Since the environment in Exhibit 1 — exposure, frequency, trend, etc. — does not change by accident year, the average of 5000 simulated samples of this dollar difference across all possible values of x should get close to the true average dollar difference by the law of large numbers, so the average bias should get close to zero. For the multiplicative estimators, the average bias will probably not be zero. Take the WAD method for example. Clearly there is a positive probability (albeit small) that $\bar{x}=0$, so the expected value of the WAD link ratio $\frac{\bar{y}}{\bar{x}}$ is infinity. The average of 5000 simulations of this ratio attempts to estimate that infinite expected value, so it should not be surprising that WAD usually overstates development — and the greater the probability that $\bar{x}=0$, the greater the overstatement.[21]

[21] This argument can be made more rigorous. The condition that the probability of the sample average of x be greater than zero is a sufficient but not necessary condition that $E(b_{WAD}) = \infty$. For a general, heuristic argument that

The average bias of the BF and SB methods should be greater than zero as well because the LDFs on which they rely are themselves overstated more often than not. The average LSM bias is a more complicated function of the probability distribution of x because the LSM link ratio involves x terms in the numerator and squared x terms in the denominator. The average bias appears to shift as an accident year matures. The LSL method as modified herein has residual average bias because it incorporates the biased LSM method when it detects outliers. It also seems to be the case that the bias of the estimated 4-5 year link ratio is driving the cumulative bias for the immature years.

Figure 9A illustrates the difference between a model that is unbiased for each possible value of x, LSL, and a model which is "unbiased" only in the average, ADD. To reiterate, the purely multiplicative and purely additive estimators will understate expected development when the current evaluation is less than average and overstate expected development when the current evaluation is greater than average.

Next, consider the variance. In simplified terms, the average bias statistic allows expected overstatements to cancel out expected understatements. This is not the case for the variance statistic. In Figure 9A it is clear that, ideally, the ADD estimate of y will be closer to the true conditional expected value of y (the idealized LSL line) than will the WAD estimate for virtually all values of x. Thus, the variance of ADD should be less than the variance of WAD. The variance of LSL should be the smallest of all. However, LSL estimates twice as many parameters than do ADD and LSM, so it needs a larger sample size to do a comparable job. For the relatively small and thin triangles simulated here, a pure unmodified LSL estimate flops around like a fish out of water — the price it must pay to be unbiased for all values of x. In other words, in actual practice, the variance of an LSL method unmodified for outliers and applied to a triangle with few degrees of freedom, will probably be horrendous. What is perhaps remarkable is the degree to which the rudimentary adjustment adopted here tames the LSL method.

Finally, let's look at what would happen if we estimated the LSL parameters under the assumption that all link ratio coefficients (a_n, b_n) are equal. We know from the previous section that this is true because the reporting pattern is exponential. The results of this model are:

A/Y	Average Bias	Std Dev Bias	Average %Bias	Std Dev %Bias	Age-Age Bias	Age-Age %Bias
1	0.000	0.000	0.000	0.000		
2	0.025	1.275	0.001	0.034	1.035	1.001
3	0.006	1.669	0.001	0.044	-0.019	0.000
4	-0.034	1.850	0.000	0.049	-0.040	-0.001
5	-0.006	1.815	0.001	0.049	0.028	0.001
Total	-0.010	5.064	0.000	0.027		

This model is the beneficiary of more degrees of freedom (eight — two parameters estimated

WAD yields biased estimates, see [Stanard].

from ten data points for each iteration) and as a result has the smallest average bias and variance yet. These results lead to a somewhat counterintuitive conclusion: information about development across immature ages sheds light on future development across mature ages. For example, the immature development just experienced by the young accident year 4 from age 1 to age 2 is a valuable data point in the estimate of the upcoming development of the old accident year 2 from age 4 to age 5. This should not be viewed simply as a bit of mathematical prestidigitation but as an example of the efficiencies that can be achieved if simplifying assumptions — even as innocuous as exponential reporting — can be justified.

Exhibit II: Random Severity, No Trend

In this case, 5000 triangles of aggregate, trend-free incurred losses were simulated and the same calculations were performed.

Rarely does the property/casualty actuary experience loss triangles devoid of trend, so this model is of limited interest. The introduction of uncertainty via the case reserves makes it more likely that negative development will appear, in which case LSL reverts to LSM. As a result, the additive models overtake LSL in accuracy.

Exhibit III: Random Severity, 8% Severity Trend Per Year

This is where it gets interesting. This could be considered the standard situation in which an actuary compiles a loss triangle that includes trend and calculates loss development factors. In this case, the environment is changing. The trending process follows the Unified Inflation Model with $\alpha=\frac{1}{2}$,[22] which is to say that half of the impact of inflation is a function of the occurrence date and half is a function of the transaction date (e.g., evaluating the case incurred or paying the claim).

At first, one might think that a multiplicative estimator would have had a better chance of catching the trend than would an additive estimator, but such does not appear to be the case. Consider Figure 9B which graphs expected 12-24 month development for the first four accident years. Trend has pushed the true development line upward at an 8% clip, illustrated by four thin lines. The LSL model tries to estimate the average of the development lines, the WAD estimator tries to pass through the average $(\bar{x},\bar{y})$ midpoint of all accident years combined, and the additive estimators try to find the line parallel to the line $y=x$ which also passes through the average midpoint. Again, ADD will probably be closer than WAD to the average LSL line for every value of x. The upward trend makes it more likely that the estimated LSL intercept will be less than zero, which makes it more likely that LSL reverts to LSM, so the modified LSL's variance gets closer yet to the variance of LSM.

Exhibit IV: Random Severity, 8% Trend, On-Level Triangle

[22] Robert P. Butsic and Rafal J. Balcarek, "The Effect of Inflation on Losses and Premiums for Property-Liability Insurers," *Inflation Implications for Property-Casualty Insurance*, 1981 Casualty Actuarial Society Discussion Paper Program, p. 58.

In this case, rows of the triangle were trended to the level of the most recent accident year assuming that the research department is perfectly prescient in its estimate of past trend. For most of the models the total bias decreases while the total variance increases. LSM and WAD are virtually unchanged, GAD and SAD are exactly unchanged (of course), and the nonlinear estimates move in opposite directions.

For the most part, working with the on-level triangle does seem to improve the accuracy of estimated ultimate loss, but perhaps not to the degree one might hope. It would be interesting to see if working with separate claim count and on-level severity triangles would successfully decompose the random effects and further improve the predictions.

7. CONCLUSION

Loss development predictions can be improved by the use of least squares estimators. In certain situations the least squares estimators coincide with the more traditional simple average development and geometric average development estimators. Under the four sets of assumptions about the loss development process considered here, the weighted average link ratio estimator is always inferior to an alternative, least squares estimator.

If the assumptions of a given model considered here can be married with the independence assumption that forms the basis of the Chain Ladder Method, the developed estimates of ultimate loss are unbiased. The variance of estimated ultimate loss can itself be estimated through relatively straightforward application of recursive formulas. A range of estimates can be given with associated approximate levels of confidence if one is willing to make some assumptions about the probability distribution of the error terms.

At this point, statistical techniques may be of some guidance in selecting one model over another, but the final choice of the most appropriate set of assumptions will probably be a judgment call depending on, among other things, the exposure and the claims operation of the book of business.

The simulation study suggests that the performance of the more general Least Squares Linear method exceeds that of the multiplicative development methods and may, in some situations, rival that of the nonlinear additive methods in common use today. It would be interesting to investigate the correlation between development estimates of ultimate loss based on incurred and paid triangles, and use that information to derive optimal, variance-minimizing weights for making final selections.

Figure 1A

Auto Liability
Incurred Loss + ALAE ($000)

	Age of Accident Year (Months)																		
A/Y	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
1973	45	111	134	149	163	164	165	165	165	165	165	165	165	165	165	165	165	165	165
1974	38	92	105	123	315	315	338	340	341	345	350	442	518	558	536	536	536	536	
1975	79	195	240	447	513	738	757	767	792	797	797	797	797	797	797	797	797		
1976	103	442	901	1,480	2,325	2,372	2,378	2,549	2,566	2,466	2,469	2,480	2,480	2,480	2,480	2,480			
1977	58	274	706	1,555	1,605	1,719	1,785	1,741	1,604	1,604	1,604	1,604	1,604	1,604	1,604				
1978	50	405	1,371	1,193	1,229	1,371	1,524	1,518	1,510	1,516	1,517	1,517	1,517	1,517					
1979	65	1,228	1,371	1,277	1,378	1,398	1,552	2,011	2,026	2,012	2,020	2,020	2,020						
1980	561	1,210	1,501	1,641	1,870	1,945	1,967	1,985	2,174	2,175	2,175	2,175							
1981	555	1,469	1,517	1,626	1,652	1,749	1,764	1,764	1,764	1,764	1,764								
1982	444	856	1,107	1,404	1,730	1,948	2,029	2,085	2,103	2,122									
1983	441	977	834	988	1,122	1,297	1,476	1,476	1,480										
1984	458	1,393	1,790	2,149	2,387	2,530	2,492	2,446											
1985	1,262	4,472	5,758	6,239	6,481	6,718	6,856												
1986	3,128	6,399	8,675	9,390	9,780	10,204													
1987	3,556	6,099	7,139	7,766	7,947														
1988	1,969	3,010	3,928	3,927															
1989	1,260	3,374	3,930																
1990	764	2,795																	
1991	1,287																		

Figure 1B

Auto Liability
Incurred Loss + ALAE Development

Development Period

A/Y	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-132	132-144	144-156	156-168	168-180	180-192	192-204	204-216	216-228
1973	2.467	1.207	1.112	1.094	1.006	1.006	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1974	2.421	1.141	1.171	2.561	1.000	1.073	1.006	1.003	1.012	1.014	1.263	1.172	1.077	0.961	1.000	1.000	1.000	
1975	2.468	1.231	1.863	1.148	1.439	1.026	1.013	1.033	1.006	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
1976	4.291	2.038	1.643	1.571	1.020	1.003	1.072	1.007	0.961	1.001	1.004	1.000	1.000	1.000	1.000			
1977	4.724	2.577	2.203	1.032	1.071	1.038	0.975	0.921	1.000	1.000	1.000	1.000	1.000	1.000				
1978	8.100	3.385	0.870	1.030	1.116	1.112	0.996	0.995	1.004	1.001	1.000	1.000	1.000					
1979	18.892	1.116	0.931	1.079	1.015	1.110	1.298	1.007	0.993	1.004	1.000	1.000						
1980	2.157	1.240	1.093	1.140	1.040	1.011	1.009	1.095	1.000	1.000	1.000							
1981	2.647	1.033	1.072	1.016	1.059	1.009	1.000	1.000	1.000	1.000								
1982	1.928	1.293	1.268	1.232	1.125	1.043	1.028	1.009	1.009									
1983	2.215	0.854	1.185	1.136	1.156	1.138	1.000	1.003										
1984	3.041	1.285	1.201	1.111	1.060	0.985	0.982											
1985	3.544	1.288	1.084	1.039	1.037	1.021												
1986	2.685	1.033	1.082	1.042	1.043													
1987	1.715	1.171	1.088	1.023														
1988	1.529	1.305	1.000															
1989	2.678	1.165																
1990	3.658																	
<u>Averages</u>																		
SAD	3.953	1.433	1.242	1.217	1.085	1.044	1.031	1.007	0.999	1.002	1.033	1.025	1.013	0.992	1.000	1.000	1.000	1.000
WAD	2.480	1.206	1.116	1.082	1.059	1.034	1.035	1.008	0.996	1.002	1.010	1.010	1.007	0.998	1.003	1.007	1.013	1.000
GAD	3.129	1.340	1.203	1.177	1.080	1.043	1.028	1.006	0.998	1.002	1.030	1.023	1.012	0.992	1.000	1.000	1.000	1.000

Figure 2A

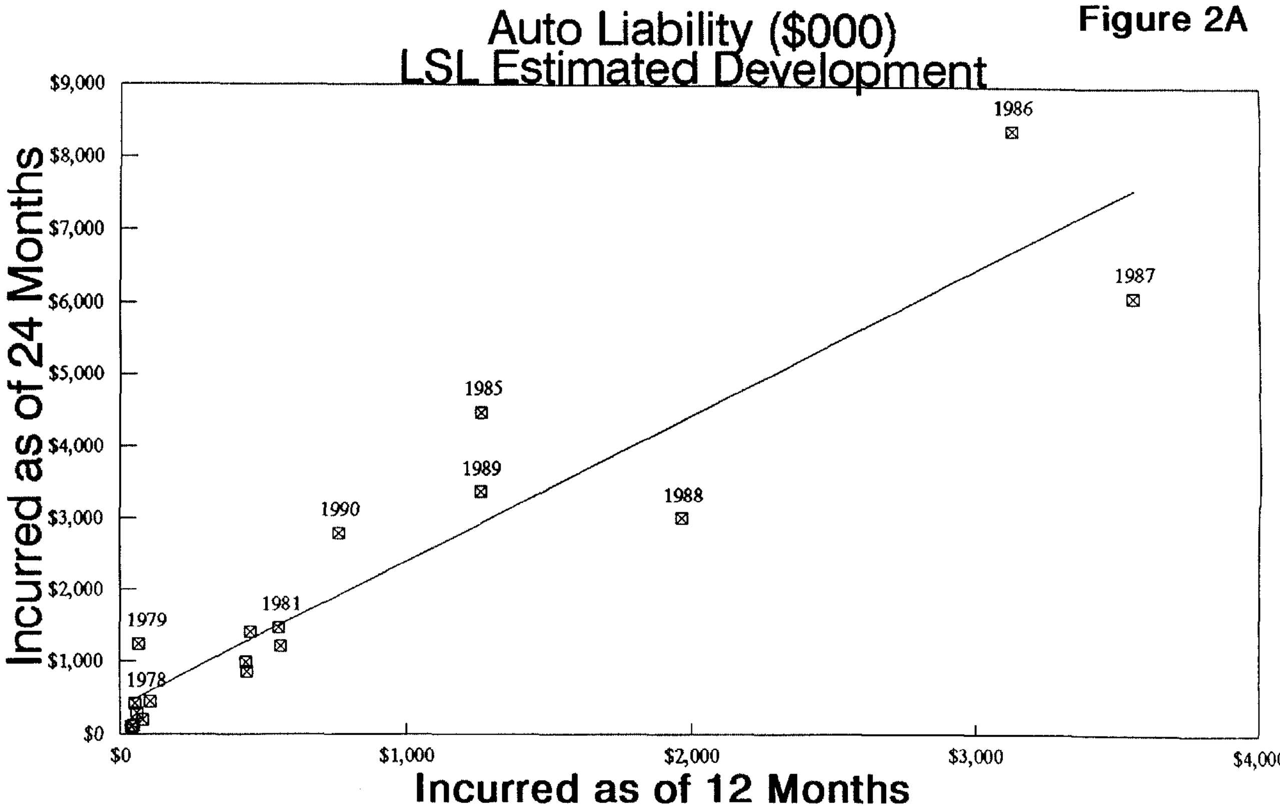

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Figure 2B

Auto Liability ($000)
LSM and LSL Estimated Development

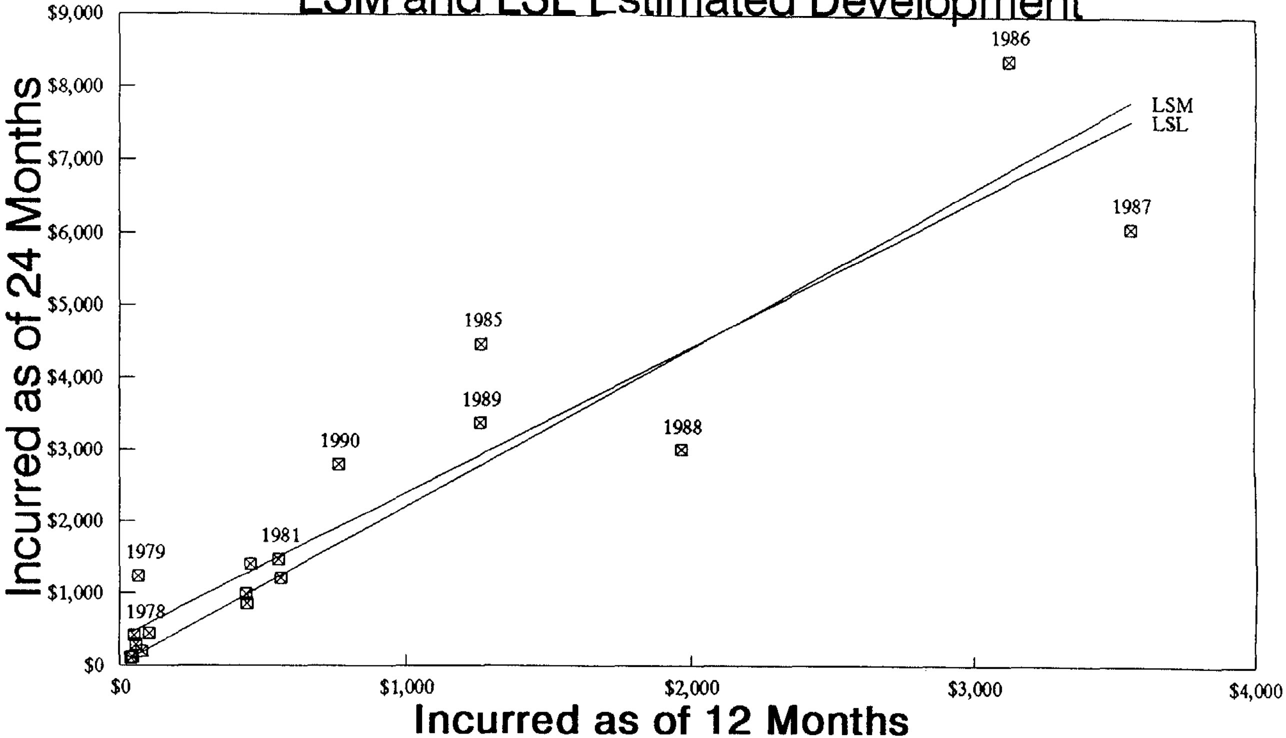

Figure 3

Notation for the "Known" and "Completed" Portions of a Loss Triangle

A/Y	0	1	2	…	n−1	n	n+1	…	N−1	N
					Age of Accident Year					
N	$x_{N,0}$	$x_{N,1}$	$x_{N,2}$	⋯	$x_{N,n-1}$	$x_{N,n}$	$x_{N,n+1}$	⋯	$x_{N,N-1}$	$x_{N,N}$
N−1	$x_{N-1,0}$	$x_{N-1,1}$	$x_{N-1,2}$	⋯	$x_{N-1,n-1}$	$x_{N-1,n}$	$x_{N-1,n+1}$	⋯	$x_{N-1,N-1}$	$x_{N-1,N} \mid x_{N-1,N-1}$
N−2	$x_{N-2,0}$	$x_{N-2,1}$	$x_{N-2,2}$	⋯	$x_{N-2,n-1}$	$x_{N-2,n}$	$x_{N-2,n+1}$	⋯	$x_{N-2,N-1} \mid x_{N-2,N-2}$	$x_{N-2,N} \mid x_{N-2,N-2}$
	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
n	$x_{n,0}$	$x_{n,1}$	$x_{n,2}$	⋯	$x_{n,n-1}$	$x_{n,n}$	$x_{n,n+1} \mid x_{n,n}$	⋯	$x_{n,N-1} \mid x_{n,n}$	$x_{n,N} \mid x_{n,n}$
n−1	$x_{n-1,0}$	$x_{n-1,1}$	$x_{n-1,2}$	⋯	$x_{n-1,n-1}$	$x_{n-1,n} \mid x_{n-1,n-1}$	$x_{n-1,n+1} \mid x_{n-1,n-1}$	⋯	$x_{n-1,N-1} \mid x_{n-1,n-1}$	$x_{n-1,N} \mid x_{n-1,n-1}$
n−2	$x_{n-2,0}$	$x_{n-2,1}$	$x_{n-2,2}$	⋯	$x_{n-2,n-1} \mid x_{n-2,n-2}$	$x_{n-2,n} \mid x_{n-2,n-2}$	$x_{n-2,n+1} \mid x_{n-2,n-2}$	⋯	$x_{n-2,N-1} \mid x_{n-2,n-2}$	$x_{n-2,N} \mid x_{n-2,n-2}$
	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
1	$x_{1,0}$	$x_{1,1}$	$x_{1,2} \mid x_{1,1}$	⋯	$x_{1,n-1} \mid x_{1,1}$	$x_{1,n} \mid x_{1,1}$	$x_{1,n+1} \mid x_{1,1}$	⋯	$x_{1,N-1} \mid x_{1,1}$	$x_{1,N} \mid x_{1,1}$
0	$x_{0,0}$	$x_{0,1} \mid x_{0,0}$	$x_{0,2} \mid x_{0,0}$	⋯	$x_{0,n-1} \mid x_{0,0}$	$x_{0,n} \mid x_{0,0}$	$x_{0,n+1} \mid x_{0,0}$	⋯	$x_{0,N-1} \mid x_{0,0}$	$x_{0,N} \mid x_{0,0}$

The shaded area highlights $S_n = \sum_{i=0}^{n-1} x_{i,n} \mid x_{i,i}$. $M_n = E(S_n)$.

Figure 4

Auto Liability
Incurred Loss + ALAE ($000)

Completed Triangle

Age of Accident Year (Months)

A/Y	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
1973	45	111	134	149	163	164	165	165	165	165	165	165	165	165	165	165	165	165	165
1974	38	92	105	123	315	315	338	340	341	345	350	442	518	558	536	536	536	536	536
1975	79	195	240	447	513	738	757	767	792	797	797	797	797	797	797	797	797	797	797
1976	103	442	901	1,480	2,325	2,372	2,378	2,549	2,566	2,466	2,469	2,480	2,480	2,480	2,480	2,480	2,480	2,480	2,480
1977	58	274	706	1,555	1,605	1,719	1,785	1,741	1,604	1,604	1,604	1,604	1,604	1,604	1,604	1,604	1,604	1,604	1,604
1978	50	405	1,371	1,193	1,229	1,371	1,524	1,518	1,510	1,516	1,517	1,517	1,517	1,517	1,517	1,517	1,517	1,517	1,517
1979	65	1,228	1,371	1,277	1,378	1,398	1,552	2,011	2,026	2,012	2,020	2,020	2,020	2,020	2,020	2,020	2,020	2,020	2,020
1980	561	1,210	1,501	1,641	1,870	1,945	1,967	1,985	2,174	2,175	2,175	2,175	2,175	2,175	2,175	2,175	2,175	2,175	2,175
1981	555	1,469	1,517	1,626	1,652	1,749	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764	1,764
1982	444	856	1,107	1,404	1,730	1,946	2,029	2,085	2,103	2,122	2,122	2,122	2,122	2,122	2,122	2,122	2,122	2,122	2,122
1983	441	977	834	988	1,122	1,297	1,476	1,476	1,480	1,480	1,480	1,480	1,480	1,480	1,480	1,480	1,480	1,480	1,480
1984	458	1,393	1,790	2,149	2,387	2,530	2,492	2,446	2,468	2,468	2,468	2,468	2,468	2,468	2,468	2,468	2,468	2,468	2,468
1985	1,262	4,472	5,758	6,239	6,481	6,718	6,856	7,075	7,139	7,139	7,139	7,139	7,139	7,139	7,139	7,139	7,139	7,139	7,139
1986	3,128	8,399	8,675	9,390	9,780	10,204	10,360	10,691	10,787	10,787	10,787	10,787	10,787	10,787	10,787	10,787	10,787	10,787	10,787
1987	3,556	6,099	7,139	7,766	7,947	8,275	8,410	8,679	8,757	8,757	8,757	8,757	8,757	8,757	8,757	8,757	8,757	8,757	8,757
1988	1,969	3,010	3,928	3,927	4,155	4,354	4,446	4,588	4,629	4,629	4,629	4,629	4,629	4,629	4,629	4,629	4,629	4,629	4,629
1989	1,260	3,374	3,930	4,288	4,522	4,734	4,829	4,984	5,028	5,028	5,028	5,028	5,028	5,028	5,028	5,028	5,028	5,028	5,028
1990	764	2,795	3,268	3,589	3,811	3,999	4,086	4,217	4,255	4,255	4,255	4,255	4,255	4,255	4,255	4,255	4,255	4,255	4,255
1991	1,287	2,982	3,470	3,802	4,028	4,223	4,313	4,451	4,491	4,491	4,491	4,491	4,491	4,491	4,491	4,491	4,491	4,491	4,491

<u>Least Squares Development Coefficients</u>

a	373.63	255.26	137.50	161.37	58.01	43.37	0	0	0	0	0	0	0	0	0	0	0	0
b	2.027	1.078	1.056	1.017	1.034	1.011	1.032	1.009	1	1	1	1	1	1	1	1	1	1

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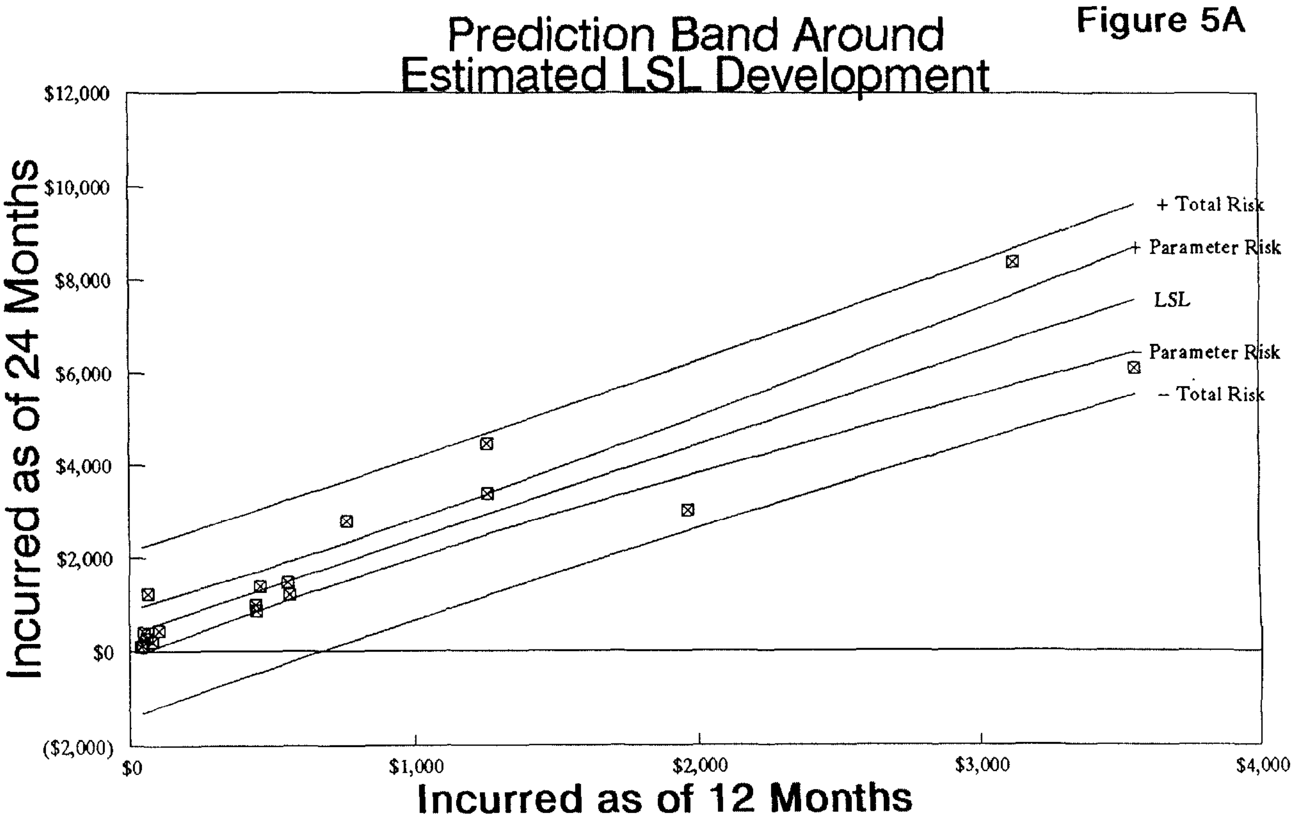
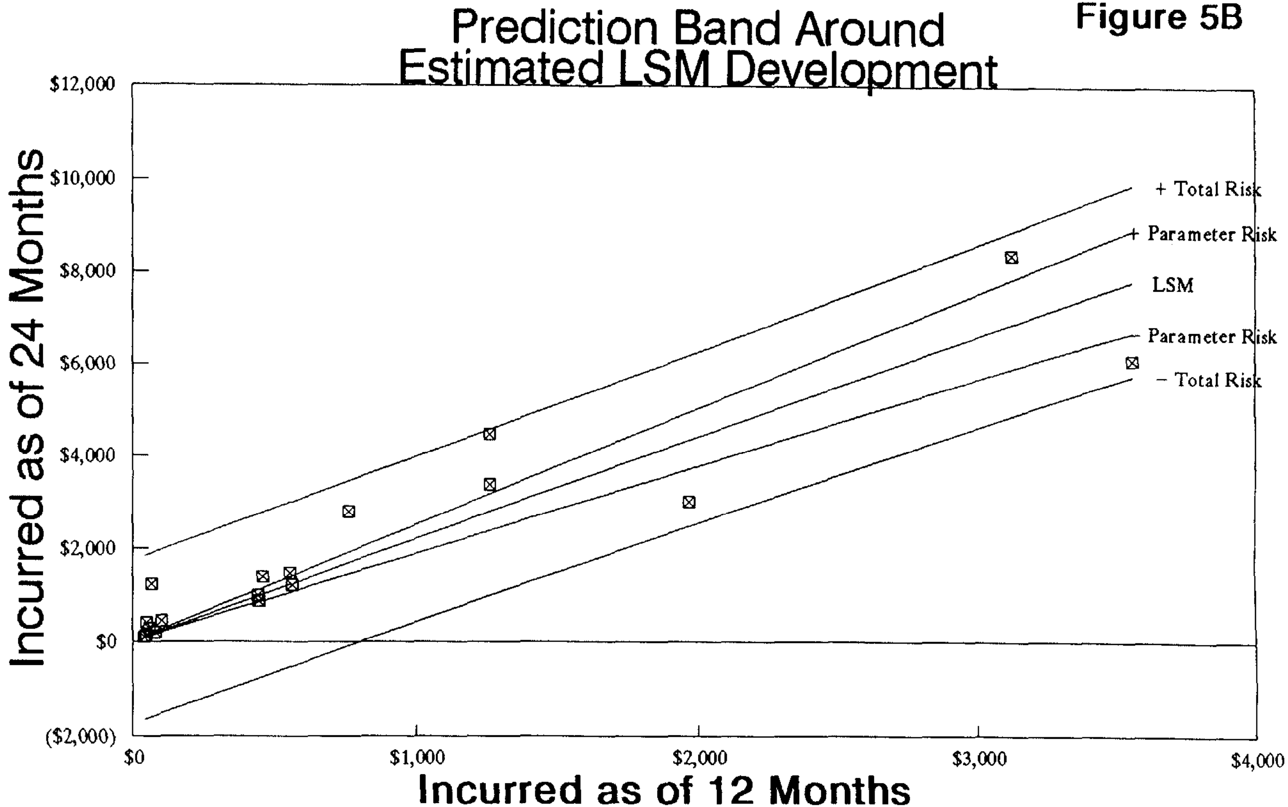
Figure 5A



Radius of band = 2 standard deviations

Associated confidence level = 93% (approx.)

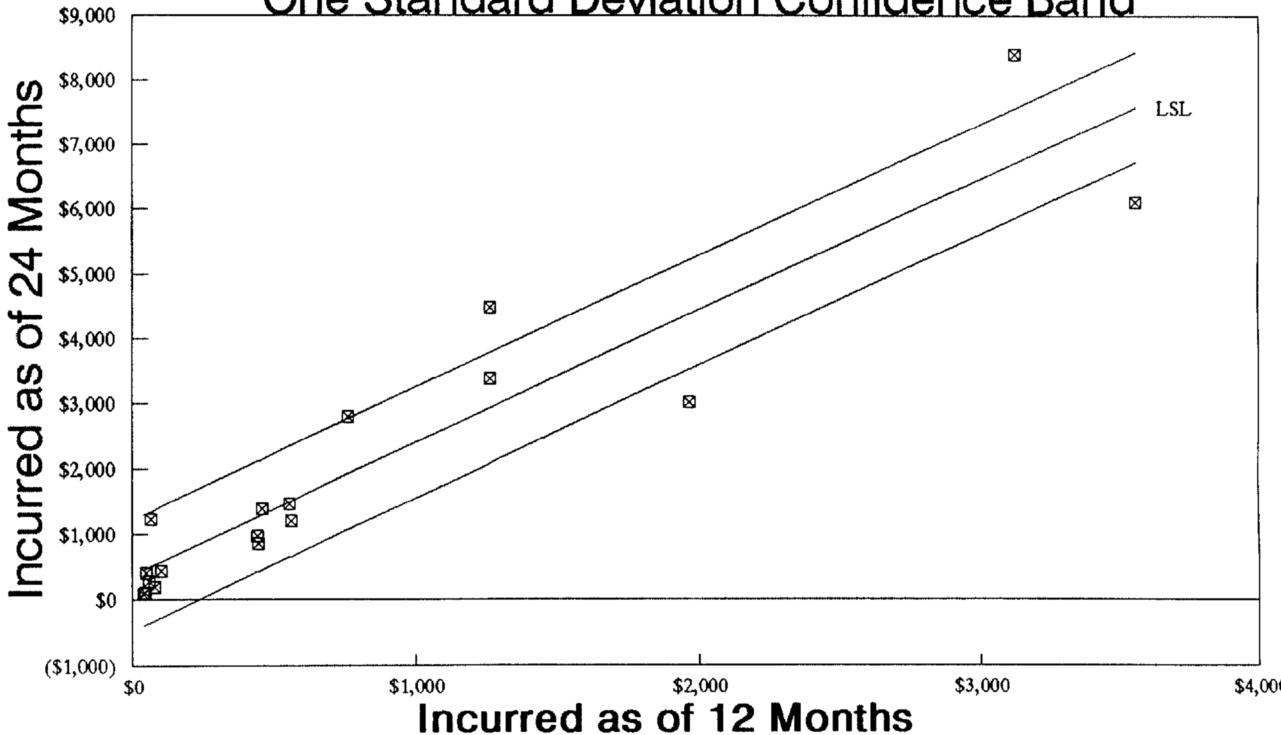
Figure 5B



Radius of band = 2 standard deviations
Associated confidence level = 93% (approx.)

Figure 5C

Identifying Outliers
One Standard Deviation Confidence Band



Radius of band = 1 standard deviation
Associated confidence level = 66% (approx.)

Figure 6A

Auto Liability
Variance of Estimated Ultimate Loss
Accident Year 1991

Age (months)	12	24	36	48	60	72	84	96	Ultimate 108	120	132	Closed 144	Paid to Date	Indicated Reserve
n	0	1	2	3	4	5	6	7	8	9	10	11		
mu hat	$1,287	$2,982	$3,470	$3,802	$4,028	$4,223	$4,313	$4,451	$4,491	$4,491	$4,491	$4,491	$202	$4,289
x bar	824	2,000	2,317	2,495	2,325	1,866								
b hat		2.027	1.078	1.056	1.017	1.034	1.011	1.022	1.032	1	1	1		
Var(b hat)		0.0377	0.0017	0.0007	0.0004	0.0001	0.0002	0.0035	0.0006	0	0	0		
s		849	384	278	212	76	72	146	139	73	31	3		
I		18	17	16	15	14	13	12	13	12	11	10		
df		16	15	14	13	12	11	10	11	10	9	8		
Parameter Risk		48,112	66,266	79,751	86,179	92,745	96,123	1.66E+05	1.88E+05	1.88E+05	1.88E+05	1.88E+05		
Process Risk		7.20E+05	9.85E+05	1.18E+06	1.26E+06	1.35E+06	1.39E+06	1.47E+06	1.59E+06	1.59E+06	1.59E+06	1.59E+06		
Total Risk		7.69E+05	1.05E+06	1.26E+06	1.35E+06	1.45E+06	1.48E+06	1.64E+06	1.77E+06	1.78E+06	1.78E+06	1.78E+06		
Std Dev		876.671	1,025.227	1,120.290	1,160.473	1,202.590	1,218.497	1,279.595	1,332.182	1,334.199	1,334.566	1,334.569		
Total df												129		
One-tailed 80% t-value (same as standard normal when df=129)												0.842		
80% Confidence Risk Load (t-value x Std Dev)												$1,124		$1,124
Upper bound on 80% Confidence Interval for Ultimate Loss												$5,615	$202	$5,413
80% Confidence Risk Load as a Percent of the Expected Value												25%		26%

Figure 6B

Auto Liability
Variance of Estimated Ultimate Loss
Accident Years 1984 – 1991

Age (months)	12	24	36	48	60	72	84	96	Ultimate 108	120	132	Closed 144	Total Paid to Date	Indicated Total Reserve
n	0	1	2	3	4	5	6	7	8	9	10	11		
M hat	$1,287	$2,982	$6,739	$11,678	$16,516	$25,585	$36,443	$44,685	$47,555	$47,555	$47,555	$47,555	$37,854	$9,701
x bar	824	2,000	2,317	2,495	2,325	1,866								
b hat		2.027	1.078	1.056	1.017	1.034	1.011	1.022	1.032	1	1	1		
Var(b hat)		0.0377	0.0017	0.0007	0.0004	0.0001	0.0002	0.0035	0.0006	0	0	0		
s		849	384	278	212	76	72	146	139	73	31	3		
I		18	17	16	15	14	13	12	13	12	11	10		
df		16	15	14	13	12	11	10	11	10	9	8		
PARAMETER RISK		48,112	95,937	1.61E+05	2.27E+05	2.63E+05	3.83E+05	5.70E+06	7.35E+06	7.35E+06	7.35E+06	7.35E+06		
PROCESS RISK		7.20E+05	1.13E+06	1.49E+06	1.72E+06	1.87E+06	1.95E+06	2.18E+06	2.48E+06	2.53E+06	2.54E+06	2.54E+06		
TOTAL RISK		7.69E+05	1.23E+06	1.65E+06	1.95E+06	2.14E+06	2.33E+06	7.88E+06	9.82E+06	9.87E+06	9.88E+06	9.88E+06		
Std Dev		876.671	1,108.321	1,286.375	1,396.973	1,461.640	1,525.937	2,806.755	3,134.438	3,142.149	3,143.706	3,143.721		

	Closed 144	Total Paid to Date	Indicated Total Reserve
Total df	129		
One–tailed 80% t–value (same as standard normal when df=129)	0.842		
80% Confidence Risk Load (t–value x Std Dev)	$2,647		$2,647
Upper bound on 80% Confidence Interval for Ultimate Loss	$50,202	$37,854	$12,348
80% Confidence Risk Load as a Percent of the Expected Value	6%		27%

Figure 7

Auto Liability
Incurred Loss + ALAE Development

Estimated Least Squares Development Coefficients

	12–24	24–36	36–48	48–60	60–72	72–84	84–96	96–108	108–120	120–132	132–144	144–156	156–168	168–180	180–192	192–204	204–216	216–228
LSL																		
a	373.63	255.26	137.50	161.37	58.01	43.37	18.67	−8.51										
b	2.027	1.078	1.056	1.017	1.034	1.011	1.022	1.013										
s	848.8	384.204	277.64	211.942	76.0792	72.0653	145.832	77.1915										
df	16	15	14	13	12	11	10	9										
Std(a)	77.35	39.7946	31.7267	25.6207	10.0074	13.4745	67.418	36.9558										
Std(b)	0.194	0.04063	0.02726	0.01978	0.00802	0.01281	0.0591	0.03224										
One−tailed 99% t−value	2.62	2.65	2.68	2.72	2.76	2.82	2.90	3.00										
Upper bound on a	202.66	105.46	85.03	69.69	27.62	38.00	195.51	110.87										
LSM																		
b	2.204	1.133	1.083	1.046	1.045	1.024	1.032	1.009	0.992	1.001	1.003	1.002	1.002	0.999	1.000	1.000	1.000	1.000
s	876.5	421.549	288.053	238.949	85.5021	74.8248	139.291	73.3336	31.2845	2.93139	34.6692	30.8364	17.687	10.822	0.000	0.000		
df	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
Std(b)	0.157	0.0336	0.02092	0.01656	0.00664	0.00853	0.02397	0.01335	0.00585	0.0006	0.00764	0.00772	0.0051	0.0035	0.000	0.000		

Figure 8A

Claim Count Development
Expected Number of Claims = 40

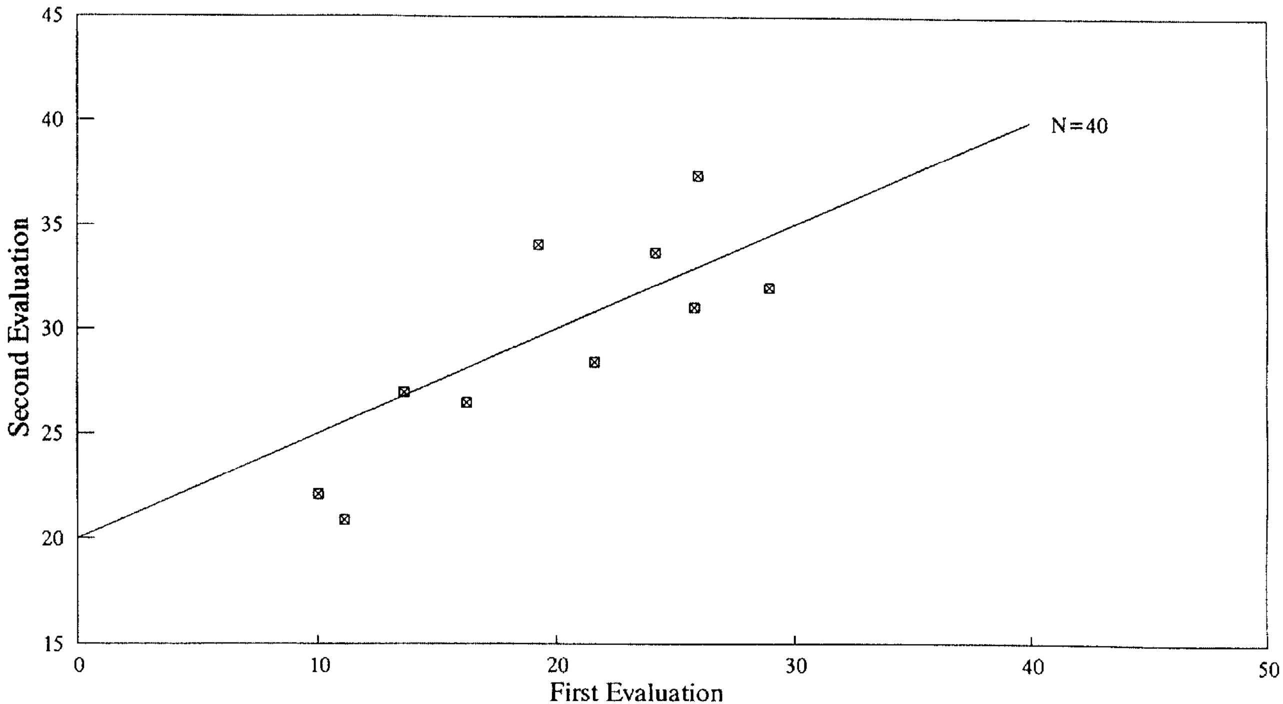
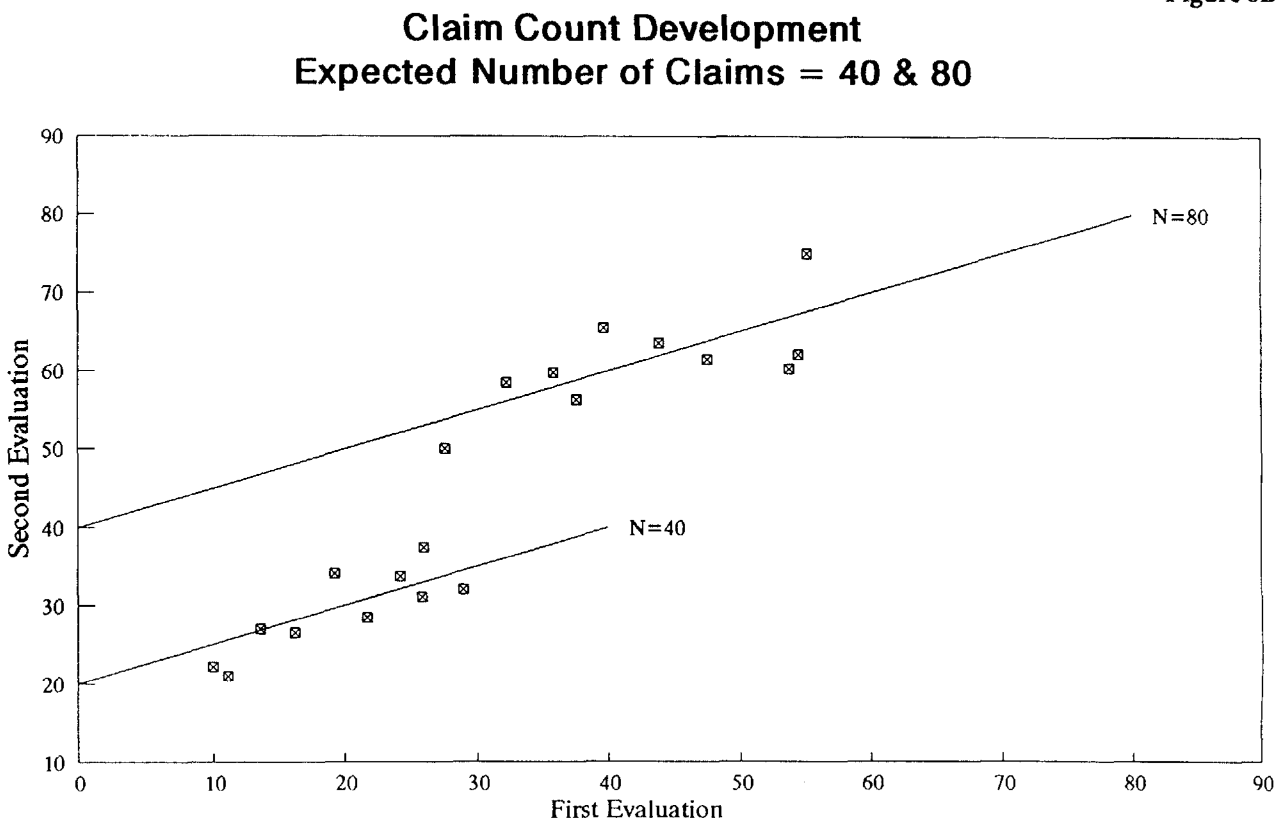


Figure 8B



Claim Count Development
Expected Number of Claims = 40 & 80
Second Evaluation
First Evaluation
N=80
N=40
90
80
70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90

Idealized Development Estimators
No Trend

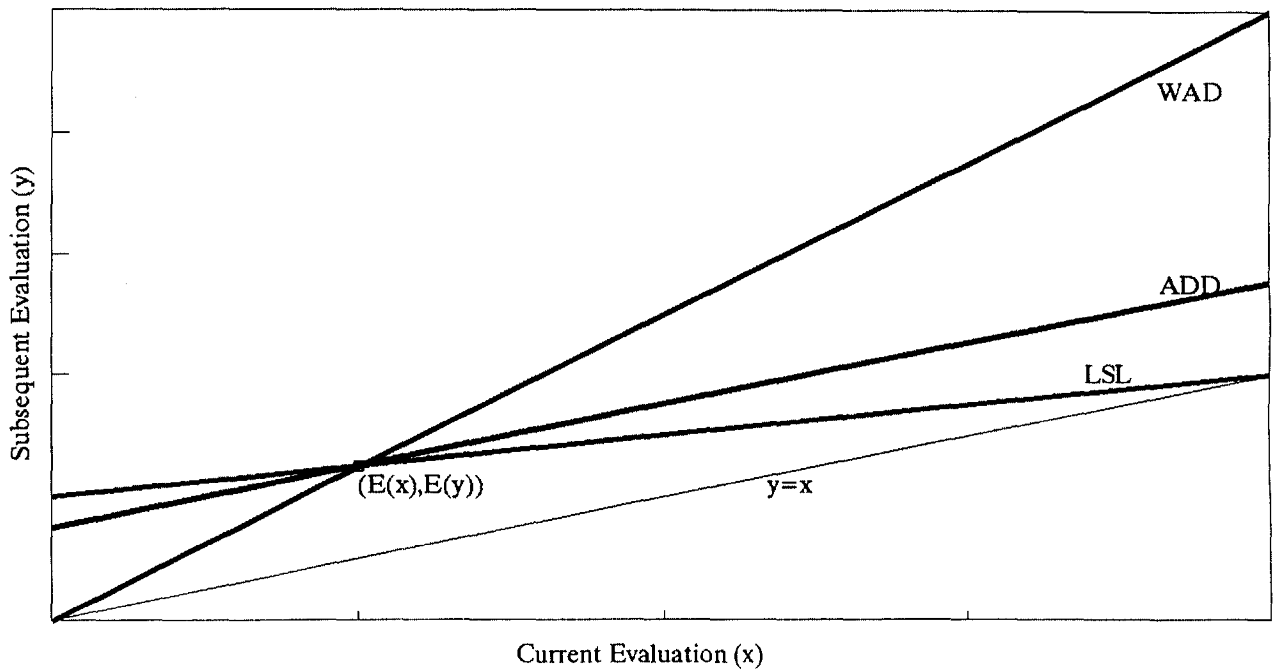


Figure 9B

Idealized Development Estimators With Trend

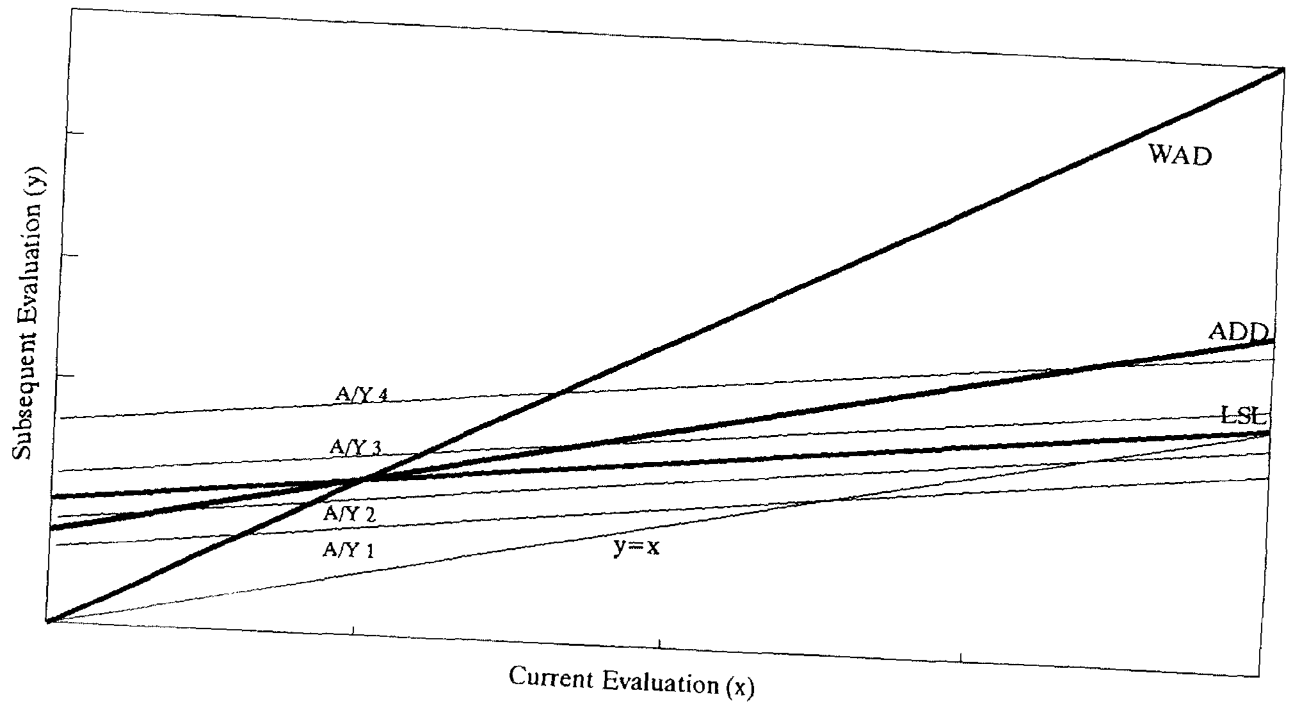


EXHIBIT I

Claim Counts Only

	A/Y	Average Bias	Std Dev Bias	Average %Bias	Std Dev %Bias	Age–Age Bias	Age–Age %Bias
Linear							
LSL							
	1	0.000	0.000	0.000	0.000		
	2	0.116	2.000	0.003	0.053	0.116	0.003
	3	0.153	2.772	0.004	0.073	0.037	0.001
	4	0.101	3.166	0.003	0.083	(0.052)	(0.001)
	5	0.080	3.780	0.003	0.100	(0.021)	0.000
	Total	0.451	8.251	0.002	0.043		
ADD							
	1	0.000	0.000	0.000	0.000		
	2	0.059	1.868	0.002	0.049	0.059	0.002
	3	0.075	2.847	0.002	0.075	0.016	0.000
	4	0.047	3.644	0.002	0.096	(0.028)	0.000
	5	0.096	3.692	0.003	0.097	0.049	0.001
	Total	0.277	8.407	0.001	0.044		
LSM							
	1	0.000	0.000	0.000	0.000		
	2	0.116	2.000	0.003	0.053	0.116	0.003
	3	0.143	3.321	0.004	0.087	0.027	0.001
	4	0.004	5.246	0.000	0.138	(0.139)	(0.004)
	5	(0.748)	10.536	(0.020)	0.277	(0.752)	(0.020)
	Total	(0.485)	14.009	(0.003)	0.074		
WAD							
	1	0.000	0.000	0.000	0.000		
	2	0.116	2.000	0.003	0.053	0.116	0.003
	3	0.203	3.336	0.005	0.088	0.087	0.002
	4	0.281	5.308	0.007	0.139	0.078	0.002
	5	0.888	11.101	0.023	0.292	0.607	0.016
	Total	1.488	14.520	0.008	0.076		
GAD							
	1	0.000	0.000	0.000	0.000		
	2	0.116	2.000	0.003	0.053	0.116	0.003
	3	0.234	3.345	0.006	0.088	0.118	0.003
	4	0.424	5.346	0.011	0.140	0.190	0.005
	5	1.873	11.585	0.049	0.305	1.449	0.038
	Total	2.647	14.943	0.014	0.079		
SAD							
	1	0.000	0.000	0.000	0.000		
	2	0.116	2.000	0.003	0.053	0.116	0.003
	3	0.265	3.354	0.007	0.088	0.149	0.004
	4	0.571	5.390	0.015	0.142	0.306	0.008
	5	2.958	12.268	0.078	0.322	2.387	0.062
	Total	3.910	15.530	0.021	0.082		
Nonlinear							
SB							
	1	0.000	0.000	0.000	0.000		
	2	0.102	1.940	0.003	0.051	0.102	0.003
	3	0.147	3.021	0.004	0.079	0.045	0.001
	4	0.137	3.997	0.004	0.105	(0.010)	0.000
	5	0.185	4.280	0.006	0.113	0.048	0.002
	Total	0.571	9.564	0.003	0.050		
BF							
	1	0.000	0.000	0.000	0.000		
	2	0.114	1.952	0.003	0.051	0.114	0.003
	3	0.184	3.064	0.005	0.081	0.070	0.002
	4	0.215	4.151	0.006	0.109	0.031	0.001
	5	0.338	5.164	0.010	0.136	0.123	0.004
	Total	0.851	10.626	0.004	0.056		

EXHIBIT II

Random Severity, No Trend

	A/Y	Average Bias	Std Dev Bias	Average %Bias	Std Dev %Bias	Age–Age Bias	Age–Age %Bias
Linear							
LSL							
	1	0	0	0.000	0.000		
	2	9,206	193,945	0.026	0.302	9,206	0.026
	3	8,749	218,463	0.069	0.420	(458)	0.042
	4	30,028	429,112	0.138	0.650	21,279	0.065
	5	39,426	535,959	0.228	1.004	9,398	0.079
	Total	87,410	888,404	0.040	0.356		
ADD							
	1	0	0	0.000	0.000		
	2	158	185,077	0.010	0.329	158	0.010
	3	(7,445)	196,201	0.023	0.472	(7,603)	0.013
	4	324	272,189	0.066	0.581	7,769	0.042
	5	(2,668)	271,443	0.140	0.680	(2,991)	0.069
	Total	(9,631)	596,942	(0.004)	0.255		
LSM							
	1	0	0	0.000	0.000		
	2	9,206	193,945	0.026	0.302	9,206	0.026
	3	6,192	221,114	0.033	0.415	(3,015)	0.007
	4	24,331	477,371	0.052	0.742	18,140	0.018
	5	12,290	825,131	0.036	1.404	(12,042)	(0.015)
	Total	52,019	1,127,243	0.020	0.453		
WAD							
	1	0	0	0.000	0.000		
	2	9,206	193,945	0.026	0.302	9,206	0.026
	3	11,815	222,675	0.048	0.421	2,608	0.021
	4	51,641	515,997	0.119	0.807	39,826	0.068
	5	116,664	894,747	0.310	1.597	65,023	0.171
	Total	189,327	1,208,220	0.088	0.487		
GAD							
	1	0	0	0.000	0.000		
	2	9,206	193,945	0.026	0.302	9,206	0.026
	3	13,873	219,115	0.054	0.412	4,666	0.027
	4	61,706	484,892	0.147	0.763	47,833	0.088
	5	184,903	854,318	0.489	1.593	123,197	0.298
	Total	269,687	1,130,473	0.130	0.469		
SAD							
	1	0	0	0.000	0.000		
	2	9,206	193,945	0.026	0.302	9,206	0.026
	3	20,621	227,597	0.072	0.440	11,415	0.045
	4	97,144	598,072	0.233	0.980	76,523	0.150
	5	405,202	1,241,904	1.063	2.516	308,058	0.673
	Total	532,174	1,552,136	0.255	0.640		
Nonlinear							
SB							
	1	0	0	0.000	0.000		
	2	6,126	184,062	0.026	0.304	6,126	0.026
	3	3,909	196,494	0.052	0.430	(2,217)	0.025
	4	15,414	291,195	0.097	0.575	11,506	0.043
	5	11,071	286,813	0.172	0.698	(4,344)	0.068
	Total	36,520	633,658	0.017	0.271		
BF							
	1	0	0	0.000	0.000		
	2	9,040	200,965	0.034	0.373	9,040	0.034
	3	10,750	221,175	0.073	0.525	1,710	0.038
	4	29,330	331,648	0.132	0.691	18,580	0.055
	5	37,124	374,743	0.225	0.886	7,794	0.082
	Total	86,244	820,177	0.040	0.342		

EXHIBIT III
Random Severity, 8% Trend

	A/Y	Average Bias	Std Dev Bias	Average %Bias	Std Dev %Bias	Age-Age Bias	Age-Age %Bias
Linear							
LSL							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	11,815	318,796	0.061	0.469	(1,034)	0.030
	4	8,339	515,561	0.080	0.629	(3,475)	0.018
	5	(23,573)	731,012	0.075	0.944	(31,912)	(0.005)
	Total	9,430	1,181,752	0.002	0.367		
ADD							
	1	0	0	0.000	0.000		
	2	(2,249)	177,229	0.008	0.337	(2,249)	0.008
	3	(15,161)	262,260	0.009	0.461	(12,912)	0.001
	4	(35,576)	335,003	0.005	0.511	(20,414)	(0.004)
	5	(92,221)	399,076	(0.028)	0.551	(56,645)	(0.033)
	Total	(145,207)	757,285	(0.053)	0.249		
LSM							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	16,307	328,599	0.043	0.475	3,458	0.013
	4	27,133	580,424	0.057	0.728	10,826	0.013
	5	8,411	1,111,762	0.035	1.360	(18,722)	(0.021)
	Total	64,698	1,504,280	0.021	0.472		
WAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	23,423	333,524	0.057	0.477	10,575	0.026
	4	62,726	608,272	0.122	0.775	39,303	0.061
	5	169,257	1,272,791	0.310	1.620	106,531	0.168
	Total	268,255	1,659,744	0.098	0.527		
GAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	26,050	331,370	0.062	0.466	13,201	0.031
	4	77,169	580,779	0.149	0.755	51,119	0.082
	5	277,757	1,295,202	0.495	1.717	200,588	0.301
	Total	393,824	1,619,314	0.148	0.534		
SAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	35,174	346,105	0.080	0.497	22,326	0.049
	4	124,456	685,305	0.235	0.924	89,282	0.144
	5	647,473	4,098,366	1.107	4.508	523,017	0.706
	Total	819,951	4,291,335	0.299	1.164		
Nonlinear							
SB							
	1	0	0	0.000	0.000		
	2	10,229	177,339	0.036	0.323	10,229	0.036
	3	7,628	272,101	0.055	0.456	(2,601)	0.018
	4	(5,009)	357,093	0.057	0.530	(12,637)	0.002
	5	(62,946)	420,117	0.021	0.590	(57,936)	(0.034)
	Total	(50,098)	825,565	(0.018)	0.269		
BF							
	1	0	0	0.000	0.000		
	2	16,575	212,872	0.052	0.421	16,575	0.052
	3	23,046	310,265	0.091	0.589	6,471	0.037
	4	25,574	422,741	0.114	0.668	2,529	0.021
	5	(9,528)	534,249	0.101	0.780	(35,103)	(0.012)
	Total	55,667	1,113,743	0.020	0.357		

EXHIBIT IV

Random Severity, 8% Trend, Estimates Based on On–Leveled (at 8%) Triangle

	A/Y	Average Bias	Std Dev Bias	Average %Bias	Std Dev %Bias	Age–Age Bias	Age–Age %Bias
Linear							
LSL							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	19,663	321,503	0.080	0.479	6,815	0.049
	4	38,827	508,047	0.147	0.637	19,164	0.062
	5	44,325	695,596	0.216	0.928	5,498	0.060
	Total	115,663	1,148,516	0.045	0.357		
ADD							
	1	0	0	0.000	0.000		
	2	(205)	182,866	0.014	0.358	(205)	0.014
	3	(4,949)	272,965	0.033	0.505	(4,744)	0.019
	4	(3,371)	352,774	0.074	0.577	1,578	0.040
	5	(7,726)	422,975	0.140	0.664	(4,355)	0.061
	Total	(16,251)	833,130	(0.003)	0.277		
LSM							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	16,069	326,583	0.043	0.473	3,220	0.013
	4	26,536	577,658	0.055	0.725	10,467	0.012
	5	3,262	1,070,100	0.027	1.316	(23,274)	(0.027)
	Total	58,715	1,459,667	0.019	0.460		
WAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	23,310	332,453	0.057	0.476	10,461	0.026
	4	62,521	607,521	0.121	0.774	39,211	0.061
	5	166,470	1,251,178	0.305	1.598	103,950	0.164
	Total	265,149	1,635,365	0.097	0.520		
GAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	26,050	331,370	0.062	0.466	13,201	0.031
	4	77,169	580,779	0.149	0.755	51,119	0.082
	5	277,757	1,295,202	0.495	1.717	200,588	0.301
	Total	393,824	1,619,314	0.148	0.534		
SAD							
	1	0	0	0.000	0.000		
	2	12,848	190,771	0.030	0.300	12,848	0.030
	3	35,174	346,105	0.080	0.497	22,326	0.049
	4	124,456	685,305	0.235	0.924	89,282	0.144
	5	647,473	4,098,366	1.107	4.508	523,017	0.706
	Total	819,951	4,291,335	0.299	1.164		
Nonlinear							
SB							
	1	0	0	0.000	0.000		
	2	8,650	175,543	0.032	0.316	8,650	0.032
	3	10,927	275,491	0.063	0.471	2,277	0.030
	4	17,818	368,370	0.106	0.570	6,891	0.040
	5	12,875	440,455	0.173	0.684	(4,943)	0.061
	Total	50,271	870,120	0.021	0.284		
BF							
	1	0	0	0.000	0.000		
	2	12,243	199,536	0.041	0.382	12,243	0.041
	3	20,320	303,669	0.084	0.567	8,078	0.041
	4	38,157	423,818	0.142	0.679	17,837	0.054
	5	51,227	547,415	0.223	0.842	13,070	0.071
	Total	121,946	1,110,267	0.046	0.356		

APPENDIX

Theorem 1: The $\mathbf{b}_{WAD}$ estimator satisfies Properties 4 and 5: the sum of the residuals is zero and the line through the origin with slope $\mathbf{b}_{WAD}$ passes through the sample average $(\bar{x}, \bar{y})$.

Proof:

$$\begin{aligned} \Sigma(y - b_{WAD}x) &= \Sigma y - \frac{\bar{y}}{\bar{x}}\Sigma x \\ &= n(\bar{y} - \frac{\bar{y}}{\bar{x}}\bar{x}) \\ &= 0 \quad . \end{aligned}$$

This proves Property 4. Next, $\bar{y} - \frac{\bar{y}}{\bar{x}}\bar{x} = 0$, so $\bar{y} = b_{WAD}\bar{x}$, demonstrating that the sample average is on the line through the origin with slope $\mathbf{b}_{WAD}$, Property 5.

Theorem 2: $\mathrm{Var}(\mathbf{b}_{WAD}) \geq \mathrm{Var}(\mathbf{b}_{LSM})$.

Proof: First, write $\mathbf{b}_{LSM} = \Sigma x_i y_i/\Sigma x^2 = \Sigma w_i y_i$ where $w_i = x_i/\Sigma x^2$. Recall that all expectations of y are conditional on x, including the variance, which means that expressions involving x, in particular w, may be manipulated as constants. Therefore,

$$\begin{aligned} \mathrm{Var}(b_{LSM}) &= \mathrm{Var}(\sum_i w_i y_i \mid (x_1, x_2, \ldots, x_n)) \\ &= \sum_i w_i^2 \mathrm{Var}(y_i \mid x_i) \\ &= \sum_i w_i^2 \sigma^2 \\ &= \sigma^2 \sum_i \left(\frac{x_i}{\Sigma x^2}\right)^2 \\ &= \frac{\sigma^2}{\Sigma x^2} \quad . \end{aligned}$$

Next,

$$\begin{aligned} \operatorname{Var} b_{WAD} &= \operatorname{Var}\left(\frac{\Sigma y}{\Sigma x}\right) \\ &= \frac{1}{(\Sigma x)^2}\Sigma \operatorname{Var} y \\ &= \frac{n\sigma^2}{(\Sigma x)^2} \quad . \end{aligned}$$

To show that $\operatorname{Var}(b_{WAD}) \geq \operatorname{Var}(b_{LSD})$ we only have left to show that

$$\frac{1}{n}(\Sigma x)^2 \leq \Sigma x^2$$

or

$$\left(\frac{1}{n}\Sigma x\right)^2 \leq \frac{1}{n}\Sigma x^2 \ .$$

But the latter is just the Schwartz Inequality.[1] QED.

Theorem 3: Under Model III, the least squares estimator is b_{SAD}.

Proof: The transformed Model III

$$\frac{y}{x} = b + e$$

is of the form

$$u = bv + e$$

where the variable v is identically equal to unity. Thus, the transformed model satisfies all the assumptions of Model II. Accordingly, its least squares estimator is

$$\hat{b} = \frac{\Sigma vu}{\Sigma v^2} = \frac{\sum \frac{y}{x}}{n} = b_{SAD} \quad .$$

[1]See for example John F. Randolph, *Basic Real and Abstract Analysis*, p. 35.

Theorem 4: Under Model IV, the least squares estimator is $\mathbf{b}_{GAD}$.

Proof: The transformed Model IV

$$\ln(y) = \ln(\mathbf{b}) + \ln(x) + \ln(e)$$

or

$$\ln(y) - \ln(x) = \ln(\mathbf{b}) + \ln(e)$$

is of the form

$$u = b'v + ve'$$

where $b' = \ln(b)$, $v = 1$, and $E(e') = 0$. Thus, the transformed model satisfies the SAD assumptions. By Theorem 3

$$\hat{b}' = \frac{1}{n}\sum \frac{u}{v} = \frac{1}{n}\Sigma(\ln(y) - \ln(x)) = \frac{1}{n}\sum \ln(\frac{y}{x}) \quad .$$

Therefore, the least squares estimator of the "untransformed" parameter $\mathbf{b}$ is

$$\hat{b} = \exp(\hat{b}') = \exp\left(\frac{1}{n}\sum \ln\frac{y}{x}\right) = \left(\exp\sum \ln\frac{y}{x}\right)^{\frac{1}{n}} = \sqrt[n]{\prod \frac{y}{x}} = b_{GAD} \quad .$$

Theorem 5: Under the assumptions of Model I, $y_{LSL} = \mathbf{a}_{LSL} + \mathbf{b}_{LSL}x$ is an unbiased estimator of y; i.e., $E(y_{LSL}) = E(y)$. Under the assumptions of Model II, $y_{LSM} = \mathbf{b}_{LSM}x$ is an unbiased estimator of y.

Proof: Model I assumes that $E(y) = \mathbf{a} + \mathbf{b}x$. Since all expectations are conditional on x and since $\mathbf{a}_{LSL}$ and $\mathbf{b}_{LSL}$ are unbiased, we have

$$\begin{aligned} E(y_{LSL}) &= E(a_{LSL} + b_{LSL}x) \\ &= E(a_{LSL}) + E(b_{LSL}x) \\ &= E(a_{LSL}) + E(b_{LSL})x \\ &= a + bx \\ &= E(y) \quad . \end{aligned}$$

The proof for LSM is similar.

Lemma 1: Under LSL, $E(x_n|x_0) = a_n + b_nE(x_{n-1}|x_0)$. Under LSM, $E(x_n|x_0) = b_nE(x_{n-1}|x_0)$.

Proof 1: The proof will be given for LSL. The proof for LSM is similar.

First,

$$f(x_n|x_0) = \frac{f(x_n, x_0)}{f(x_0)} = \frac{\int_{x_{n-1}} f(x_n, x_{n-1}, x_0)\, dx_{n-1}}{f(x_0)} \quad .$$

Next, the "Multiplication Rule" of conditional density functions[2] states that

$$f(x_n, x_{n-1}, x_0) = f(x_n|(x_{n-1}, x_0))f(x_{n-1}|x_0)f(x_0).$$

Therefore,

$$f(x_n|x_0) = \frac{\int_{x_{n-1}} f(x_n|(x_{n-1}, x_0))f(x_{n-1}|x_0)f(x_0)\, dx_{n-1}}{f(x_0)} = \int_{x_{n-1}} f(x_n|(x_{n-1}, x_0))f(x_{n-1}|x_0)\, dx_{n-1} \quad .$$

By the CLIA, the random variable $x_n|x_{n-1}$ is independent of x_0. Therefore $f(x_n|(x_{n-1},x_0))$ does not depend on x_0, so $f(x_n|(x_{n-1},x_0))=f(x_n|x_{n-1})$. The rest of the proof hinges on our ability to interchange the order of integration. We will make whatever assumptions are necessary about the form of the density functions

[2]See Robert V. Hogg and Allen T. Craig, *Introduction to Mathematical Statistics*, p. 64.

to justify that step. Then

$$
\begin{aligned}
E(x_n|x_0) &= \int_{x_n} x_n f(x_n|x_0)\,dx_n \\
&= \int_{x_n} x_n \left(\int_{x_{n-1}} f(x_n|(x_{n-1},x_0))\,f(x_{n-1}|x_0)\,dx_{n-1} \right) dx_n \\
&= \int_{x_{n-1}} \left(\int_{x_n} x_n f(x_n|(x_{n-1},x_0))\,dx_n \right) f(x_{n-1}|x_0)\,dx_{n-1} \qquad (1) \\
&= \int_{x_{n-1}} \left(\int_{x_n} x_n f(x_n|x_{n-1})\,dx_n \right) f(x_{n-1}|x_0)\,dx_{n-1} \\
&= \int_{x_{n-1}} (a_n + b_n x_{n-1})\,f(x_{n-1}|x_0)\,dx_{n-1} \\
&= a_n + b_n \int_{x_{n-1}} x_{n-1} f(x_{n-1}|x_0)\,dx_{n-1} \\
&= a_n + b_n E(x_{n-1}|x_0) \quad .
\end{aligned}
$$

Proof 2: Recall the well-known identity $E(X)=E_Y[E(X|Y)]$.[3] Consider the following variation reiterated in equation (1) above:

$$E(x_n|x_0) = E_{x_{n-1}|x_0}[E(x_n|(x_{n-1},x_0))] \quad .$$

For LSL we have

$$
\begin{aligned}
E(x_n|x_0) &= E_{x_{n-1}|x_0}[E(x_n|(x_{n-1},x_0))] \\
&= E_{x_{n-1}|x_0}[E(x_n|x_{n-1})] \qquad \text{by CLIA} \\
&= E_{x_{n-1}|x_0}[a_n + b_n x_{n-1}] \\
&= a_n + b_n E(x_{n-1}|x_0) \quad .
\end{aligned}
$$

Theorem 6: $E(\beta_n|x_0) = E(x_n|x_0)$.

Proof: By induction on n. The proof will be given for LSL; the proof for LSM is similar.

---

[3] See for example I. B. Hossack, J. H. Pollard, and B. Zehnworth, *Introductory Statistics with Applications in General Insurance*, 1983, p. 63.

For n=1 the theorem is simply a restatement of Theorem 5.

Assume that $E(\beta_{n-1}|x_0) = E(x_{n-1}|x_0)$. We have that $\beta_n = \hat{a}_n + \hat{b}_n\beta_{n-1}$ where $\hat{a}_n$ and $\hat{b}_n$ are functions of the random variables $x_n|x_{n-1}$, and β_{n-1} is a function of the random variables $x_{n-1}|x_{n-2}, \ldots, x_1|x_0$ and x_0. The CLIA implies that $x_n|x_{n-1}$ is independent of $x_{n-1}|x_{n-2}, \ldots, x_1|x_0$, and x_0, so $\hat{a}_n$ and $\hat{b}_n$ are independent of β_{n-1}. Therefore,

$$
\begin{aligned}
E(\beta_n|x_0) &= E(\hat{a}_n|x_0) + E(\hat{b}_n|x_0)E(\beta_{n-1}|x_0) && \because \hat{b}_n \text{ and } \beta_{n-1} \text{ are independent} \\
&= E_{x_{n-1}|x_0}[E(\hat{a}_n|(x_{n-1},x_0))] + E_{x_{n-1}|x_0}[E(\hat{b}_n|(x_{n-1},x_0))]\,E(\beta_{n-1}|x_0) \\
&= E_{x_{n-1},x_0}[E(\hat{a}_n|x_{n-1})] + E_{x_{n-1}|x_0}[E(\hat{b}_n|x_{n-1})]\,E(\beta_{n-1}|x_0) \\
&= E_{x_{n-1}|x_0}[a_n] + E_{x_{n-1}|x_0}[b_n]\,E(\beta_{n-1}|x_0) \\
&= a_n + b_n E(\beta_{n-1}|x_0) \\
&= a_n + b_n E(x_{n-1}|x_0) && \text{by the induction hypothesis} \\
&= E(x_n|x_0) && \text{by Lemma 1.}
\end{aligned}
$$

Theorem 7:

	<u>LSL</u>	<u>LSM</u>
For n=1:		
	$\operatorname{Var}\beta_1 = \dfrac{\sigma_1^2}{I_1} + (x_0-\bar{x}_0)^2\operatorname{Var}\hat{b}_1$	$\operatorname{Var}\beta_1 = x_0^2\operatorname{Var}\hat{b}_1$

For n>1:

$$\mathrm{Var}\,\beta_n = \frac{\sigma_n^2}{I_n} + (\mu_{n-1}-\overline{x}_{n-1})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\beta_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\beta_{n-1}$$

$$\mathrm{Var}\,\beta_n = \mu_{n-1}^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\beta_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\beta_{n-1}$$

Proof: We will prove the LSM case first. We saw in Theorem 6 that $\hat{b}_n$ and β_{n-1} are independent random variables. The formula[4] for the variance of the product of two independent random variables x and y is

$$\mathrm{Var}(xy) = \sigma_x^2\sigma_y^2 + \mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 \ .$$

This proves the theorem for LSM because $\hat{b}_n$ is unbiased.

For LSL,

$$\mathrm{Var}\,\beta_n = \mathrm{Var}\,\hat{a}_n + 2\,\mathrm{Cov}(\hat{a}_n, \hat{b}_n\beta_{n-1}) + \mathrm{Var}(\hat{b}_n\beta_{n-1})\,.$$

It is well known[5] that the random variables $\overline{x}_n$ and $\hat{b}_n$ are uncorrelated when $\hat{b}_n$

[4]See Hogg and Craig, p. 178, problem 4.92.

[5] See R. Miller and D. Wichern, *Intermediate Business Statistics*, 1977, p.202, for example.

is determined by least squares; since all expectations are conditional, we have that

$$\operatorname{Var}\hat{a}_n = \operatorname{Var}(\bar{x}_n - \bar{x}_{n-1}\hat{b}_n)$$

$$= \operatorname{Var}\bar{x}_n + \bar{x}_{n-1}^2 \operatorname{Var}\hat{b}_n$$

$$= \frac{\sigma_n^2}{I_n} + \bar{x}_{n-1}^2 \operatorname{Var}\hat{b}_n \quad . \qquad (2)$$

Next,

$$\operatorname{Cov}(\hat{a}_n, \hat{b}_n \beta_{n-1}) = E\beta_{n-1} \operatorname{Cov}(\hat{a}_n, \hat{b}_n) \qquad \because \beta_{n-1} \text{ is independent of } \hat{a}_n \text{ and } \hat{b}_n$$

$$= \mu_{n-1} \operatorname{Cov}(\hat{a}_n, \hat{b}_n)$$

and

$$\operatorname{Cov}(\hat{a}_n, \hat{b}_n) = \operatorname{Cov}(\bar{x}_n - \bar{x}_{n-1}\hat{b}_n, \hat{b}_n)$$

$$= \operatorname{Cov}(-\bar{x}_{n-1}\hat{b}_n, \hat{b}_n)$$

$$= -\bar{x}_{n-1} \operatorname{Var}\hat{b}_n \quad . \qquad (3)$$

Putting these together with the formula for $\operatorname{Var}(\hat{b}_n \beta_{n-1})$ from the LSM derivation above we have

$$\operatorname{Var}\beta_n = \frac{\sigma_n^2}{I_n} + \bar{x}_{n-1}^2 \operatorname{Var}\hat{b}_n - 2\mu_{n-1}\bar{x}_{n-1} \operatorname{Var}\hat{b}_n + \beta_{n-1}^2 \operatorname{Var}\hat{b}_n + \hat{b}_n^2 \operatorname{Var}\beta_{n-1} + \operatorname{Var}\beta_{n-1}$$
$$= \frac{\sigma_n^2}{I_n} + (\mu_{n-1} - \bar{x}_{n-1})^2 \operatorname{Var}\hat{b}_n + b_n^2 \operatorname{Var}\beta_{n-1} + \operatorname{Var}\hat{b}_n \operatorname{Var}\beta_{n-1} \quad .$$

Theorem 8:

$$\mathrm{Var}(x_n | x_0) = \sigma_n^2 + b_n^2 \,\mathrm{Var}(x_{n-1} | x_0) \quad .$$

Proof:

$$\begin{aligned}
\mathrm{Var}(x_n | x_0) &= E_{x_{n-1}|x_0}[\mathrm{Var}(x_n | (x_{n-1}, x_0))] + \mathrm{Var}_{x_{n-1}|x_0}[E(x_n | (x_{n-1}, x_0))] \\
&= E_{x_{n-1}|x_0}[\mathrm{Var}(x_n | x_{n-1})] + \mathrm{Var}_{x_{n-1}|x_0}[E(x_n | x_{n-1})] && \text{by CLIA} \\
&= E_{x_{n-1}|x_0}(\sigma_n^2) + \mathrm{Var}_{x_{n-1}|x_0}(a_n + b_n x_{n-1}) && \text{under LSL} \\
&= \sigma_n^2 + b_n^2 \mathrm{Var}(x_{n-1} | x_0) && \text{under LSL or LSM.}
\end{aligned}$$

Lemma 2: $E(S_n) = na_n + b_n(E(S_{n-1}) + x_{n-1,n-1})$.

Proof:

$$
\begin{aligned}
E(S_n) &= E\left(\sum_{i=0}^{n-1} x_{i,n} \mid x_{i,i}\right) \\
&= \sum_{i=0}^{n-1} E(x_{i,n} \mid x_{i,i}) \\
&= \sum_{i=0}^{n-1} E_{x_{i,n-1} \mid x_{i,i}} [E(x_{i,n} \mid (x_{i,n-1}, x_{i,i}))] \\
&= \sum_{i=0}^{n-1} E_{x_{i,n-1} \mid x_{i,i}} [E(x_{i,n} \mid x_{i,n-1})] \qquad \text{by CLIA} \\
&= \sum_{i=0}^{n-1} E_{x_{n-1} \mid x_0} (a_n + b_n x_{i,n-1}) \\
&= na_n + b_n\left(\sum_{i=0}^{n-2} E(x_{i,n-1} \mid x_{i,i}) + x_{n-1,n-1}\right) \\
&= na_n + b_n\,(E(S_{n-1}) + x_{n-1,n-1}).
\end{aligned}
$$

Theorem 9: Let $XD_n=(x_{0,0},x_{1,1},\ldots,x_{n-1,n-1})$ denote the current diagonal of the triangle for the n youngest accident years. Then

$$E(\hat{M}_n|XD_n) = E(S_n).$$

Proof: By induction on n. The proof will be given for LSL; the proof for LSM is similar. For n=1, we know that

$$\begin{aligned} E(\hat{M}_1|XD_1) &= E(\hat{\beta}_{0,1}|x_{0,0})) \\ &= E(x_{0,1}|x_{0,0}) && \text{by Theorem 7} \\ &= E(S_1) \end{aligned}$$

Now, assume $E(\hat{M}_{n-1}|XD_{n-1}) = E(S_{n-1})$. Under LSL, $\hat{M}_n=n\hat{a}_n+\hat{b}_n(\hat{M}_{n-1}+x_{n-1,n-1})$ where $\hat{a}_n$ and $\hat{b}_n$ are functions of the random variables $x_{i,n}|x_{i,n-1}$ and $\hat{M}_{n-1}$ is a function of random variables $x_{i,j}|x_{i,j-1}$ and of $x_{j,j}$ for j<n. By the CLIA $\hat{a}_n$ and $\hat{b}_n$ are independent of $\hat{M}_{n-1}$. Therefore

$$\begin{aligned} E(\hat{M}_n|XD_n) &= E(n\hat{a}_n+\hat{b}_n(\hat{M}_{n-1}+x_{n-1,n-1})|XD_n) \\ &= E(n\hat{a}_n|XD_n) + E(\hat{b}_n|XD_n)\,E(\hat{M}_{n-1}+x_{n-1,n-1}|XD_n) \\ &= na_n + b_n\,(E(\hat{M}_{n-1}|XD_{n-1}) + x_{n-1,n-1}) \\ &= na_n + b_n\,(E(S_{n-1}) + x_{n-1,n-1}) && \text{by the induction hypothesis} \\ &= E(S_n) && \text{by Lemma 2.} \end{aligned}$$

Theorem 10: PARAMETER RISK

	LSL	LSM
For n=1:	$\mathrm{Var}\,\hat{M}_1 = \dfrac{\sigma_1^2}{I_1} + (x_{0,0}-\bar{x}_0)^2\,\mathrm{Var}\,\hat{b}_1$	$\mathrm{Var}\,\hat{M}_1 = x_{0,0}^2\,\mathrm{Var}\,\hat{b}_1$
For n>1:	$\mathrm{Var}\,\hat{M}_n = n^2\dfrac{\sigma_n^2}{I_n} + (M_{n-1} + x_{n-1,n-1} - n\bar{x}_{n-1})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{M}_{n-1}$	$\mathrm{Var}\,\hat{M}_n = (M_{n-1} + x_{n,n})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\,\mathrm{Var}\,\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{M}_{n-1}$

Proof:

We will prove the LSM case first. Since $\hat{M}_n = \hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1})$, the proof is immediate by virtue of the formula for the variance of the product of two independent random variables once we note that

$$\mathrm{Var}(\hat{M}_{n-1} + x_{n-1,n-1}) = \mathrm{Var}(\hat{M}_{n-1})$$

because $x_{n-1,n-1}$ can be treated as a constant with respect to this conditional variance.

For LSL,

$$\mathrm{Var}\,\hat{M}_n = \mathrm{Var}(n\hat{a}_n + \hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1}))$$

$$= \mathrm{Var}(n\hat{a}_n) + 2\,\mathrm{Cov}(n\hat{a}_n, \hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1})) + \mathrm{Var}(\hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1}))\ .$$

In the proof of Theorem 7 we saw that (equation (2))

$$\mathrm{Var}\,\hat{a}_n = \frac{\sigma_n^2}{I_n} + \bar{x}_{n-1}^2 \mathrm{Var}\,\hat{b}_n$$

and that (equation (3))

$$\mathrm{Cov}(\hat{a}_n, \hat{b}_n) = -\bar{x}_{n-1}\,\mathrm{Var}\,\hat{b}_n \quad .$$

Since $\hat{M}_{n-1}$ is independent of $\hat{a}_n$ and $\hat{b}_n$ and since all expectations are conditional on the current diagonal,

$$\mathrm{Cov}(n\hat{a}_n, \hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1})) = n\,E(\hat{M}_{n-1} + x_{n-1,n-1})\,\mathrm{Cov}(\hat{a}_n, \hat{b}_n)$$

therefore

$$\begin{aligned}
\mathrm{Var}\,\hat{M}_n &= n^2\left(\frac{\sigma_n^2}{I_n} + \bar{x}_{n-1}^2\mathrm{Var}\,\hat{b}_n\right) - 2nE(\hat{M}_{n-1} + x_{n-1,n-1})\,\bar{x}_{n-1}\mathrm{Var}\,\hat{b}_n \\
&\quad + (M_{n-1} + x_{n-1,n-1})^2\,\mathrm{Var}\,\hat{b}_n + b_n^2\mathrm{Var}\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\hat{M}_{n-1} \\
&= n^2\frac{\sigma_n^2}{I_n} + (M_{n-1} + x_{n-1,n-1} - n\bar{x}_{n-1})^2\,\mathrm{Var}\,\hat{b}_n \\
&\quad + b_n^2\,\mathrm{Var}\hat{M}_{n-1} + \mathrm{Var}\,\hat{b}_n\,\mathrm{Var}\,\hat{M}_{n-1} \quad .
\end{aligned}$$

Theorem 12: $\mathrm{Var}(S_n) = n\sigma_n^2 + b_n^2\mathrm{Var}(S_{n-1})$.

Proof:

$$\mathrm{Var}(S_n) = E_{X_{n-1}|XD_n}[\mathrm{Var}(S_n|(X_{n-1}, XD_n))] + \mathrm{Var}_{X_{n-1}|XD_n}[E(S_n|(X_{n-1}, XD_n))]$$

$$= E_{X_{n-1}|XD_n}[\mathrm{Var}(\sum_{i=0}^{n-1} x_{i,n}|(x_{i,i}, x_{i,n-1}))] + \mathrm{Var}_{X_{n-1}|XD_n}(a_n + b_n(S_{n-1} + x_{n-1,n-1}))$$

$$= E_{X_{n-1}|XD_n}[\mathrm{Var}(\sum_{i=0}^{n-1} x_{i,n}|x_{i,n-1})] + b_n^2\,\mathrm{Var}_{X_{n-1}|XD_{n-1}}(S_{n-1} + x_{n-1,n-1})$$

$$= n\sigma_n^2 + b_n^2\,\mathrm{Var}(S_{n-1})\,.$$

Theorem 13: Under the transformed GAD model

$$x_n' = b_n' + x_{n-1}' + e_n'$$

where we assume that $\sigma_j'^2 = \mathrm{Var}(e_j')$ are identical for every j, the estimate of the variance of the prediction of ultimate (transformed) loss

$$\hat{\mu}'_U = x_0' + \sum_{j=1}^{U} \hat{b}_j' \quad .$$

is

$$(C + \sum_{j=1}^{U} \frac{1}{I_j})\, s'^2$$

where s'^2 denotes the MSE of the simultaneous solution of the link ratios of the transformed model.

Proof: Since we assume equal variances by development age, we may solve for all parameters b_j simultaneously with the equation (refer to Figure 4)

$$
\begin{pmatrix}
x'_{N,1}-x'_{N,0} \\
x_{N-1,1}-x'_{N-1,0} \\
\vdots \\
x'_{1,1}-x'_{1,0} \\
x'_{N,2}-x'_{N,1} \\
\vdots \\
x'_{2,2}-x'_{2,1} \\
\vdots \\
x'_{N,N-1}-x'_{N,N-2} \\
x'_{N-1,N-1}-x'_{N-1,N-2} \\
x'_{N,N}-x'_{N,N-1}
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & \dots & 0 & 0 \\
1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
1 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
b'_1 \\ b'_2 \\ \vdots \\ b'_{N-1} \\ b'_N
\end{pmatrix}
+
\begin{pmatrix}
e'_1 \\ e'_1 \\ \vdots \\ e'_1 \\ e'_2 \\ \vdots \\ e'_2 \\ \vdots \\ e'_{N-1} \\ e'_{N-1} \\ e'_n
\end{pmatrix},
$$

or, in more concise format, $Y=X\beta + E$. It is well known that the least squares estimator of β is $\hat{\beta}=(X'X)^{-1}X'Y$ and that the variance-covariance matrix of this estimator is $(X'X)^{-1}\sigma'^2$. In this case, it is clear by inspection that $X'X$ is a diagonal matrix whose j^{th} entry equals I_j, the number of data points in the estimate of the j^{th} link ratio, and whose off-diagonal elements are zero. Thus, $\operatorname{Var}\hat{b}'_j = \dfrac{\sigma'^2}{I_j}$ and $\operatorname{Cov}(\hat{b}'_i,\hat{b}'_j)=0$ for $i\neq j$. Therefore, the Parameter Risk $\operatorname{Var}(C+\sum_{j=1}^{U}\hat{b}'_j)$ is exactly equal to $\sigma'^2\sum_{j=1}^{U}\dfrac{1}{I_j}$. The Process Risk is equal to

$$\sum_{j=1}^{C}\operatorname{Var}(e'_n) = C\,\sigma'^2 .$$

These variances are estimated by substituting the estimate s'^2 for σ'^2.

Correlation and the Measurement of Loss Reserve Variability

by Randall D. Holmberg

CORRELATION AND THE MEASUREMENT OF LOSS RESERVE VARIABILITY

Randall D. Holmberg, FCAS

Loss reserves are the largest liability on the balance sheet of an insurance company, yet they are only estimates. Even the actuary responsible for making the estimates is often unable to quantify the inherent uncertainty. This is partly a consequence of the complexity of estimating the variability of the reserve estimates. Correlation across several dimensions makes statistical measurement of uncertainty difficult. Most insurers have only a limited number of historical data points available with which to make estimates of the multiple correlations, making estimation of correlation problematic. This paper presents a mathematically simple model of loss development variability which allows the inclusion of several types of correlation. It can also be adapted to deal with other complexities which may arise in the analysis of reserves. The paper also presents methods which make it easier to estimate correlations in practical applications.

CORRELATION AND THE MEASUREMENT OF LOSS RESERVE VARIABILITY

Loss reserves, generally the largest liability on the balance sheet of an insurer, are only estimates of ultimate loss payments. Even if these estimates as carried on the balance sheet are unbiased, neither deliberately redundant nor inadequate, they are subject to uncertainty. Furthermore, the magnitude of the uncertainty of these estimates is generally unknown, even to the actuary who reviews reserves and certifies their adequacy in statutory financial statements. Considering the importance of reserve estimates to an insurer's reported net worth, it is important to quantify the uncertainty of these estimates.

Attempts to quantify the uncertainty of loss reserve estimates can easily get stymied by the complexity of reserve issues. There is potential for substantial correlation across many dimensions. There are usually relatively few historical data points from which to estimate the multiple correlations that are possible. Yet it is unlikely that correlation is insignificant in the variability of the total reserve estimate. Therefore, actuaries need a model which can deal with correlation but which allows reasonable estimation of the correlations involved.

This paper presents a model for measuring the uncertainty of loss reserve estimates. Its main virtue is the directness and simplicity of the approach. It includes adjustments to account for many of the kinds of correlation effects which arise in analyzing reserves. The data for this measurement will be available in one form or another at any insurance company. The relevance of the items used in the measurement to the question being asked is easy to see. The model is simple enough that it is relatively easy to add features to cope with complications that the model as presented herein does not consider.

While estimation of the correlations involved is difficult in practice, this paper presents several approaches which have proved helpful in making such estimates.

Even when the parameters required by the model are difficult to estimate, the model may be used in sensitivity testing to get a greater appreciation of the importance (or lack of importance) of correlation to the accuracy of reserve estimates. The parameters have clear-cut intuitive interpretations, so sensitivity testing should prove fruitful to a knowledgeable reserve actuary.

The paper will present the model in a relatively simple form and then suggest adaptations to deal with situations of greater complexity. An example of applying the approach is integrated into the description of the model.

BASIC APPROACH

In the property-casualty insurance industry in the United States, actuaries generally rely on a link ratio loss development approach to determine their estimates of accident-year ultimates and hence the adequacy of carried loss reserves. It seems natural to consider the way reserve adequacy is estimated in determining the variance of the resulting estimate. We take a very direct approach. We measure the variance of historical link ratios which the actuary examines when determining projections of future development patterns. From these variances, the variance of the resulting estimate of ultimate is computed. The variance of the estimated ultimate for a single exposure period is equal to the variance of the estimate of reserve shortage or redundancy for that period. The exposure period variance for a single period is then combined with those of other periods to arrive at the variance of total reserve need at a valuation date.

Variance of Link Ratios

In this description of the model, we phrase all discussion in terms of incurred loss development methods. However, this same approach works in a paid loss development context. Similarly, although all references are to "accident year" this model can be used with other exposure periods such as report years or accident quarters.

We will first establish some notation. Let R denote the total IBNR reserve need as of the valuation date in question. In our formulation, R includes provision for adverse or favorable development on known cases. Case reserves are treated as a constant. Therefore, the variability of R is equal to the variability of total reserve need. Let n be the number of accident years and the maximum number of valuations included in our development triangle. Define $l_{i,j}$ as the incurred loss for accident year i as valued j years after the beginning of the accident year. Both i and j are numbered sequentially beginning with 1. Let L_i be the ultimate loss for accident year i. Let $d_{i,j}$ be the link ratio for accident year i between valuations j and $j+1$. Finally, define $D_{i,j}$ as the development factor for accident year i from age j to ultimate. In this formulation, $d_{i,n} = D_{i,n}$ is the tail factor for accident year i. The latest available historical valuation of year i is $l_{i,n-i+1}$. Note the following:

$$d_{i,j} = \frac{l_{i,j+1}}{l_{i,j}} \qquad (1)$$

$$D_{i,j} = \prod_{k=j}^{n} d_{i,k} \qquad (2)$$

$$D_{i,j} = d_{i,j} \cdot D_{i,j+1} \text{ for } j < n \qquad (3)$$

$$L_i = D_{i,n-i+1} \cdot l_{i,n-i+1} \qquad (4)$$

$$E(L_i) = E(D_{i,n-i+1}) \cdot l_{i,n-i+1} \qquad (5)$$

$$R = \sum_i L_i - \sum_i l_{i,n-i+1} \qquad (6)$$

$$E(R) = \sum_i E(L_i) - \sum_i l_{i,n-i+1} \qquad (7)$$

In the traditional link ratio approach,

$$E(D_{i,j}) = \prod_{k=j}^{n} E(d_{i,k}), \text{ and}$$

$$E(d_{i,j}) = E(d_{k,j}) \text{ for all } i\ ,\ k \text{ for which } j > n-i \text{ and } j > n-k.$$

We will not require that these two relationships hold in our model.

$E(L_i)$ is the estimated ultimate loss. The variance of L_i around this mean is what we will measure to arrive at the variability of loss reserve estimates for a single accident year.

The first step in calculating the variance of accident year ultimates is to calculate the variance of the historical link ratios for each stage of development. Exhibit 1 shows the triangle of incurred losses we will use in our examples. Exhibit 2 is the resulting triangle of link ratios. This is a ten-year triangle, so from it we calculate the variance of all $d_{i,j}$ for a fixed j, for values of $i \le 10-j$. These variances, as well as average link ratios and standard deviations, are also displayed in Exhibit 2. Note that since the variance of the link ratio at age j, $Var(d_{i,j})$, is calculated across all $i \le 10-j$ for a fixed j, we have $Var(d_{i,j}) = Var(d_{k,j})$ even if $i \ne k$ so the first subscript is not needed. In our example, the same is true of the mean at age j, $E(d_{i,j})$. However, we will carry the first subscript throughout for consistency with other notation.

The model treats historical link ratios at a given stage of development as a sample from independent identically distributed random variables. The sample variance calculated from this sample is used as an estimator of the variance of the random variable's distribution.

The actuary's chosen projection of loss development may not match historical averages. Even in such instances, the sample variance is used to estimate the variance of future development, as it represents our best estimate of the variability of future development. However, the chosen development factor is treated as the expected value of that development. The example used in this paper includes some selected link ratios which are not equal to the historical average, in order to illustrate how these selections are treated in this method.

An issue that arises is what variance to assign to the link ratios where there are few historical points, and to the tail development factor. Unless there is reason to do otherwise, in practice we usually rely on the sample variance for all ages where there are two or more historical link ratios. In many cases, assigning zero variance to the last one-year link ratio and to the tail factor is reasonable. In other cases, regressing the standard deviation of link ratios against the quantity $\left|1-E(d_{i,j})\right|$ gives a fitted line which can be used to read off the standard deviation of the link ratio or tail factor (limiting standard deviations to non-negative values, of course). Alternate approaches could be used, such as using a parallelogram of link ratios rather than a triangle, or regressing standard deviations against the stage of development j, or simply judgmentally choosing a number for these stages of development. If the data used produce standard deviations which are sufficiently "bumpy" some of these techniques might be required even for stages of development with relatively many historical link ratios available. In the example used here, we have used sample variances where available and have assumed zero variance for the last stage of development seen in the experience and for the tail factor. This is seen in Exhibits 2 and 3.

Following equation (4), since $l_{i,n-i+1}$ is a known constant, we have:

$$Var(L_i) = Var(D_{i,n-i+1}) \cdot (l_{i,n-i+1})^2. \qquad (8)$$

Therefore, determining $Var(D_{i,n-i+1})$ for each i will determine the variance of each accident year ultimate. $Var(D_{i,n-i+1})$ is a function of the $Var(d_{i,j})$ for all $j \geq n-i+1$. However, $Var(D_{i,n-i+1})$ also depends in part on the correlation between link ratios at different stages of development within a single accident year.

Correlation Between Stages of Development

There are different reasons we might expect development at different stages to be correlated. For instance, if unusually high loss development in one period were the result of accelerated reporting, subsequent development would be lower than average as the losses that would ordinarily be reported in those later periods would have already been reported. In this instance, correlation between one stage and subsequent stages would be negative. Positive correlation would occur if there were a tendency for weaker-than-average initial reserving to be corrected over a period of several years. In that case, an unusually high degree of development in one period would be a warning of more to come. These examples do not exhaust the possible reasons for correlation.

The usual link ratio approach, constructing development factors to ultimate as $E(D_{i,j}) = \prod_{k=j}^{n} E(d_{i,k})$, implicitly assumes the stages of development are uncorrelated. If the $d_{i,j}$ were independent for different values of j within a fixed i, we would have the following:

$$Var(D_{i,j}) = E^2(d_{i,j}) \cdot Var(D_{i,j+1}) + Var(d_{i,j}) \cdot E^2(D_{i,j+1}) + Var(d_{i,j}) \cdot Var(D_{i,j+1}) \qquad (9)$$

Since for the tail factor ($d_{i,10}$ in our example) $d_{i,n} = D_{i,n}$, we could start with the tail factor and use equation (9) with our variances of link ratios to chain backward and build $Var(D_{i,n-i+1})$ for all i. Here we generalize to a situation where correlation among stages of development may exist.

Hayne approached the problem of correlated link ratios using an assumption of multivariate lognormality. We propose an approach which is perhaps more intuitive and which is certainly simpler. As a consequence, our approach lacks much of the elegance of Hayne's, but it provides a model which can yield significant insights into the effects of correlation between link ratios.

Our mathematically convenient model for correlation treats a single link ratio, $d_{i,j}$, and the following age-to-ultimate factor, $D_{i,j+1}$, as correlated. We postulate a distribution for $d_{i,j}$ and a relationship between $d_{i,j}$, and $D_{i,j+1}$ which allow relatively easy calculation of means and variances while still permitting the inclusion of various correlation effects. For mathematical tractability, we assume $d_{i,j}$ is uniformly distributed with known mean and variance. Assume a uniformly distributed random variable $X_{i,j}$, stochastically independent of $d_{i,j}$ with a relation as follows:

$$D_{i,j+1} = a \cdot d_{i,j} + b \cdot X_{i,j} \tag{10}$$

$$\text{where } a + b = 1 \tag{11}$$

If the correlation coefficient between $d_{i,j}$ and $D_{i,j+1}$, ρ, is known[1], we have enough information to solve for a and b, and for the lower and upper bounds of the range of $X_{i,j}$,, which we will call $A_{X_{i,j}}$ and $B_{X_{i,j}}$. We can then calculate $E(D_{i,j})$ and $Var(D_{i,j})$.

[1]We will discuss the estimation of ρ later. For the moment, assume the value of ρ is known.

The derivation of these results follows. We have the following as a consequence of (10):

$$E(D_{i,j+1}) = a \cdot E(d_{i,j}) + b \cdot E(X_{i,j}) \qquad (12)$$

$$Var(D_{i,j+1}) = a^2 \cdot Var(d_{i,j}) + b^2 \cdot Var(X_{i,j}) \qquad (13)$$

As a consequence of (10), we have the following for the correlation coefficient between $d_{i,j}$ and $D_{i,j+1}$ (see e.g. Sachs):

$$\rho = a \cdot \sqrt{\frac{Var(d_{i,j})}{Var(D_{i,j+1})}}$$

$$\therefore \quad a = \rho \cdot \sqrt{\frac{Var(D_{i,j+1})}{Var(d_{i,j})}} \qquad (14)$$

Having determined a we can further calculate:

$$b = 1 - a \qquad (15)$$

From (12):

$$E(X_{i,j}) = \frac{E(D_{i,j+1}) - a \cdot E(d_{i,j})}{b} \qquad (16)$$

From (13):

$$Var(X_{i,j}) = \frac{Var(D_{i,j+1}) - a^2 \cdot Var(d_{i,j})}{b^2} \qquad (17)$$

We want to know $E(d_{i,j} \cdot D_{i,j+1}) = E(D_{i,j})$, and $Var(d_{i,j} \cdot D_{i,j+1}) = Var(D_{i,j})$. We will calculate these by specifying the distributions of $X_{i,j}$ and $d_{i,j}$ and integrating the appropriate expressions over the relevant domains. First we determine these domains.

For a random variable Y, uniformly distributed on the interval between A_Y and B_Y,

$$E(Y) = \frac{A_Y + B_Y}{2} \qquad (18)$$

$$Var(Y) = \frac{(B_Y - A_Y)^2}{12} \qquad (19)$$

So we can derive $A_{X_{i,j}}, B_{X_{i,j}}, A_{d_{i,j}}$, and $B_{d_{i,j}}$. For notational convenience, we will use A_X to denote $A_{X_{i,j}}$, B_X to denote $B_{X_{i,j}}$, A_d to denote $A_{d_{i,j}}$, and B_d to denote $B_{d_{i,j}}$ in what follows. We can determine the bounds of $X_{i,j}$ using formulae following from (18) and (19):

$$A_X = E(X_{i,j}) - \frac{[12 \cdot Var(X_{i,j})]^{1/2}}{2} \tag{20}$$

$$B_X = E(X_{i,j}) + \frac{[12 \cdot Var(X_{i,j})]^{1/2}}{2} \tag{21}$$

Analogous formulae give the values of A_d and B_d. Now we can set up integrals and calculate $E(D_{i,j})$ and $Var(D_{i,j})$.

$$E(D_{i,j}) = E(d_{i,j} \cdot D_{i,j+1}) = \frac{1}{(B_X - A_X)\cdot(B_d - A_d)} \cdot \int_{A_d}^{B_d}\int_{A_X}^{B_X} d_{i,j} \cdot (a \cdot d_{i,j} + b \cdot X_{i,j}) dX_{i,j}\, dd_{i,j}$$

$$= \frac{\frac{a}{3}\cdot(B_X - A_X)\cdot(B_d^3 - A_d^3) + \frac{b}{4}\cdot(B_X^2 - A_X^2)\cdot(B_d^2 - A_d^2)}{(B_X - A_X)\cdot(B_d - A_d)} \tag{22}$$

$$E(D_{i,j}^2) = E(d_{i,j}^2 \cdot D_{i,j+1}^2) = \frac{1}{(B_X - A_X)\cdot(B_d - A_d)} \cdot \int_{A_d}^{B_d}\int_{A_X}^{B_X} d_{i,j}^2 \cdot (a \cdot d_{i,j} + b \cdot X_{i,j})^2 dX_{i,j}\, dd_{i,j}$$

$$= \frac{\frac{a^2}{5}(B_X - A_X)(B_d^5 - A_d^5) + \frac{a \cdot b}{4}(B_X^2 - A_X^2)(B_d^4 - A_d^4) + \frac{b^2}{9}(B_X^3 - A_X^3)(B_d^3 - A_d^3)}{(B_X - A_X)\cdot(B_d - A_d)} \tag{23}$$

$$Var(D_{i,j}) = E(D_{i,j}^2) - E^2(D_{i,j}). \tag{24}$$

We chain backward to calculate $E(D_{i,n-i+1})$ and $Var(D_{i,n-i+1})$ for all i, allowing us to calculate $Var(L_i)$. The way we do this is as follows. Start with the tail factor $D_{i,n}$ so that $E(D_{i,n}) = E(d_{i,n})$ and $Var(D_{i,n}) = Var(d_{i,n})$, quantities we have estimated or assumed. We also know $E(d_{i,n-1})$ and $Var(d_{i,n-1})$. Use (14) and (15) to calculate a and b. Use (16) and (17) to compute $E(X_{i,n-1})$ and $Var(X_{i,n-1})$. Calculate A_X, B_X, A_d, and B_d

using (20) and (21) and the analogous equations for A_d, and B_d. Finally, calculate $E(D_{i,n-1})$ and $Var(D_{i,n-1})$ using (22), (23), and (24). We can repeat this process, using $E(D_{i,n-1})$ and $Var(D_{i,n-1})$ to estimate $E(D_{i,n-2})$ and $Var(D_{i,n-2})$. We continue backward in this fashion until we reach $E(D_{i,n-i+1})$ and $Var(D_{i,n-i+1})$, allowing us to calculate $Var(L_i)$.

Correlation affects both $E(D_{i,j})$ and $Var(D_{i,j})$. Beyond this, there are important conditional expectations and variances, $E(D_{i,j}|d_{i,j-1})$ and $Var(D_{i,j}|d_{i,j-1})$ for $1<j\leq n-i+1$. If we believe that link ratios and the following age-to-ultimate development factors are correlated, then knowledge of the last historical link ratio for each accident year should affect both our expectation of future development on that year and the variance of our estimated ultimate. It is internally consistent if in projecting ultimate losses and in estimating the variance of those ultimates we use conditional expectations and variances per the following:

from (5): $$E(L_i)=l_{i,n-i+1}\cdot E(D_{i,n-i+1}|d_{i,n-i}),\text{ if } i<n \tag{25}$$

from (8): $$Var(L_i)=(l_{i,n-i+1})^2\cdot Var(D_{i,n-i+1}|d_{i,n-i}),\text{ if } i<n \tag{26}$$

from (5): $$E(L_n)=l_{n,1}\cdot E(D_{n,1}) \tag{27}$$

from (8): $$Var(L_n)=(l_{n,1})^2\cdot Var(D_{n,1}) \tag{28}$$

Note that for $i<n$:

from (12): $$E(D_{i,n-i+1}|d_{i,n-i})=a\cdot d_{i,n-i}+b\cdot E(X_{i,n-i}) \tag{29}$$

from (13): $$Var(D_{i,n-i+1}|d_{i,n-i})=b^2\cdot Var(X_{i,n-i}) \tag{30}$$

Exhibit 3 shows a calculation of conditional expectations and variances. For informational purposes, it also shows unconditional expectations and variances including correlation, and expectations and variances excluding correlation. Exhibit 3 uses the same value of ρ for all i and all j, but this is not a requirement of the model. However, when the question of estimating ρ arises, there is a benefit to having a single value to estimate, and there may be some intuitive appeal to having a single value of ρ.

At this point it is helpful to go through Exhibit 3 step by step to clarify how the model is used in practice. At the top of the exhibit, we show for each stage of development j the expected link ratio $E(d_{i,j})$ and the variance of that link ratio $Var(d_{i,j})$. $E(d_{i,j})$ is equal to our selected link ratio. $Var(d_{i,j})$ is estimated using the sample variance as shown in Exhibit 2. In this example, neither $E(d_{i,j})$ nor $Var(d_{i,j})$ vary with i, as has been noted previously. However, the model could cope with different $E(d_{i,j})$ and $Var(d_{i,j})$ for different i.

The next item in Exhibit 3 is our value of ρ, which in this example is the same for all i, j. From this point on, the exhibit is easier to interpret if we start on the right of the exhibit and work our way to the left. At each j, we determine a using equation (14) and b using (15). For $j \geq 8$, we have $a = 0$, since $Var(D_{i,j+1}) = 0$. For lesser j, we use the value of $Var(D_{i,j+1})$ from the next column to the right to calculate a and b. $E(X_{i,j})$ and $Var(X_{i,j})$, calculated using (16) and (17) respectively, are shown next, followed by A_X, B_X, which follow using (20) and (21). The values of $E(d_{i,j})$ and $Var(d_{i,j})$ from the top of the exhibit similarly allow us to calculate A_d, and B_d which are the next values shown in Exhibit 3. Then, using (22) and (23), we can calculate $E(D_{i,j})$ and $Var(D_{i,j})$. These values flow into the calculation for the next column to the left. For each j, once we have completed the column to this point we have enough information to proceed with the calculations for the column for $j-1$.

Below these unconditional means and variances, we calculate the conditional values $E(D_{n-j+1,j}|d_{n-j+1,j-1})$ and $Var(D_{n-j+1,j}|d_{n-j+1,j-1})$ using equations (25) through (28), which will be used to project the ultimate and the variance of that ultimate for accident year

$n-j+1$. These calculations require some parameters from the column for $j-1$, namely a, b, $E(X_{n-j+1,j-1})$, and $Var(X_{n-j+1,j-1})$.

Note that there are no conditional expectation and variance for $j=1$ or equivalently for $i=n$. This is of course because there is no $d_{n,0}$.

The effect of correlation on unconditional $E(D_{i,11-i})$ is relatively small, but the effect on unconditional $Var(D_{i,11-i})$ is significant, when compared to the values ignoring correlation. When conditional expectations and variances are used, both expectations and variances are significantly affected by correlation.

The assumption of a uniformly distributed $X_{i,j}$ and $d_{i,j}$ is primarily for mathematical convenience in determining variances of the product of correlated random variables. It is not intended to represent a realistic model of the probability distribution of the link ratios or age-to-ultimate development factors. Thus, the actuary may decide that using conditional probabilities and variances is putting too much reliance on a model which was chosen largely for convenience. In such a case, the actuary might base estimated ultimates on the traditional age-to-ultimate factor as the product of projected link ratios (implicitly ignoring correlation for the purpose of projecting ultimates), but use the variances including correlation, either conditional or unconditional. Alternatively, he or she might use the unconditional $E(D_{i,j})$ including correlation (recognizing correlation among future development, but ignoring the correlation to historical link ratios) instead of relying on conditional $E(D_{i,j}|d_{i,j-1})$ from this model. Note that as a consequence of (13) and (30), $Var(D_{i,j}|d_{i,j-1}) \le Var(D_{i,j})$. Hence, the use of the correlated unconditional $Var(D_{i,j})$ would be conservative. The correlated $E(D_{i,j})$ is greater than the age-to-ultimate excluding correlation if correlation is positive, less than the uncorrelated age-to-ultimate if correlation is negative (assuming that loss development is positive). $E(D_{i,j}|d_{i,j-1})$ may be greater or smaller than $E(D_{i,j})$ depending on whether $d_{i,j-1}$ is greater or smaller than the average link ratio at that stage, and whether correlation is positive or negative.

Exhibit 4 shows a calculation where $E(D_{i,n-i+1}|d_{i,n-i})$ and $Var(D_{n-i+1}|d_{i,n-i})$ are used for calculating expected ultimates and the variance of those ultimates, based on equations (25) through (28). In our example we use conditional expectations and variances in the interest of internal consistency, as discussed earlier. For convenience in later calculations, Exhibit 4 shows variances converted to standard deviations.

Following from equation (6), we have $Var(R) = Var\left(\sum_i L_i\right)$. If the L_i for different i were independent, we could calculate the variance of the estimated total reserve need R as

$$Var(R) = \sum_i Var(L_i) \qquad (31)$$

However, the model does not require this assumption of independence, as will be seen below.

Correlation Between Accident Years

There are reasons that the estimated ultimates for different accident years as of a given valuation date might be positively correlated. If current case reserves are stronger (or weaker) at the valuation date than assumed implicitly in the projected development pattern for one accident year, it is likely to be true for all accident years. If claim processing has been disrupted in some way, that may very well affect all accident years. If a judicial decision changes the likelihood of paying out on certain types of claims, that could affect all accident years. There are doubtless other examples of contingencies which could cause positive correlation. It is less clear, at least to us, what realistic contingencies in property-casualty insurance could result in negative correlation, although hypothetical examples can be created.

One concern which must be noted at this point is that some of the situations which can cause correlation between accident years can also cause correlation between stages of development within an accident year. The prior section dealt with measuring the correlation between stages of development within an accident year. It is important in adding consideration of correlation between accident years that we avoid "double-counting" the correlation which results from the same cause as the correlation within an accident year. A method we propose to avoid (or at least ameliorate) the potential double-counting, without going into the complexities of multivariate analysis, is described in the next section on estimating correlation coefficients. The current section describes the mechanics of including correlation between accident years in our measure of the variability of total reserves.

We start with a formula for the variance of the sum of generalized random variables Y_k.

$$Var\left(\sum_{k=1}^{n} Y_k\right) = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} Cov\left(Y_{k_1}, Y_{k_2}\right)$$

Note that $Cov(Y_k, Y_k) = Var(Y_k)$, and that each term $Cov\left(Y_{k_1}, Y_{k_2}\right)$, for $k_1 \neq k_2$ appears twice in this sum. Thus, for example, this formula would agree with the familiar formula for the variance of the sum of two random variables,

$$Var(Y+Z) = Var(Y) + Var(Z) + 2 \cdot Cov(Y,Z).$$

We approach the calculation of the variance of R through use of a correlation matrix. Since $Cov(L_k, L_m) = \rho_{k,m} \cdot Var(L_k)^{1/2} \cdot Var(L_m)^{1/2}$, where $\rho_{k,m}$ is the correlation coefficient between L_k and L_m, defining a matrix of the values of $\rho_{k,m}$ is the first step to calculating the variance of R.

Set up an $n \times n$ matrix C, with $c_{km} = c_{mk} = \rho_{k,m}$ for all k, m.[2] $Var(L_k)$ is known for each k based on the work done in the previous section. Therefore, we calculate

$$Var(R) = \sum_{k=1}^{n} \sum_{m=1}^{n} Cov(L_k, L_m) = \sum_{k=1}^{n} \sum_{m=1}^{n} \rho_{k,m} \cdot Var(L_k)^{1/2} \cdot Var(L_m)^{1/2} \quad (32)$$

Exhibit 5 shows the calculation of $Var(R)$ in this manner. The exhibit shows the matrix of correlation coefficients first, i.e. the matrix of $\rho_{k,m}$. It then illustrates the calculation of the matrix of covariances using the accident year standard deviations calculated in Exhibit 4. Each element in the second matrix is a term from equation (32). The sum of all elements in the second matrix is equal to the displayed "Variance of Estimated Reserve Need" at the bottom of Exhibit 5. This is converted to a standard deviation for the reader's convenience.

Estimating Correlation Coefficients

The inclusion of correlation has significant effects on both estimated IBNR reserve need and the variance of that estimate. Exhibits 4 and 5 show an estimated needed IBNR of \$69,879 and a standard deviation of that estimate of \$21,492. If we had used zero correlation everywhere in the model, the corresponding numbers would have been \$68,325 estimated IBNR need with a standard deviation of \$14,717. Clearly, the existence of correlation is an important factor in measuring the variability of reserve estimates.

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[2] Again, take $\rho_{k,m}$ as given. The problem of estimating these correlations is treated later.

How do we determine the correlation ρ within each accident year and the correlations $\rho_{k,m}$ between accident years at a given valuation point? How do we account for the fact that some of the correlation captured in ρ is caused by the same factors that result in correlation measured in the $\rho_{k,m}$? The first step in our recommended approach is to estimate ρ, without any consideration of collinearity, as described below.

One approach which can be used to estimate ρ is an iterative approach based on the incurred loss triangle being analyzed. In our experience convergence is usually pretty rapid, taking 3 to 5 iterations.

On the first pass, treat ρ as if it were zero. Take the resulting estimated ultimates $E(L_i) = l_{i,n-i+1} \cdot \prod_{k=n-i+1}^{n} E(d_{i,k})$ and use them to calculate implied $D'_{i,j} = E(L_i)/l_{i,j}$ for $j \le n-i$ (that is for all j with at least one historical link ratio in addition to the projected development). Transform all $D'_{i,j}$ and all $d_{i,j-1}$ to random variables with mean zero and standard deviation one using these formulae:

$$D''_{i,j} = \frac{(D'_{i,j} - E(D'_{i,j}))}{(Var(D'_{i,j}))^{1/2}} \qquad (33)$$

$$d''_{i,j-1} = \frac{(d_{i,j-1} - E(d_{i,j-1}))}{(Var(d_{i,j-1}))^{1/2}}$$

where expectations and variances are calculated across varying i within a fixed j. Calculate the sample correlation coefficient between these $d''_{i,j-1}$ and $D''_{i,j}$ for all i,j such that $2 \le j \le n-i$. Use this sample correlation coefficient as ρ for all i,j in all calculations. If the ultimates are being projected using $E(L_i) = l_{i,n-i+1} \cdot \prod_{k=n-i+1}^{n} E(d_{i,k})$, this is the final estimate of ρ. If ultimates are being projected using $E(D_{i,j}|d_{i,j-1})$, or using

$E(D_{i,j})$ including the effect of correlation, as seen in Exhibits 3 and 4, we must go through the calculation of new ultimates $E(L_i)$ since these ultimates depend on ρ. For the second iteration, base $D'_{i,j}$ on the new estimate of $E(L_i)$ and calculate new $D''_{i,j}$ using (33). Calculate a new sample correlation coefficient between $d''_{i,j-1}$ and $D''_{i,j}$. Repeat this process until it converges on a value for ρ. This iterative approach maximizes use of the most relevant data for the determination of ρ, by putting all available data points into the determination.

Other methods which would determine a different ρ for each j would require substantially more historical points than are usually available to an insurer. We have tried using accident quarter by calendar quarter triangles to expand the number of historical points, but the data are so variable when cut this fine that the approach did not work well.

The question of determining the elements of the accident year correlation matrix C presents similar challenges to the determination of ρ. There are additional complexities due to the need to avoid the effects of collinearity.

In practice, we have found it difficult to determine values for the non-diagonal elements of this matrix from company data. When we look at homogeneous lines of insurance, calculated correlation coefficients are often not significant if a reasonable standard is used. This is particularly true for stages of development where there are relatively few data points. An approach we have used with some success is similar in respects to that outlined above for estimating ρ.

Start with our incurred loss triangle and estimated ultimates $E(L_i)$. In this instance, the $E(L_i)$ include whatever correlation effects based on ρ the actuary has decided to include. Calculate implied $D'_{i,j} = E(L_i)/l_{i,j}$ for $j \leq n-i$. Transform all $D'_{i,j}$ to random variables

$D''_{i,j}$ with mean zero and standard deviation one using the formula:

$$D''_{i,j} = \frac{(D'_{i,j} - E(D'_{i,j}))}{(Var(D'_{i,j}))^{1/2}} \qquad (34)$$

Calculate the sample correlation coefficient between $D''_{i,j}$ and $D''_{i-1,j+1}$ for all values i, j where $j \leq n-i$ and both $D''_{i,j}$ and $D''_{i-1,j+1}$ exist. This represents the correlation between age-to-ultimate development at a given valuation date for accident years which are separated by one year. Use this sample correlation coefficient for all c_{km} where $|k-m| = 1$. Use an analogous approach for $|k-m| = 2$, basing c_{km} on the sample correlation between $D''_{i,j}$ and $D''_{i-2,j+2}$. Continue in this fashion for $|k-m| = 3, 4, \ldots,$ until correlation is negligible or until there are too few points with the proper spacing to calculate a sample correlation coefficient.

The adjustment to this procedure to remove collinearity is to restate (34) as:

$$D''_{i,j} = \frac{(D'_{i,j} - E(D'_{i,j}|d_{i,j-1}))}{(Var(D'_{i,j}|d_{i,j-1}))^{1/2}} \qquad (35)$$

Thus, the correlation between $D_{i,j}$ and $d_{i,j-1}$ is considered and is "reduced out" of the measure of correlation between accident years. This way, any factor which contributes to correlation both between accident years and within accident years is not double-counted. In practice, we can estimate the correlation between accident years using both (34) and (35). If the results flowing from using (34) without eliminating collinearity and ignoring correlation within accident years show a stronger correlation than the combination of using (35) with correlation within accident year, we can use (34) and ignore the correlation within accident years. Otherwise, we use the combination of correlation within accident years and the correlation between accident years measured using the results of (35).

We have found that it is often helpful to restrict these calculations to relatively low values of j, for instance, look only at pairs $D''_{i,j}$ and $D''_{i-1,j+1}$ for which $j \le 3$. It appears that in many cases the correlation between accident years becomes insignificant for accident years which are beyond the earliest few stages of development at the valuation date under consideration.

Another approach which has proved useful is looking at higher levels of aggregation for determination of this correlation, rather than looking at a homogeneous line of insurance. Combinations of lines often show more correlation between accident years than can be seen through the "noise" at a finer level of detail. If correlations from aggregated data are to be applied at a finer level of detail, the actuary should make sure that the lines combined to reach the aggregate are expected to behave similarly in terms of loss development, so that the correlations might be reasonable for use at the detail level. A further consideration is that the collinearity adjustment described above must then be done at the aggregated level. While this adjustment is possible, the description of the calculation is not given here. In practice, when we have used aggregated data to determine correlation between accident years, we have not included correlation within accident years in our variance measure. Then we can rely on equation (34) in making our estimate of the correlation.

Exhibit 5 shows a situation where correlation was believed to be significant only between years falling in the last 4 accident years as of a given valuation date, and where correlation was negligible for $|k-m|>2$.

OTHER ISSUES AND ADDITIONAL COMPLICATIONS

The model as described above deals with the variability of reserve estimates including assumptions of two varieties of correlation. This section discusses additional concerns that arise in estimating reserve need and in some cases describes how the model could be adapted to address those concerns.

Homogeneity of Data

The model as presented depends on the data in the loss development triangle being homogeneous. If the data are not homogeneous, but the mix is constant through time, the model may still provide useful information. If the data are not homogeneous and the mix is not constant through time, the model as presented will not give representative results.

Correlation Between Lines of Insurance

The model as described above deals with one homogeneous line of insurance. When analyzing reserves for an insurer, we are usually concerned with the variability of the estimated reserve need for the insurer as a whole as well as on a line-by-line basis. If the R_s are independent, the formula for the variability of the total reserve estimate is

$$Var\left(\sum_{s=1}^{r} R_s\right) = \sum_{s=1}^{r} Var(R_s) \qquad (36)$$

where R_s is the reserve estimate for line s, and r is the number of lines of insurance.

Intuitively, it would be expected that some of the R_s are not independent. However, with measurement of the correlation within accident years and between accident years as described earlier in this paper, we have generally felt comfortable that the great majority of the effects which in practical application cause correlation between lines of insurance are already captured in the measures of correlation already described. Therefore, we have used (36) to estimate the variance of reserve estimates for combinations of lines of insurance. The substantial enhancements to this model to accommodate a further measure of correlation are beyond the scope of this paper.

Effect of Inflation

Variability in loss development could be the result of changes in inflation rates. If the actuary feels the effect of inflation on the loss development triangle would distort the measurement of reserve variability described in this paper, the triangle should be adjusted to a constant dollar basis before this variability model is used. It must be noted that in order for the resulting variability measure to be complete, consideration of the variability of estimates of future inflation would have to be included separately. Such consideration is beyond the scope of this paper. Failing to remove the effects of inflation from the triangle before applying our model of variability implicitly assumes that future variability of inflation will have the same effects on reserve estimates that the historical triangle shows. The actuary may feel this assumption is justified, but at any rate the choice of such an assumption should be a conscious decision.

Varying Volume of Data Through Time

In practice, the triangle of loss development data we are analyzing may have significant changes in the volume of business through the period of time covered. In such a situation, the calculation of sample $Var(d_{i,j})$ in the manner shown in Exhibit 2 could be distorted by points showing unusual development but backed by very little data. To cope with such situations, we have used a "dollar-weighted variance" approach which is shown in

Appendix A. The essence of this approach is defined by:

$$E_w(d_{i,j}) = \frac{\sum_{i=1}^{n-j} l_{i,j+1}}{\sum_{i=1}^{n-j} l_{i,j}} \qquad \text{for } 1 \le j \le n-1 \text{ and} \tag{37}$$

$$Var_w(d_{i,j}) = \frac{\sum_{i=1}^{n-j} l_{i,j} \cdot \left(\frac{l_{i,j+1}}{l_{i,j}} - E_w(d_{i,j}) \right)^2}{\sum_{i=1}^{n-j} l_{i,j}} \qquad \text{for } 1 \le j \le n-1 \tag{38}.$$

We use the weighted variance $Var_w(d_{i,j})$ in exactly the same way we used $Var(d_{i,j})$ in the earlier description of the model. Appendix A shows a revised version of Exhibits 1 through 5 (renumbered as 1A through 5A) substituting this weighted approach. In this particular example the effect is not large because the triangle used in our examples does not have extreme volume changes. In practice we have encountered many situations where the volume adjustment is important. In fact, we almost always use the weighted variance approach in practice since in situations where it is unimportant it has little effect, and in situations where it is important it gives a better representation of the variability of loss development.

Paid Versus Incurred Development

The model has been described in terms of incurred loss development and IBNR estimates. However, there is nothing in the formulation which requires that it be used in this way. All formulae and relationships would hold equally well for paid loss development analysis. Interpretation of correlation coefficients might vary, however. Depending on the reason for correlation between accident years on an incurred development basis, it might be expected that on a paid development basis there would be no correlation at all. Correlation between link ratios within a given accident year might also be zero when viewed in a paid loss development context. The actuary should consider these issues, and where possible test to see if correlation does exist. If correlation is eliminated by using paid development data, the actuary could rely on equations (8), (9) and (31) and greatly simplify the calculation of variability of reserve estimates.

Trends in Loss Development Patterns

As presented in the example in this paper, the model included no consideration of changes through time in loss development patterns. Some simple kinds of changes could be included relatively easily. For example, if a regression curve were fitted to historical values of $d_{i,j}$ and projected values were read off that curve, the appropriate adaptation of the model would be to substitute the curve values for all projected $E(d_{i,j})$ and substitute historical variance around the fitted curve for all $Var(d_{i,j})$ in the formulae describing the model. A volume-weighting scheme would be possible in this context if desired. The complications for using fitted curves in the analysis of correlation should also be considered. When normalizing the variables to arrive at $d''_{i,j-1}$ and $D''_{i,j}$ the expected values and variances should be measured considering the fitted curves rather than "raw" means and variances.

CONCLUSIONS

The model presented in this paper uses the variance of link ratios to estimate the variance of reserve estimates. It allows the inclusion of correlation effects of several varieties. It can be elaborated to cope with a number of concerns which may be important in specific situations.

While the estimation of correlation is often difficult in practice, we have presented some approaches which maximize the use of historical data in making such estimates. If estimates prove impractical, this model can be applied in a sensitivity-testing manner to demonstrate that the effects of correlation can be important as regards both the estimated reserve need and the variance of that estimate. Whether or not correlations can be estimated with much accuracy, this model gives actuaries an approach to better quantifying the uncertainty of reserve estimates.

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Triangle of Incurred Loss Valuations

Exhibit 1A

Stage of Development (Years) j

Accident Year i	1	2	3	4	5	6	7	8	9	10
1	32,223	48,439	54,284	58,146	61,305	63,739	63,604	62,721	63,247	62,159
2	42,588	65,239	77,329	82,064	85,260	85,226	80,944	79,577	80,614	
3	44,960	69,989	75,140	79,019	80,548	80,864	79,341	79,525		
4	33,145	56,088	60,732	66,551	66,857	68,395	66,806			
5	30,754	46,587	54,855	57,645	56,249	54,560				
6	33,594	47,576	52,870	59,598	58,715					
7	31,064	54,187	63,529	73,791						
8	33,831	48,453	62,742							
9	44,772	72,814								
10	48,307									

Weighted Average Development and Weighted Variance $\quad l_{i,j}\cdot\left(\frac{l_{i,j+1}}{l_{i,j}}-E_w(d_{i,j})\right)^2$

Exhibit 2A

j

Accident Year i	1	2	3	4	5	6	7	8	9
1	97	38	13	91	64	34	1	0	0
2	29	87	50	48	5	53	5	0	
3	0	395	93	2	1	3	11		
4	597	244	5	7	16	0			
5	57	39	71	87	78				
6	676	67	87	52					
7	1078	30	355						
8	536	1035							
9	209								
$E_w(d_{i,j})$	1.558	1.149	1.087	1.015	1.007	0.975	0.991	1.011	0.983
$Var_w(d_{i,j})$	0.010029	0.004433	0.001536	0.000713	0.000472	0.000305	0.000075	0.000005	
$Var_w(d_{i,j})^{1/2}$	0.100145	0.066581	0.039193	0.026699	0.021719	0.0174634	0.008638	0.002306	

Selected Link Ratios

Modeling Correlation Within an Accident Year

Exhibit 3A

Stage: (j)	1	2	3	4	5	6	7	8	9	10
$E(d_{i,j})$	1.558	1.180	1.120	1.015	1.007	0.975	0.991	1.011	0.983	1.000
$Var_w(d_{i,j})$	0.010029	0.004433	0.001536	0.000713	0.000472	0.000305	0.000075	0.000005	0.000000	0.000000
ρ	0.100									
a	0.110096	0.095952	0.107745	0.114186	0.092599	0.0520322	0.026238	0	0	0
b	0.889904	0.904048	0.892255	0.885814	0.907401	0.9479678	0.973762	1	1	1
$E(X_{i,j})$	1.265	1.090	0.964	0.960	0.955	0.985	0.994	0.983	1.000	1.000
$Var(X_{i,j})$	0.015197	0.004944	0.002218	0.001173	0.000486	0.000091	0.000005	0.000000	0.000000	0.000000
A_X	1.051	0.968	0.883	0.901	0.917	0.968	0.990	0.983	1.000	1.000
B_X	1.478	1.212	1.046	1.020	0.993	1.001	0.998	0.983	1.000	1.000
A_d	1.385	1.065	1.052	0.968	0.970	0.945	0.976	1.007	0.983	1.000
B_d	1.731	1.295	1.188	1.061	1.045	1.005	1.006	1.015	0.983	1.000
Unconditional, including correlation:										
$E(D_{i,j})$	2.022	1.297	1.099	0.981	0.967	0.960	0.984	0.994	0.983	1.000
$Var(D_{i,j})$	0.050964	0.012156	0.004081	0.001783	0.000929	0.000404	0.000083	0.000005	0.000000	0.000000
$E(D_{i,j}^2)$	4.139	1.694	1.211	0.964	0.935	0.921	0.969			
Conditional Expectation and Variance:										
$E(D_{i,j}\|d_{i,j-1})$	2.022	1.305	1.110	0.985	0.963	0.956	0.985	0.994	0.983	1.000
$Var(D_{i,j}\|d_{i,j-1})$	0.050964	0.012035	0.004041	0.001765	0.000920	0.000400	0.000082	0.000005	0.000000	0.000000
If no correlation:										
$E(D_{i,j})$	2.020	1.296	1.098	0.981	0.967	0.960	0.984	0.994	0.983	1.000
$Var(D_{i,j})$	0.041337	0.010046	0.003363	0.001501	0.000810	0.000370	0.000079	0.000005	0.000000	0.000000

Projected Ultimates and Standard Deviations
By Accident Year, Including Correlation

Exhibit 4A

Accident Year	Current Valuation	Expected LDF	Variance of LDF	Expected Ultimate	Needed IBNR	Variance of Ult	Std. Dev. of Ult.
1	62,159	1.000	0.000000	62,159	0	0	0
2	80,614	0.983	0.000000	79,227	−1,387	0	0
3	79,525	0.994	0.000005	79,040	−485	32163	179
4	66,806	0.985	0.000082	65,773	−1,033	364808	604
5	54,560	0.956	0.000400	52,166	−2,394	1191987	1,092
6	58,715	0.963	0.000920	56,560	−2,155	3172039	1,781
7	73,791	0.985	0.001765	72,713	−1,078	9613035	3,100
8	62,742	1.110	0.004041	69,632	6,890	15905997	3,988
9	72,814	1.305	0.012035	94,987	22,173	63806418	7,988
10	48,307	2.022	0.050964	97,671	49,364	118928464	10,905
Total	660,033			729,929	69,896		

Projected Reserve Need -- All Years Combined
Including Correlation Between Accident Years

Exhibit 5A

Matrix of correlation coefficients $\rho_{k,m}$

	Year	1	2	3	4	5	6	7	8	9	10
						m					
	1	1	0	0	0	0	0	0	0	0	0
	2	0	1	0	0	0	0	0	0	0	0
	3	0	0	1	0	0	0	0	0	0	0
	4	0	0	0	1	0	0	0	0	0	0
k	5	0	0	0	0	1	0	0	0	0	0
	6	0	0	0	0	0	1	0	0	0	0
	7	0	0	0	0	0	0	1	0.5	0.2	0
	8	0	0	0	0	0	0	0.5	1	0.5	0.2
	9	0	0	0	0	0	0	0.2	0.5	1	0.5
	10	0	0	0	0	0	0	0	0.2	0.5	1

Matrix of Covariances (= Correlation Coefficient X Std Dev for Year (Horiz) X Std Dev for Year (Vert))

$$\rho_{k,m} \cdot Var(L_k)^{1/2} \cdot Var(L_m)^{1/2}$$

	Year		1	2	3	4	5	6	7	8	9	10
							m					
		Std. Dev.	*0*	*0*	*179*	*604*	*1,092*	*1,781*	*3,100*	*3,988*	*7,988*	*10,905*
	1	*0*	0	0	0	0	0	0	0	0	0	0
	2	*0*	0	0	0	0	0	0	0	0	0	0
	3	*179*	0	0	32163	0	0	0	0	0	0	0
	4	*604*	0	0	0	364808	0	0	0	0	0	0
k	5	*1,092*	0	0	0	0	1191987	0	0	0	0	0
	6	*1,781*	0	0	0	0	0	3172039	0	0	0	0
	7	*3,100*	0	0	0	0	0	0	9613035	6182736	4953275	0
	8	*3,988*	0	0	0	0	0	0	6182736	15905997	15928784	8698680
	9	*7,988*	0	0	0	0	0	0	4953275	15928784	63806418	43555709
	10	*10,905*	0	0	0	0	0	0	0	8698680	43555709	118928464

Variance of Estimated Reserve Need: 371653280
Standard Deviation of Estimated Reserve Need: 19,278

Variability of Loss Reserves

by Robert L. Brown

VARIABILITY OF LOSS RESERVES

ABSTRACT

This paper addresses the issue of biases in the loss reserving process, some of which may be intentional. Using an empirical analysis of data from 169 companies over a seventeen year period, it is observed that the level of loss reserves exhibits cyclical behaviour, is different for companies of different sizes and is different for reinsurers than for direct insurers. Furthermore, after these factors are accounted for, the differences in levels between individual companies accounts for about three-quarters of the explainable variation.

The paper suggests that greater independence on the part of the loss reserve specialist could lead to more objective estimation and could reduce historical variability by about 40%.

1 Introduction

Loss reserve estimates are made at least annually by one or more loss reserve specialists (now normally actuaries) in each insurance company writing property-casualty business. The degree of difficulty in estimating loss reserves depends on a number of factors. For example, some liability and other lines of business have long average time delays before settlement; the size of the ultimate loss may be positively correlated with the length of such delay; and the size of the ultimate loss may be highly variable.

The size of the loss reserve has an immediate impact on the income statement of the insurer since the year-to-year increase in loss reserves is a direct charge to income in the company's income statement. In theory, the size of loss reserves for a particular block of insurance business in successive years has no impact on the ultimate profitability of that block. The reserve only allocates portions of the profit to the successive years. In practice, however, the size of the loss reserves in successive years may influence the setting of premiums since incurred losses include estimates of loss reserves. Establishing inadequate reserves may lead to inadequate future premium income. Furthermore, a sharp increase in loss reserves has a direct influence on the income statement, shareholder confidence and stock prices.

Because the size of loss reserves affects income and hence taxes and stock prices, it is possible that the loss reserves in financial statements may not be objective estimates of future losses (see, for example, the article by Loomis, 1984). Grace (1990) hypothesizes that insurers are influenced by the desire to maximize earnings each year while maintaining a smooth progression of earnings in order to minimize investor uncertainty. This suggests that a company that is under-reserved in one year is likely to be under-reserved in the next year. In various empirical studies, Anderson (1973), Smith (1980) and Weiss (1985) all found that reserve errors had the effect of smoothing the underwriting income of insurers. This implies that the 'true' underwriting income stream is more volatile than that obtained using reserve estimates.

Because the insurance business is inherently risky and the estimation process is imperfect, variability in loss reserves is inevitable. The objective of this paper is to examine and test empirically various sources of variability, and test the hypothesis that reserve errors are nonrandom.

2 Sources of Reserve Error

The deviation of a reserve estimate from its 'true' value is termed the reserve error. The 'true' value of the reserve established at the end of an accounting year can only be established (or more accurately estimated) after all (or almost all) claims have been settled.

Some errors in reserving may be deliberate. Any attempt to smooth earnings is in this category. Forbes (1970), Smith (1980) and Weiss (1985) all confirmed that financial results for property-casualty companies were consistent with management's deliberate attempt to influence income smoothing through the reserving process. To the extent that a firm should be viewed from an on-going basis, this smoothing is considered a desirable characteristic of the loss reserving process by Pentikäinen and Rantala (1992). Of course, from a break-up perspective, such smoothing would be undesirable.

Deliberate over-reserving may be desirable to make the balance sheet of a company somewhat conservative. Reserves for property-casualty insurers have historically not included discounting of future cash flows even though reserves (or more correctly, assets offsetting reserve liabilities) are invested. The classic argument is that such conservatism in reserving provides a margin of error against deviations in claims experience and against the effect of unanticipated inflation for claims that are not yet settled. This deliberate over-reserving is based on the belief that the balance sheet is more important than the income statement and that solvency considerations are paramount. The amount of any over-reserving plays the same role as surplus, but is hidden and thus protected from distribution to policy-owners or shareholders. Generally accepted accounting principles (GAAP) argue in favor of explicit recognition of the various sources of profit and a more 'accurate' income statement and hence, a more accurate balance sheet. Current tax regulation requires methods consistent with GAAP.

The Conference of Consulting Actuaries (1992) report that for the period 1987-1991, a total of 164 U.S. property-casualty insurance companies were declared insolvent and that the ultimate total surplus deficiency for these companies could reach $5 billion. They report (p.91) that:

> "The stated leading cause of insolvency is 'underreserving'. But in many cases, further analysis will show that this is a symptom rather than a cause. Underreserving can be a form of deferring the real problem. The practice of underreserving can lead to more easily defined causes.
>
> When management recognizes there are serious problems, the easiest immediate solution is to seek justification for lower reserves. The loss reserve is an estimate of future costs for events that have occurred previously. Payments resulting from past events will be made over an extended period of time. The inherent delay in the loss reserve payout is often the basis for deliberate underreserving. In the case of deliberate underreserving, a further cause must usually be sought.
>
> There may also be inadvertent underreserving, because future events may be hard to anticipate, or there may be a lack of understanding of the extent of loss. In instances where events giving rise to liabilities occur over an extended period of time, inadvertent underreserving can indeed be a cause of an insolvency."

As indicated by the last paragraph above, some sources of variation in loss reserves are non-deliberate. There are a variety of sources of such error. Some may be non-random and some may be random. Random errors occur when, for example, more claim-causing events ('accidents') than anticipated occur and when the sizes of losses associated with accidents are different from expected. This is often termed 'stochastic error'.

Another source of error is 'model error'. Model error arises when the underlying mathematical model and associated method of estimation are inconsistent with reality. For example, many ratio-based methods, such as the *chain ladder method* (see, for example, van Eeghen, 1981) assume implicitly that the ratios of paid (or incurred) claims for successive years of development are constant. Estimation error is introduced to the extent that such ratios may vary from accident year to accident year. It is well-known that the chain ladder method under-estimates reserves when the outcomes are stochastic (Stanard, 1985).

Further error is introduced when a model is calibrated. This error is often called 'parameter error' since most models and hence methods are described in terms of parameters that require estimation based on a sample of previous years' data.

Pentikäinen and Rantala (1992) describe reserve volatility when the chain ladder method is used as well as when a premium-based loss reserving method is used. The premium based-method uses a percentage of the earned premium for each accident year in the calculation of reserves. To the extent that premiums are not good predictors of actual claims, model errors are introduced. Because it is well-known that an underwriting cycle exists, one would expect a similar cycle in loss reserves if they are based on premiums. Recently, Lamm-Tennant, Starks and Stokes (1992) analyzed loss ratios recognizing the nature of the cycle. They cite many references to the underwriting cycle. To the extent that there is systematic over- and under-pricing through the cycle, corresponding errors in loss reserves can occur when the methods used in loss reserving are linked to premium income rather than 'true' expected claims.

Stochastic error arises when 'the unexpected occurs'. Although many loss reserve methods are based on models that ignore stochastic variation in claims, there have been several methods that incorporate the stochastic component. Many are described in Taylor (1986) and the Institute of Actuaries' *Claims Reserving Manual* (1989). De Jong and Zehnwirth (1983) describe a general state-space model for loss reserving. This approach is used by Zehnwirth (1985). Verrall (1989a, 1989b, 1990) and Renshaw (1989) describe stochastic versions of the chain ladder method.

Modelling the stochastic error allows for the development of probability statements about the adequacy of a particular level of reserves. These probability statements can take into account both the stochastic error and the estimation error. However, they do not take any account of model error. The estimates are based on the assumption that the model (and hence the method) is appropriate.

Different methods yield different results because they are based on different model assumptions and/or different calibration methods (statistical estimation criteria). Methods are considered robust if they are rather insensitive to the model assumptions. The more robust a method is, the less sensitive it is to systematic variation that needs to be reflected in the reserve estimate. All reserving methods (and statistical estimation procedures) compromise robustness and sensitivity. Pentikäinen and Rantala (1992) try to address the issue of interpreting the different results from different methods, i.e. the model error.

3 The Approach of this Paper

Development of reserve estimates should involve selection of a model, calibration of the selected model and validation of the calibrated model. The estimate of loss reserves is a forecast. For models that include formal assumptions about the variability of the claims process (i.e. stochastic error), estimates of the likely variability of the forecast value can be obtained using statistical theory. In the practice of loss reserving, the forecasts are based on past payments or incurred claims, usually set up in the form of the standard 'runoff triangle'. This is done separately for each line of business. There is no standard way of combining the estimates of variability of various lines of business. If it is assumed that the experience of lines of business are stochastically independent, then the variance associated with a reserve estimate for all lines combined is the sum of the variances associated with each line. When the standard deviation is used as a measure of volatility, it is clear that variability decreases as lines of business are combined. However, if there is a strong positive correlation between lines, this does not hold.

The approach of this paper is to treat the whole company as the business entity, rather than treat each line of business separately. From the point of view of either the income statement (underwriting results) or the balance sheet (solvency) of an insurer, all lines of business are always combined. We will examine historical loss reserves and compare them with the corresponding subsequent runoff of claims. This will allow us to assess the performance of loss reserving in the past and to study the sources of variability in order to better explain the entire loss reserve process.

Since loss reserve specialists use different methods, estimates of variability for that specialist depend on the individual methods used. We will examine variability empirically by studying the variability for all companies combined, for each company over time, and for various subclasses of companies to provide a better explanation of the actual variability that should be anticipated regardless of the methods used.

We believe that it is inappropriate to impose solvency margins developed on a theoretical basis for any specific loss reserving method (e.g. chain ladder). Because of the different nature of claims information for different lines of business and for claims at different stages in the claim settlement process (e.g. incurred but not reported, reported but not yet settled), it is probably best to measure uncertainty of loss reserves on the basis of historical variability. In order to best understand historical variability, empirical studies are necessary.

Previous empirical studies of accuracy of loss reserves have been done by Forbes (1969, 1970), Anderson (1973), Ansley (1979) and Smith (1980), Aiuppa and Treischmann (1987), Grace (1990) and Panjer and Brown (1992) who each studied a collection of companies over a period of time. In particular, Grace examines the desire of companies to maximize firm value through the reserving process considering the tax status of the company. She also considers the desire of companies to smooth income by minimizing the variability of earnings. These are considered deliberate attempts to distort the true income picture of a company. This variation is specific to the individual company.

There are sources of variability that are not specific to a single company. For example, the effect of inflationary increases in costs affects all insurers in a particular line of business such as automobile insurance. Such industry-wide influences should be observable as an annual effect across all companies in historical studies. Ansley (1979) studied the effect of inflation of reserve estimates.

Finally the level of conservatism inherent in reserves also varies from company to company. Small companies writing a few small lines of business have less ability to diversify variability than large companies writing more lines of business over which variability is not (or negatively) correlated. Hence, one would expect smaller companies to hold relatively larger reserves than large insurers.

Similarly, one might expect reinsurers that are part of world-wide reinsurance groups to hold relatively smaller reserves than small domestic insurers who do not have the ability to diversify risk internationally.

4 The Data

In this paper we try to identify the influences of various sources on the loss reserves by examining historical data from a set of 169 companies operating under federal regulation in Canada. This study is significantly larger than any of the cited previous studies, most of which examine only U.S. companies. The data were obtained from the annual statements over the period 1975-1991.

Companies are categorized by 'size' (Small, Medium and Large) on the basis of 1991 premium income, and by 'type' (Domestic or Foreign) on the basis of ownership. Reinsurers are separately identified.

For each year from 1975 to 1986, the aggregate loss reserve for all lines of business (including loss adjustment expenses) for all prior accident years combined is compared with the runoff in the subsequent five years and any remaining loss reserve at that time. The difference is measured as a percentage excess or deficiency (see section 5).

Using the runoff for five years for the most recent accident year means that, for the prior accident years included in the estimate, the runoff will be more than five years old. For Canadian insurers, for most (but certainly not all) lines of business, the vast majority of claims will have been settled within five years of occurrence. The reserve established after five years will still contain some error. However, since the reserve after five years is generally quite small, the error in estimating the 'true' required reserve should also be small relative to the initial error.

The study is conducted for all lines of business combined since the solvency of the company, the value of the firm, and investor and public confidence are dependent on the overall performance of the company. No adjustment is made for discounting since reserves for the period 1975-1986 were established on a basis which ignored discounting.

In an environment in which interest earned on reserves is accounted for, actual loss reserves could be smaller. Similarly, to the extent that there is an implicit offset of interest and future inflation, any inflationary increases in subsequent payments make reserves appear deficient.

The number of companies in each category is given below:

	Domestic	Foreign	Total
Large Insurers	24	25	49
Medium Insurers	23	24	47
Small Insurers	23	21	44
Reinsurers	6	23	29
Total	76	93	169

5 The Model

Let E_i denote the estimate in year i of outstanding losses in respect of all accident years i and prior. Let U_i denote the estimate made in year $i+5$ of outstanding losses at the end of year i for accident years i and prior. In the analysis in this paper, U_i is treated as the 'true' level of outstanding losses at the end of accident year i and E_i is an estimate of this true value. Of course U_i is itself an estimate; but, at year $i + 5$, all accident year values are at least 5 years mature. For lines of business that are not too long-tailed, the estimate at year $i + 5$ will be reasonably accurate, or at least significantly more accurate than the estimate E_i made at year i.

The excess/deficiency of the estimate E_i is defined as $(E_i - U_i)/U_i$. It is measured as a fraction of the 'true' value. For positive values of E_i and U_i, the excess/deficiency only takes on values greater than -1. For the purpose of the statistical analysis described below we transform the excess/deficiency to obtain values taking on all possible values of the real line.

Let $X_i = 100 \log (E_i/U_i)$. Then $E_i = U_i e^{X_i/100}$, resulting in a simple multiplicative model for the estimate E_i. Explanatory variables are now introduced and a statistical analysis of the values of Xi for all 12 years and for all 169 companies in the data is carried out. The explanatory (categorical) variables in the analysis are:

Year:	y_i,	i	=	1975, 1976, ..., 1986	12
Size:	s_j,	j	=	Small, Medium, Large, Reinsurer	4
Type:	t_k,	k	=	Domestic, Foreign	2
Company:	c_l,	l	=	company identifiers	

169
12
338
169
2028

Using a standard analysis of variance (ANOVA) procedure we examine the model

1 12 4 2 48 24 8 96 195

$$X_{ijkl} = \mu + y_i + s_j + t_k + \alpha_{ij} + \alpha_{ik} + \alpha_{jk} + \alpha_{ijk} + \epsilon_{ijkl}$$

where μ is the overall mean level of X_{ijkl}, y_i is the effect of year i, s_j is the effect of size j, t_k is the effect of type k. The quantities α_{ij}, α_{ik}, α_{jk} and a_{ijk} represent the interaction terms of year, type and size. Finally ϵ_{ijkl} represents the residual 'error' and has mean 0 and variance σ^2. It represents that part of X_{ijkl} that cannot be explained by the above mentioned factors and their interactions.

The results of the analysis of variance are shown in Table 1. It shows that, at a 5% significance level:

i) each of year and size are statistically significant explanatory variables;

ii) type is not significant,

iii) only one two-factor interaction is significant, namely type and size; and

iv) the three-way interaction is not significant.

Nonsignificance of some of the interaction terms means that the factor 'year' is independent of size and type. However, the interaction between size and type is significant.

Using the reduced model

$$X_{ijkl} = \mu + y_i + s_j + t_k + \alpha_{jk} + \epsilon_{ijkl}$$

results in the analysis of variance table given in Table 2. From Table 2, it can be seen that the R^2 = 10.7%, meaning that only 10.7% of the total variance can be explained by the effects of year, type and size.

TABLE 1
ANALYSIS OF VARIANCE FOR EXCESS/DEFICIENCY

Source of Variation	Sum of Squares	%	d.f.	F-ratio	Sig. Level
Year	213,846	6.0	11	11.46	0.00
Size	126,675	3.6	3	24.89	0.00
Type	2,886	0.1	1	1.70	0.19
Year Type	12,345	0.3	11	0.66	0.78
Year Size	33,728	1.0	33	0.66	0.96
Type Size	34,494	1.0	3	6.78	0.00
Year Type Size	31,698	0.9	33	0.57	0.98
Model	455,673	12.9			
Error	3,083,677	87.1			
Total	3,539,351	100.0			

TABLE 2
ANALYSIS OF VARIANCE FOR EXCESS/DEFICIENCY

Source of Variation	Sum of Squares	%	d.f.	F-ratio	Sig. Level
Year	213,846	6.0	11	11.65	0.000
Size	126,675	3.6	3	25.31	0.00
Type	2, 886	0.1	1	1.73	0.19
Type Size	34,417		3	6.88	0.00
Model	377,824	10.7			
Error	3,539,351	89.3			
Total	3,539,351	100.0			

TABLE 3
ANALYSIS OF VARIANCE FOR EXCESS/DEFICIENCY

Source of Variation	Sum of Squares	%	d.f.	F-ratio	Sig. Level
Year	213,846	6.0	11	16.00	0.000
Size	126,675	3.6	3	34.76	0.00
Type	2, 886	4.6	1	2.38	0.19
Type Size	34,417	1.0	3	9.44	0.00
Company	1,055,063	29.8	161	5.39	0.00
Model	1,432,888	40.5			
Error	2,106,463	59.5			
Total	3,539,351	100.0			

Table 2 indicates that although the variables 'year', 'type' and 'size' play a significant role in explaining the variation of loss reserves, they collectively explain only 10.7% of the total observed variability.

In order to test the hypothesis that individual companies consistently over- or under-reserve, the variable 'company' was introduced as an additional explanatory variable. The result are given in Table 3.

Table 3 indicates that an additional 29.8% of the total variability can be explained by the variable 'company'. This suggests that individual companies are consistently over- or under-reserved (after account is taken of type and size) year after year. Although some consistent over- or under-reserving should be expected since the reserves in successive years are correlated, the data indicate the degree of consistency is high.

The model in Table 3 explains 40.5% of the variation leaving 59.5% unexplained. This unexplained variation is due to stochastic error and possible non-stochastic error for reasons that are not (but could possibly be) incorporated into a model.

6 The Results

The least squares estimates for 'year', 'size' and 'type' in the model used in Table 3 are given in Table 4. Because of the significant interaction between 'size' and 'type', the (apparently nonsignificant) main effect 'type' remains in the model.

Table 4 indicates that the average level of reserves established during the period 1975-1985 was almost 1% (exp(0.0093)-1) in excess in the level required. It also indicates that there was a dramatic cyclical effect on reserve levels.

Strazewski (1984) reports that the Insurance Services Office estimated that property-liability companies in the United States were under-reserved by 10%. Our analysis, based on data through 1987, indicates that for Canada, reserves in 1982 were 9.3% (exp(0.0093-0.1067)-1) deficient.

Although there was no apparent difference between domestic and foreign insurers overall, statistically significant interactions between 'type' and 'size' arose as a result of the large variations shown.

TABLE 4
RESULTS OF ANALYSIS OF VARIANCE
Means of Main Effects

Overall	**No. of Observations**	**Overall Mean**
	1914	0.93%

Year of Reserve	No. of Observations	Mean Effect
1975	150	-6.58%
1976	145	6.19%
1977	149	19.27%
1978	151	22.63%
1979	156	11.88%
1980	163	-1.02%
1981	165	-5.67%
1982	167	-10.67%
1983	166	-3.73%
1984	167	-7.58%
1985	166	-8.64%
1986	169	-0.53%
Type of Company		
Domestic	835	0.36%
Foreign	1079	1.35%
Size of Company		
Small	486	13.13%
Medium	543	0.50%
Large	577	-1.97%
Reinsurer	308	-12.20%

Means of Interaction Terms

Type	**Size**	**No. of Observations**	**Mean**
Domestic	Small	218	3.88%
	Medium	265	-0.15%
	Large	284	1.66%
	Reinsurer	68	-14.39%
Foreign	Small	268	20.65%
	Medium	278	1.13%
	Large	293	-5.49%
	Reinsurer	240	-11.58%

We have not shown the 169 individual company effects after accounting for the group effects. Listing the companies and their individual effects serves no useful purpose for us. However, knowledge of the individual effects can be very useful to individual companies' managements and loss reserve specialists and others, as discussed below.

7 Conclusions

The results of this study show that a significant amount (40.5%) of the variability can be easily explained. First, reserve levels for property-casualty companies follow a cyclical pattern. Furthermore there are general differences in reserve levels for companies of different sizes and between direct insurers and reinsurers.

Almost three-quarters of the explained variation comes from the individual companies, irrespective of type, size, or year of valuation. The most important observation is that the individual loss reserve specialist in a given company has consistently over- or under-reserved.

This suggests that any efforts by managements, professions or regulatory authorities should be aimed at the individual company level. Consistent under- or over-reserving may be a result of intentionally trying to improve the apparent financial situation of the company. Greater independence of the loss reserve specialist may provide more objective estimates. The 'appointed actuary' position created through the new act governing insurers and other financial institutions in Canada may improve the situation.

Similarly a tendency for reserve excesses and deficiencies to follow a cyclical pattern suggests that insurers strengthen reserves when they can afford it. This is inconsistent with an objective assessment of loss reserves. Again, independence of the loss reserve specialist may help this situation. Finally, methods of loss reserves that are linked to loss ratios would appear to be inappropriate since premiums are subject to cyclical behaviour as a result of competitive pressure.

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A Method to Estimate Probability Level for Loss Reserves

by Roger M. Hayne

A METHOD TO ESTIMATE PROBABILITY
LEVELS FOR LOSS RESERVES

by

Roger M. Hayne

Abstract

This paper explores the collective risk model as a vehicle for estimating the probability distribution for reserves. Though this basic model has been suggested in the past and it provides a direct means to estimate process uncertainty, it does not directly address the potentially more significant problem of parameter uncertainty. This paper presents some techniques to estimate parameter uncertainty and, to some extent, also uncertainty regarding projection model selection inherent in reserve estimates.

A METHOD TO ESTIMATE PROBABILITY

LEVELS FOR LOSS RESERVES

1. Introduction

The collective risk model, see for example Beard, Pentikäinen and Pesonen [1], provides a conceptually simple framework to model total claims in the insurance process. In its simplest form this model calculates the total loss from an insurance portfolio as the sum of N random claims chosen from a single claim size distribution where the number N is itself a random variable. With some fairly broad assumptions regarding the number and size of claims we can draw conclusions regarding the various moments of distribution of total claims. Thus this model seems to be a reasonable choice as a starting point in estimating the distribution of reserves for an insurer.

The distribution resulting from this simple collective risk model provides an estimate of the potential variation in total payments assuming all distributions are correct. We often refer to this variation as process variation, that inherent due to the random nature of the process itself. Not directly addressed in this simple collective risk model is the possibility that the estimates of the parameters for the underlying distributions, are incorrect. Variation due to this latter uncertainty is often called parameter variation.

Parameter variation is itself an important aspect in assessing the variability inherent in insurance related estimates. Meyers and Schenker [2] discuss this aspect of collective risk applications. They conclude, not surprisingly, that for a "large" volume of claims, that expected to be experienced by most insurers, parameter uncertainty is a much more significant contributor to overall variability than the random, or process, portion.

As indicated above, the collective risk model does not directly address parameter uncertainty nor does it address the methodology used in obtaining reserve estimates themselves. In practice actuaries often apply several methods, based on different underlying assumptions, to derive different projections of required reserves. The actuary then selects a "best estimate" of required reserves, based on the various projections used, keeping in mind the nature of the data and the assumptions inherent in each of the methods. Complicating matters further is the fact that most of the generally accepted actuarial projection methods currently in use are not stochastic in nature, that is, they do not have specific assumptions regarding underlying probability distributions. Thus, in many cases, they only provide "point estimates" without any indication as to the statistical nature of those estimates.

Even if the actuary uses stochastic methods, methods that make assumptions regarding the underlying distributions, the result will usually be a single distribution of total losses or reserves. It is possible that different methods may lead to different estimates of the distribution of reserves. This raises another area of uncertainty that should be considered in estimating probability levels for loss reserves; that of uncertainty that the model applied is indeed the correct one. This is sometimes termed specification uncertainty.

Though many of the stochastic methods we have seen attempt to provide estimates of process variation and sometimes even parameter variation within the framework of the particular model those methods do not provide a convenient means of measuring the possibility that the model itself may be incorrect. Even regression related approaches with regimens in selecting which independent variables to include can only claim to provide the "best" estimate within a particular family of models and do not generally address whether another family is indeed better for a particular situation.

For these reasons this paper will deal with an application of collective risk theory to estimate probability levels in loss reserves. Though the method that we present follows the general approach described in

Hayne [3] we cover ground not covered there, especially in the area of estimating the impact of parameter uncertainty in probability levels.

2. The Collective Risk Model

The basic collective risk model, as described above, can probably be seen best as the implementation of the following algorithm:

Algorithm 2.1

1. Randomly select N, the number of claims.
2. Randomly select N claims, $X_1, X_2, \ldots, X_N$ from the claim size distribution.
3. Calculate aggregate loss as $T = X_1 + X_2 + \ldots + X_N$.
4. Repeat steps 1 through 3 "many" times.

The distribution of T then represents the distribution of total losses given the distributions of the individual claims X_i and the distribution of N, the number of claims. Assuming these distributions are correct the result of this algorithm provides an estimate of the inherent process variation. It does not, however, provide a means of incorporating parameter uncertainty.

We will follow Heckman and Meyers [4] and consider a revised collective risk algorithm that incorporates parameter uncertainty in both the claim count and claim size distributions. We assume that the number of claims N has a Poisson distribution with mean λ, and hence variance $\mathrm{Var}(N) = \lambda$. We also assume that χ is a random variable with $E(\chi) = 1$, and $\mathrm{Var}(\chi) = c$. The variable χ then will be used to reflect the uncertainty with the selection of the expected claim count parameter λ. If χ is assumed to have a

Gamma distribution then Heckman and Meyers show that the resulting N will have a negative binomial distribution with

$$\mathrm{E}(N) = \lambda, \text{ and}$$
$$\mathrm{Var}(N) = \lambda + c\lambda^2$$

In this case $\mathrm{Var}(N) \geq \mathrm{E}(N)$, with equality only if $c = 0$.

As Heckman and Meyers point out, the Poisson distribution assumes that claims during two disjoint time periods are independent, that the expected claims in a time interval is dependent only on the length of the interval and not on the starting point of that interval and that no more than one claim can occur at a time. They introduce the contagion parameter c to allow for dependence of the number claims in one time interval on claims in prior interval(s). The above modification with $c > 0$ assumes that the number of claims in one interval is positively correlated with the number in past intervals. For example, a successful liability claim may lead to an increased number of future claims.

Similarly it is possible that the existence of past claims may decrease the possibility of future claims. An example that Heckman and Meyers point out in this situation is with a group of life insurance policies where claims in an earlier period reduces the number of claims in a later period. They model this by assuming that the final claim count distribution will be Binomial. In this case $\mathrm{Var}(N) < \mathrm{E}(N)$, which can be accomplished with an appropriate negative value for c, even though a negative value does not make sense in the original derivation of the distribution for N. We will thus assume that N has either a Binomial distribution ($c < 0$), a Poisson distribution ($c = 0$), or a Negative Binomial distribution ($c > 0$).

The modification of Algorithm 2.1 also reflects uncertainty in the overall mean of the claim size distribution. For this we assume that β is a random variable with $\mathrm{E}\left(1/\beta\right) = 1$ and $\mathrm{Var}\left(1/\beta\right) = b$. With these added distributions Heckman and Meyers present the following modified collective risk algorithm:

Algorithm 2.2

1. Randomly select a number N from the assumed claim count distribution.

2. Select N claims $X_1, X_2, \ldots, X_N$ from the assumed claim size distribution.

3. Randomly select a number β from the assumed distribution.

4. Calculate the aggregate loss as $T = \frac{1}{\beta}(X_1 + X_2 + \ldots + X_N)$.

5. Repeat steps 1 through 4 "many" times.

We note that in the case that $b = c = 0$, that is, no parameter uncertainty, Algorithm 2.2 simply reduces to Algorithm 2.1 with an assumed Poisson claim count distribution.

Following Heckman and Meyers we will assume that β has a Gamma distribution. We follow their caution that this is selected for its mathematical convenience rather than for a specific property of parameter uncertainty. We refer readers to page 31 of [4] for a further discussion of this assumption.

The collective risk model has some useful properties, for example, if we know the moments of the claim count and claim size distributions, assuming independence of the various distributions, we can determine the corresponding moments of the final aggregate distribution. These properties hold for both the formulation in Algorithm 2.1 and the formulation in Algorithm 2.2. In particular under the above conventions we have:

$$\mathrm{E}(T) = \lambda \mathrm{E}(X) \tag{2.1}$$

$$\mathrm{Var}(T) = \lambda \mathrm{E}(X^2)(1+b) + \lambda^2 \mathrm{E}^2(X)(b+c+bc) \tag{2.2}$$

Since Algorithm 2.1 is a special case of Algorithm 2.2 with $b = c = 0$, equations (2.1) and (2.2) will still hold. In this case, however, the last term in the formula for Var(T) disappears and equation (2.2) becomes:

$$\mathrm{Var}(T) = \lambda \mathrm{E}(X^2) \tag{2.3}$$

The difference between these two variance equations is notable. In the case of equation (2.3), the variance of the average claim, i.e. $\mathrm{Var}(T/\lambda)$, will approach 0 as λ gets large. However, in the case of equation (2.3), if either b or c is non-zero, $\mathrm{Var}(T/\lambda)$ approaches $\mathrm{E}^2(X)(b+c+bc)$. Thus introduction of parameter uncertainty introduces uncertainty in the average that cannot be overcome by increasing the number of claims, or by diversifying the risk. In financial terms, parameter uncertainty in this manner introduces undiversifyable risk.

Heckman and Meyers present an algorithm for approximating the distribution of T in the case that the cumulative density function for the claim size distribution is a step function. Since any smooth function can be approximated within any required tolerance by a step function, this is not a restrictive assumption. We will use that algorithm in the method presented here.

3. Point Estimates of Reserves

Exhibit 1 presents summaries of various medical malpractice loss statistics that were derived from the data used by Berquist and Sherman [5]. To keep the numbers to a manageable size, all losses and claim counts in that paper were divided by 10 and the dates were changed to make the exhibits here appear more current. In addition, page 2 of Exhibit 1 shows projected ultimate reported claims. This projection is based on a development factor method applied to reported counts using volume weighted averages as selected factors. Though the data are hypothetical, they do reflect characteristics of actual loss data.

In addition, we included another example of our calculations and estimates of probability levels in the appendix to this paper. That example is based on the data set used in the Advanced Case Study session of the 1992 Casualty Loss Reserve Seminar.

As pointed out by Berquist and Sherman a comparison of the trends in average case reserves and average loss payments, as shown in Exhibit 2, indicates a potential change in relative reserve adequacy. This change, if it is occurring, could affect the incurred loss projections.

In addition, reference to ratios of closed to projected ultimate claims, as shown in Exhibit 3, seems to indicate a change in the rate at which claims are being closed. This could affect projections based on paid losses.

Since there appear to be occurrences that can influence forecasts based on either paid or incurred data we considered two sets of forecasts; one based on the data shown in Exhibit 1 without any adjustment and the second based on data adjusted in an attempt to remove the influences of these apparent changes. The resulting adjusted paid and incurred loss data appear in Exhibit 4.

We used methods similar to those presented in [5] to adjust the paid losses for apparent changes in the rate of claims closing. We calculated the adjusted incurred as the sum of the adjusted paid losses plus the product of adjusted average reserves times adjusted claims open. We calculated the adjusted reserves as suggested in [5].

Exhibit 3 also shows the triangle of adjusted closed claims. We obtained this triangle as the product of the forecast ultimate reported claims for an accident times the most recent percentage of ultimate claims closed at that particular valuation point. For example, the estimate of 210 claims closed for 1989 at 36 months is the product of 42.3%, the percent of ultimate closed at 36 months for the most recent accident year (1990) times 497, the projected ultimate claims for 1989.

We used four different projection methods on each set of data; paid loss development, incurred loss development, a severity projection method and a hindsight average outstanding loss method. In both of the development factor methods we used an exponential curve fit to the difference of selected development factors minus 1 to estimate development after 96 months. In the severity projection method we reviewed the average costs per ultimate claim and inherent trends in those averages at the various stages of development to "square the triangle" of average payments, see, for example [5] for examples of this technique.

For the hindsight average outstanding loss method we calculated the average unpaid loss per open and incurred but not reported (IBNR) claim at various stages of development. We calculated these averages as the ratios of the difference of initial forecast ultimate losses minus paid losses to date divided by the difference of forecast ultimate claims minus claims closed to date. We used the unweighted average of the other three projections as the initial selection in this case. We then reviewed these averages and inherent trends at each stage of development and selected a representative average for the accident year currently at that age. We then used the product of that average and the number of open and IBNR claims as an estimate of the future payments for that year. Our ultimate loss projection for this method was then the sum of this outstanding loss estimated and the amount paid to date.

Exhibit 5 then shows a summary of the various projections and our weighted average selection, based on the weights shown in the bottom portion of that exhibit. We judgmentally selected the weights shown but they reflect our view of the extent that the hypotheses of the indicated projection method fit with what has been occurring in the data.

We recognize that these methods and selections are based on judgment and that different actuaries may have different opinions than we do. However, we believe that the method to estimate variation that we will present is sufficiently adaptable to accommodate different selections or even different underlying forecasting methods.

If we had estimates of the variances of the different projection methods another weighting presents itself. If we assume the various projections are independent then the weighted average with the least variance is that which assigns a weight to a random variable proportional to the inverse of its variance. This is intuitively appealing since, in this case, uncertain projections, identified by high variances, are given relatively less weight than more precise ones.

4. Estimate of Process Variation

We will estimate the process variation, that which is due only to random fluctuation, using the unadjusted collective risk model as described in Algorithm 2.1. Later we will examine an approach to include parameter uncertainty in the estimates and to use Algorithm 2.2.

Since we will be using the collective risk model we will need estimates of the distributions of the number of claims and of the size of individual claims. We will use the results of our reserve forecasts as a starting point.

Columns (1) through (7) of Exhibit 6 shows the calculation of indicated reserves and resulting indicated average loss per outstanding and IBNR claim by accident year. We will assume that the total outstanding claims have a lognormal distribution and that the loss data, and corresponding reserves, represent losses at $500,000 policy limits. We make these assumptions to maintain simplicity in the presentation. In practice the actuary will need to make appropriate estimates for these distributions.

We have also selected the coefficient of variation (ratio of the standard deviation to the mean) for the lognormal distribution, as shown in column (8). Though the selections here are judgmental they are based on two assumptions:

1. In ratemaking for this line of business we have selected a lognormal distribution with a coefficient of variation of 5.0 in calculating our increased limits distributions.

2. As time progresses the book of open and IBNR claims become more homogeneous and thus we would expect the coefficient of variation to decrease.

In practice we would have to derive estimates for these parameters too. One approach would be to consider the distribution of open and IBNR claims at various stages of development for older accident years that are completely, or at least nearly completely, closed out. Such a review would provide better insight in the selection of the coefficient of variation.

We have selected a lognormal distribution here primarily for its computational convenience. All of the concepts we will present will apply for most commonly used claim size distributions, though some of the specific formulae we will use may need to be modified.

Also, for convenience, we will assume that open claims and IBNR claims have the same claim size distribution and that they are independent. A potential refinement would be to separately estimate the distributions for open and IBNR claims. Again, this could be accomplished by reviewing distributions for older accident years, but we will not explore this further here.

There may be some argument with the assumption of independence. It is possible that settlement of open claims, and resulting precedent, may influence the distribution of IBNR claims, or even that of other open claims. The inclusion of the mixing parameter by Heckman and Meyers will essentially affect all claims in the same way, adjusting the aggregate losses either up or down uniformly, thereby building in some dependence. We recognize that notwithstanding the use of a mixing parameter our assumptions may slightly understate the spread of reserves if the distributions for open and IBNR claims are not independent.

Columns (9) and (10) of Exhibit 6 show the μ and σ parameters for the selected lognormal distribution. In this case we selected the following parameterization for the lognormal probability density function:

$$f(x)=\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}$$

With this parameterization, if X is the lognormal variable, μ and σ represent the mean and standard deviation respectively of the normal distribution of ln(X). In addition, the coefficient of variation (c.v.) for the unlimited distribution and expected loss limited to L respectively are given by:

$$\text{c.v.}=\sqrt{e^{\sigma^2}-1}$$

$$E(X|L)=e^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\frac{\ln L-\mu}{\sigma}-\sigma\right)+L\left[1-\Phi\left(\frac{\ln L-\mu}{\sigma}\right)\right]$$

Here $\Phi(X)$ denotes the probability that a standard normal variable will not exceed X. This and other formulae regarding the lognormal distribution can be found in [6] among other sources. We solved the first of these equations directly for σ. Given σ, then, we used numerical methods to estimate the value of μ that would yield a mean limited to $500,000 equal to the selected average reserve shown in column (7). Many commercially available software and spreadsheet packages contain such algorithms, one could also write a simple algorithm using interval halving since the function $E(X|L)$ is an increasing function of μ for a fixed L.

Exhibit 7 shows the selected step function approximations for the claim size distributions. Since these distributions will be used as input for the Heckman and Meyers algorithm, the probability for an indicated amount does not correspond to the probability that the limited mean will not exceed that amount. Rather these represent step function approximations for the lognormal distribution which have means equal to the expected limited losses.

We will assume that the number of open claims is certain, that is, it has 0 variance. This is equivalent to a contagion parameter $c = -1/\lambda$. We will assume that the IBNR claims have a Poisson distribution. Claims that close without payment may add some technical complexity to the selection of these distributions. We can include this in a number of ways. Probably the most straight-forward would be to include a positive probability of \$0 losses in the claim size distribution. We note that the positive probability of a \$0 loss may present problems with the algorithm presented in [4]. This practical problem can be overcome by using a small loss amount such as \$0.01 instead of \$0 for the claim size distribution input. Again, in order to keep these discussions relatively simple we will not make this refinement here, although the example we present in the appendix to this paper does deal with such a situation.

Another potentially complicating factor with these assumptions is the presence of reopened claims. We have assumed that the claim count data includes a reopened claim as a separate count and we have thus included provision for reopened counts in our estimates for IBNR claims. Again, we could adjust the claim count distribution for open claims to accommodate reopens. Another option would be to model reopened claims separately, similar to the way we treat IBNR claims.

We note another option in representing the combined distribution of open and IBNR claims. Let λ_O denote the number of open claims and λ_I the number of expected number of IBNR claims. We have assumed that the number of open claims is certain and that the number of IBNR claims has a Poisson distribution. Then the number of combined claims has mean $\lambda_O + \lambda_I$ and variance λ_I. We see that a claim distribution with mean $\lambda_O + \lambda_I$ using contagion parameter

$$c = -\frac{\lambda_O}{(\lambda_O + \lambda_I)^2}$$

will also have variance equal to λ_I. This is one potential short-cut in the calculations. If one assumes that open and IBNR claims have the same distributions then this assumed claim count distribution could replace the two separate distributions in the calculations.

We note, however, that this value of c is negative, resulting in the use of a binomial distribution which has a maximum number of possible claims. This may be undesirable in applications. However, we calculated aggregate loss distributions using both this single distribution and using separate distributions for open and IBNR claims and we found no discernible difference in the results.

Making use of the algorithm in [4] we calculated the resulting distribution of aggregate reserves for each accident year separately. We then used the same algorithm to calculate the aggregate distribution for all years combined, using the output of the algorithm to estimate the aggregate reserves for individual accident years. In this case we assumed 1 "claim" and used contagion factors of -1 for each year (implying a zero claim count variance) to estimate the distribution for aggregate reserves.

The user of this algorithm should be aware that the output provides estimates of the value of the cumulative density function at selected values of the aggregate reserves. These correspond to the valuation of that function at those points. Though this is valuable information, it does not directly provide a step function approximation to the aggregate reserve function that maintains expected values. We thus modified the output, similar to the modification for the individual claim size distributions, to obtain better step function approximations to the indicated cumulative density function before using them as input for the final calculations.

Exhibit 8 shows the estimated distribution of aggregate reserves for each accident year and for all accident years combined. To facilitate comparison between the years we show the estimated probability levels for various multiples of the expected values (shown in the first line). Heckman and Meyers refer to these ratios as "entry ratios."

As can be seen from this exhibit, the distributions of reserves for earlier accident years appear to be more disperse than those for later years. In addition, the distribution of aggregate reserves for all accident years is quite tight. This is a result of the law of large numbers. Even with this substantial narrowing of the ranges, in this case random fluctuation alone could result in reserves of more than 110% of the expected value approximately 5% of the time, with an approximate 0.1% chance of exceeding 120%. In this case roughly 90% of the aggregate reserve distribution falls between ±10% of the expected value. We stress that only accounts for random fluctuations assuming all our hypotheses are correct. We have not yet addressed uncertainty in these assumptions.

5. Estimate of the Contagion Parameter

We first address uncertainty in the expected claim count parameter, λ. For this we consider projected ultimate frequencies by accident year as shown in Exhibit 9. A review such as this may be conducted in conjunction with a periodic rate review and all factors considered in such a review should be included in these projections. Here we selected an average annual frequency trend of 2.3% as indicated by an exponential fit through the frequencies for all years.

Assuming that 1993 will have an estimated 8,700 earned exposures column (6) shows the indicated 1993 claims assuming the respective historical frequencies, adjusted to 1993 level using the 2.3% assumed trend. We see that this results in an average of 516 claims per year with an unbiased estimate of the variance of 3,158 as compared with the expected variance of 516 if the distribution were Poisson. We thus assume a contagion parameter of 0.0099 by solving the equation $3{,}158 = 516 + c \times 516^2$ for c. We will then assume that the distributions of IBNR claims for all accident years have this same factor to reflect parameter uncertainty.

6. Estimates of Mixing Parameters

Returning to our ultimate loss, and hence reserve, selections described in section 3 (*Point Estimates of Reserves*) we note that our selected weights can be thought of as providing our subjective judgment regarding the likelihood that the underlying assumptions for the various methods are met in this particular data set. This may be thought of as a form of Bayesian á-priori probability estimate.

Following this thought, we can calculate the variance of the projection methods about the weighted average, using the same weights as used in the selections. In particular, if, for a fixed accident year, Z_i denotes the projection for method i and w_i denotes the relative weight given to method i then our selection and corresponding variance can be calculated as:

$$E(Z) = \sum_{i=1}^{n} w_i Z_i$$

$$\text{Var}(Z) = \sum_{i=1}^{n} w_i \left(Z_i - E(Z)\right)^2$$

These estimates are shown in column (8) of Exhibit 10. If we then assume that the methods that we applied consider all different sets of alternative hypotheses then the variance in the methods is an indication of the overall variance of the estimates, and hence reserves, for a particular year.

As indicated above, we can explain a portion of the variance experienced by process variation and in uncertainty in the claim counts. In particular, using formula (2.2) separately for open and IBNR claims we derive:

$$\begin{aligned}\mathrm{Var}(Z_O) &= \lambda_O\left(\mathrm{E}\left(X_O^2 \mid L\right) - \mathrm{E}^2\left(X_O \mid L\right)\right)\\ \mathrm{Var}(Z_I) &= \lambda_I \mathrm{E}\left(X_I^2 \mid L\right) + c\lambda_I^2 \mathrm{E}^2\left(X_I \mid L\right)\end{aligned} \qquad (6.1)$$

The first of these equations assumes a contagion parameter $c = -1/\lambda_O$, and both follow directly from equation (2.2) with b = 0. With our assumption that the reserves for open and IBNR claims are independent then the total variance is the sum of the variances.

Columns (1) through (5) of Exhibit 10 summarize estimates from Exhibits 1 and 6. Column (6) shows the value of $\mathrm{E}\left(X^2 \mid L\right)$ using the following formula (see, for example, [6]):

$$\mathrm{E}\left(X^2 \mid L\right) = e^{2\sigma^2+2\mu}\Phi\left(\frac{\ln L - \mu}{\sigma} - 2\sigma\right) + L^2\left[1 - \Phi\left(\frac{\ln L - \mu}{\sigma}\right)\right]$$

Using these values and equations (6.1) we calculated the amount of variance that can be explained by process variation and the contagion parameter. This explained variance is shown in column (7).

As can be seen there, the explained variance exceeds the variance in the selection in accident years 1985 and 1986, but is less for the other years. Thus there is variance in the projections that is not explained by process variation or by uncertainty in the claim count projections. We will assume that this remaining uncertainty is explained by a non-zero mixing parameter, b. For this, we solve the following equation for b:

$$Var(T) = Y + b\left[\lambda_o E\left(X_o^2 \middle| L\right) + \lambda_o(\lambda_o - 1)E^2\left(X_o \middle| L\right) + \lambda_i E\left(X_i^2 \middle| L\right) + \lambda_i^2(c+1)E^2\left(X_i \middle| L\right)\right] \quad (6.2)$$

Where Var(*T*) denotes the variance in selected in column (8) and *Y* denotes the explained variance in column (7). Column (9) shows the resulting b values. The b values we selected to estimate uncertainty in the expected value are shown in column (10).

We note that the indicated b parameter increases from 1985 through 1991 but decreases in 1992. This is primarily due to the decrease in the variance in the selected between 1991 and 1992 because of the wider range of forecasts for 1991 than 1992. Though it may seem counterintuitive for parameter uncertainty to decrease, it is possible that the wider range in 1991 may indicate that changes that appear to have influenced the 1991 forecasts more.

These b parameter estimates provide for parameter uncertainty regarding severity within each accident year. As yet unanswered is the question of uncertainty affecting all accident years. For this we chose an approach similar to that taken in estimating the c parameter.

As is often done in ratemaking applications, we used the trend inherent in the historical pure premiums to adjust historical pure premiums to present separate "observations" of 1993 pure premiums. We then used the variation inherent in these "observations" as an indication of the amount of overall uncertainty we have in the 1993 severity estimate. We then assumed, as in our estimates of the contagion parameter, that this uncertainty will apply to our total reserve estimates for historical years.

Calculations shown in Exhibit 11 derive estimates similar to those in Exhibits 9 and 10. Column (1) shows the limited severity implied by our projections while column (2) simply repeats our assumption that the losses will have a coefficient of variation of 5.0. Of course, if there were reason to believe that this coefficient will change over time we could modify the values in column (2). Column (3) then shows the unlimited severity for a lognormal distribution with the coefficient of variation shown in column (2) that would yield the severities limited to $500,000 shown in column (1).

Column (4) shows our selected frequency as shown in Exhibit 9 and column (5) shows the indicated unlimited pure premium. We then calculated an annual pure premium trend of 18.6% based on all observations of unlimited pure premiums in column (5). Similar to the analysis in Exhibit 9 we adjusted these observed pure premiums to our expected 1993 level using this indicated 18.6% trend. We elected to base our projections on the unlimited pure premium due to the damping effects of a fixed limit on limited severities.

We note that the usual arguments of additional variability in the unlimited averages that are cited as a reason for basing ratemaking analysis on limited data do not necessarily apply here. Since the unlimited loss estimates are based on the limited losses and a smooth distribution that does not change drastically from year to year, there is little additional fluctuation introduced in considering unlimited losses in this case.

Column (7) then shows the various indications of 1993 total losses, using the assumed 8,700 exposures as used in Exhibit 9. Using the estimated 516 claims for accident year 1993 from Exhibit 9, we derive the indicated unlimited severities shown in column (8). Column (9) then shows the resulting 1993 level severities limited to $500,000 per claim, again using the lognormal distribution, the coefficients of variation in column (2) and the unlimited means in column (8).

Finally the various observations of indicated 1993 total limited losses are shown in column (10). Based on these observations we expect $13,054 thousand in losses in 1993 with a variance of 3,082,167 million, assuming the observations are independent. This corresponds to an average of $25,298 per claim limited to $500,000 and an unlimited average of $29,346. This latter amount is the unlimited severity necessary for a lognormal distribution with coefficient of variation 5.0 to have a mean limited to $500,000 equal to $25,298.

These assumptions, including our selected contagion parameter, then result in an expected variance of 4,027,361 million. This in turn results in a negative value for b when we solve equation 6.2. Thus we conclude that our assumptions are sufficient to account for observed variation in these estimates and we will select an overall b parameter equal to zero.

As with calculations without parameter uncertainty, we calculated the aggregate distributions for reserves for each year separately. In this case we used the selected contagion parameter and selected b parameters shown in Exhibit 10. We then convoluted the resulting distributions with a mixing parameter set to zero.

Similar to Exhibit 8, Exhibit 12 shows the estimated distributions of reserves including these estimates of parameter uncertainty. Comparing these two exhibits shows the significant impact of including parameter uncertainty as described here. For example, without parameter uncertainty 97% of the estimated 1991 reserves fall within 30% of the expected value whereas less than 56% fall in this range if parameter uncertainty is included.

A similar observation, though not as dramatic, also holds for the aggregate distributions. Without parameter uncertainty 90% of losses are within 10% of the expected. With parameter uncertainty only 51% of the losses are in that range. Another comparison shows that the 90% probability level is approximately $45 million without parameter uncertainty but is approximately $50 million when parameter uncertainty is considered. Exhibits 12 and 13 graphically show this comparison for the cumulative density functions and probability density functions respectively.

7. Conclusions

Now that our presentation is complete, we once again point out that the methodology we presented does not depend on the choice of the underlying claim size distribution, nor does it require the use of the same distributions for both open and IBNR claims. Of course, calculations of the limited mean and variance would change with different claim size distributions but all concepts and methodology still apply.

We note that this methodology attempts to recognize uncertainty arising from the process, in the selection of parameters, and, to some extent, in the selection of reserve forecasting model. We also recognize that much more work is necessary before we have a comprehensive approach to measure all these sources of uncertainty. However, echoing, Meyers and Schenker, we conclude that parameter uncertainty can be have a significant impact on the distribution of reserves.

EXAMPLE MEDICAL MALPRACTICE DATA

Incurred Losses

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	$290	$516	$1,071	$1,461	$1,666	$2,090	$2,289	$2,351
1986	483	1,071	1,691	2,284	2,621	3,197	3,222	
1987	546	1,194	2,073	3,093	4,240	4,838		
1988	873	1,863	3,214	5,720	6,114			
1989	1,123	1,997	5,014	7,373				
1990	871	3,346	6,348					
1991	1,293	4,890						
1992	1,579							

Cumulative Paid Losses

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	$13	$41	$144	$299	$447	$818	$1,264	$1,582
1986	4	53	202	364	752	1,430	1,898	
1987	30	115	248	507	1,140	1,771		
1988	5	79	381	977	1,852			
1989	21	83	360	1,129				
1990	17	159	627					
1991	21	157						
1992	21							

<u>NOTE:</u>
1. All dollar amounts are in thousands.

EXAMPLE MEDICAL MALPRACTICE DATA

Reported Claim Count

Accident Year	Months of Development 12	24	36	48	60	72	84	96	Projected Ultimate
1985	107	168	219	252	256	259	261	263	263
1986	102	185	231	269	275	278	280		282
1987	130	251	314	375	387	392			398
1988	135	273	352	421	446				458
1989	138	283	367	467					497
1990	136	277	362						463
1991	155	279							459
1992	160								500

Cumulative Closed Claim Count

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	32	84	119	137	153	182	208	227
1986	36	89	116	134	165	202	226	
1987	42	118	142	195	244	286		
1988	31	117	169	232	294			
1989	29	144	213	279				
1990	33	135	196					
1991	41	132						
1992	40							

Exhibit 2

COMPARISON OF AVERAGE PAYMENT AND AVERAGE RESERVE TRENDS

Average Reserve per Open Claim

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	$3,693	$5,655	$9,270	$10,104	$11,835	$16,519	$19,340	$21,361
1986	7,258	10,604	12,948	14,222	16,991	23,250	24,519	
1987	5,864	8,113	10,610	14,367	21,678	28,934		
1988	8,346	11,436	15,481	25,095	28,039			
1989	10,110	13,770	30,221	33,213				
1990	8,291	22,444	34,464					
1991	11,158	32,197						
1992	12,983							
Indicated Trend	15.6%	29.5%	31.1%	34.3%	32.7%	32.3%	26.8%	

Average Payment per Closed Claim

Accident Year	Months of Development 0 - 12	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96
1985	$402	$539	$2,971	$8,620	$9,199	$12,669	$17,084	$16,634
1986	110	919	5,487	9,129	12,403	18,452	19,533	
1987	706	1,115	5,644	4,928	12,994	14,948		
1988	161	862	5,782	9,477	14,085			
1989	724	541	4,003	11,709				
1990	518	1,394	7,635					
1991	517	1,494						
1992	525							
Indicated Trend	12.9%	12.0%	11.5%	6.7%	14.2%	8.6%	14.3%	

Exhibit 3

EXAMPLE MEDICAL MALPRACTICE DATA

Ratios of Closed to Projected Ultimate Claims

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	12.2%	31.9%	45.2%	52.1%	58.2%	69.2%	79.1%	86.3%
1986	12.8%	31.6%	41.1%	47.5%	58.5%	71.6%	80.1%	
1987	10.6%	29.6%	35.7%	49.0%	61.3%	71.9%		
1988	6.8%	25.5%	36.9%	50.7%	64.2%			
1989	5.8%	29.0%	42.9%	56.1%				
1990	7.1%	29.2%	42.3%					
1991	8.9%	28.8%						
1992	8.0%							

Adjusted Cumulative Closed Claim Count

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	21	76	111	148	169	189	211	227
1986	23	81	119	158	181	203	226	
1987	32	115	168	223	256	286		
1988	37	132	194	257	294			
1989	40	143	210	279				
1990	37	133	196					
1991	37	132						
1992	40							

Exhibit 4

EXAMPLE MEDICAL MALPRACTICE DATA

Cumulative Paid Losses Adjusted for Closure Rates

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	$2	$34	$111	$396	$623	$919	$1,317	$1,582
1986	0	32	210	620	966	1,417	1,898	
1987	7	106	354	817	1,287	1,771		
1988	6	123	554	1,272	1,852			
1989	24	82	337	1,129				
1990	19	153	627					
1991	12	157						
1992	21							

Incurred Losses Adjusted for Closure Rates and Reserve Changes

Accident Year	Months of Development 12	24	36	48	60	72	84	96
1985	$422	$1,315	$1,962	$2,371	$2,227	$2,450	$2,383	$2,351
1986	443	1,697	2,417	3,044	2,959	3,304	3,222	
1987	640	2,610	3,663	4,634	4,481	4,838		
1988	733	3,108	4,671	6,008	6,114			
1989	861	3,490	5,042	7,373				
1990	991	4,185	6,348					
1991	1,344	4,890						
1992	1,579							

NOTE:
1. All dollar amounts are in thousands.

Exhibit 5

EXAMPLE MEDICAL MALPRACTICE DATA

Ultimate Loss Projections

Accident	Unadjusted Methods				Adjusted Methods				Weighted
	Development		Severity	Hindsight	Development		Severity	Hindsight	
Year	Incurred	Paid	Projection	Method	Incurred	Paid	Projection	Method	Average
1985	$2,414	$2,300	$2,300		$2,351	$1,902	$1,901		$2,242
1986	3,399	3,454	3,354		3,180	2,741	2,674		3,075
1987	5,317	4,536	4,865		4,649	3,519	3,714		4,279
1988	7,979	8,149	7,586	$6,797	6,438	5,254	5,249	$5,413	5,806
1989	11,222	9,697	9,818	8,862	7,631	4,878	6,107	6,430	6,783
1990	14,746	13,215	11,247	11,049	8,671	7,326	6,877	6,838	7,999
1991	22,083	12,250	13,372	14,924	9,814	7,591	7,763	8,126	9,263
1992	19,360	10,141	17,740	20,673	12,419	9,964	9,717	10,273	11,335

Selected Weights

Accident	Unadjusted Methods				Adjusted Methods			
	Development		Severity	Hindsight	Development		Severity	Hindsight
Year	Incurred	Paid	Projection	Method	Incurred	Paid	Projection	Method
1985	2	1	1		2	1	1	
1986	2	1	1		8	4	2	
1987	2	1	1		9	6	3	
1988	1	1	1	1	4	4	8	8
1989	1	1	1	1	4	4	8	8
1990	1	1	1	1	4	4	8	8
1991	1	1	1	1	4	4	8	8
1992	1	1	1	1	4	4	8	8

NOTE:
1. All dollar amounts are in thousands.

Exhibit 6

ESTIMATED TOTAL RESERVES

Accident Year	(1) Selected Ultimate Losses	(2) Losses Paid to Date	(3) Indicated Reserves (1) - (2)	(4) Estimated Ultimate Claims	(5) Claims Closed to Date	(6) Indicated Open & IBNR Claims (4) - (5)	(7) Indicated Average Reserve (3)/(6)	(8) Selected Coefficient of Variation	(9) Indicated Lognormal Parameters μ	(10) Indicated Lognormal Parameters σ
1985	$2,242	$1,582	$660	263	227	36	$18,333	3.4	8.5995	1.5908
1986	3,075	1,898	1,177	282	226	56	21,018	3.6	8.7009	1.6236
1987	4,279	1,771	2,508	398	286	112	22,393	3.8	8.7279	1.6544
1988	5,806	1,852	3,954	458	294	164	24,110	4.0	8.7702	1.6832
1989	6,783	1,129	5,654	497	279	218	25,936	4.2	8.8152	1.7104
1990	7,999	627	7,372	463	196	267	27,610	4.4	8.8520	1.7360
1991	9,263	157	9,106	459	132	327	27,847	4.6	8.8294	1.7602
1992	11,335	21	11,314	500	40	460	24,596	4.8	8.6557	1.7832
Total	$50,782	$9,037	$41,745							

NOTE:
1. Amounts in columns (1), (2), and (3) are in thousands of dollars.

SELECTED CLAIM SIZE DISTRIBUTIONS

Loss	Accident Year							
Amount	1985	1986	1987	1988	1989	1990	1991	1992
$50	0.00139	0.00138	0.00156	0.00169	0.00181	0.00194	0.00229	0.00346
100	0.00549	0.00535	0.00590	0.00625	0.00653	0.00685	0.00786	0.01125
250	0.02612	0.02476	0.02602	0.02657	0.02688	0.02737	0.03002	0.03951
500	0.06697	0.06290	0.06446	0.06461	0.06435	0.06453	0.06895	0.08595
750	0.10693	0.10021	0.10155	0.10097	0.09990	0.09952	0.10498	0.12719
1,000	0.14405	0.13495	0.13588	0.13452	0.13262	0.13162	0.13775	0.16384
1,250	0.17806	0.16688	0.16730	0.16516	0.16247	0.16085	0.16740	0.19640
1,500	0.21001	0.19694	0.19682	0.19392	0.19047	0.18824	0.19508	0.22644
2,000	0.26564	0.24955	0.24837	0.24413	0.23933	0.23603	0.24310	0.27766
2,500	0.31402	0.29555	0.29337	0.28795	0.28201	0.27775	0.28484	0.32148
3,500	0.39369	0.37186	0.36799	0.36071	0.35297	0.34718	0.35397	0.39280
5,000	0.48100	0.45649	0.45082	0.44171	0.43222	0.42489	0.43097	0.47050
6,000	0.52587	0.50043	0.49391	0.48399	0.47373	0.46570	0.47128	0.51050
7,500	0.58142	0.55520	0.54772	0.53693	0.52585	0.51703	0.52191	0.56031
8,500	0.61120	0.58482	0.57690	0.56573	0.55429	0.54512	0.54959	0.58723
10,000	0.65070	0.62430	0.61584	0.60425	0.59242	0.58283	0.58671	0.62315
12,500	0.70072	0.67482	0.66585	0.65394	0.64182	0.63186	0.63494	0.66931
15,000	0.74028	0.71516	0.70595	0.69396	0.68176	0.67164	0.67406	0.70639
20,000	0.79521	0.77194	0.76270	0.75097	0.73903	0.72896	0.73043	0.75921
25,000	0.83318	0.81180	0.80280	0.79156	0.78009	0.77030	0.77110	0.79686
35,000	0.88178	0.86369	0.85546	0.84533	0.83495	0.82593	0.82590	0.84700
50,000	0.91994	0.90547	0.89839	0.88977	0.88087	0.87299	0.87238	0.88889
60,000	0.93492	0.92224	0.91583	0.90804	0.89996	0.89274	0.89195	0.90632
75,000	0.95108	0.94053	0.93497	0.92822	0.92117	0.91480	0.91385	0.92573
85,000	0.95822	0.94876	0.94366	0.93748	0.93099	0.92509	0.92410	0.93475
100,000	0.96685	0.95877	0.95429	0.94885	0.94311	0.93784	0.93681	0.94589
125,000	0.97598	0.96956	0.96586	0.96136	0.95656	0.95211	0.95109	0.95833
150,000	0.98167	0.97640	0.97328	0.96947	0.96539	0.96155	0.96057	0.96654
175,000	0.98559	0.98119	0.97852	0.97525	0.97171	0.96836	0.96745	0.97246
200,000	0.98837	0.98464	0.98232	0.97947	0.97638	0.97343	0.97256	0.97685
225,000	0.99043	0.98722	0.98519	0.98268	0.97995	0.97732	0.97651	0.98022
250,000	0.99200	0.98921	0.98741	0.98519	0.98275	0.98039	0.97963	0.98288
275,000	0.99321	0.99076	0.98915	0.98716	0.98497	0.98283	0.98213	0.98500
300,000	0.99421	0.99204	0.99061	0.98882	0.98685	0.98491	0.98425	0.98680
350,000	0.99563	0.99390	0.99273	0.99126	0.98962	0.98800	0.98742	0.98948
400,000	0.99658	0.99517	0.99419	0.99295	0.99156	0.99018	0.98966	0.99137
450,000	0.99727	0.99610	0.99526	0.99421	0.99302	0.99182	0.99136	0.99281
500,000	0.99777	0.99677	0.99605	0.99514	0.99410	0.99305	0.99263	0.99388

Exhibit 8

ESTIMATED PROBABILITY LEVELS FOR RESERVES

Without Parameter Uncertainty

	Accident Year								
	1985	1986	1987	1988	1989	1990	1991	1992	Total
	Expected Reserve								
	$660	$1,177	$2,508	$3,954	$5,654	$7,372	$9,106	$11,314	$41,745
Ratio to Expected	Estimated Probability Level								
0.3	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.0115	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	0.0519	0.0202	0.0017	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.6	0.1322	0.0743	0.0174	0.0051	0.0004	0.0007	0.0003	0.0001	0.0000
0.7	0.2424	0.1710	0.0748	0.0376	0.0095	0.0123	0.0075	0.0031	0.0000
0.8	0.3635	0.2955	0.1918	0.1366	0.0710	0.0792	0.0626	0.0421	0.0006
0.9	0.4794	0.4278	0.3567	0.3134	0.2491	0.2576	0.2378	0.2095	0.0479
1.0	0.5815	0.5541	0.5359	0.5281	0.5200	0.5200	0.5179	0.5162	0.5074
1.1	0.6670	0.6665	0.6960	0.7213	0.7667	0.7596	0.7749	0.7981	0.9452
1.2	0.7375	0.7599	0.8182	0.8579	0.9140	0.9070	0.9230	0.9434	0.9990
1.3	0.7962	0.8330	0.9001	0.9369	0.9757	0.9719	0.9805	0.9892	1.0000
1.4	0.8449	0.8874	0.9492	0.9753	0.9946	0.9932	0.9962	0.9985	1.0000
1.5	0.8842	0.9262	0.9760	0.9914	0.9990	0.9987	0.9994	0.9999	1.0000
1.6	0.9150	0.9530	0.9894	0.9973	0.9999	0.9998	0.9999	1.0000	1.0000
1.7	0.9384	0.9708	0.9956	0.9992	1.0000	1.0000	1.0000	1.0000	1.0000
1.8	0.9558	0.9823	0.9983	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
1.9	0.9685	0.9895	0.9993	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.0	0.9777	0.9939	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.1	0.9844	0.9965	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.2	0.9892	0.9981	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.3	0.9926	0.9989	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.4	0.9950	0.9994	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.5	0.9967	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.6	0.9978	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.7	0.9985	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.8	0.9990	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.9	0.9994	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.0	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.1	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.2	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.3	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

NOTE:

1. Reserve estimates are in thousands.

Exhibit 9

ESTIMATE OF CONTAGION PARAMETER

(1)	(2)	(3)	(4)	(5)	(6)
Accident Year	Estimated Ultimate Claims	Earned Exposures	Indicated Frequency (2)/(3)	Selected On-Level Frequency	Indicated 1993 Claims (5)x8,700
1985	263	5,907	4.45%	5.34%	465
1986	282	4,965	5.68%	6.66%	579
1987	398	7,719	5.16%	5.91%	514
1988	458	7,922	5.78%	6.48%	564
1989	497	11,361	4.37%	4.79%	417
1990	463	7,525	6.15%	6.58%	572
1991	459	8,376	5.48%	5.73%	499
1992	500	8,649	5.78%	5.91%	514

Indicated Trend			2.3%		
Arithmetic Average					516
Variance Estimate					3,158
Indicated c Value					0.0099

ESTIMATES OF PARAMETER UNCERTAINTY FOR MEANS

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Indicated Lognormal Parameters		Estimated Number of Claims		Expected			Variance		
Accident					Average		Explained	in	Implied	Selected
Year	μ	σ	Open	IBNR	Reserve	$E(X^2\|L)$	Variance	Selected	b value	b value
1985	8.5995	1.5908	36	0	$18,333	2,267	69,525	40,192	-0.0581	0.0000
1986	8.7009	1.6236	54	2	21,018	2,920	139,662	71,526	-0.0477	0.0000
1987	8.7279	1.6544	106	6	22,393	3,322	319,139	373,623	0.0091	0.0091
1988	8.7702	1.6832	152	12	24,110	3,821	539,092	746,291	0.0147	0.0147
1989	8.8152	1.7104	188	30	25,936	4,366	831,265	2,277,671	0.0574	0.0574
1990	8.8520	1.7360	166	101	27,610	4,890	1,256,128	4,180,470	0.0974	0.0974
1991	8.8294	1.7602	147	180	27,847	5,044	1,784,293	9,390,867	0.1742	0.1742
1992	8.6557	1.7832	120	340	24,596	4,280	2,588,688	8,436,909	0.0720	0.0720

Selected Contagion Parameter: 0.0099

NOTE:

1. Amounts in columns (6), (7) and (8) are in millions.

Exhibit 11

ESTIMATE OF OVERALL MIXING PARAMETER

Accident Year	(1) Indicated Limited Severity	(2) Selected Coefficient of Variation	(3) Indicated Unlimited Severity	(4) Selected Frequency	(5) Indicated Unlimited Pure Premium (3)x(4)	(6) Unlimited Pure Premium at 1993 Level	(7) Indicated 1993 Unlimited Loss (6)x8,700	(8) Indicated 1993 Unlimited Severity (7)/516	(9) Indicated 1993 Limited Severity	(10) Indicated 1993 Limited Loss (4)x (9)x516
1985	$8,525	5.0	$8,913	4.45%	$397	$1,554	$13,520	$26,202	$22,916	$11,825
1986	10,904	5.0	11,572	5.68%	657	2,168	18,866	36,562	30,547	15,762
1987	10,751	5.0	11,399	5.16%	588	1,636	14,237	27,591	23,976	12,372
1988	12,677	5.0	13,605	5.78%	786	1,844	16,046	31,097	26,599	13,725
1989	13,648	5.0	14,736	4.37%	644	1,274	11,085	21,483	19,219	9,917
1990	17,276	5.0	19,081	6.15%	1,173	1,957	17,024	32,992	27,987	14,441
1991	20,181	5.0	22,692	5.48%	1,244	1,750	15,223	29,502	25,415	13,114
1992	22,670	5.0	25,882	5.78%	1,496	1,774	15,436	29,915	25,723	13,273
Indicated Trend					18.6%					
Average (000)										$13,054
Variance Estimate (000,000)										3,082,167
Average Limited Severity										$25,298
Corresponding Unlimited Severity										29,346
$E(X^2 \mid L)$ (000,000)										4,536
Selected 1993 Claim Counts										516
Explained Variance (000,000)										4,027,361
Implied b value										-0.00542
Selected Overall b value										0

NOTE:

1. Columns (7) and (10) are in thousands.

Exhibit 12

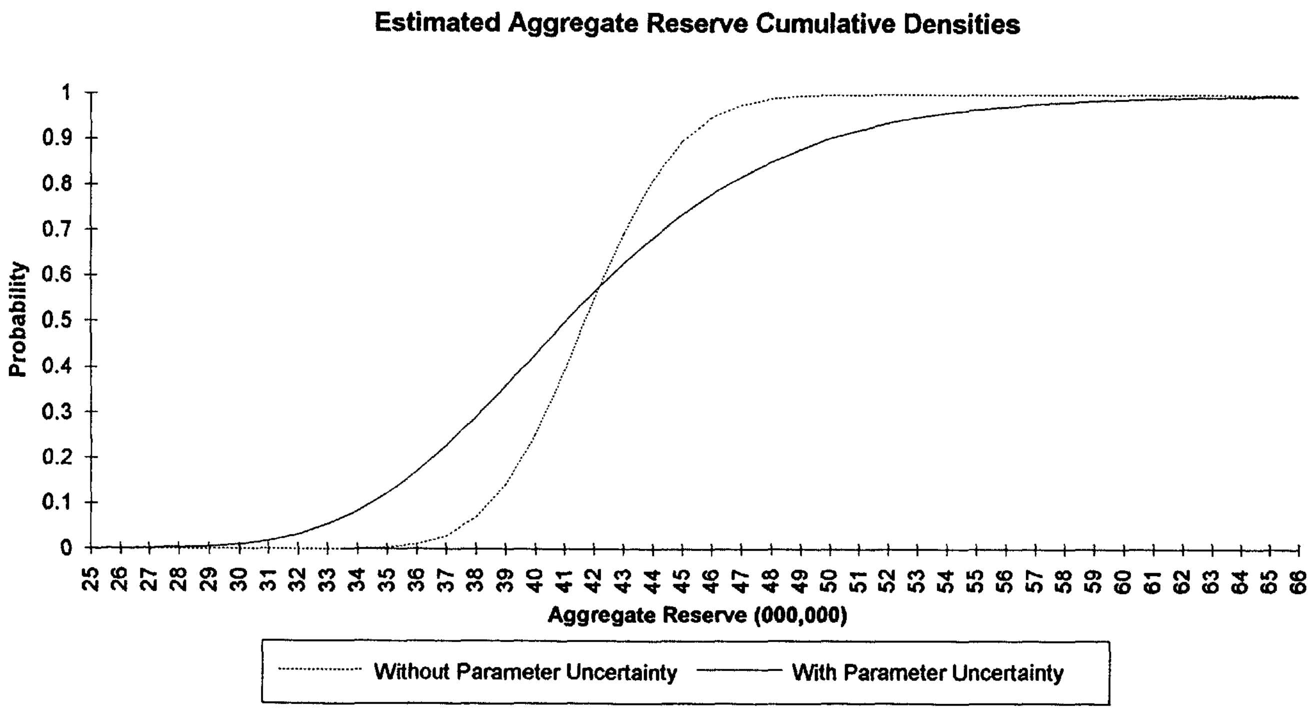
ESTIMATED PROBABILITY LEVELS FOR RESERVES

With Parameter Uncertainty

	Accident Year								
	1985	1986	1987	1988	1989	1990	1991	1992	Total
	Expected Reserve								
	$660	$1,177	$2,508	$3,954	$5,854	$7,372	$9,106	$11,314	$41,745
Ratio to Expected	Estimated Probability Level								
0.3	0.0008	0.0001	0.0000	0.0000	0.0000	0.0001	0.0008	0.0000	0.0000
0.4	0.0115	0.0024	0.0001	0.0000	0.0006	0.0026	0.0117	0.0008	0.0000
0.5	0.0519	0.0202	0.0037	0.0015	0.0083	0.0211	0.0541	0.0101	0.0000
0.6	0.1322	0.0743	0.0264	0.0151	0.0439	0.0779	0.1382	0.0502	0.0001
0.7	0.2424	0.1710	0.0936	0.0686	0.1284	0.1798	0.2527	0.1400	0.0052
0.8	0.3635	0.2955	0.2152	0.1851	0.2597	0.3120	0.3775	0.2733	0.0638
0.9	0.4794	0.4278	0.3749	0.3549	0.4137	0.4511	0.4965	0.4248	0.2630
1.0	0.5815	0.5541	0.5421	0.5401	0.5630	0.5789	0.6007	0.5688	0.5476
1.1	0.6670	0.6865	0.6899	0.7028	0.6898	0.6861	0.6874	0.6900	0.7769
1.2	0.7375	0.7599	0.8043	0.8239	0.7879	0.7706	0.7570	0.7840	0.9051
1.3	0.7962	0.8330	0.8840	0.9032	0.8589	0.8347	0.8118	0.8527	0.9626
1.4	0.8449	0.8873	0.9350	0.9500	0.9080	0.8818	0.8543	0.9011	0.9856
1.5	0.8842	0.9262	0.9652	0.9755	0.9409	0.9159	0.8870	0.9341	0.9944
1.6	0.9150	0.9530	0.9822	0.9885	0.9623	0.9402	0.9122	0.9564	0.9977
1.7	0.9384	0.9708	0.9912	0.9948	0.9761	0.9575	0.9315	0.9712	0.9990
1.8	0.9558	0.9823	0.9958	0.9977	0.9849	0.9697	0.9464	0.9810	0.9995
1.9	0.9685	0.9895	0.9980	0.9990	0.9904	0.9783	0.9578	0.9874	0.9998
2.0	0.9777	0.9939	0.9991	0.9996	0.9939	0.9845	0.9667	0.9916	0.9999
2.1	0.9844	0.9965	0.9996	0.9998	0.9961	0.9888	0.9735	0.9944	0.9999
2.2	0.9892	0.9981	0.9998	0.9999	0.9975	0.9919	0.9788	0.9962	1.0000
2.3	0.9926	0.9989	0.9999	1.0000	0.9984	0.9941	0.9830	0.9975	1.0000
2.4	0.9950	0.9994	1.0000	1.0000	0.9990	0.9957	0.9863	0.9983	1.0000
2.5	0.9967	0.9997	1.0000	1.0000	0.9993	0.9968	0.9889	0.9988	1.0000
2.6	0.9978	0.9998	1.0000	1.0000	0.9996	0.9976	0.9910	0.9992	1.0000
2.7	0.9985	0.9999	1.0000	1.0000	0.9997	0.9982	0.9926	0.9994	1.0000
2.8	0.9990	1.0000	1.0000	1.0000	0.9998	0.9987	0.9939	0.9996	1.0000
2.9	0.9994	1.0000	1.0000	1.0000	0.9999	0.9990	0.9950	0.9997	1.0000
3.0	0.9996	1.0000	1.0000	1.0000	0.9999	0.9992	0.9959	0.9998	1.0000
3.1	0.9997	1.0000	1.0000	1.0000	0.9999	0.9994	0.9966	0.9999	1.0000
3.2	0.9998	1.0000	1.0000	1.0000	1.0000	0.9996	0.9971	0.9999	1.0000
3.3	0.9999	1.0000	1.0000	1.0000	1.0000	0.9997	0.9976	0.9999	1.0000

NOTE:
1. Reserve estimates are in thousands.

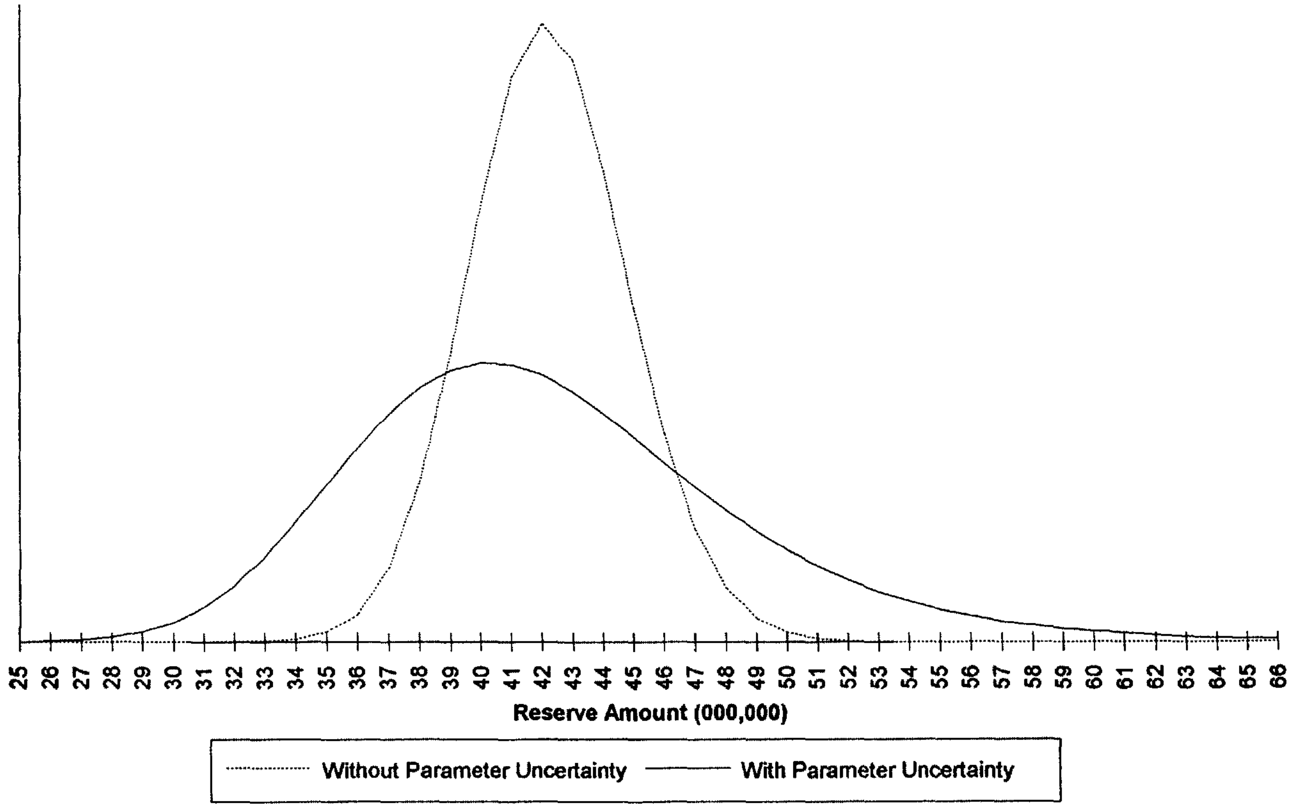
Exhibit 13



Estimated Aggregate Reserve Cumulative Densities
Probability
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66
Aggregate Reserve (000,000)
Without Parameter Uncertainty
With Parameter Uncertainty

Exhibit 14

Estimated Aggregate Probability Density Functions



APPENDIX

This appendix summarizes the analysis of another data set using the methods presented in this paper. The data used are those provided to the panelists for the Advanced Case Study session of the 1992 Casualty Loss Reserve Seminar, as summarized in Exhibit A-1. The first two pages of that exhibit give a summary background information regarding the data source while the last three pages give summary triangles and exposure information. Included are eighteen years of development for eighteen accident years including data on paid and outstanding losses, claims closed with payment, reported claims, open claims and earned exposures.

Our analysis indicated that there seemed to be changes in the percentage of reported claims that are paid for the various accident years. It appears that the court decision cited in the background material resulted in a higher proportion of reported claims being paid than the levels prior to that decision. We noted other changes in these ratios in the data. We thus selected paid counts, as opposed to reported, as the denominator in calculating severities in our severity and hindsight projection methods.

We used four projection methods to estimate ultimate reported counts. The first two were development factor methods applied to historical paid claims and historical incurred claims (paid claims plus outstanding claims). The third method estimated ultimate paid claims as the product of the number of ultimate reported claims and the forecast percentage of ultimate claims that will be paid. We used development factor methods applied to the historical ratios of paid to closed (defined to be reported minus open) claims. We considered trends in both the resulting reported frequencies and indicated percentages paid to temper the leveraging effect of development factor methods for more immature years.

The fourth method was a hindsight method based on frequencies. This method is similar to what we used to estimate losses, as described in the main portion of this paper.

Exhibit A-2 summarizes these projections and shows our selections and various diagnostics. These projections indicate an increase in estimated ultimate reported frequency in 1987 after a general decrease in prior years, as shown in column (12), and a marked increase in the percentage of reported that are estimated to be paid as shown in column (13).

After an analysis similar to that for the sample medical malpractice data, we noted that there appears to be a change in the rate at which claims are being closed. We thus considered loss projections based on paid loss data adjusted to remove this apparent change. Exhibit A-3 then shows a summary of our ultimate loss projections similar to Exhibit 5.

Exhibit A-4 then summarizes the assumptions we used to estimate the distribution of aggregate reserves before consideration of parameter uncertainty. In this case we assumed that claims closing with payment would have lognormal distributions with unlimited means equal to the average reserve per estimated future paid claim, shown in column (3). We assumed that all claims closing with payment would have a coefficient of variation equal to 1.25 and judgmentally scaled this back as shown in column (7). Though 1.25 may seem arbitrary and possibly low, its selection was based on discussions with the source of these sample data.

We have also elected to combine accident years 1984 and prior. This is due primarily to the relative scarcity of data for those years and the resulting "noise" in estimates for individual accident years.

As with the analysis in the main section of this paper, we assumed that open and IBNR claims both had the same loss distribution. Again, this is more of a convenience than a requirement of this approach. In this case, however, we assumed that the distribution of claims closing with payment would be lognormal and included $0.01 losses in the input distribution with the complement of the probability of a claim closing with payment. We then adjusted the remaining distribution accordingly. Exhibit A-5 shows an example using accident year 1986.

Exhibit A-6 shows the resulting aggregate distributions for the reserves without consideration of parameter uncertainty, similar to Exhibit 8. As can be seen from this exhibit, the rather large number of claims results in relatively little variation in aggregate amounts. Virtually all of the distribution is within 5% of the expected value of $203.2 million.

Exhibit A-7 corresponds to Exhibit 9 and results in an estimate for the overall contagion parameter of 0.0097. As shown in Exhibit A-2, however, due to changes that appeared in the data we used several different forecasting methods to estimate ultimate paid claims with variance among the methods as shown in column (10) of Exhibit A-2 and summarized in column (2) of Exhibit A-8.

Assuming our forecasts of the percentage of ultimate reported claims that will be paid, we can translate these variance estimates for ultimate paid claims to variance estimates for reported claims, as shown in column (4) of Exhibit A-8. We calculated the amount shown for 1984 and Prior as the sum of the corresponding amounts for the individual accident years.

We then solved for the contagion parameter, using the ultimate reported count estimates in column (1) and the variance estimates in column (4) to derive the estimates in column (5). In most accident years, the variance in the estimates is greater than what would be expected from a Poisson distribution. In addition to this variance for individual accident years, there is additional variation from year to year as shown in Exhibit A-7. We thus selected our contagion parameters as the sum of the indicated parameters in column (5) and the overall indicated parameter shown in Exhibit A-7.

Exhibit A-9 shows our estimates of the mixing parameters for the individual years. Since we assume that the losses are unlimited we can easily determine the indicated standard deviation, and hence variance using the unlimited mean and assumed coefficient of variation. Column (10) then shows the variance explained using the selected contagion parameters from Exhibit A-8 and the claim counts and claim size variances. Column (11) shows the variance among methods and shows that, except for accident years 1985 and 1991, the variance in methods exceeds what can be explained by our other assumptions. Column (12) gives the resulting implied values for the mixing parameter b while column (13) shows our selections.

As with Exhibit 11, we also calculated the variation in ultimate losses over the accident years, shown in Exhibit A-10. In this case the observed variance exceeds the amount that can be explained with the overall contagion parameter and our estimates of claim count and claim size distributions. This then implies an additional mixing parameter of 0.00069 shown at the bottom of Exhibit A-10.

We then calculated the individual distributions for each of the accident years separately, using the estimates of contagion and mixing parameters shown in Exhibits A-8 and A-9. We used the overall mixing parameter from Exhibit A-10 to reflect additional uncertainty in our final convolution of the distributions for individual accident years.

Exhibit A-11 then presents a summary of our estimates for the individual years and for the aggregate reserves. As with the analysis in the main section of this paper, the introduction of parameter uncertainty markedly widens the aggregate distribution. Whereas without parameter uncertainty, 90% of the losses were within 2.5% of the expected, with parameter uncertainty this percentage drops to 33%. Without parameter uncertainty 99.9% of the reserves were within 5% of the expected while with parameter uncertainty 60% fall in this range and we would have to widen the range to 20% to capture more than 99% of the indicated values. Exhibits A-12 and A-13 show these comparisons graphically.

BACKGROUND INFORMATION RELATING TO SAMPLE DATA

These data are based on actual bodily injury liability experience for an insurer, though we have randomly disturbed the true data to protect the identity of the insurer. The liability coverage is not particularly long-tailed and does not contain exposure to continuing damage or latent exposure claims such as asbestos or pollution.

For your information, the incremental paid counts and amounts and the incremental reported counts as well as outstanding counts and amounts were all multiplied by values selected randomly from a lognormal distribution. The corresponding normal distribution [that of ln(X)] had a mean of 0 and a standard deviation of 0.05. Thus the data should be close to "real." The exposures shown have also been modified from the actual data, however the underlying frequencies and pure premiums remain unchanged from that which would have arisen from the randomly perturbed data.

We have included five summary triangles:

1. Cumulative Paid Losses. Total loss payments at annual valuations for each accident year.

2. Outstanding Losses. Carried case reserves, without any actuarial or bulk adjustments, valued at successive year-ends.

3. Cumulative Paid Claims. Total claims closed with payment at annual valuations.

4. Outstanding Claims. Total claims open at year-end valuation dates whether or not the claim subsequently closes with payment.

5. Reported Claims. Total claims reported to the insurer, whether or not the claim subsequently closes with payment.

The accident years shown are real. Losses included are total direct losses and the insurer has experienced some drift to higher policy limits over time. This drift has been gradual and somewhat consistent over the time period under consideration. The exposure counts are not inflation-sensitive but do not reflect changes in the mix of exposures between lower and higher risk insureds that may have occurred over time. Similar to the drift in policy limits there has been a general, and gradual, drift to a greater proportion of lower risk insureds in this book.

The exposures are relatively homogeneous over time and contain no claims from outside the United States. There have been no changes in the overall mix of legal jurisdictions affecting these claims. There was, however, a notable legal decision near the end of 1986 affecting claims under this coverage. You can assume that this change made it easier to initiate claims and more difficult for the insurer to settle those claims early as compared to the situation prior to that time.

You may note a decrease in payments and reported claims during calendar year 1991. This is not the result of the random disturbances we introduced in the data but is present in the actual data. The Company is unable to provide a specific explanation as to the reason for this decrease.

Cumulative Paid Losses

Accident Year	Months of Development																	
	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$267	$1,975	$4,587	$7,375	$10,661	$15,232	$17,888	$18,541	$18,937	$19,130	$19,189	$19,209	$19,234	$19,234	$19,246	$19,246	$19,246	$19,246
1975	310	2,809	5,686	9,386	14,884	20,654	22,017	22,529	22,772	22,821	23,042	23,060	23,127	23,127	23,127	23,127	23,159	
1976	370	2,744	7,281	13,287	19,773	23,888	25,174	25,819	26,049	26,180	26,268	26,364	26,371	26,379	26,397	26,397		
1977	577	3,877	9,612	16,962	23,764	26,712	28,393	29,656	29,839	29,944	29,997	29,999	29,999	30,049	30,049			
1978	509	4,518	12,067	21,218	27,194	29,617	30,854	31,240	31,598	31,889	32,002	31,947	31,965	31,986				
1979	630	5,763	16,372	24,105	29,091	32,531	33,878	34,185	34,290	34,420	34,479	34,498	34,524					
1980	1,078	8,066	17,518	26,091	31,807	33,883	34,820	35,482	35,607	35,937	35,957	35,962						
1981	1,646	9,378	18,034	26,652	31,253	33,376	34,287	34,985	35,122	35,161	35,172							
1982	1,754	11,256	20,624	27,857	31,360	33,331	34,061	34,227	34,317	34,378								
1983	1,997	10,628	21,015	29,014	33,788	36,329	37,446	37,571	37,681									
1984	2,164	11,538	21,549	29,167	34,440	36,528	36,950	37,099										
1985	1,922	10,939	21,357	28,488	32,982	35,330	36,059											
1986	1,962	13,053	27,869	38,560	44,461	45,988												
1987	2,329	18,086	38,099	51,953	58,029													
1988	3,343	24,806	52,054	66,203														
1989	3,847	34,171	59,232															
1990	6,090	33,392																
1991	5,451																	

Claims Closed with Payment

Accident Year	Months of Development																	
	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	268	607	858	1,090	1,333	1,743	2,000	2,076	2,113	2,129	2,137	2,141	2,143	2,143	2,145	2,145	2,145	2,145
1975	294	691	913	1,195	1,620	2,076	2,234	2,293	2,320	2,331	2,339	2,341	2,343	2,343	2,343	2,343	2,344	
1976	283	642	961	1,407	1,994	2,375	2,504	2,549	2,580	2,590	2,596	2,600	2,602	2,603	2,603	2,603		
1977	274	707	1,176	1,688	2,295	2,545	2,689	2,777	2,809	2,817	2,824	2,825	2,825	2,826	2,826			
1978	269	658	1,228	1,819	2,217	2,475	2,613	2,671	2,691	2,706	2,710	2,711	2,714	2,717				
1979	249	771	1,581	2,101	2,528	2,816	2,930	2,961	2,973	2,979	2,986	2,988	2,992					
1980	305	1,107	1,713	2,316	2,748	2,942	3,025	3,049	3,063	3,077	3,079	3,080						
1981	343	1,042	1,608	2,260	2,596	2,734	2,801	2,835	2,854	2,859	2,860							
1982	350	1,242	1,922	2,407	2,661	2,834	2,887	2,902	2,911	2,915								
1983	428	1,257	1,841	2,345	2,683	2,853	2,908	2,920	2,925									
1984	291	1,004	1,577	2,054	2,406	2,583	2,622	2,636										
1985	303	1,001	1,575	2,080	2,444	2,586	2,617											
1986	318	1,055	1,906	2,524	2,874	2,958												
1987	343	1,438	2,384	3,172	3,559													
1988	391	1,671	3,082	3,771														
1989	433	1,941	3,241															
1990	533	1,923																
1991	339																	

Cumulative Reported Claims

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,912	2,854	3,350	3,945	4,057	4,104	4,149	4,155	4,164	4,167	4,169	4,169	4,169	4,170	4,170	4,170	4,170	4,170
1975	2,219	3,302	3,915	4,462	4,618	4,673	4,696	4,704	4,708	4,711	4,712	4,716	4,716	4,716	4,716	4,716	4,717	
1976	2,347	3,702	4,278	4,768	4,915	4,983	5,003	5,007	5,012	5,012	5,013	5,014	5,015	5,015	5,015	5,015		
1977	2,983	4,346	5,055	5,696	5,818	5,861	5,884	5,892	5,896	5,897	5,900	5,900	5,900	5,900	5,900			
1978	2,538	3,906	4,633	5,123	5,242	5,275	5,286	5,292	5,298	5,302	5,304	5,304	5,306	5,306				
1979	3,548	5,190	5,779	6,206	6,313	6,329	6,339	6,343	6,347	6,347	6,348	6,348	6,348					
1980	4,583	6,106	6,656	7,032	7,128	7,139	7,147	7,150	7,151	7,153	7,154	7,154						
1981	4,430	5,967	6,510	6,775	6,854	6,873	6,883	6,889	6,892	6,894	6,895							
1982	4,408	5,849	6,264	6,526	6,571	6,589	6,594	6,596	6,600	6,602								
1983	4,861	6,437	6,869	7,134	7,196	7,205	7,211	7,212	7,214									
1984	4,229	5,645	6,053	6,419	6,506	6,523	6,529	6,531										
1985	3,727	4,830	5,321	5,717	5,777	5,798	5,802											
1986	3,561	5,045	5,656	6,040	6,096	6,111												
1987	4,259	6,049	6,767	7,206	7,282													
1988	4,424	6,700	7,548	8,105														
1989	5,005	7,407	8,287															
1990	4,889	7,314																
1991	4,044																	

Outstanding Claims

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,381	1,336	1,462	1,660	1,406	772	406	191	98	57	23	13	3	4	0	0	0	0
1975	1,289	1,727	1,730	1,913	1,310	649	358	167	73	30	9	6	4	2	2	1	1	
1976	1,605	1,977	1,947	1,709	1,006	540	268	166	79	48	32	16	14	10	10	7		
1977	2,101	2,159	2,050	1,735	988	582	332	139	66	38	27	21	21	8	3			
1978	1,955	1,943	1,817	1,384	830	460	193	93	56	31	15	9	7	2				
1979	2,259	2,025	1,548	1,273	752	340	150	68	36	24	18	13	4					
1980	2,815	1,991	1,558	1,107	540	228	88	55	28	14	8	6						
1981	2,408	1,973	1,605	954	480	228	115	52	27	15	11							
1982	2,388	1,835	1,280	819	354	163	67	44	21	10								
1983	2,641	1,765	1,082	663	335	134	62	34	18									
1984	2,417	1,654	896	677	284	90	42	15										
1985	1,924	1,202	941	610	268	98	55											
1986	1,810	1,591	956	648	202	94												
1987	2,273	1,792	1,059	626	242													
1988	2,403	1,966	1,166	693														
1989	2,471	2,009	1,142															
1990	2,642	2,007																
1991	2,366																	

Outstanding Losses

Accident Year	Months of Development: 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$5,275	$8,867	$12,476	$11,919	$8,966	$5,367	$3,281	$1,524	$667	$348	$123	$82	$18	$40	$0	$0	$0	$0
1975	6,617	11,306	13,773	14,386	10,593	4,234	2,110	1,051	436	353	93	101	10	5	5	3	3	
1976	7,658	11,064	13,655	13,352	7,592	4,064	1,895	1,003	683	384	216	102	93	57	50	33		
1977	8,735	14,318	14,897	12,978	7,741	4,355	2,132	910	498	323	176	99	101	32	14			
1978	8,722	15,070	15,257	11,189	5,959	3,473	1,531	942	547	286	177	61	67	7				
1979	9,349	16,470	14,320	10,574	6,561	2,864	1,328	784	424	212	146	113	38					
1980	11,145	16,351	14,636	11,273	5,159	2,588	1,290	573	405	134	81	54						
1981	10,933	15,012	14,728	9,067	5,107	2,456	1,400	584	269	120	93							
1982	13,323	16,218	12,676	6,290	3,355	1,407	613	398	192	111								
1983	13,899	16,958	12,414	7,700	4,112	1,637	576	426	331									
1984	14,272	15,806	10,156	8,005	3,604	791	379	159										
1985	13,901	15,384	12,539	7,911	3,809	1,404	827											
1986	15,952	22,799	16,016	8,964	2,929	1,321												
1987	22,772	24,146	18,397	8,376	3,373													
1988	25,216	26,947	17,950	8,610														
1989	24,981	30,574	19,621															
1990	30,389	34,128																
1991	28,194																	

Accident Year	Earned Exposures
1974	11,000
1975	11,000
1976	11,000
1977	12,000
1978	12,000
1979	12,000
1980	12,000
1981	12,000
1982	11,000
1983	11,000
1984	11,000
1985	11,000
1986	12,000
1987	13,000
1988	14,000
1989	14,000
1990	14,000
1991	13,000

Exhibit A-2

SAMPLE BODILY INJURY LIABILITY LOSS DATA

Projections of the Ultimate Number of Claims Closed with Payment

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
					Selected Weights					Indicated Variance
Accident Year	Development Paid	Development Incurred	Percent Paid	Hindsight Frequency	Development Paid	Development Incurred	Percent Paid	Hindsight Frequency	Weighted Average	in Selected Methods
1974	2,145	2,145	2,143		1	0	0		2,145	0.0
1975	2,344	2,345	2,345		1	0	0		2,344	0.0
1976	2,603	2,610	2,608		1	0	0		2,603	0.0
1977	2,826	2,827	2,828		1	0	0		2,826	0.0
1978	2,718	2,715	2,716		1	0	0		2,718	0.0
1979	2,994	2,987	2,996		1	0	0		2,994	0.0
1980	3,085	3,075	3,083		1	1	1		3,081	18.7
1981	2,865	2,857	2,864		1	1	1		2,862	12.7
1982	2,924	2,907	2,911		1	1	1		2,915	53.7
1983	2,941	2,919	2,930		1	1	1		2,930	80.7
1984	2,661	2,620	2,640	2,647	1	1	1	1	2,642	218.5
1985	2,660	2,626	2,643	2,639	1	1	1	1	2,642	147.5
1986	3,056	2,978	3,023	3,018	1	1	1	1	3,019	766.8
1987	3,879	3,676	3,813	3,728	1	2	2	3	3,755	4,596.6
1988	4,718	4,279	4,585	4,373	1	2	2	3	4,446	23,048.9
1989	5,233	4,540	5,014	4,641	1	2	2	3	4,783	60,976.5
1990	5,398	4,516	5,137	4,821	1	2	2	3	4,896	84,230.1
1991	3,903	3,990	4,574	4,447	1	2	2	2	4,275	76,972.0

	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
Accident Year	Estimated Ultimate Reported	Indicated Reported Frequency	Indicated Percent Paid	Number Reported to Date	Number Open	Number IBNR (11)-(14)	Number Paid to Date	Indicated Future Paid (9)-(17)	Future Percent Paid (18)/[(15)+(16)]
1974	4,170	0.379	51.4%	4,170	0	0	2,145	0	--
1975	4,719	0.429	49.7%	4,717	1	2	2,344	0	0.0%
1976	5,016	0.456	51.9%	5,015	7	1	2,603	0	0.0%
1977	5,904	0.492	47.9%	5,900	3	4	2,826	0	0.0%
1978	5,306	0.442	51.2%	5,306	2	0	2,717	1	50.0%
1979	6,348	0.529	47.2%	6,348	4	0	2,992	2	50.0%
1980	7,154	0.596	43.1%	7,154	6	0	3,080	1	16.7%
1981	6,900	0.575	41.5%	6,895	11	5	2,860	2	12.5%
1982	6,602	0.600	44.2%	6,602	10	0	2,915	0	0.0%
1983	7,216	0.656	40.6%	7,214	18	2	2,925	5	25.0%
1984	6,534	0.594	40.4%	6,531	15	3	2,636	6	33.3%
1985	5,808	0.528	45.5%	5,802	55	6	2,617	25	41.0%
1986	6,120	0.510	49.3%	6,111	94	9	2,958	61	59.2%
1987	7,319	0.563	51.3%	7,282	242	37	3,559	196	70.3%
1988	8,232	0.588	54.0%	8,105	693	127	3,771	675	82.3%
1989	9,002	0.643	53.1%	8,287	1,142	715	3,241	1,542	83.0%
1990	8,918	0.637	54.9%	7,314	2,007	1,604	1,923	2,973	82.3%
1991	7,982	0.614	53.6%	4,044	2,366	3,938	339	3,936	62.4%

Exhibit A-3

SAMPLE BODILY INJURY LIABILITY LOSS DATA

Projections of the Ultimate Losses

	Unadjusted Paid Methods			Incurred	Paid Methods Adjusted for Claims Closing Changes			
Accident Year	Devel-opment	Severity Method	Hindsight	Devel-opment	Devel-opment	Severity Method	Hindsight	Weighted Average
1974	$19,246	$19,245		$19,246	$19,246	$19,245		$19,246
1975	23,159	23,159		23,162	23,161	23,159		23,160
1976	26,397	26,397		26,430	26,400	26,397		26,406
1977	30,049	30,049		30,054	30,061	30,063		30,057
1978	31,996	31,994		31,971	32,021	32,023		32,003
1979	34,559	34,563		34,510	34,572	34,572		34,554
1980	36,012	36,023		35,955	36,012	36,011		35,999
1981	35,221	35,231		35,131	35,221	35,217		35,199
1982	34,478	34,464		34,344	34,426	34,423		34,416
1983	37,941	37,864		37,811	37,768	37,765		37,812
1984	37,474	37,371		36,979	37,214	37,205		37,205
1985	36,715	36,505	$36,409	36,543	36,394	36,407	$36,429	36,463
1986	47,818	47,338	47,044	46,916	47,083	47,054	47,055	47,117
1987	63,861	62,577	62,799	60,585	61,685	61,571	62,844	62,173
1988	83,555	80,717	79,763	74,708	78,748	78,001	79,268	78,809
1989	99,338	94,900	90,936	84,444	91,348	89,375	91,514	90,845
1990	110,157	105,279	94,068	92,617	102,640	95,849	96,509	98,101
1991	127,250	104,212	94,090	87,770	312,670	91,947	96,203	94,044

Selected Weights

	Unadjusted Paid Methods			Incurred	Paid Methods Adjusted for Claims Closing Changes			Indicated Variance
Accident Year	Devel-opment	Severity Method	Hindsight	Devel-opment	Devel-opment	Severity Method	Hindsight	in Selected Methods
1974	1	1		2	2	2		0
1975	1	1		2	2	2		2
1976	1	1		2	2	2		194
1977	1	1		2	2	2		31
1978	1	1		2	2	2		453
1979	1	1		2	2	2		659
1980	1	1		2	2	2		655
1981	1	1		2	2	2		1,547
1982	1	1		2	2	2		2,102
1983	1	1		2	2	2		3,455
1984	1	1		2	2	2		25,279
1985	1	1	2	2	2	2	3	7,936
1986	1	1	2	2	2	2	3	50,268
1987	1	1	2	2	2	2	3	876,278
1988	1	1	2	2	2	2	3	4,889,756
1989	1	1	2	2	2	2	3	13,592,826
1990	1	1	2	2	2	2	3	26,807,766
1991	0	1	2	2	0	2	3	20,489,727

NOTES:
1. Dollar amounts are in thousands.
2. Variance amounts are in millions.

Exhibit A-4

SAMPLE BODILY INJURY DATA

Summary Reserve and Claim Indications

Accident Year	(1) Indicated Reserves	(2) Indicated Future Paid Claims	(3) Indicated Average Claim to be Paid (1)/(2)	(4) Total Number Open	(5) Total Number IBNR	(6) Selected Percent to be Paid (2)/[(4)+(5)]	(7) Selected Coefficient of Variation
1984 & Prior	$404	17	$23,765	77	17	18.1%	1.050
1985	404	25	16,160	55	6	41.0%	1.075
1986	1,129	61	18,508	94	9	59.2%	1.100
1987	4,144	196	21,143	242	37	70.3%	1.125
1988	12,606	675	18,676	693	127	82.3%	1.150
1989	31,613	1,542	20,501	1,142	715	83.0%	1.175
1990	64,709	2,973	21,766	2,007	1,604	82.3%	1.200
1991	88,593	3,936	22,508	2,366	3,938	62.4%	1.225
Total	$203,198	9,408	$21,598	6,599	6,436	72.2%	

NOTE:
1. Amounts in column (1) are in thousands.

Exhibit A-5

SAMPLE BODILY INJURY DATA

Severity Input for Accident Year 1986

Loss Amount	Step Function Approximation for Lognormal	Selected Input Distribution .408 + .592 x (1)
$0.01	–	0.40800
950	0.00007	0.40804
2,316	0.02575	0.42324
4,358	0.11754	0.47758
7,117	0.26685	0.56598
10,625	0.43335	0.66454
14,909	0.58465	0.75411
19,994	0.70653	0.82627
25,902	0.79769	0.88023
32,651	0.86274	0.91874
40,259	0.90778	0.94541
48,743	0.93837	0.96352
58,118	0.95890	0.97567
68,399	0.97260	0.98378
79,598	0.98170	0.98917
91,728	0.98774	0.99274
104,801	0.99176	0.99512
118,829	0.99444	0.99671
133,822	0.99623	0.99777
149,791	0.99743	0.99848
166,746	0.99824	0.99896
184,696	0.99879	0.99928
203,651	0.99916	0.99950
223,619	0.99942	0.99966
244,608	0.99959	0.99976
266,629	0.99971	0.99983
289,687	0.99980	0.99988
313,791	0.99986	0.99992
338,949	0.99990	0.99994
365,168	0.99993	0.99996
392,455	0.99995	0.99997
420,817	0.99996	0.99998
450,261	0.99997	0.99998
480,793	0.99998	0.99999
512,420	0.99999	0.99999
545,148	0.99999	0.99999
578,984	0.99999	0.99999
613,932	0.99999	0.99999
650,000	1.00000	1.00000

NOTE:

1. The amounts in column (1) are based on a lognormal distribution with mean 18,508 and coefficient of variation 1.100.

Exhibit A-6

SAMPLE BODILY INJURY DATA

Estimated Probability Levels for Reseves Without Parameter Uncertainty

	Accident Year								
	1984 & Prior	1985	1986	1987	1988	1989	1990	1991	Total
	Expected Reserve								
	$404	$404	$1,129	$4,144	$12,806	$31,613	$64,709	$88,593	$203,198
Ratio to Expected	Estimated Probability Level								
0.300	0.0030	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.500	0.0433	0.0117	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.600	0.0996	0.0445	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.700	0.1864	0.1169	0.0208	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.750	0.2396	0.1703	0.0502	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000
0.800	0.2976	0.2342	0.1024	0.0088	0.0000	0.0000	0.0000	0.0000	0.0000
0.825	0.3281	0.2693	0.1383	0.0205	0.0000	0.0000	0.0000	0.0000	0.0000
0.850	0.3590	0.3062	0.1813	0.0424	0.0005	0.0000	0.0000	0.0000	0.0000
0.875	0.3903	0.3443	0.2305	0.0792	0.0034	0.0000	0.0000	0.0000	0.0000
0.900	0.4217	0.3833	0.2849	0.1346	0.0166	0.0013	0.0000	0.0000	0.0000
0.925	0.4530	0.4228	0.3433	0.2098	0.0584	0.0132	0.0013	0.0013	0.0000
0.950	0.4842	0.4622	0.4046	0.3026	0.1533	0.0727	0.0242	0.0242	0.0009
0.975	0.5149	0.5014	0.4668	0.4079	0.3116	0.2388	0.1665	0.1665	0.0458
1.000	0.5451	0.5398	0.5285	0.5177	0.5104	0.5069	0.5050	0.5050	0.4951
1.025	0.5744	0.5770	0.5883	0.6236	0.7019	0.7663	0.8340	0.8340	0.9488
1.050	0.6028	0.6130	0.6449	0.7187	0.8462	0.9222	0.9715	0.9715	0.9994
1.075	0.6303	0.6475	0.6973	0.7985	0.9329	0.9819	0.9975	0.9975	1.0000
1.100	0.6567	0.6802	0.7449	0.8618	0.9752	0.9970	0.9999	0.9999	1.0000
1.125	0.6820	0.7110	0.7876	0.9090	0.9921	0.9997	1.0000	1.0000	1.0000
1.150	0.7063	0.7397	0.8251	0.9422	0.9979	1.0000	1.0000	1.0000	1.0000
1.175	0.7291	0.7665	0.8572	0.9646	0.9995	1.0000	1.0000	1.0000	1.0000
1.200	0.7507	0.7913	0.8846	0.9792	0.9999	1.0000	1.0000	1.0000	1.0000
1.225	0.7710	0.8140	0.9077	0.9880	1.0000	1.0000	1.0000	1.0000	1.0000
1.250	0.7902	0.8348	0.9268	0.9933	1.0000	1.0000	1.0000	1.0000	1.0000
1.275	0.8082	0.8537	0.9423	0.9964	1.0000	1.0000	1.0000	1.0000	1.0000
1.300	0.8250	0.8709	0.9549	0.9981	1.0000	1.0000	1.0000	1.0000	1.0000
1.350	0.8548	0.9002	0.9730	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000
1.400	0.8803	0.9236	0.9842	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
1.500	0.9202	0.9563	0.9948	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.600	0.9478	0.9757	0.9984	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.800	0.9786	0.9928	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.000	0.9915	0.9978	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.500	0.9991	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

NOTE:
1. Dollar amounts are in thousands.

Exhibit A-7

SAMPLE BODILY INJURY DATA

Selection of Overall Contagion Parameter

Accident Year	Indicated Ultimate Reported Frequency	Selected On-Level Frequency	Indicated 1992 Claims (2)x13,000
1974	0.379	0.571	7,423
1975	0.429	0.631	8,203
1976	0.456	0.656	8,528
1977	0.492	0.692	8,996
1978	0.442	0.608	7,904
1979	0.529	0.711	9,243
1980	0.596	0.783	10,179
1981	0.575	0.738	9,594
1982	0.600	0.753	9,789
1983	0.656	0.805	10,465
1984	0.594	0.713	9,269
1985	0.528	0.619	8,047
1986	0.510	0.585	7,605
1987	0.563	0.631	8,203
1988	0.588	0.644	8,372
1989	0.643	0.688	8,944
1990	0.637	0.667	8,671
1991	0.614	0.628	8,164

Indicated Trend	2.3%

Arithmetic Average	8,756
Estimate of Variance	753,367
Indicated Overall Contagion Parameter	0.0097

Exhibit A-8

SAMPLE BODILY INJURY DATA

Selected Contagion Parameters

Accident Year	(1) Estimated Ultimate Reported	(2) Indicated Variance in Selected Methods	(3) Estimated Proportion Paid	(4) Estimated Variance in Reported (2)/[(3)x(3)]	(5) Indicated Individual Contagion Parameter	(6) Selected Contagion Parameter
1984 & Prior	65,869	384.2	--	1,338.7	0.0000	0.0097
1985	5,808	147.5	45.5%	712.5	-0.0002	0.0096
1986	6,120	766.8	49.3%	3,154.7	-0.0001	0.0096
1987	7,319	4,596.6	51.3%	17,466.4	0.0002	0.0099
1988	8,232	23,048.9	54.0%	79,042.8	0.0010	0.0108
1989	9,002	60,976.5	53.1%	216,258.6	0.0026	0.0123
1990	8,918	84,230.1	54.9%	279,462.0	0.0034	0.0131
1991	7,982	76,972.0	53.6%	267,918.8	0.0041	0.0138

Exhibit A-9

SAMPLE BODILY INJURY DATA

Estimates of Mixing Parameters

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Estimates Based on Claims With Payment						$E(x^2)$
Accident Year	Estimated Average Reserve	Selected Coefficient of Variation	Indicated Standard Deviation (1)x(2)	Indicated Variance (3)x(3)	$E(x^2)$ (4)+(1)x(1)	Indicated Percent Paid	Based on Reported Claims (5)x(6)
1984 & Prior	$23,765	1.050	$24,953	622.665	1,187.440	18.1%	214.927
1985	16,160	1.075	17,372	301.786	562.932	41.0%	230.802
1986	18,508	1.100	20,359	414.481	757.027	59.2%	448.160
1987	21,143	1.125	23,786	565.768	1,012.794	70.3%	711.994
1988	18,676	1.150	21,477	461.279	810.072	82.3%	666.689
1989	20,501	1.175	24,089	580.264	1,000.555	83.0%	830.461
1990	21,766	1.200	26,119	682.213	1,155.971	82.3%	951.364
1991	22,508	1.225	27,572	760.232	1,266.842	62.4%	790.509

	(8)	(9)	(10)	(11)	(12)	(13)
	Estimated Number of Claims					
Accident Year	Open	IBNR	Explained Variance	Variance in Selected	Implied b Value	Selected b Value
1984 & Prior	77	17	18,830	34,377	0.1161	0.1161
1985	55	6	11,680	7,936	-0.0256	0.0000
1986	94	9	34,969	50,268	0.0138	0.0138
1987	242	37	148,177	876,278	0.0544	0.0544
1988	693	127	423,955	4,889,756	0.0379	0.0379
1989	1,142	715	3,027,679	13,592,826	0.0200	0.0200
1990	2,007	1,604	13,618,136	26,807,766	0.0062	0.0062
1991	2,366	3,938	46,708,007	20,489,727	-0.0062	0.0000

NOTE:
1. Amounts in columns (4), (5), (7), (10), and (11) are in millions.

Exhibit A-10

SAMPLE BODILY INJURY DATA

Estimate of Overall Mixing Parameter

Accident Year	(1) Estimated Ultimate Losses	(2) Earned Exposures	(3) Indicated Pure Premium (1)/(2)	(4) Estimated Pure Premium at 1992 Level	(5) Indicated 1992 Loss (4)x13,000
1974	$19,246	11,000	$1,750	$6,764	$87,932
1975	23,160	11,000	2,105	7,547	98,111
1976	26,406	11,000	2,401	7,985	103,805
1977	30,057	12,000	2,505	7,728	100,464
1978	32,003	12,000	2,667	7,633	99,229
1979	34,554	12,000	2,880	7,646	99,398
1980	35,999	12,000	3,000	7,388	96,044
1981	35,199	12,000	2,933	6,701	87,113
1982	34,416	11,000	3,129	6,631	86,203
1983	37,812	11,000	3,437	6,757	87,841
1984	37,205	11,000	3,382	6,168	80,184
1985	36,463	11,000	3,315	5,608	72,904
1986	47,117	12,000	3,926	6,161	80,093
1987	62,173	13,000	4,783	6,963	90,519
1988	78,809	14,000	5,629	7,602	98,826
1989	90,845	14,000	6,489	8,129	105,677
1990	98,101	14,000	7,007	8,143	105,859
1991	94,044	13,000	7,234	7,798	101,374

A. Indicated Trend	7.8%	
B. Average (000)		$93,421
C. Variance Estimate (000,000)		93,442,417
D. Estimated 1992 Claims Reported		8,756
E. Indicated Severity (000) (A/C)		$10.669
F. Selected Coefficient of Variation		1.250
G. Indicated Standard Deviation (000) (ExF)		$13.336
H. Indicated Variance (000,000) (GxG)		177.849
I. Indicated $E(x^2)$(000,000) (H+ExE)		291.677
J. Selected Overall Contagion Parameter		0.0097
K. Explained Variance (000,000)		87,317,281
L. Indicated Overall Mixing Parameter		0.00069
M. Selected Overall Mixing Parameter		0.00069

NOTE:

1. Amounts in columns (1) and (5) are in thousands of dollars.

Exhibit A-11

SAMPLE BODILY INJURY DATA

Estimated Probability Levels for Reseves With Parameter Uncertainty

	Accident Year								
	1984 & Prior	1985	1986	1987	1988	1989	1990	1991	Total
	Expected Reserve								
	$404	$404	$1,129	$4,144	$12,606	$31,613	$64,709	$88,593	$203,198
Ratio to Expected	Estimated Probability Level								
0.300	0.0128	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.500	0.1059	0.0118	0.0005	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000
0.600	0.1925	0.0446	0.0081	0.0192	0.0039	0.0002	0.0000	0.0000	0.0000
0.700	0.2940	0.1170	0.0486	0.0848	0.0375	0.0084	0.0001	0.0000	0.0000
0.750	0.3468	0.1705	0.0930	0.1425	0.0817	0.0292	0.0014	0.0001	0.0000
0.800	0.3993	0.2344	0.1573	0.2152	0.1489	0.0758	0.0111	0.0021	0.0000
0.825	0.4252	0.2695	0.1963	0.2559	0.1904	0.1116	0.0250	0.0071	0.0009
0.850	0.4508	0.3064	0.2397	0.2988	0.2367	0.1561	0.0502	0.0194	0.0033
0.875	0.4754	0.3445	0.2864	0.3428	0.2866	0.2089	0.0908	0.0455	0.0127
0.900	0.5001	0.3834	0.3357	0.3874	0.3388	0.2686	0.1494	0.0925	0.0394
0.925	0.5235	0.4228	0.3865	0.4321	0.3924	0.3335	0.2259	0.1658	0.0982
0.950	0.5467	0.4623	0.4380	0.4761	0.4462	0.4013	0.3175	0.2657	0.2007
0.975	0.5690	0.5014	0.4893	0.5190	0.4993	0.4700	0.4187	0.3859	0.3449
1.000	0.5904	0.5398	0.5395	0.5603	0.5506	0.5374	0.5225	0.5150	0.5113
1.025	0.6115	0.5770	0.5877	0.5998	0.5995	0.6017	0.6219	0.6395	0.6715
1.050	0.6310	0.6130	0.6338	0.6371	0.6454	0.6616	0.7113	0.7484	0.8018
1.075	0.6505	0.6475	0.6768	0.6722	0.6880	0.7160	0.7875	0.8351	0.8927
1.100	0.6685	0.6801	0.7169	0.7049	0.7273	0.7646	0.8489	0.8985	0.9478
1.125	0.6862	0.7109	0.7533	0.7352	0.7626	0.8070	0.8961	0.9413	0.9770
1.150	0.7030	0.7396	0.7866	0.7630	0.7946	0.8434	0.9308	0.9673	0.9907
1.175	0.7188	0.7664	0.8164	0.7885	0.8231	0.8742	0.9552	0.9834	0.9966
1.200	0.7344	0.7911	0.8429	0.8117	0.8481	0.8998	0.9718	0.9915	0.9988
1.225	0.7486	0.8139	0.8662	0.8326	0.8703	0.9210	0.9828	0.9960	0.9996
1.250	0.7627	0.8347	0.8867	0.8515	0.8896	0.9381	0.9898	0.9980	0.9999
1.275	0.7755	0.8536	0.9045	0.8686	0.9063	0.9519	0.9940	0.9990	1.0000
1.300	0.7880	0.8708	0.9198	0.8839	0.9208	0.9629	0.9966	0.9990	1.0000
1.350	0.8109	0.9001	0.9442	0.9098	0.9437	0.9783	0.9990	1.0000	1.0000
1.400	0.8315	0.9235	0.9617	0.9303	0.9604	0.9875	0.9997	1.0000	1.0000
1.500	0.8665	0.9562	0.9826	0.9588	0.9808	0.9961	1.0000	1.0000	1.0000
1.600	0.8942	0.9756	0.9924	0.9758	0.9909	0.9988	1.0000	1.0000	1.0000
1.800	0.9334	0.9927	0.9986	0.9918	0.9980	0.9999	1.0000	1.0000	1.0000
2.000	0.9578	0.9978	0.9998	0.9972	0.9996	1.0000	1.0000	1.0000	1.0000
2.500	0.9859	0.9999	1.0000	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
3.000	0.9949	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

NOTE:
1. Dollar amounts are in thousands.

Exhibit A-12

Estimated Aggregate Reserve Cumulative Densities

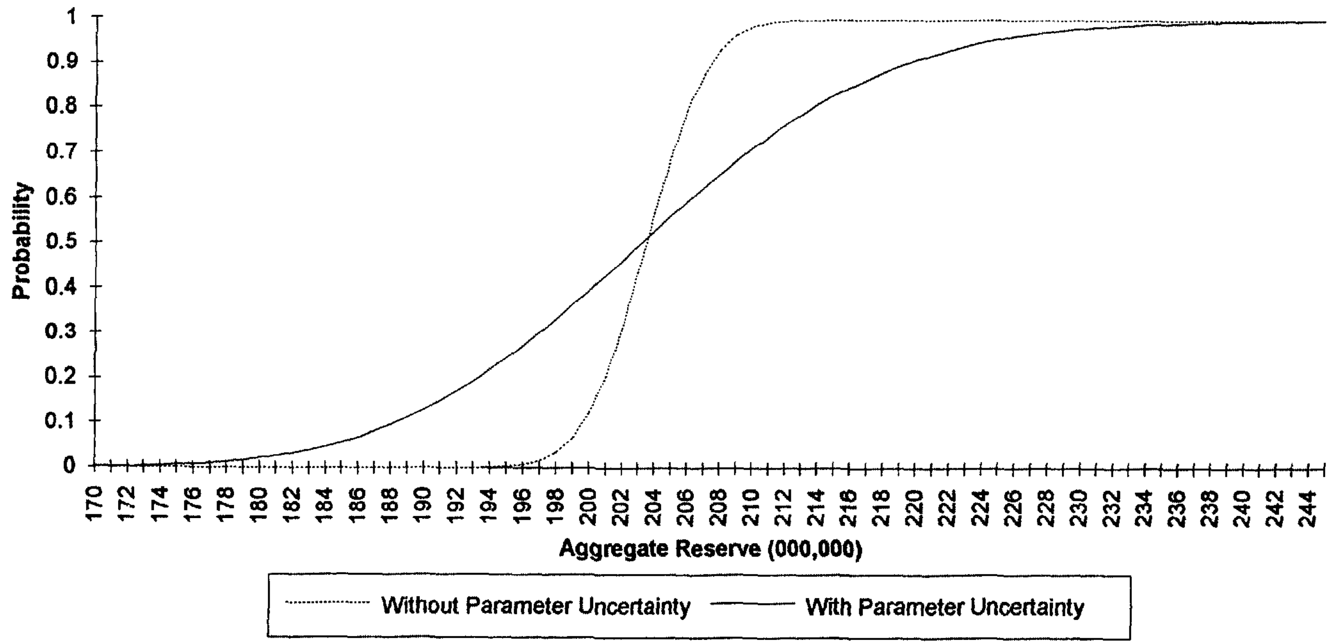
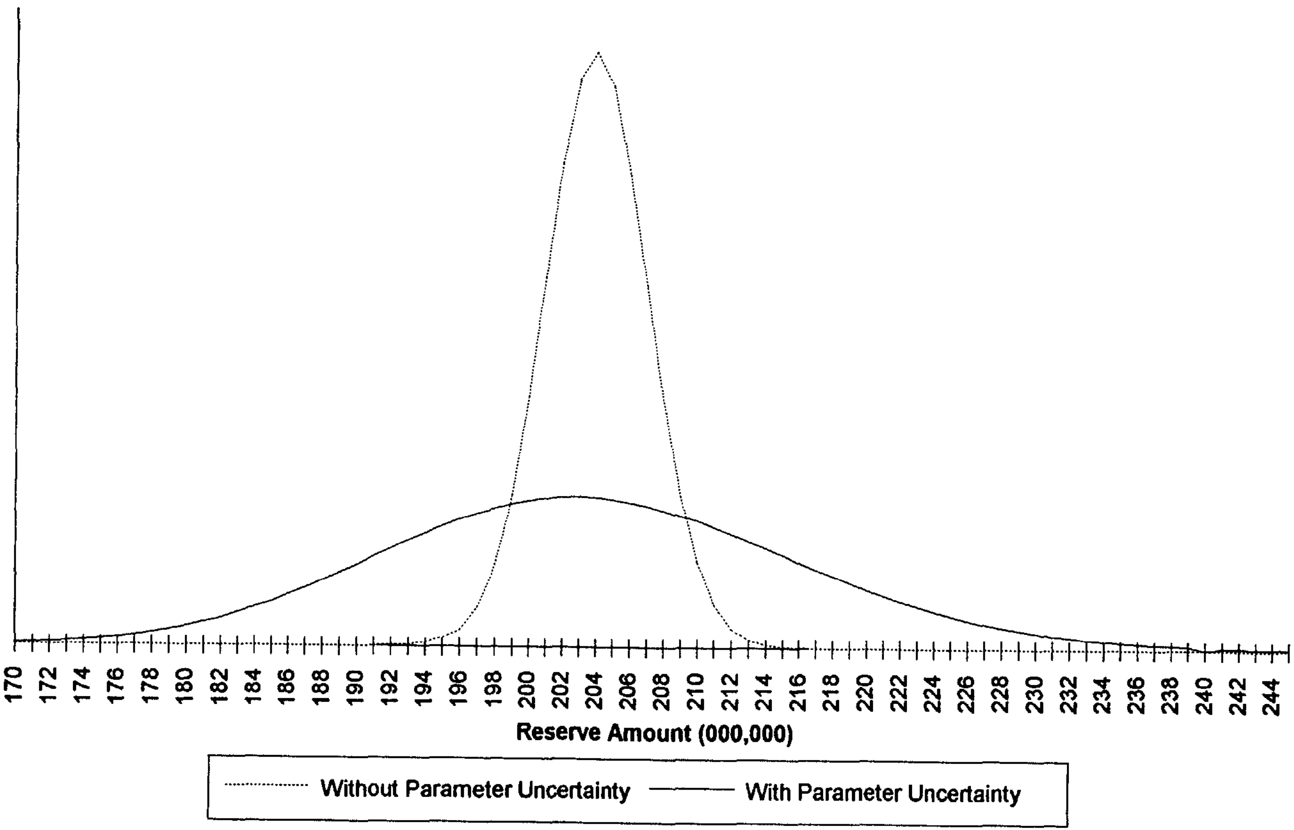


Exhibit A-13

Estimated Aggregate Probability Density Functions



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A Note on Simulation of Claim Activity for Use in Aggregate Loss Distributions

by Daniel K. Lyons

A NOTE ON SIMULATION OF CLAIM ACTIVITY FOR USE IN AGGREGATE LOSS DISTRIBUTIONS

Abstract
Aggregate loss distributions have been used in a number of different applications over the last few years. These applications have usually focused on the distribution of losses at ultimate or final values and have not studied how losses move to ultimate values over time. The approach outlined in this note models claim activity through the use of transition matrices. Individual claim activity is then incorporated into an aggregate loss simulation model to determine a number of distributions of interest.

This note will present an overview of how to determine the distribution of paid, case, and incurred but not reported (IBNR) losses over time in a manner consistent with the determination of aggregate loss distributions. The method is based on determining severity distributions for both paid and case incurred losses at different valuations, determining transition matrices to model claim changes over time, and simulating many years of claim activity. This method may require much computer time and, if it is to be company specific, detailed loss stratification data. While these requirements may by burdensome the method also permits an analysis of the distribution of loss development factors and of run off ratios.

In 1988 Hayne outlined an approach[1] using collective risk theory to measure the variability of loss reserves. The approach used in this note is an application of the use of collective risk theory such that claim development may be introduced into the process.

When the Insurance Services Office (ISO) prepares a review of increased limits factors they track the severity distribution over time. This is done because ISO is interested in the distribution of losses at their ultimate values. ISO's supplementary exhibits show triangles of pareto parameters obtained from fitting curves to accident year case incurred losses at various valuations. These fits and the relationship between the curves are used to determine the final severity curve upon which indicated increased limits factors are based. This material generally shows the average size of loss increases as the accident year matures.

Severity distributions are needed in determining aggregate loss distributions. Much has been written about the use of aggregate distributions and there are a few methods to use to calculate an aggregate distribution.[2,3] In a recent paper[4] Bear and Nemlick use aggregate loss distributions to quantify the expected impact of swing rated reinsurance contracts. In 1980 Patrik and John[5] used the notion of supporting surplus as measured by the use of an aggregate loss distribution to determine the appropriate load for working cover reinsurance treaties. All of these methods use severity distributions at ultimate or final values. ISO uses severity distributions at different valuations in their increased limit reviews but do not measure how individual claims change from one valuation to another. Transition matrices could be used to model this activity. In the formulation of the algorithm used in this paper I am using the severity distribution format as used by Heckman and Meyers (probability of loss in certain intervals is specified, the loss within an interval is uniform). Appendix A sheets one and five show the severity distributions for paid and incurred losses at twelve, twenty four, thirty six, and forty eight month valuations. The average loss is shown at the bottom of each column. Sheets two through four and six through eight show the transition matrices to go from one valuation to the next. Since I am using severity distributions consisting of twenty intervals each transition matrix is twenty by twenty. The second column in sheet 2, the column labeled "0", shows the movement of claims in the first loss interval ($0 to $5,000) at twelve months to other loss intervals at twenty four months. In this example 45% of claims remain in the first interval. Twenty five percent of claims move up an interval($5,000 to $10,000), 15% move up two intervals, 10% three intervals, and 5% four intervals. Other columns show how losses in other intervals are expected to move during the course of the development period. You will note; entries in each column sum to one, amounts beneath the diagonal represent positive development (claims get larger), and entries above the diagonal represent negative development (claims get smaller). In terms of matrix notation if S_1 is the severity vector at the first valuation and T_{12} is the first to second valuation transition matrix then S_2, the severity distribution at the second valuation, equals $T_{12} \cdot S_1$. This can be extended so that $S_3 = T_{23} \cdot S_2$, $S_4 = T_{34} \cdot S_3$ and so on. The ultimate severity distribution can be obtained from the initial severity distribution at twelve months and all the transition matrices.

This approach can be used for paid losses as well as case incurred losses. If paid and incurred transactions are used from the same set of losses you should be able to produce the same ultimate severity distribution in both instances. The illustrative paid and case incurred material (strictly hypothetical and not based on any data set) contained in Appendix A produce roughly the same severity distribution at forty eight months.

These initial severity distributions and transition matrices are used to model the paid and case incurred activity on a claim by claim basis. This routine is then used in a simulation program to calculate an aggregate loss distribution. The final aggregate loss distribution is similar to one produced using the Heckman - Meyers algorithm. This approach extends the aggregate loss distribution over development and payment periods in a way which is consistent with the ideas underlying the collective risk model.

To illustrate this I used the following algorithm to produce aggregate losses:

1. Randomly select the number of claims for a year from a negative binomial distribution with mean equal to 126 (approximately) and variance 378. The mean number of claims was selected so that the expected ultimate loss amount is about $5,000,000.
2. For each claim randomly select a report lag from a poisson distribution with mean equal to one half. If the lag is greater than two, cap the lag at two. This was done so that all years would be at ultimate values at the end of six development years. For purposes of simplification the initial severity distributions and transition matrices do not vary as a function of lag. In this example if the lag for a claim is one year the twelve month severity distribution is used as the twenty four month severity distribution and all transition matrices are adjusted accordingly. In practice the initial severity distributions and transition matrices would likely vary as a function of lag because claims which are reported later usually have higher average values.
3. For each claim randomly select a loss interval from the case incurred loss severity distribution at twelve months. Within the interval randomly select a loss amount on the assumption that losses are uniformly distributed in the interval. This is the value of the claim at twelve months.

4. For each claim at twelve months enter the appropriate column of the twelve to twenty four month transition matrix (based on the loss interval) and randomly select a loss interval for the twenty four month valuation (determined by the row). If the loss interval does not change use the twelve month loss value at the value of the claim at twenty four months. If the loss interval changes randomly select a loss amount on the assumption that losses are uniformly distributed in the new interval.
5. Repeat step four for the other development periods until the claim is at ultimate. This produces a series of case incurred claim amounts for an individual claim at different loss valuations.
6. When the final or ultimate loss interval is determined work backwards using the paid transition matrices and paid severity distributions to determine the payment history for the claim. For example, suppose a claim is in the tenth loss interval at development period four (this is ultimate). It is possible to determine what loss intervals the claim could have been in at period three (i.e., those columns that have a non-zero entry in the tenth row of the transition matrix) and to randomly select a period three loss interval based on the relevant transition matrix and the period three paid severity distribution. That is, the probability of being in the tenth interval at period four equals $\Sigma(t_{10j} \cdot s_j)$, j = 1 to 20 where s_j is the probability of being in the j^{th} interval of the period three severity distribution and t_{10j} is the tenth row of the period three to four transition matrix. Randomly assign a column j based on the ratio of $t_{10j} \cdot s_j$ to $\Sigma(t_{10j} \cdot s_j)$.
7. When all payment values for a claim are determined accumulate the paid and case incurred values and repeat steps two through six until all claims as specified in step one are finished.
8. Repeat steps one through seven for the desired number of simulation years.

I ran this procedure for 10,000 years using the material in Appendix A. The table below shows the average paid and case incurred values generated by the simulation:

Valuation	Paid Loss	Case Incurred Loss
12	697,224	1,160,996
24	1,768,930	2,729,345
36	3,289,626	4,095,227
48	4,486,742	4,783,310
60	4,927,417	4,982,895
72	5,010,529	5,010,529

More importantly I accumulated various distributions about average values. Rather than show tables of the resultant distributions I will illustrate them graphically. (The program output can be used to calculate means, variances, deciles, etc.) Exhibits A through G show the graphs of a number of distributions.

Exhibit A, Sheet two shows the distribution of losses at ultimate values. I have labeled this "Outstanding Losses at Time 0" because it represents the a priori distribution of loss before any experience has been registered. This graph was prepared using losses at their ultimate values after the simulation had worked through all of the transition matrices. Using the accumulated loss arrays by year it is also possible to determine the distribution of outstanding losses (case outstanding and IBNR) at the end of any valuation. The distribution of outstanding losses is obtained by subtracting paid losses from ultimate losses. Exhibit A, Sheets three through five shows the distribution of outstanding losses at the end of the first, second, and third valuation respectively. Exhibit A, Sheet one shows these distributions on the same graph. This illustrates the reduction in average outstanding loss as well as variance over time. It is important to realize that these distributions are on an a priori basis. To determine the variability of reserves given a particular amount of reported losses at a specified valuation it would be necessary to determine the outstanding loss distribution on a conditional basis. If variability arouse only from claim counts and the first valuation severity distribution (i.e., there were no IBNR claims or case development) the conditional variance of outstanding losses at the first valuation would be zero.

Exhibit B shows similar graphs for IBNR reserves. These distributions were calculated by subtracting case incurred losses from ultimate losses. Exhibit B, Sheet one, as in Exhibit A, Sheet one, shows the reduction in average IBNR reserves and variance over time.

Exhibits C and D were determined from the accumulated loss arrays too. These graphs show the distribution of incremental (calendar year) paid and case reported losses respectively for a variety of valuations.

Exhibits E and F show the distribution of paid and case incurred loss development factors. The substantial reduction in loss development factor variance as losses mature is particularly noticeable in the sheet one of both exhibits. This type of analysis could be helpful in establishing credibility standards for development factors or to help select the underlying curve to use to model loss development factors for other variability of loss reserve approaches.[6,7]

Exhibit G shows the distribution of run off ratios of loss reserves as of twelve months. I used case incurred loss development factors to estimate ultimate losses and calculated the run off ratio by dividing ultimate losses less paid losses at twelve months by estimated ultimate losses at twelve months less paid losses at twelve months. I did not allow for sufficient room in the program output to show the tail of the distribution - in this example it appears there is continued risk of adverse run off in excess of 50% of carried reserves. This type of analysis might be used to test different IBNR reserving methods under different claim department reserving practices.[8]

I have tried to outline a straightforward approach that might be used to help quantify the variability of a number of different reserve amounts or loss development measures. I am aware that specifying the transition matrices for different development periods on both a paid and case basis could be time consuming and that once accomplished the simulations could take a great deal of computer time. However, there is no substitute for data and it is appealing that such transition matrices could be tailored to individual claim department practices and empirical severity distributions. In addition computer performance continues to improve making large simulation exercises more practical.

I am also aware that this method does not address parameter risk. This is an important source of risk and the variance indications obtained from this approach should be viewed accordingly.

REFERENCES

[1] Hayne, R., "Application of Collective Risk Theory to Estimate Variability in Loss Reserves", 1988 Casualty Actuarial Society Discussion Paper Program, p. 275.

[2] Heckman, P. and Meyers, G., "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions", Proceedings of the Casualty Actuarial Society, LXX (1983), p. 22.

[3] Panjer, H. "Recursive Evaluation of a Family of Compound Distributions", ASTIN Bulletin, v.12 (1981),p. 22.

[4] Bear, R. and Nemlick, K. "Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Reinsurance Treaties", Proceedings of the Casualty Actuarial Society, LXXVII (1990), p. 60.

[5] Patrik, G. and John, R., "Pricing Excess of Loss Casualty Working Cover Reinsurance Treaties", Casualty Actuarial Society 1980 Discussion Paper Program, p. 399.

[6] Kelly, M. "Practical Loss Reserving Method with Stochastic Development Factors", 1992 Casualty Actuarial Society Discussion Paper Program, p. 355.

[7] Hayne, R., "An Estimate of Statistical Variation in Development Factor Methods", Proceedings of the Casualty Actuarial Society LXXII (1985), p. 25.

[8] Stanard, J. "A Simulation Test of Prediction Errors of Loss reserve Estimation Techniques", Proceedings of the Casualty Actuarial Society LXXII (1985), p. 124.

Exhibit A
Sheet 1

Aggregate Outstanding Losses

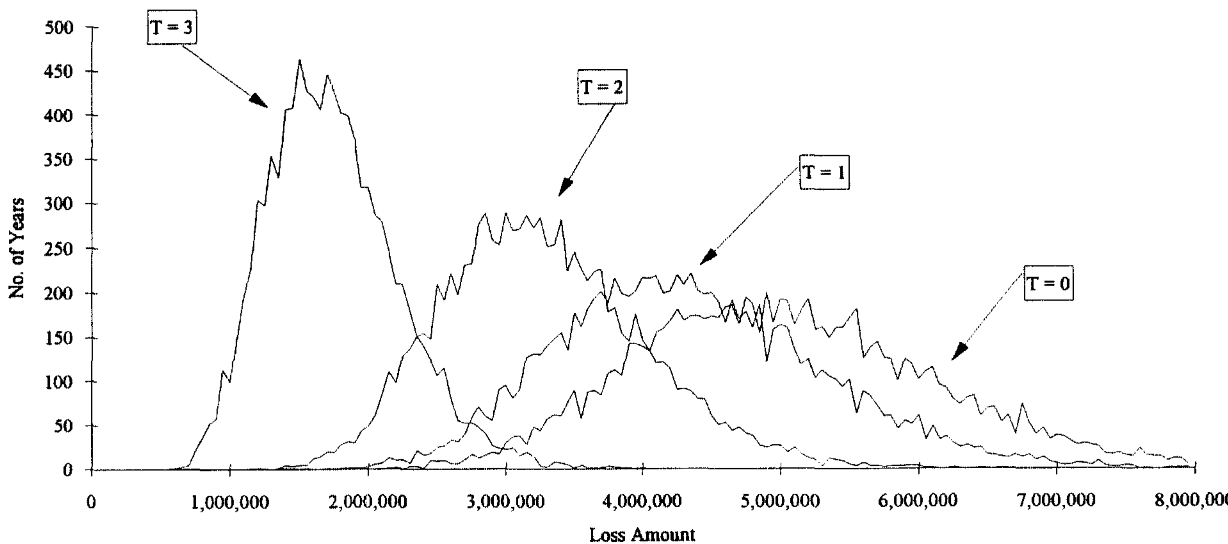


Exhibit A
Sheet 2

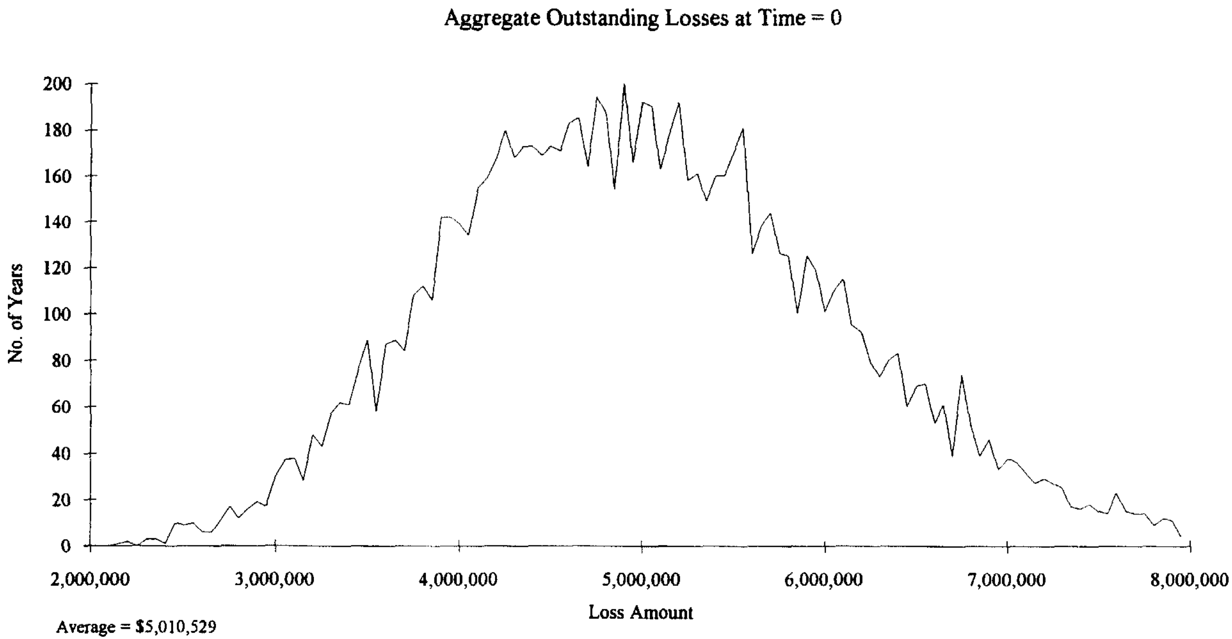
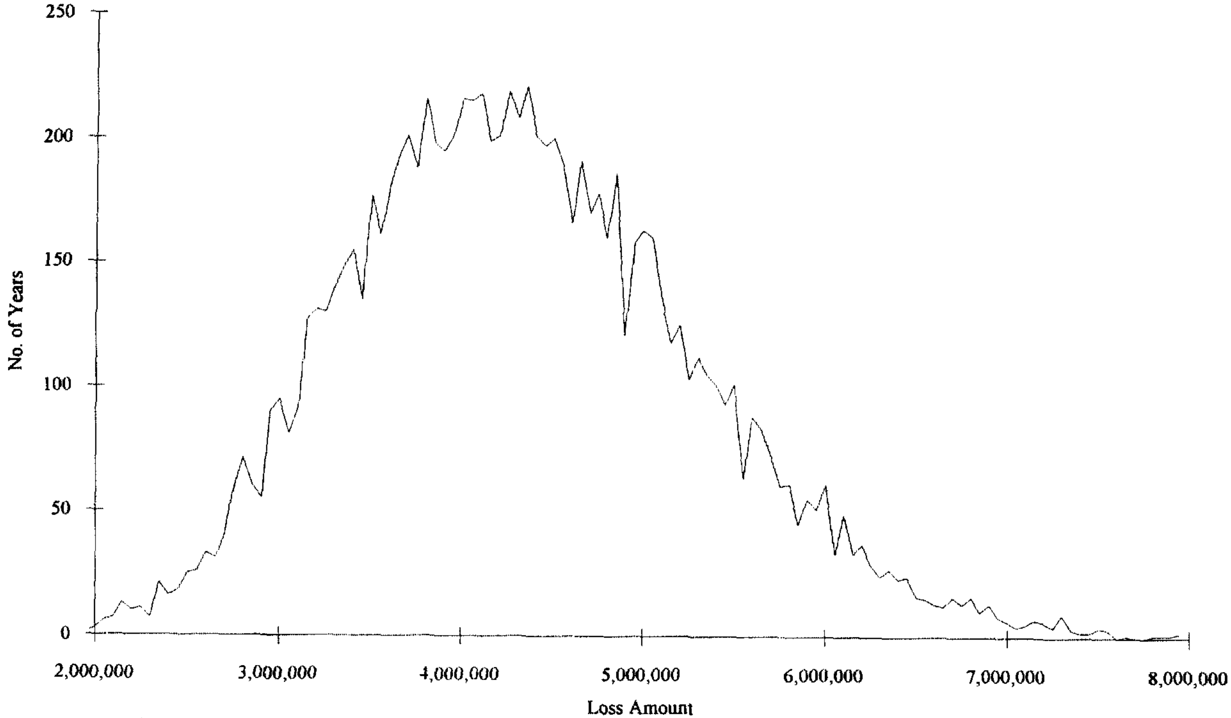


Exhibit A
Sheet 3

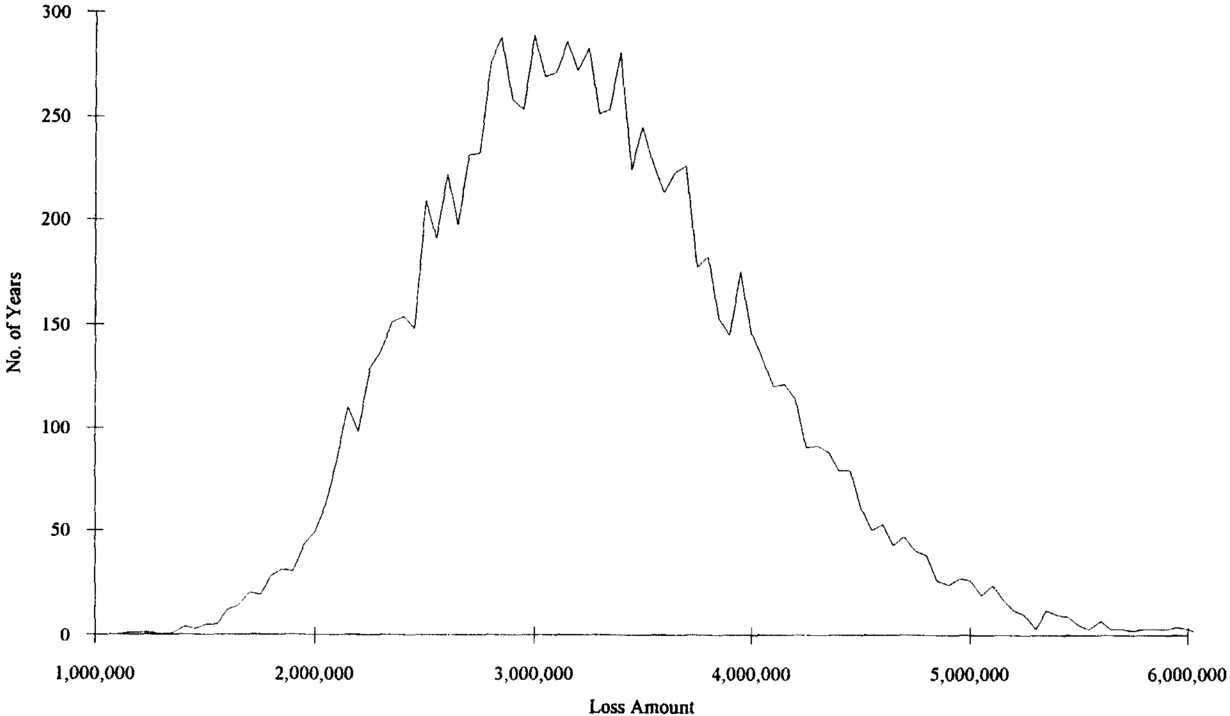
Aggregate Outstanding Losses at Time = 1



Average = $4,313,306

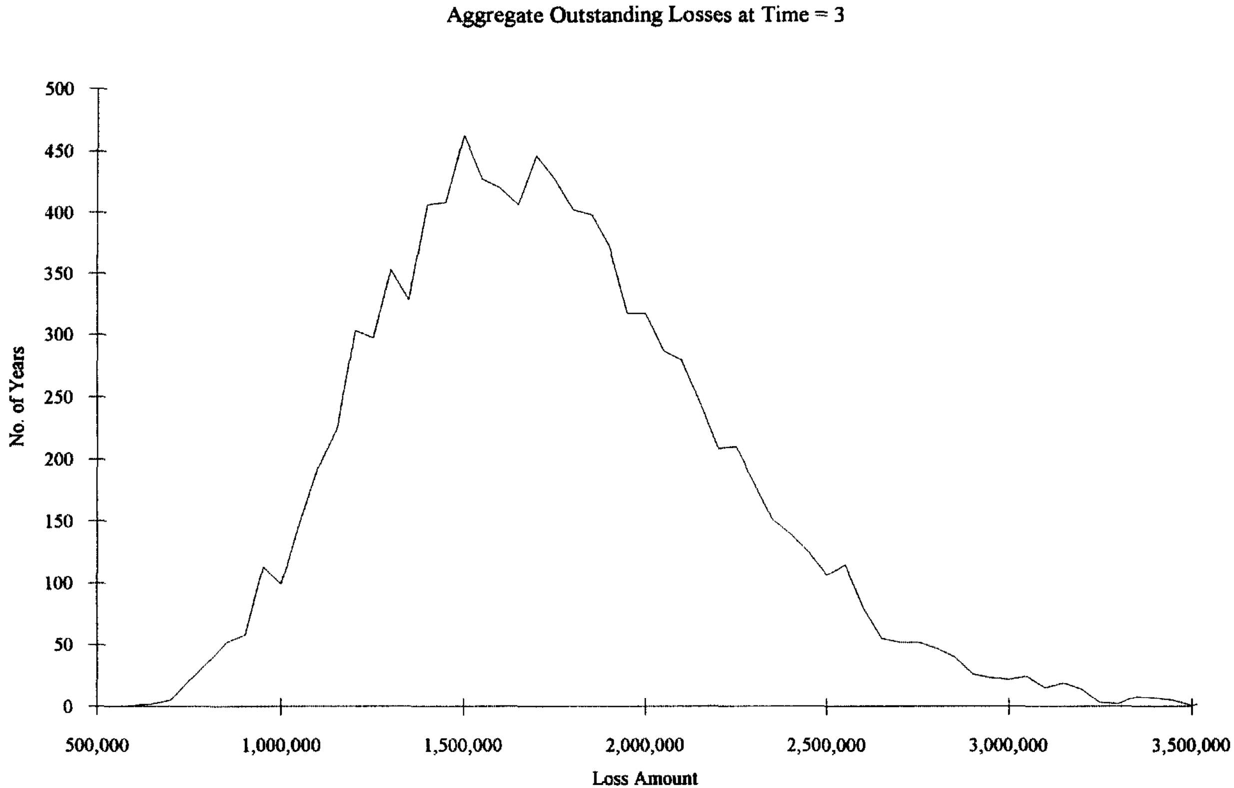
Exhibit A
Sheet 4

Aggregate Outstanding Losses at Time = 2



Average = $3,241,599

Exhibit A
Sheet 5



Average = $1,720,903

Exhibit B
Sheet 1

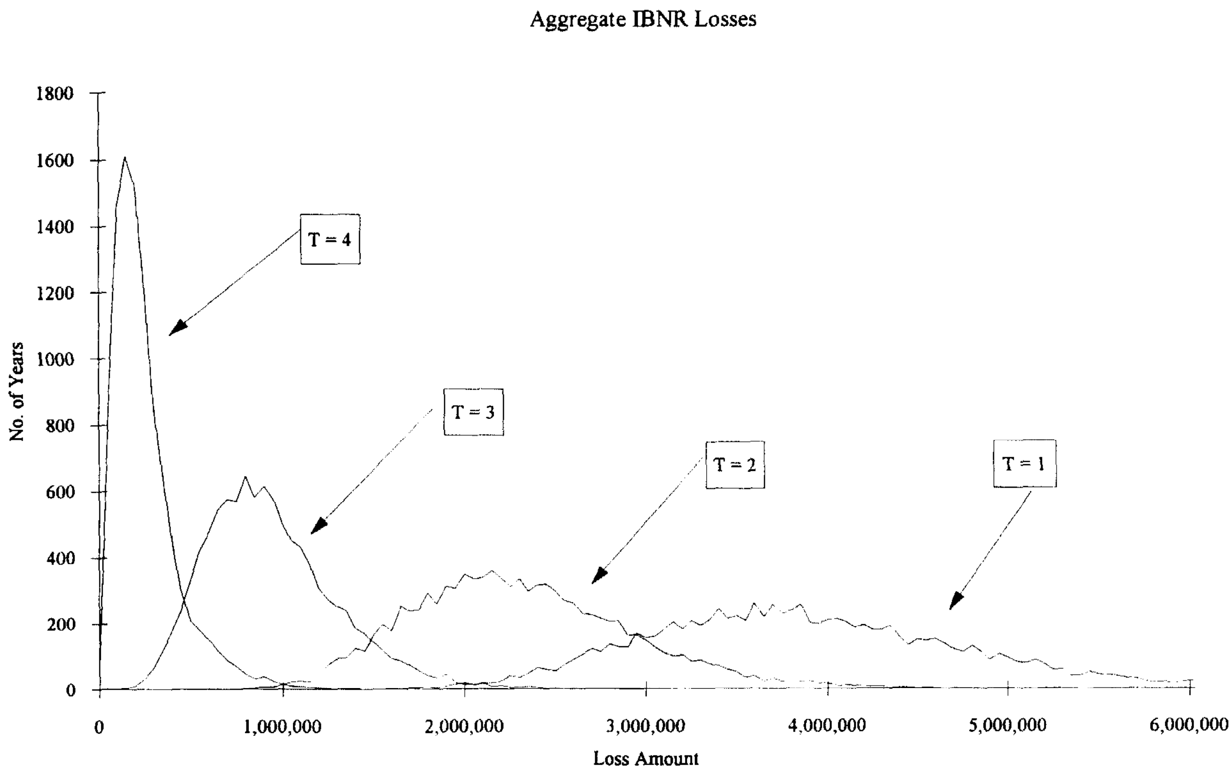
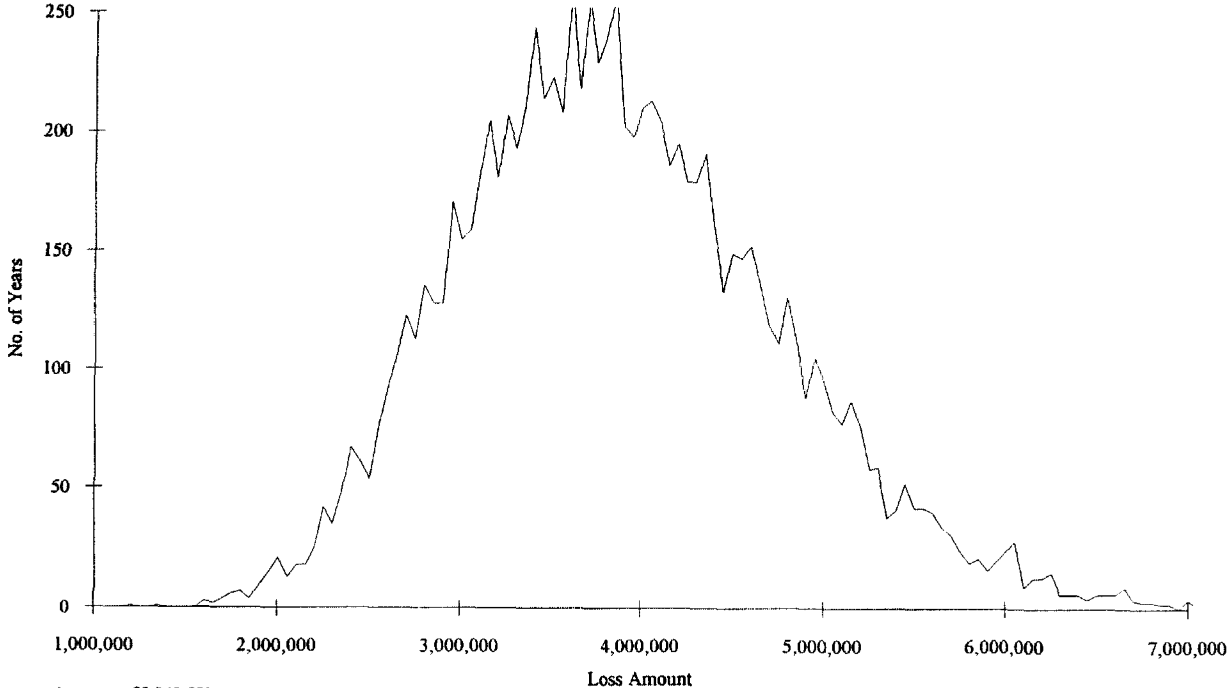


Exhibit B
Sheet 2

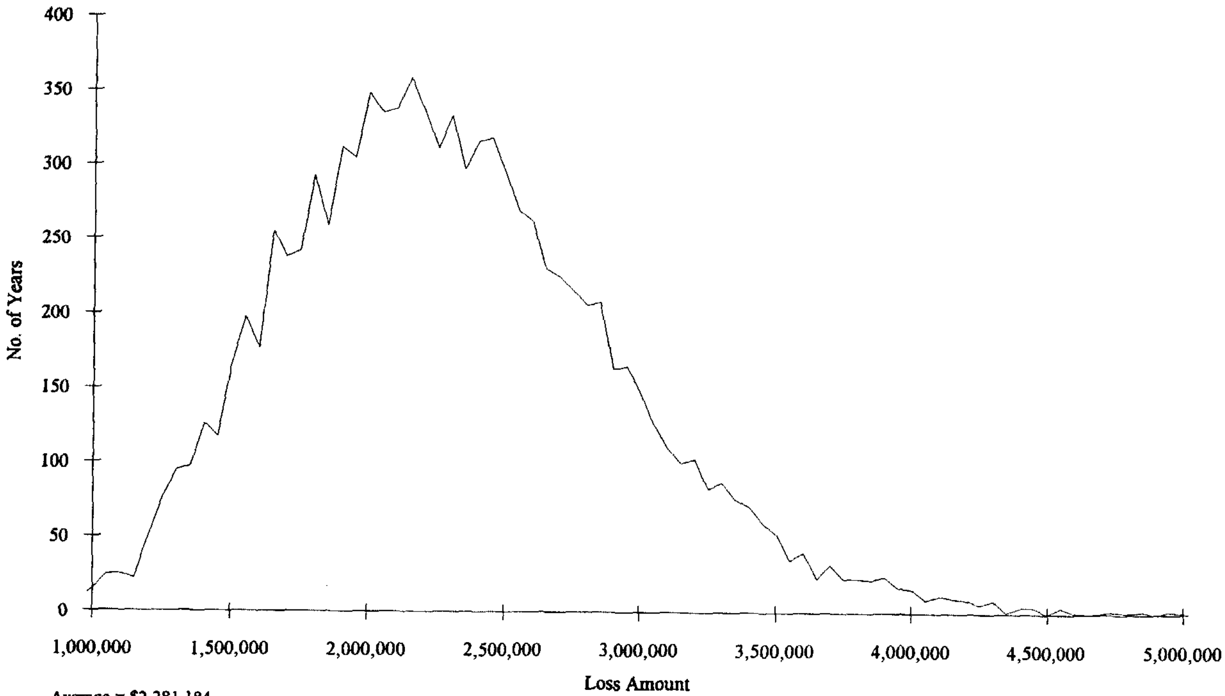
Aggregate IBNR Losses at Time = 1



Average = $3,849,533

Exhibit B
Sheet 3

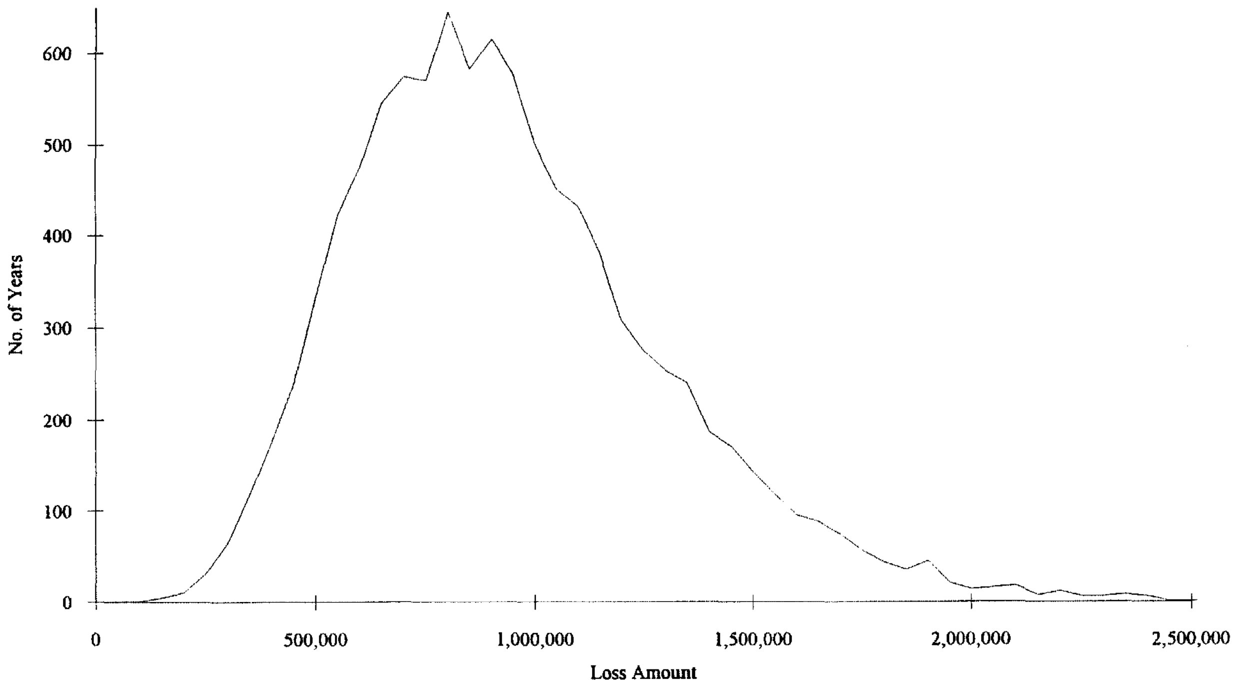
Aggregate IBNR Losses at Time = 2



Average = $2,281,184

Exhibit B
Sheet 4

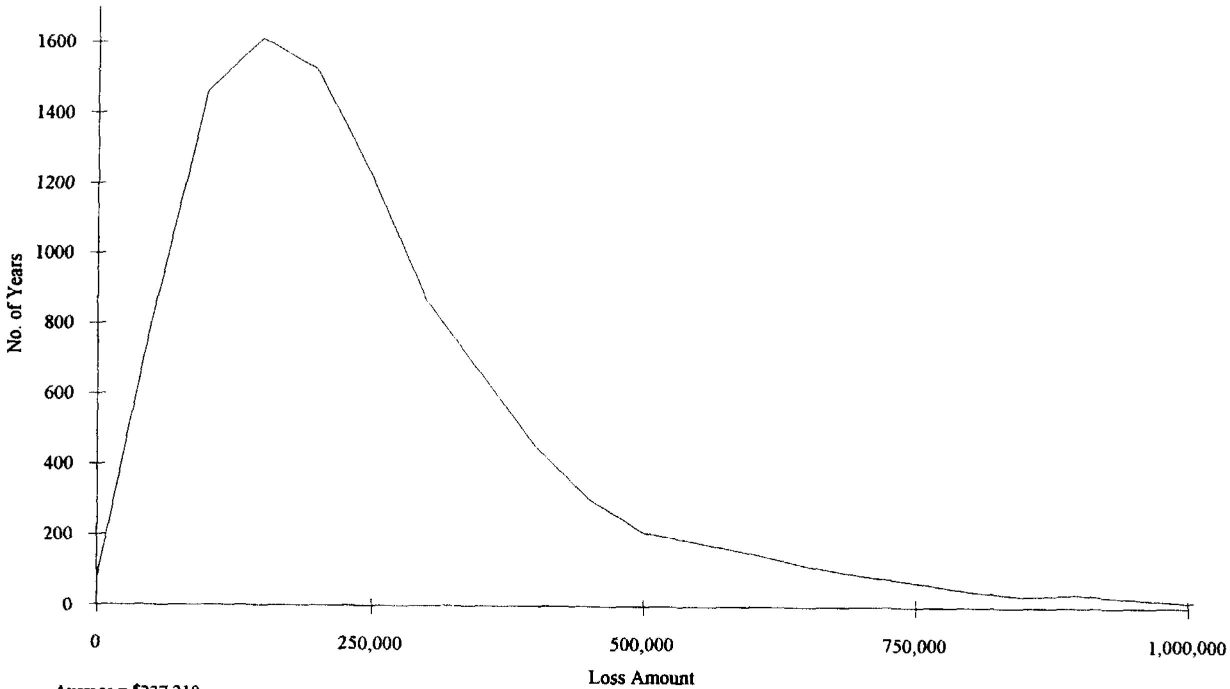
Aggregate IBNR Losses at Time = 3



Average = $915,302

Exhibit B
Sheet 5

Aggregate IBNR Losses at Time = 4



Average = $227,219

Exhibit C

Incremental Paid Losses

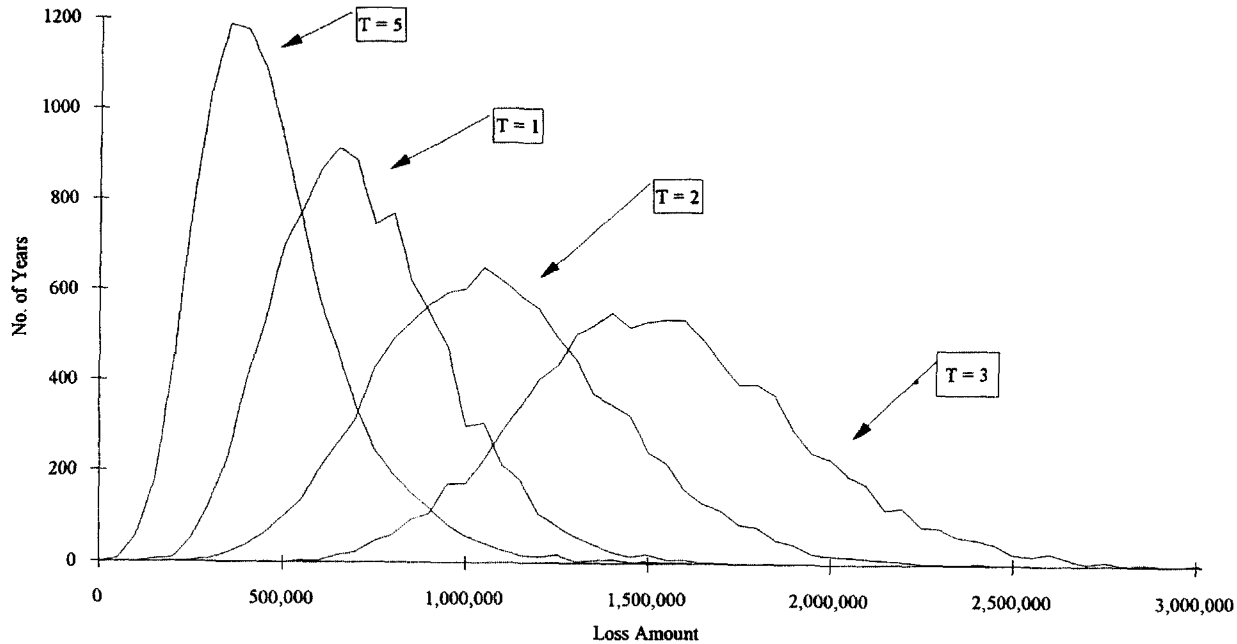


Exhibit D

Incremental Incurred Losses

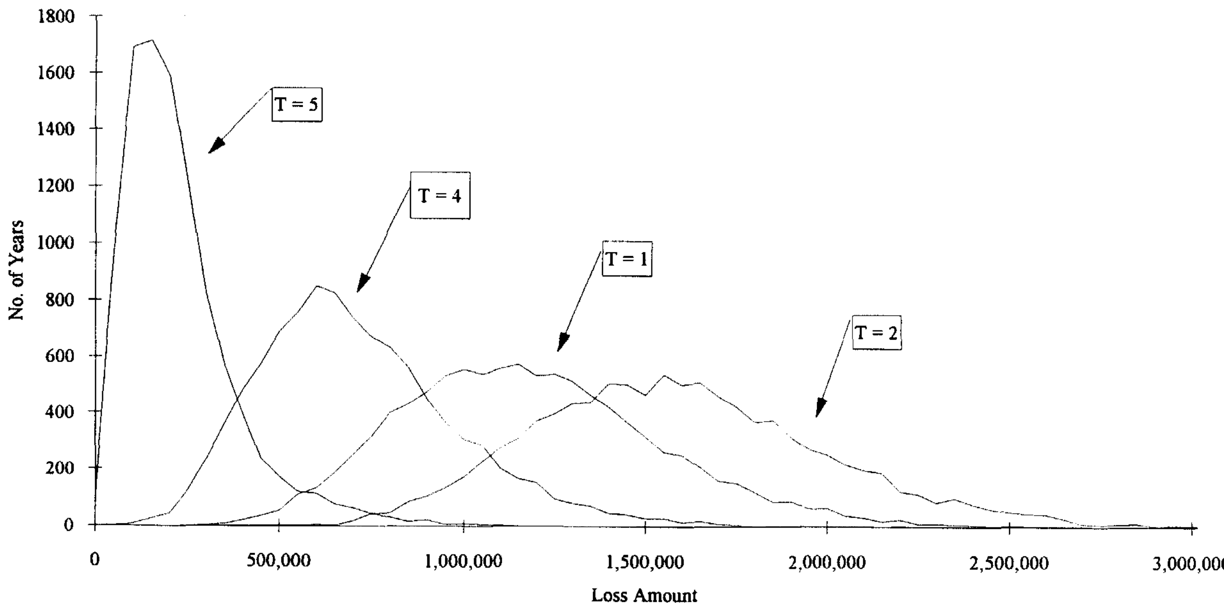
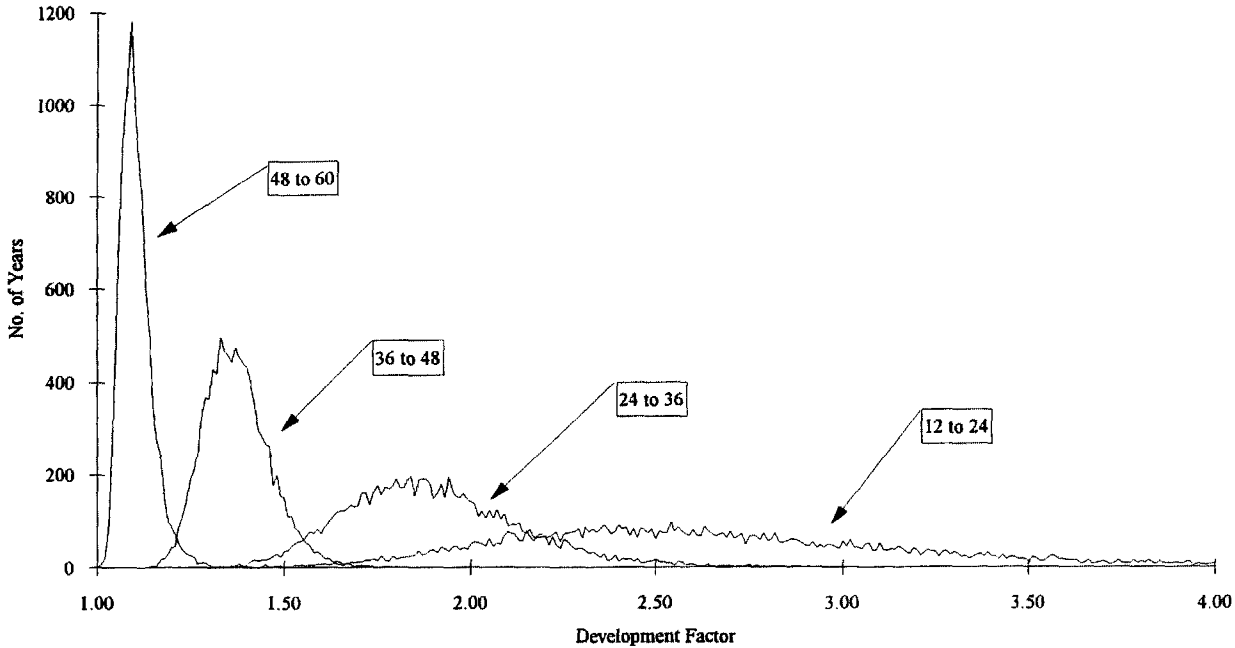
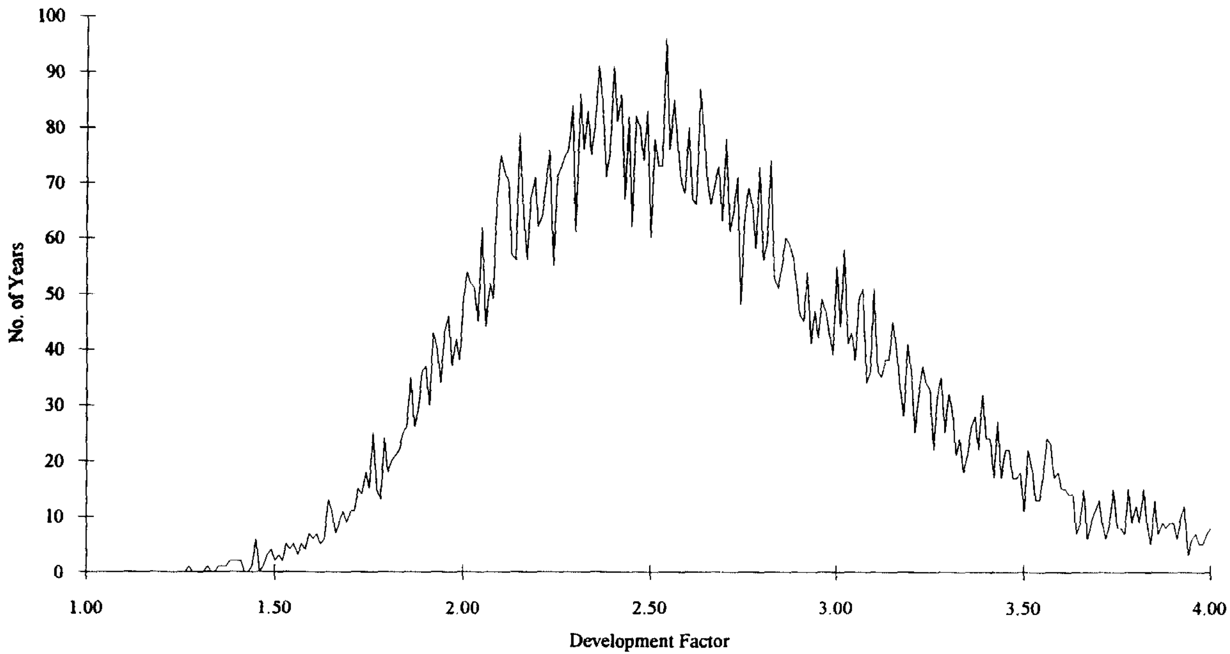


Exhibit E
Sheet 1

Paid Loss Development Factors



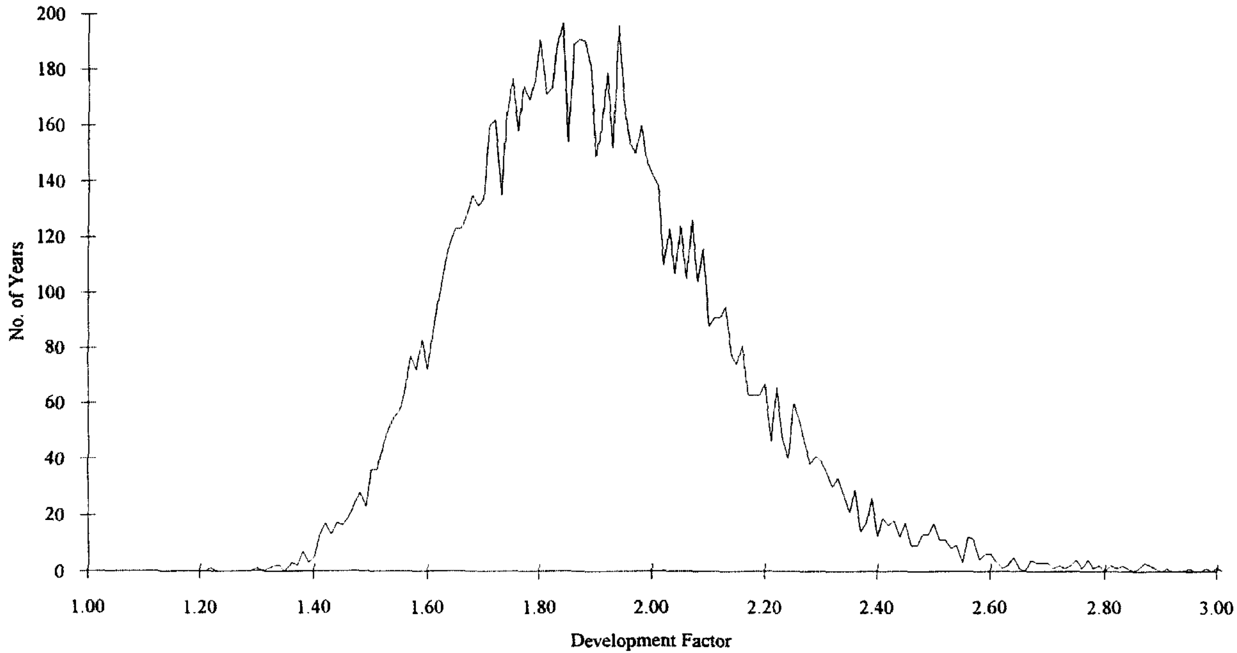
Paid Loss Development Factor - 12 to 24



Arithmetic Average = 2.65

Exhibit E
Sheet 3

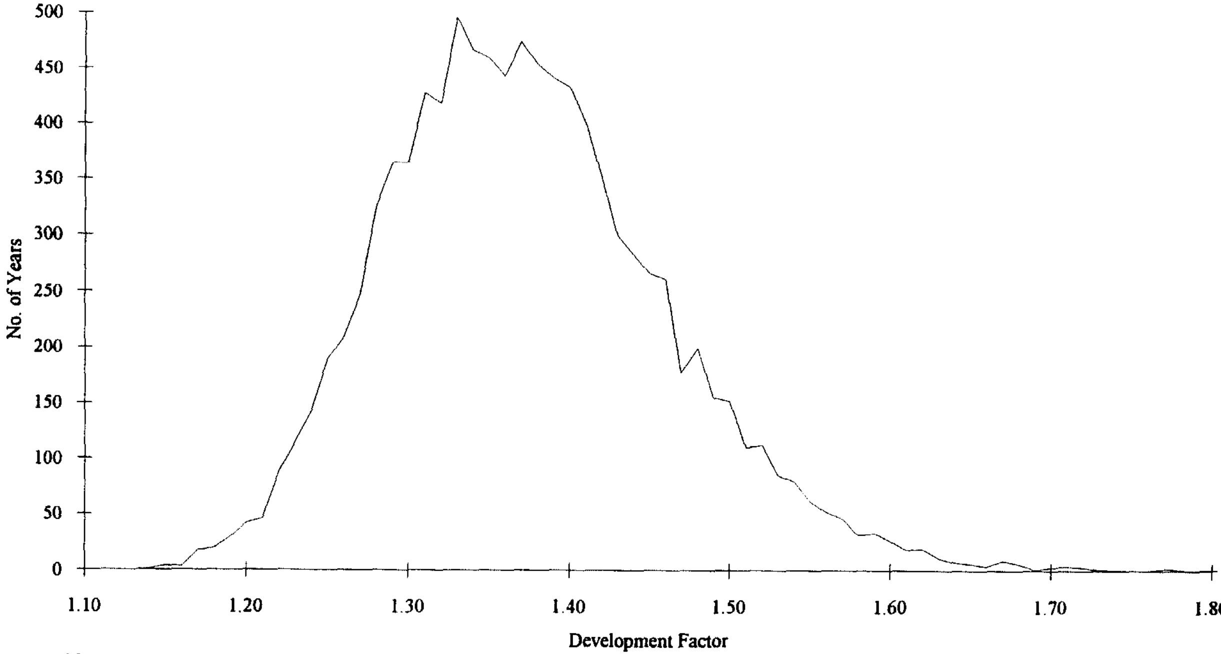
Paid Loss Development Factor - 24 to 36



Arithmetic Average = 1.89

Exhibit E
Sheet 4

Paid Loss Development Factor - 36 to 48



Arithmetic Average = 1.37

Exhibit F

Incurred Loss Development Factors

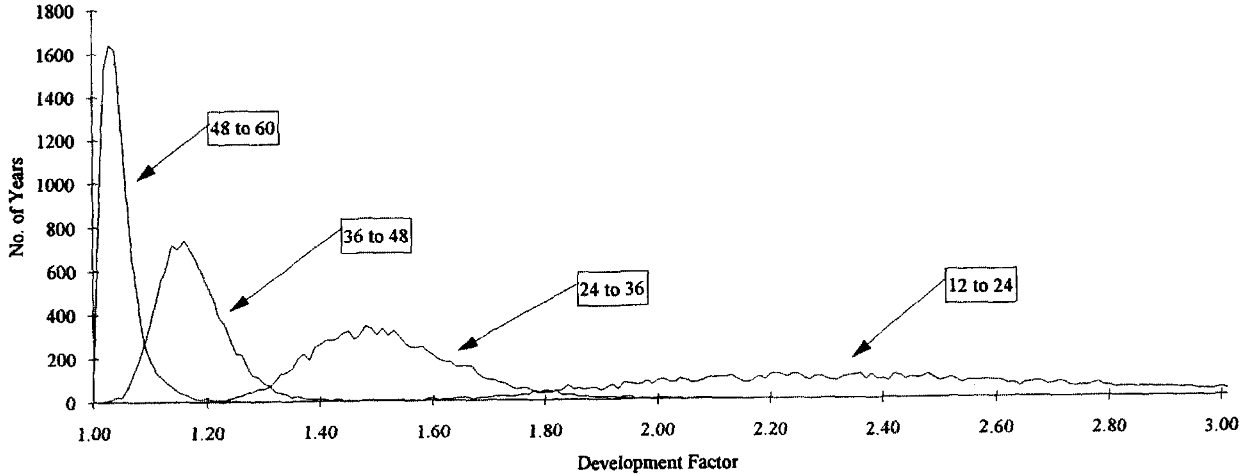
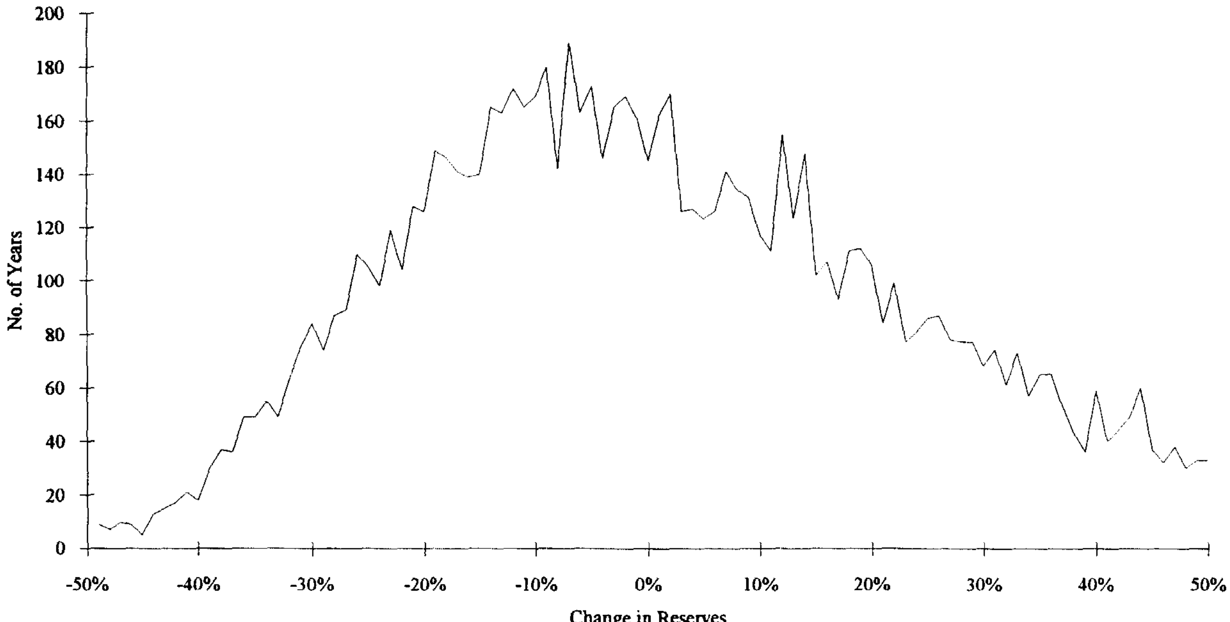


Exhibit G

Run Off Ratio Analysis of Reserves at First Valuation



Arithmetic Average = 6.3%

Appendix A
Sheet 1

Distribution of Case Incurred Losses

Loss Range			Case Incurred Loss Severity Distribution at ... Months			
			12	24	36	48
0	-	5,000	.67000000	.32550000	.25080000	.22572000
5,000	-	10,000	.12000000	.22950000	.20262500	.21757375
10,000	-	25,000	.07000000	.16100000	.17547500	.17683250
25,000	-	50,000	.05000000	.11900000	.14227500	.14393500
50,000	-	75,000	.04000000	.07500000	.09372500	.09615250
75,000	-	100,000	.02000000	.03300000	.05100000	.05313625
100,000	-	150,000	.01000000	.02150000	.03345000	.03432750
150,000	-	200,000	.01000000	.01450000	.01999000	.01866400
200,000	-	250,000	.01000000	.01250000	.01435000	.01419650
250,000	-	300,000		.00500000	.00760000	.00699450
300,000	-	350,000		.00200000	.00382500	.00377000
350,000	-	400,000		.00100000	.00251000	.00258875
400,000	-	450,000		.00050000	.00130000	.00178900
450,000	-	500,000			.00045000	.00143350
500,000	-	600,000			.00028000	.00090475
600,000	-	700,000			.00017500	.00072225
700,000	-	800,000			.00009000	.00052500
800,000	-	900,000			.00005500	.00031450
900,000	-	1,000,000			.00002000	.00027800
1,000,000	-	1,000,000			.00000500	.00014175
Average			15,175.00	28,040.00	37,228.00	39,693.29

Appendix A
Sheet 2

Case Incurred Loss Transition Matrix - 12 to 24

Loss Range at 24 mos.	Loss Range at 12 months: 0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.450	.200																		
5,000	.250	.400	.200																	
10,000	.150	.200	.450	.100																
25,000	.100	.100	.150	.550	.050															
50,000	.050	.050	.100	.150	.500	.050														
75,000		.050	.050	.100	.200	.500	.050													
100,000			.050	.050	.150	.200	.500	.050												
150,000				.050	.050	.150	.200	.500												
200,000					.050	.050	.150	.200	.600											
250,000						.050	.050	.150	.200	.600										
300,000							.050	.050	.100	.200	.600									
350,000								.050	.050	.100	.200	.600								
400,000									.050	.050	.100	.200	.600							
450,000										.050	.050	.100	.200	.600						
500,000											.050	.050	.100	.200	.600					
600,000												.050	.050	.100	.200	.600				
700,000													.050	.050	.100	.200	.650	.100	.100	.050
800,000														.050	.050	.100	.150	.650	.200	.100
900,000															.050	.050	.100	.150	.500	.200
1,000,000																.050	.100	.100	.200	.650

Case Incurred Loss Transition Matrix - 24 to 36

Loss Range at 36 mos.	Loss Range at 24 months																			
	0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.700	.100																		
5,000	.150	.600	.100																	
10,000	.100	.150	.600	.100																
25,000	.050	.100	.150	.600	.100															
50,000		.050	.100	.150	.600	.100														
75,000			.050	.100	.150	.600														
100,000				.050	.100	.150	.700													
150,000					.050	.100	.150	.670												
200,000						.050	.100	.150	.670											
250,000							.050	.100	.150	.640										
300,000								.050	.100	.150	.550									
350,000								.030	.050	.100	.200	.550								
400,000									.030	.050	.100	.200	.550							
450,000										.030	.050	.100	.200	.500						
500,000										.020	.040	.050	.100	.200	.550					
600,000										.010	.030	.040	.050	.100	.200	.600				
700,000											.020	.030	.040	.050	.100	.200	.650	.100	.100	.050
800,000											.010	.020	.030	.050	.050	.100	.150	.650	.200	.100
900,000												.010	.020	.050	.050	.050	.100	.150	.500	.200
1,000,000													.010	.050	.050	.050	.100	.100	.200	.650

Appendix A
Sheet 4

Case Incurred Loss Transition Matrix - 36 to 48

Loss Range at 48 mos.	Loss Range at 36 months																			
	0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.900																			
5,000	.100	.950																		
10,000		.050	.950																	
25,000			.050	.950																
50,000				.050	.950															
75,000					.050	.950														
100,000						.050	.950													
150,000							.050	.850												
200,000								.100	.850											
250,000								.050	.100	.600										
300,000									.050	.150	.500									
350,000										.100	.150	.500								
400,000										.050	.100	.150	.500							
450,000										.050	.100	.100	.150	.500						
500,000										.030	.050	.050	.100	.200	.500					
600,000										.020	.050	.050	.050	.100	.200	.500				
700,000											.050	.050	.050	.050	.100	.200	.600	.050	.050	
800,000												.050	.050	.050	.100	.100	.200	.650	.100	.050
900,000												.050	.050	.050	.050	.100	.100	.200	.650	.100
1,000,000													.050	.050	.050	.100	.100	.100	.200	.850

Distribution of Paid Losses

Loss Range			Paid Loss Severity Distribution at ... Months			
			12	24	36	48
0	-	5,000	.83900000	.51179000	.28148450	.22518760
5,000	-	10,000	.02200000	.17924000	.20094000	.21704890
10,000	-	25,000	.03400000	.13807000	.18165150	.17642663
25,000	-	50,000	.03200000	.05549000	.14032750	.14364213
50,000	-	75,000	.03800000	.04215000	.07695400	.09596605
75,000	-	100,000	.01500000	.01655000	.04276350	.05302065
100,000	-	150,000	.01100000	.02880000	.03588100	.03435765
150,000	-	200,000	.00600000	.00126000	.00727910	.01871986
200,000	-	250,000	.00300000	.01753000	.01828910	.01425819
250,000	-	300,000		.00673000	.00739710	.00947793
300,000	-	350,000		.00173000	.00331240	.00536202
350,000	-	400,000		.00063000	.00231130	.00199542
400,000	-	450,000		.00003000	.00057470	.00149668
450,000	-	500,000			.00035740	.00100664
500,000	-	600,000			.00023830	.00063843
600,000	-	700,000			.00014590	.00069786
700,000	-	800,000			.00005470	.00037797
800,000	-	900,000			.00003080	.00013853
900,000	-	1,000,000			.00000690	.00011645
1,000,000	-	1,000,000			.00000030	.00006443
Average			10,845.00	21,630.13	33,768.83	39,673.48

Appendix A
Sheet 6

Paid Loss Transition Matrix - 12 to 24

Loss Range at 24 mos.	Loss Range at 12 months																			
	0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.610																			
5,000	.200	.520																		
10,000	.150	.200	.230																	
25,000	.030	.200	.320	470																
50,000	.010	.050	.200	250	.470															
75,000		.030	.050	.070	.200	.290														
100,000			.200	.200	.170	.250	.490													
150,000				.010	.010	.010	.010	.050												
200,000					.150	.250	.200	.500	.960											
250,000						.200	.200	.250	.010	600										
300,000							.100	.100	.010	.200	.600									
350,000								.100	.010	.100	.200	600								
400,000									.010	.050	.100	.200	.600							
450,000										.050	050	.100	.200	.600						
500,000											.050	.050	.100	.200	600					
600,000												050	.050	.100	.200	600				
700,000													050	.050	.100	200	.650			
800,000														.050	050	.100	.150	750		
900,000															.050	.050	.100	150	800	
1,000,000																.050	.100	.100	.200	1.000

Paid Loss Transition Matrix - 24 to 36

Loss Range at 36 mos.	Loss Range at 24 months																			
	0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.550																			
5,000	.200	.550																		
10,000	.150	.200	.500																	
25,000	.100	.150	.250	.500																
50,000		.100	.150	.250	.580															
75,000			.100	.150	.250	.610														
100,000				.100	.150	.250	.690													
150,000					.020	.070	.150	.760												
200,000						.070	.150	.150	.720											
250,000							.010	.030	.150	.660										
300,000								.030	.070	.150	.600									
350,000								.030	.050	.100	.200	.600								
400,000									.010	.030	.050	.150	.550							
450,000										.030	.050	.100	.200	.500						
500,000										.020	.040	.050	.100	.200	.550					
600,000										.010	.030	.040	.050	.100	.200	.600				
700,000											.020	.030	.040	.050	.100	.200	.650			
800,000											.010	.020	.030	.050	.050	.100	.150	.750		
900,000												.010	.020	.050	.050	.050	.100	.150	.800	
1,000,000													.010	.050	.050	.050	100	.100	.200	1.000

Appendix A
Sheet 8

Paid Loss Transition Matrix - 36 to 48

Loss Range at 48 mos.	Loss Range at 36 months																			
	0	5,000	10,000	25,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000	350,000	400,000	450,000	500,000	600,000	700,000	800,000	900,000	1,000,000
0	.800																			
5,000	.200	.800																		
10,000		.200	.750																	
25,000			.250	.700																
50,000				.300	.700															
75,000					.300	.700														
100,000						.300	.600													
150,000							.400	.600												
200,000								.200	.700											
250,000								.200	.200	.590										
300,000									.100	.200	.620									
350,000										.050	.100	.560								
400,000										.050	.100	.200	.580							
450,000										.050	.050	.100	.200	.350						
500,000										.030	.030	.040	.080	.200	.450					
600,000										.030	.050	.050	.050	.100	.210	.550				
700,000											.050	.030	.050	.100	.100	.150	.600			
800,000												.010	.020	.100	.100	.100	.150	.700		
900,000												.010	.010	.100	.070	.100	.150	.200	.900	
1,000,000													.010	.050	.070	.100	.100	.100	.100	1.000

Statistical Methods for the Chain Ladder Technique

by Richard J. Verrall

Abstract

This paper considers the application of loglinear models to claims reserving. The models encompass the chain ladder technique and extend the range of the possible analyses. By bringing the methods within a statistical framework, a coherent strategy for testing goodness of fit and for forecasting outstanding claims is produced. Improvements to the basic chain ladder technique are given which use Bayesian methods.

Key Words Claims Reserving, Linear Models, Bayes and Empirical Bayes Methods, State Space Models, The Chain Ladder Technique.

1. Introduction

Forecasting outstanding claims and setting up suitable reserves to meet these claims is an important part of the business of a general insurance company. Indeed, the published profits of these companies depend not only on the actual claims paid, but on the forecasts of the claims which will have to be paid. It is essential, therefore, that a reliable estimate is available of the reserve to be set aside to cover claims, in order to ensure the financial stability of the company and its profit and loss account. There are a number of methods which have proved useful in practice, one of which is extensively used and is known as the chain ladder technique. In recent years, a statistical framework for analysing this data has been built up, which encompasses the actuarial method, extending and consolidating it. The aim of this paper is to bring together these results and to illustrate how the chain ladder technique can be improved and extended, without altering the basic foundations upon which it has been built. These improvements are designed to overcome two problems with the chain ladder technique. Firstly, that not enough connection is made between the accident years, resulting in an over-parametrised model and unstable forecasts. Secondly, that the development pattern is assumed to be the same for all accident years. No allowance is made by the chain ladder technique for any change in the speed with which claims are settled, or for any other factors which may change the shape of the run-off pattern. Before describing the methods for overcoming these problems, we first define the chain ladder linear model, and show how it can be used to give upper prediction bounds on total outstanding claims.

2. The Data

It will be assumed throughout this paper that the data is in the form of a triangle. It should be emphasised that this is for notational convenience only: there are no problems in extending the methods to other shapes of data. The year in which the policy is written will be called the underwriting year, accident year or year of business. In the years after the policy was written the company may receive claims related to that policy, and these claims are indexed by their business year and the delay. The following data set, which is taken from Taylor and Ashe (1983) will be used for illustrative purposes. The data is given in the form of incremental claims in each delay year.

357848	766940	610542	482940	527326	574398	146342	139950	227229	67948
352118	884021	933894	1183289	445745	320996	527804	266172	425046	
290507	1001799	926219	1016654	750816	146923	495992	280405		
310608	1108250	776189	1562400	272482	352053	206286			
443160	693190	991983	769488	504851	470639				
396132	937085	847498	805037	705960					
440832	847631	1131398	1063269						
359480	1061648	1443370							
376686	986608								
344014									

The data may take a slightly different shape if one or more of the corners is truncated, but this paper will consider triangles of data (without loss of generality). The first column will be labelled delay year 1, rather than delay year 0.

Sometimes, the rows are standardised by dividing by a measure of the volume of business, such as the premium income. This is reversed when predictions of outstanding claims are made. For the above triangle the exposure factors are:

610 721 697 621 600 552 543 503 525 420.

The incremental claims relating to business year i and delay year j will be denoted Z_{ij}, so that the set of data observed is

$$\{ Z_{ij} : i=1,...,t; j=1,...,t-i+1\}$$

The statistical approach uses the incremental claims, but the chain ladder technique is applied to the cumulative claims, which are defined by:

$$C_{ij} = \sum_{k=1}^{j} Z_{ik}$$

The problem is to forecast outstanding claims on the basis of past experience. In other words to fill in the lower right hand triangle of claims. Sometimes it is also useful to extend the forecasts beyond the latest delay year (i.e. to the right of the claims run-off triangle). The standard actuarial technique does not attempt to do this.

3. Linear Models and the Chain Ladder Technique

This paper will concentrate on the chain ladder technique. It its familiar form, this assumes that the cumulative claims for each business year develop similarly by delay year, and estimates development factors as ratios of sums of cumulative claims with the same delay index. Thus the estimate of the development factor for column j is

$$\frac{\sum_{i=1}^{t-j+1} C_{ij}}{\sum_{i=1}^{t-j+1} C_{i,j-1}}. \tag{3.1}$$

The model on which this is based is

$$E\,(\,C_{ij}\mid C_{i1}\,,C_{i2}\,,\ldots,C_{i,j-1}\,) \;=\; \lambda_j\,C_{i,j-1} \qquad j=2,\ldots,t. \tag{3.2}$$

and (3.1) is an estimate of λ_j . It has the advantage that it is relatively straightforward to calculate, but there is no clear basis on which to examine the properties in greater detail. It can be seen as a useful "rough-and-ready" estimation method.

The expected ultimate loss, E (C_{it}) , is estimated by multiplying the latest loss, $C_{i,t-i+1}$, by the appropriate estimated λ-values :

$$\text{estimate of } E\,(\,C_{it}\,) = (\prod_{j=t-i+2}^{t} \tilde{\lambda}_j\,)\,C_{i,t-i+1}\,. \tag{3.3}$$

The chain ladder technique produces forecasts which have a row effect and a column effect. The column effect is obviously due to the parameters { λ_j ; j=2,...,t }. There is also a row effect since the estimates for each row depend not only on the parameters { λ_j ; j=2,...,t }, but also on the row being considered. The latest cumulative claims, $C_{i,t-i+1}$, can be considered as the row effect. This leads to consideration of other models which have row and column effects, in particular the two-way analysis of variance model. The connection is first made with a multiplicative model. This uses the non-cumulative data, Z_{ij} , and models them according to:

$$E (Z_{ij}) = U_i S_j \tag{3.4}$$

where U_i is a parameter for row i,

S_j is a parameter for column j

A multiplicative error structure is assumed.

Also $$\sum_{j=1}^{t} S_j = 1 \tag{3.5}$$

S_j is the expected proportion of ultimate claims which occur in the jth development year.

U_i is the expected total ultimate claim amount for business year i (neglecting any tail factor).

Kremer(1982) showed that the following relationships between the parameters hold:

$$S_j = \frac{\lambda_j - 1}{\prod_{l=j}^{t} \lambda_l} \qquad (j \geq 2) \tag{3.6}$$

$$S_1 = \frac{1}{\prod_{l=2}^{t} \lambda_l} \tag{3.7}$$

and $$U_i = E (C_{it}). \tag{3.8}$$

Estimators of { S_i ; i=1,...,t } and { U_j ; j=1,...,t } can be obtained by applying a linear model to the logged incremental claims data. Taking logs of both sides of equation (3.4), and assuming that the incremental claims are positive:

$$E (Y_{ij}) = \mu + \alpha_i + \beta_j \tag{3.9}$$

where $Y_{ij} = \log Z_{ij}$

and the errors now have an additive structure and are assumed to have mean zero.

The errors will also be assumed to be identically distributed with variance σ^2, although this distributional assumption can be relaxed.

The usual restriction is placed on the parameters to ensure a non-singular design matrix, in this case

$$\alpha_1 = \beta_1 = 0.$$

Now equation (3.9) can be written in the form of a linear model. Suppose, for example, there are three years of data.

Then

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{13} \\ y_{22} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_2 \\ \alpha_3 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{13} \\ e_{22} \\ e_{31} \end{bmatrix} \qquad (3.10)$$

Kremer (1982) derived the normal equations for the chain ladder linear model and also examined the relationships between the linear model and the crude chain ladder technique. By reversing the transformation it can be shown that

$$U_i = e^{\alpha_i} e^{\mu} \sum_{j=1}^{i} e^{\beta_j}$$

Kremer showed that if the estimate of U_i is obtained by "hatting" the parameters in the above identity, the result is very similar to that obtained from the chain ladder technique. The resulting estimate of U_i is not the maximum likelihood estimate, neither is it unbiased, but it does serve the purpose of illustrating the similarity between the chain ladder technique and the two-way analysis of variance.

Furthermore, if all the geometric means are replaced by arithmetic means the estimators of the parameters of the models are equivalent. Thus the two estimation methods, the chain ladder method and the linear model, will produce identical results. The structure of the models is identical and the only difference is the estimation technique. It can be argued that the linear model estimates are best in a statistical sense, but it should be emphasised that in using the linear model instead of the crude chain ladder technique, there are no radical changes.

In general, a loglinear model applied to this data takes the form

$$\underline{y} = X\underline{\beta} + \underline{e}$$

where $\underline{y}$ is the vector of logged incremental claims,

X is the design matrix,

$\underline{\beta}$ is the parameter vector and

$\underline{e}$ is a vector of errors.

Apart from the chain ladder linear model, other models which have been suggested as suitable for claims data include a gamma curve (suggested by Zehnwirth (1985))

$$y_{ij} = \nu_i + \delta_i \log j + \gamma_i (j - 1) + e_{ij}$$

and an exponential tail (suggested by Ajne (1989)) in which the first few delay years follow the chain ladder model and the later delay years follow an exponential curve.
The statistical treatment facilitates the production of standard errors as well as point forecasts. This is a considerable advantage over the ad hoc methods, and allows 'safe' upper limits on outstanding claims to be set. The statistical analysis is more comprehensive and allows a greater study of the models, their fit to the data and any unusual features in the data. Also, Bayesian methodology can be incorporated to allow the structured input of other information, and to extend the range of the analysis by including empirical Bayes and state space methods. This has beneficial consequences for the stability of the predictions.

With reference to the computing aspects, Renshaw (1989) has shown how these models can be implemented in GLIM, and Christofides (1990) has used the spread-sheet package SuperCalc5.