Duration, Hiding in A Taylor Series

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Introduction

Duration has been touted as a tool for measuring the sensitivity of the price, or value, of an asset, or liability, whose cash flows are fairly determinable, to changes in interest rates. This paper seeks to describe the above relationship in a concrete fashion by expressing the value of an asset or liability as a function of the current interest rate. This function is then expanded in a Taylor series to illustrate just where the duration concept fits in. After this presentation is made, the Taylor series is further employed to illustrate that one may obtain a level of immunization as close to complete as desired by essentially matching successive terms in the Taylor series, the second of which reflects duration.

The Fundamental Relationships

The formula below presents the price of a known stream of cash flows given an interest rate *i*. This paper will assume a flat yield curve for ease of presentation.

$$P(i) = \sum CF_t / (1+i)^t$$

$P(i)$ is the price of this cash flow and is expressed as a function of the interest rate i. CF_t is the cash flow at time t.

The Taylor series for the price at a new interest rate may be expressed as follows:

$$P(i + \Delta i) = P(i) + P'(i)\,\Delta i + \frac{P''(i)(\Delta i)}{2!} + \ldots$$

The change in the interest rate, Δi, has produced a change in the price of $P(i + \Delta i) - P(i)$. It is this change in price that is frequently estimated using duration.

The duration, $D(i)$, of a stream of cash flows as a function of the interest rate i is:

$$D(i) = \frac{\sum t\,CF_t/(1+i)^t}{\sum CF_t/(1+i)^t}$$

Note the denominator is the price of the cash flow. The second term in the Taylor series, $P'(i)\Delta i$, can be shown to consist of duration multiplied by a constant and the change in i.

$$P'(i) = \frac{d}{di}\sum CF_t(1+i)^{-t} = \frac{-1}{(1+i)}\sum tCF_t/(1+i)^t$$

$$P'(i) = -D(i)P(i)/(1+i)$$

Therefore, using only the first two terms of the Taylor series; the change in the price of the instrument, $P(i+\Delta i) - P(i)$, is often approximated by $-D(i)\,P(i)\,\Delta i/(1+i)$.

This approximation is refined when the third term is considered. However, this term essentially reflects the quantity known as convexity. Convexity is defined as:

$$C(i) = \frac{\sum t^2 CF_t/(1+i)^t}{\sum CF_t/(1+i)^t}$$

The relation to the Talyor series is revealed by determining the second derivative of the price as follows:

$$P''(i) = \frac{d}{di}(P'(i)) = -\frac{d}{di}\sum tCF_t(1+i)^{-(t+1)}$$

This equals:

$$P''(i) = \frac{1}{(1+i)^2} \text{ X } [\sum t^2 CF_t /(1+i)^t + \sum t \ CF_t /(1+i)^t \ ]]$$

OR

$$P''(i) = [C(i) + D(i)] \ P(i)/(1+i)^2$$

Therefore, the price of the instrument after a change in interest rates of Δi can be approximated by:

Original Price X {1 - Duration x$\Delta i/(1+i)$ + [Convexity + Duration] x $(\Delta i)^2$ X .5/$(1+i)^2$}

The use of duration, in the second term of the Taylor series, to determine the change in the instrument value is only an approximation. As more terms of the Taylor series are added the accuracy improves (note the limit of the series must exist).

By matching the cash flows of an asset to the cash flow of a liability one is assured that the gain or loss on the asset due to changes in the interest rate, will be exactly offset by changes in the value of the liability. The assets and liabilities are said to be completely matched or immunized against changes in interest rates. This assumes that the cash flows of the asset and the liability are fixed.

Often the duration of assets is matched to the duration of liabilities in an attempt to gain a level of immunization when the cash flows are not exactly matched. One of the primary purposes of matching duration rather than the entire cash flows is that some of the assets held can be of longer maturities to take advantage of the higher yields. When this is done it is often not realized that there is a trade-off. As duration matching principally accounts for only the first two terms in the Taylor series, the immunization is not complete. Therefore, the price of the investment gain from the higher yields is the potential loss resulting from the asset liability mismatch.

Immunization, to any Desired Level

By matching successive levels of "duration", thereby matching successive terms of the Taylor series, one may gain any given level of desired partial immunization. If one matches all of the "duration" terms, then all of the terms of the Taylor series are matched. When this occurs complete immunization is achieved and the cash flows are exactly matched.

To prove the first statement we must assume that the current price of an asset, $P_A(i)$, equals the price of the liability, $P_L(i)$, and that the price of these items at the interest rates $i + \Delta i$ exist. They can then be represented by:

$$P_A(i+\Delta i) = P_A(i) - D_A(i)\, P_A(i)\Delta i/(1+i) + [C_A(i) + D_A(i)]\, P_A(i)\, (\Delta i)^2 \times .5/(1+i)^2 \ldots$$

$$P_L(i+\Delta i) = P_L(i) - D_L(i)\, P_L(i)\Delta i/(1+i) + [C_L(i) + D_L(i)]\, P_L(i)\, (\Delta i)^2 \times .5/(1+i)^2 \ldots$$

Let us assume that the (n+1)st term of the Taylor series equals $(-1)^n K_n(i)\, P(i)\, (\Delta i)^n/n!/(1+i)^n$ where $K_n(i)$ is a linear function of the first n duration terms. The jth duration term is represented by:

$$D_j(i) = \frac{\sum t^j CF_t/(1+i)^t}{\sum CF_t/(1+i)^t}$$

Then given a desired maximum level of mismatch $\epsilon>0$ there exists an integer m such that:

$$\left| \sum_{n>m} \frac{K_n(i)P(i)(\Delta i)^n}{n!(1+i)^n} \right| < \epsilon/2$$

for both the asset and the liability. Hence, if we match the first m-1 duration terms of the asset and the liability, thereby matching the first m-1 $K_n(i)$ terms, we see that the absolute value of P_A

$(i + \Delta i)$ - P_L $(i + \Delta i)$ is less than ϵ. That is to say the desired level of immunization has been achieved.

The Lemma

The lemma can be proven by induction. The earlier discussion on duration already illustrates the case where n=1. Let us assume that it is true for n+1 and prove the assertion for n+2 then the proof of the lemma will be complete. Thus the assumption for n+1 can be stated as:

$$\frac{(-1)^n K_n(i)P(i)(\Delta i)^n}{n!(1+i)^n} = P^n(i)\frac{(\Delta i)^n}{n!}$$

For n+2 let us begin with the right side of the equation

$$P^{n+1}(i)\,\frac{(\Delta i)^{n+1}}{(n+1)!} =$$

$$\frac{d}{di}P^n(i) \text{ X } \frac{(\Delta i)^{n+1}}{(n+1)!} =$$

$$\frac{d}{di}\frac{(-1)^n K_n(i)P(i)}{(1+i)^n} \text{ X } \frac{(\Delta i)^{n+1}}{(n+1)!} =$$

$$\frac{(-1)^n}{(1+i)^{n+1}} \text{ X } [-nK_n(i)P(i)+(1+i)P(i)\frac{d}{di}K_n(i)+(1+i)K_n(i)\frac{d}{di}P(i)\ ] \text{ X } \frac{(\Delta i)^{n+1}}{(n+1)!} =$$

$$\frac{(-1)^n P(i)(\Delta i)^{n+1}}{(1+i)^{n+1}(n+1)!} \text{ X } [-nK_n(i)+(1+i)\frac{d}{di}K_n(i)-K_n(i)D(i)\ ] =$$

Unfortunately, we must now trail off on a further aside to confirm that the expression in the brackets is in fact equivalent to $K_{n+1}(i)$. In order to make this aside more presentable, we will not include the interest rate variable. This has been included, up until now, to stress the view that duration and price are functions of the interest rate.

One of our assumptions is that $K_n = a_1D + a_2C + a_3D_3 + \ldots + a_nD_n$. Therefore in order to differentiate K_n we must differentiate D_j.

$$\frac{d}{di}D_j = \frac{d}{di}\sum t^j CF_t(1+i)^{-t}P^{-1}$$

$$= \frac{-1}{1+i}\sum t^{j+1}CF_t(1+i)^{-t}P^{-1} + \frac{1}{1+i}\sum t^j CF_t(1+i)^{-t}DPP^{-2}$$

$$\frac{d}{di}D = \frac{1}{1+i}[-D_{j+1} + DD_j]$$

We can now restate the derivative of K_n as follows.

$$\frac{d}{di}K_n = \frac{1}{1+i}[(-a_1C + a_1D^2) + (-a_2D_3 + a_2DC) + (-a_3D_4 + a_3DD_3) + \ldots +$$
$$(-a_{n-1}D_n + a_{n-1}DD_{n-1}) + (-a_nD_{n+1} + a_nDD_n)]$$

$$= \frac{-1}{1+i}[a_1C + a_2D_3 + a_3D_4 + \ldots + a_nD_{n+1}] + \frac{D}{1+i}K_n$$

Returning to the stage of the proof just before this aside, substituting the result of the aside for the derivative of K_n, and simplifying we see that:

$$P^{n+1}(i)\,\frac{(\Delta i)^{n+1}}{(n+1)!} = \frac{(-1)^n P(i)(\Delta i)^{n+1}}{(1+i)^{n+1}(n+1)!}\left[-nK_n(i) - (a_1C + a_2D_3 + a_3D_4 + \ldots + a_nD_{n+1})\right]$$

$$= \frac{(-1)^{n+1} P(i)(\Delta i)^{n+1}}{(1+i)^{n+1}(n+1)!}\,[na_1D + (na_2 + a_1)C + (na_3 + a_2)D_3 + \ldots + (na_n + a_{n-1})D_n + a_nD_{n+1}\ ]$$

$$= \frac{(-1)^{n+1} P(i)(\Delta i)^{n+1}}{(1+i)^{n+1}(n+1)!}\,K_{n+1}(i)$$

and we have proved the lemma.

The Recursion Formula

There is an interesting recursive formula that exists for the duration term coefficients above. I would liken it to a type of Pascal's triangle. The triangle is built as follows:

$K_1 = D$ coefficient of 1

$K_2 = D+C$ coefficients of 1 and 1

The trick begins with K_3.

$K_3 = a_{31}D + a_{32}C + a_{33}D_3$

The coefficients for the previous term are:

1	1

We multiply these by n-1, which is in 2 in this case. Then we shift the previous coefficients and add.

2	2	
	1	1

This produces the coefficients.

$K_3 = 2D + 3C + 1D_3$

The process would proceed for K_4 as follows.

2	3	1		multiplied by 3 yields
6	9	3		shift and add the prior coefficients
	2	3	1	and voila

$K_4 = \quad 6D + 11C + 6D_3 + D_4$

The first term in the K_n expansion is always (n-1)!. The second last term is always the sum of the first n-1 positive integers. Finally, the last term is always 1.

The reader may have noticed that the expression of K_n as a linear combination of duration terms was not illustrated. This can be shown by induction as well.

A Little Hand Waving

I would like to diverge briefly from the mathematics and discuss the earlier statement that if all the duration terms of an asset and a liability are matched then the cash flows are matched. It has been pointed out to me that the history of asset liability matching proceeded from a primary level at which the goal was to match market values of assets ($P_A(i)$) and liabilities ($P_L(i)$). The next phase was the pairing of assets and liabilities with equivalent yields. Duration matching is the first step beyond this level. Duration matching begins to consider the probabilistic nature of the price of a stream of cash flows. Duration is often described as the mean timing of the cash flows. This interpretation is obtained by examining the definition of duration and assuming that the probability associated with a cash flow at time t is $[CF_t/(1+i)^t]/P(i)$. The matching of asset and liability durations may be thought of as the matching of the mean or first moment of the random variable associated with the timing of each cash flow.

Continuing along this line of reasoning, convexity can be viewed as a variance or second moment type quantity. Convexity is an indicator of the level of dispersion of the timing of the cash flows. By accounting for duration and convexity one has matched the mean and the variance of the timing of the cash flows.

Matching successive duration terms is equivalent to matching successive moments of t. Once all of the moments of t are matched, the timing of the cash flows will be matched. Hence, the cash flows are matched.

This last discussion is by no means "concrete", but rather a "kind of" type discussion. It is provided to suggest an additional view regarding duration and asset liability matching.

Conclusion

Some people are aware of the relation, as expressed by the Taylor series, between the price of an asset or liability and its duration. For those people who are just being introduced to the concept of duration, I hope the relations presented in this paper will provide some additional insight.