CONSISTENCY OF RISK LOADED PREMIUMS

John M. Cozzolino

Consistency of Risk Loaded Premiums

By John M. Cozzolino, Ph.D.

Director

Underwriting Education Institute

The Lubin Schools

Pace University

ABSTRACT

The meaning of consistency of increased limit factors (ILF) is reconsidered and a new test of the consistency condition is proposed. It is shown that the three major measures of risk satisfy the new consistency test with no restrictions. The problems of specifying consistent risk-loaded rates for high limits are discussed and a revised subtraction formula is given for the case where risk is measured by the certainty equivalent of an exponential utility function. Risk "profile" curves are suggested as a method to emphasize the objective aspects of risk load. A new practical meaning is suggested for the old consistency condition.

John M. Cozzolino is Director of The Underwriting Education Institute and Associate Professor of Management at Pace University.

INTRODUCTION

The paper begins with the statement of two different types of consistency which are discussed in the literatures of utility theory and the literature of layer pricing, as discussed in [2] by Miccolis. The meaning of the consistency test is reconsidered and a new test is proposed. The new consistency test is shown not to impose any limitations upon either utility, variance or standard deviation as measures of risk. It is suggested that the error in the old consistency test is caused by the assumption that the rate for a excess layer can be found by subtraction of the corresponding two rates for ground-up covers. This subtraction rule is a problem for risk-loaded rates but not for expected value rates. Miccolis showed the rate reduction due to layer splitting. It is shown similarly here for exponential utility.

The next part of the paper provides a new formula for the premium of an excess layer when the measure of risk employed is Risk Adjusted Cost. The paper suggests that the old test for consistency is useful for detecting cases where the layer being tested ought to be split so that a lower pricing can be achieved.

Finally, the paper provides a formal proof in appendix I of the new formula. It gives a separate discussion of the application of the exponential utility functions in appendix II. The last part, appendix III, is a lengthy illustration of the use of the exponential utility functions and a comparison with the variance measure of risk shown by Miccolis.

Having provided a complete road map for the parts of the paper, I now add one point which helps complete the connection. The research began with the goal of applying utility to the task of calculating risk loads. Consistency was realized to be a roadblock. Therefore the formal content of the paper begins with consistency.

THE CONCEPT OF RISK LOAD

The uncertainty in the cost of insurance is its distinctive feature. Furthermore, different insurance products have different degrees of cost uncertainty, and therefore different degrees of risk to the insurer. In particular, policies with different upper limits of insurance coverage have very different degrees of risk in spite of their similarity in the type of risk. One idea of a risk load can be expressed as that amount which when added to the pure premium of each policy makes a risk averse insurance company indifferent between the alternatives which the buyer might select. Another, perhaps more fundamental purpose for the risk load is to create an "adequate" rate which holds the chance of insolvency down to an acceptable level . If the business of insurance had to exist on only expected value rates, the nature of the business would be much like gambling, where the outcomes are prone to runs of both good and bad luck. Management skills would matter little compared to the role of luck. A risk-load improves the chance of solvency by giving a positive expected growth rate to the surplus, which is the cushion against insolvency.

CONSISTENCY

There are two kinds of consistency, whose discussion would logically fit within this topic; the consistency of Increased Limits Factors (ILF's) and the consistency of choice under uncertainty are both relevant. The first is defined in the paper by Miccolis [2]. The second is in the literature of Utility Theory.

Consistency of risk loads among lines

Some would say that the risk load for automobile liability ought to be less than the risk-load for products liability. Such comparisons are based upon intuition about which things are more risky than others. Utility theory, a structure based upon axioms of consistency, is designed to give consistency of rankings by risk. Variance, as a measure of risk, may not give this kind of consistency and standard deviation also could fail in some situations. Among these later two risk load choices, variance is more likely to give this kind of consistency because it has a closer relationship to a utility function than does standard deviation. Pratt [3] shows that variance is the first order approximation to certainty equivalent when variance is small.

Consistency of increased limits factors

Risk loads, as discussed by Miccolis in his 1977 PCAS paper [2], and by Sheldon Rosenberg in his review of that paper, is the subject of our interest here. All risk load methods, including utility, variance, and standard deviation, can be inconsistent, but it happens less often for standard deviation. Standard deviation increases less rapidly as a function of loss size than does the variance which is the expectation of loss size squared. Therefore one would expect this statement is true also for policy limit which is the top loss size in the expectation integral. One purpose of this paper is to question wherein it is wrong for a rating bureau to publish rates which are inconsistent. What was the original motivation for the concern with consistency and what role does it play today? An example from the Rosenberg discussion of the Miccolis paper is presented next. All policies in this example have a $250,000 aggregate limit.

The table giving "Increased Limits Factors" for various per occurrence limits follows:

TABLE 1

P.O. LIMIT*	ILF
$25,000	2.00
$50,000	2.25
$100,000	2.80
$250,000	3.20

* DENOTES PER OCCURRENCE LIMIT

The test for inconsistency examines the ratio of the differences in ILFs to the differences in limits. It shows that the ratio based upon the change from the \$25,000 to the \$50,000 limits is .01. This is computed as the ratio of (2.25-2.00) to the difference (50-25). The scale factor change of leaving off the factor of one thousand in the premium figures does not matter if done consistently. A similar calculation also shows that the corresponding ratio based upon the change from the \$50,000 to the \$100,000 limits is .011. This is the ratio of (2.80-2.25) to (100-50). Such an increase in the ratio identifies the condition of inconsistency. While this test is simple, the meaning is not so clear. It is not clear why this ratio ought to be declining. One possible motivation is to think of the premium difference as the price of coverage for the layer going from the lower limit to the higher limit. Perhaps it is testing the condition that the price per unit of coverage declines as the layer is moved up the loss size scale. Where this motivation would be wrong is that the proper price of a layer of coverage is not the difference in premiums, when the premiums include a risk-load. One purpose of this paper is to revisit this idea of inconsistency and to suggest that it is no longer relevant.

It will be shown that a new statement of the condition of consistency is almost always true for rates which are calculated based upon a probability distribution. This paper does suggest an important warning to those who base rates for excess layers upon differences computed from tables of risk-loaded increased limits factors. I suggest that the reason behind inconsistency is half forgotten, and no longer relevant. Miccolis gives an example of inconsistencies on page 33. He wrote "The marginal premium per $1000 of coverage should decrease as the limit of coverage increases. If not, this implies negative probabilities." Miccolis shows that consistency is a property obeyed by expected value premiums; he does not claim that it is a property of risk-load based premiums. Apparently, consistency is a test of whether the increased limits factors are based upon the use of a probability distribution. If the risk-loaded premium would necessarily obey the consistency test, then Miccolis would likely have shown it! His work on the risk reduction due to layering suggests that an inequality condition exists instead. This will be discussed later.

THE NEW CONSISTENCY TEST

What ought to be true is that the cost of a layer be a decreasing function of its starting (attachment) point. A higher layer ought not cost more than a lower layer when both have the same width. This will be shown to be true for any probability distribution and for any utility function of loss. If true, for a layer of any fixed size, then the price of the layer per unit of coverage will also decline because the division by size of the layer merely scales the function. The reason that this is so generally true is that the insurer pays something (all or part) for all losses above the attachment point. The higher the attachment point, the fewer losses get that high. Miccolis states essentially the same thing at the top of page 34. The sentence "Aside from the mathematical interpretation of this consistency test, it has a very practical meaning. In general, it does not make sense to the insurance buyer to have to pay more for each additional $1000 of coverage since the probability of losses larger than some limit should be less than for a lower limit."

PROOF OF THE NEW CONSISTENCY

Let's begin with the basic formulas for expected loss, risk adjusted cost, and the variance (actually the second moment) for the case where the frequency is assumed to follow the Poisson distribution. The symbol F represents frequency; it is the parameter of the Poisson distribution, and is also the mean number of claims. Here the f(x) is the density function for the severity distribution. F(x) is the integral of the density, called the cumulative distribution function. Let U(x) be an increasing function of the individual loss size x. Consider a layer which starts at an attachment point "a", and has size "h". The largest loss completely covered is of size (a+h). Let U(x) be an increasing function of the individual loss size x. The expected value of U(X) will be denoted EU. It is found as:

$$EU = U(0)\times\int_0^a f(x)dx + \int_a^{a+h} U(x-a)f(x)dx + U(h)\times\int_{a+h}^{\infty} f(x)dx \quad (1)$$

The first integral is just F(a). It represents all the cases of individual loss where the loss is less than the attachment point. The second integral counts all loss cases within the insured layer, and the third integral counts losses above the limit. The expression for EU is a function of the attachment point, a, and the layer width, h. Those variables also appear in the limits of the three integrals, as well as in the integrand of the middle integral. We are interested in the derivative of the function EU with respect to the attachment point a. The result is:

$$\frac{\partial EU(a,a+h)}{\partial a} = -\int_{a}^{a+h} U'(x-a)f(x)dx \qquad (2)$$

From the formula for the derivative of definite integrals, one finds that all the terms coming from derivatives of the limits of integration happen to cancel each other. The remaining term, as shown above is the integral of the derivative of the former integrand. The negative sign in front results from the derivative of the argument of the function evaluated at (x-a) with respect to a. The U prime (U'(x)) stands for the derivative of the function U with respect to its argument. If and only if this derivative is positive, then the derivative of the function EU with respect to a is negative, and this is so for all positive values of h. To interpret this result, consider first the case where the function U(x) is just x itself.

$$EL = F * \left\{ \int_a^{a+h} (x-a) f(x) dx + h (1 - F(a+h)) \right\} \quad (3)$$

The expected aggregate loss from this layer is equation (3). When the function U(x) is the exponential function exp(rx), then the Risk Adjusted Cost, which is the risk loaded Premium based upon exponential utility with risk aversion level r, is given by the

$$RAC(a, a+h) = (F/r) x \left\{ \int_0^a e^0 f(x) dx - 1 + \int_a^{a+h} e^{r x (x-a)} f(x) dx + e^{(r*h)} * (1 - F(a+h)) \right\} \quad (4)$$

Its derivative with respect to a is a negative quantity, as shown in (5).

$$\frac{\partial RAC}{\partial a} = -F * \left[\int_a^{a+h} e^{r*(x-a)} * f(x) dx \right] \quad (5)$$

Thus the result is that the risk loaded premium for the layer is a decreasing function of the attachment point, a, for any positive value of h. The "new" consistency is true for all exponential utility functions regardless of the degree of positive risk aversion. With regard to variance as a measure of risk, it is well known that for Poisson frequency, the variance of the aggregate loss distribution is equivalent to the expected frequency multiplied bythe second noncentral moment of the severity distribution. When the function U(x) is x squared, the result is:

$$VAR(a, a+h) = F x \left\{ \int_a^{a+h} (x-a)^2 f(x) dx + h^2 x (1 - F(a+h)) \right\} \quad (6)$$

This function also fits into the pattern of the first two cases and will have a negative first derivative. Thus we can conclude that the premium for an excess layer, which contains a risk load based upon variance or upon exponential utility with non-negative risk aversion, is a decreasing function of its attachment point regardless of the size of the width of the layer, h. Therefore, the "new" consistency holds, with no restrictive conditions, for both variance and for exponential utility. The condition is likely to be true also for other utility functions and for the standard deviation. The only condition upon the function is that it be an increasing function; this is also required of a function for it to be a utility function. However, the starting point for this proof, equation (1), which is essentially the expected utility on a per occurrence basis, while true for exponential utility and for variance, may not be true for other utility functions. Equation (8) is the real starting point. Apparently all of these possible bases for risk load will give premiums which have decreasing premium per unit of coverage as the attachment point is moved up the loss size scale.

CONCLUSIONS ABOUT CONSISTENCY

The result of this analysis so far is the conclusion that the old definition of consistency is flawed in the way it has been applied. In the case of risk-loaded premiums for excess layers, it must be replaced by the new definition and the new test for consistency. Perhaps the old consistency should be forgotten because its reason for existence is wrong when the pricing includes a risk-load. Its practical meaning to many is just that the premium increases "too fast" as a function of the limit. It survives, giving the appearance of precision, but serving only as a vague condition for the expression of "too fast". Another view of the use of the inconsistency condition is that it may be useful to detect situations where one carrier ought not to price so wide a layer because the price can easily be reduced by splitting that layer into two or more layers. It would detect some such situations, but would it detect all such? Which ought to be detected?

RISK REDUCTION DUE TO LAYERING

The more positive thrust of the Miccolis paper is to show the risk reduction due to layering of coverage. For risk-loaded ratemaking, we have an inequality in risk-loaded premiums. It is:

$$P(x,y) \geq P(x,z) + P(z,y) \quad \text{for } x < z < y \qquad (7)$$

The inequality simply says that the premium for the coverage from x to y is more expensive than the coverage structured into two layers; the first layer is from x to z and the second layer is from z to y. An important condition is that the two layers are not insured by the same insurer. The spreading, or subdividing of risk would not then be achieved. This is fundamental for risk reduction to exist. Often, we will consider that the two insurers writing the two layers have the same risk aversion. This is not necessary, but might be convenient for illustrations.

IMPLICATIONS FOR PRICING

The most important conclusion is that there does not exist a unique risk-loaded price of coverage between X, and Y, unless you define it as the coverage provided by only one policy and only if the degree of risk aversion is fixed. Once layering is allowed, the premium depends upon the layer details. There are some implications here for the pricing operations of both insurers and reinsurers. The problems raised for a rating bureau are larger because of some uncertainty about how its products will be used by its member companies. Some alternative choices for a rating bureau are the following:

1. No Risk Load-Compute increased limits factors based upon expected value. This would give rates for excess layers also since the differences are correct for excess layers when there is no risk-load. This will not satisfy those who believe that risk-load is very important to the stability of the industry and that rating bureaus ought to maintain their practice of including it.

Objectively, the function of computing risk load fits within the function of the rating bureau because that calculation is dependent upon the historic loss data from which the degree of variability is measured. Without this measurement of actual variability, the risk-load would be entirely subjective and its theoretical connection to rate adequacy would not be easily demonstrable.

2. Publish risk-loaded ILF tables and also publish risk-loaded excess layer rates for some commonly used layers. This would be popular and would bring out the fact that layer premiums cannot be calculated by subtraction, but it could give rise to some cases of old inconsistency. This appearance of old inconsistency is considered undesirable even if the meaning of old consistency is not what it was previously thought to be.

3. Assume Standard Layers-There could be layer breakpoints at every limit which is a whole number of half-million dollar units, for example. This would probably eliminate the occurrence of old inconsistency. If the use of half-million dollar layers did not achieve this elimination, then there would be some layer sizes which would accomplish this. Another point for discussion is whether there is a limit as the process of layer subdivision is carried to the extreme of infinitely many layers of infinitesimal width. This is somewhat similar to the case of fractional participation, the fundamental basis for pro rata forms of insurance, as well as for most forms of risk sharing of investment projects. Paul Samuelson discussed the limits of risk sharing in 1963 (4). In "Risk and Uncertainty: A Fallacy of Large Numbers", his simple and elegant argument showed that the value of a small share approaches its expected value as the share gets very small.

The same argument works also for layers just as it does for shares. A layer of coverage of size dx in excess of the attachment point x can be evaluated using a series expansion for the utility function as was shown in (3) by Pratt. With the expected value as the limit of subdividing, an interesting question would be how close to this limit does the industry operate. Those familiar with the costly nature of reinsurance brokerage would be inclined to believe that the practical world of insurance operates at significantly different rates than the expected value rates except during the extremes of the soft market. Then the extreme competition does exist and drives the rates even lower than expected value rates. In other words, the limit of expected value pricing does not seem highly relevant in light of the actual behavior.

LAYER RATES BY SUBTRACTION PLUS DIVISION

Let us now concentrate upon computing the risk loaded premium for an excess layer, but using exponential utility in place of variance. The risk adjusted cost, RAC, is the certainty equivalent defined in the theory of utility but specialized here to the family of all exponential utility functions. Cozzolino [1978], "A Method for the Evaluation of Retained Risk", shows that for a Poisson frequency with parameter F and a risk aversion level denoted by r, the RAC, which represents a risk loaded premium, can be found from equation 8.

$$\mathrm{RAC} = \frac{F}{r} \times \left[e^{(r * \mathrm{RAS})} - 1 \right] \qquad (8)$$

The expression RAS stands for the Risk Adjusted Severity and it is computed from the severity distribution according to the formula:

$$RAS = \frac{1}{r} * LN\left[\int_0^{\infty} e^{r \times l} f(l)\,dl \right] \tag{9}$$

Here, LN stands for the natural logarithm. The result of the above two expressions is the simpler expression, equation (10) which will be the basis of the subsequent equations.

$$RAC = \frac{F}{r} \times \left\{ \int_0^{\infty} e^{r \times l} f(l)\,dl \quad -1 \right\} \tag{10}$$

RAC is the "certainty Equivalent" of utility theory. It is the premium for the risk represented by the severity probability distribution f(l), in combination with the Poisson frequency with parameter F. Notice that F appears only as a multiplicative factor. RAC/F then represents a rate. The next step is to study this in more detail. The symbol l represents the individual loss from the severity distribution. The symbol L represents the loss to the insurer if he insures the layer from x to y. An alternative description is that there is coverage of amount (y-x) in excess of x, also called the attachment point. Thus L is a function of l, and of x, and of y. It is shown in equation 11..

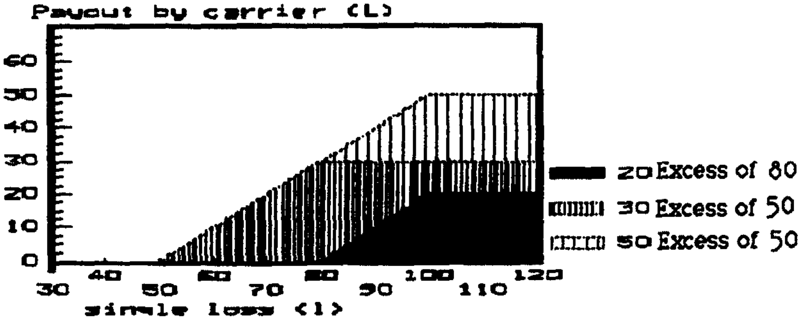
$$L(l,x,y) = \begin{cases} = 0 & \text{for } 0 < l \leq x \\ = (l-x) & \text{for } x < l \leq y \\ = (y-x) & \text{for } y < l < \infty \end{cases} \qquad (11)$$

It can easily be shown that such payout functions are additive. Thus the claim is that:

$$L(l,x,y) = L(l,x,z) + L(l,z,y) \text{ for all } z \text{ in } (x,y) \qquad (12)$$

The loss from the policy of amount (y-x) in excess of x can be expressed as the sum of the losses from two policies. They are the coverage of amount (z-x) in excess of x plus the coverage of amount (y-z) in excess of z. This can also be seen in terms of the graph of figure I which shows the three loss functions being discussed.

FIGURE I



In the example of the graph, the x value is 50, the y value is 100, and the z value chosen is 80. The sum of losses from the 30 excess of 50 plus the 20 excess of 80 equals the loss from the policy for

50 excess of 50. This is true at any individual loss size l, the abscissa of the graph.

With the loss function L now established, and its additivity demonstrated, we can now express the RAC equations in terms of L as follows:

$$RAC(x,y) = \frac{F}{r}*\left\{ \int_0^\infty e^{r*L} f(l)\,dl \quad -1 \right\} \qquad (13)$$

Note that the loss function specified in equation (11) is the L in equation (13). That is why the RAC is a function of both x, and y. The real working equation is with the definition of L(l,x,y) substituted into the last equation. It is given in equation (14). The additivity of the L loss function, equations 11, and 12, and the RAC equation (13), are all used in Appendix I to shows how to derive the last equation. Equation 14 can be further expressed in

$$RAC(x,y) = \frac{F}{r}*\left\{ e^{0}*\int_0^x f(l)\,dl + \int_x^y e^{r*(l-x)} f(l)\,dl \right. \qquad (14)$$
$$\left. + \quad e^{r*(y-x)} \int_y^\infty f(l)\,dl \quad -1 \right\}$$

terms of the premium functions of the sub layers from x to z and from z to y. The final result is equation 15 which follows:

$$RAC(x,y) = RAC(x,z) + RAC(z,y)*e^{r*(z-x)} \qquad (15)$$

Equation 15 is the main result. It is a useful tool for layer pricing. Notice that when $r = 0$ it implies that additivity is corrrect in the case of premiums based upon expected value, since e to the zero power is unity. This corresponds to expected value pricing, and is in agreement with the Miccolis results. Notice that the factor r in the exponent is positive for a risk averse decision maker, and the factor (z-x) is also positive, so that the exponential factor in the second term of the equation is a positive number greater than one. Therefore, deletion of this exponential factor would decrease the right-hand side of the equation. The result is a fundamental inequality, stated first, without proof, as equation 7. It is equation 16. This inequality also shows, by turning it around algebraically, that the price of a layer, when computed by subtraction, is

$$RAC(x,y) > RAC(x,z) + RAC(z,y) \quad \text{for all } r > 0 \quad (16)$$

overestimated. This is shown by the following revised form of equation 16, shown next:

$$RAC(z,y) < RAC(x,y) - RAC(x,z) \quad \text{For any } x < z \text{ and any } z < y. \quad (17)$$

The two premiums in the subtraction shown in equation 17 are for premiums for coverages in excess of attachment point x. More often, these terms would be representing ground up coverages and so x would then be zero. Since subtraction of risk loaded premiums gives an overestimate of the correct premium, we can turn equation 15 around to obtain a very useful correction form. It is equation 18.

$$RAC(z,y) = \left[\frac{RAC(x,y)-RAC(x,z)}{e^{r*(z-x)}}\right] \qquad (18)$$

This says that the premium computed by subtraction must be divided by a number greater than one to yield a correct result. The implication is that the correct answer is smaller than the answer obtained from subtraction. The difference,

$$\begin{aligned} &RAC\ (x,y)-RAC(X,Z)-\ RAC(Z,Y) \\ &= \text{RISK REDUCTION DUE TO SPLITING } (x,y) \end{aligned} \qquad (19)$$

is the risk reduction due to layering. This was first computed by Miccolis, shown on his p. 49, for variance as a measure of risk. How similar are the results? An example given by Miccolis is repeated here in appendix III so that the comparison can be seen.

CONCLUSIONS

An important implication of equation 18 is that there is no need for tables of increased limit factors for excess layers; the equation makes that information directly computable from the ground-up rates. It is interesting that the correction factor is not a function of the probability distribution but a function only of the risk aversion level and of the layer endpoints. All of the results shown so far which involve the family of exponential utility functions are valid for all risk aversion levels greater than or equal to zero. The user of these results should be aware, however, that for high enough risk aversion and/or high enough limits, the old inconsistency will always occur. This is not a manifestation of some obscure flaw in the theory of utility. Instead, it is simply a warning that layer splitting is essential to enable reasonable pricing. It simply demonstrates the need for layering the coverage, just as is usually done.

Experience with applications of utility analysis suggests that every company ought to have its own utility function which serves to represent the attitude toward risk of that company. Larger companies ought to be less risk averse than smaller companies, although the choice is the prerogative of management.

The risk aversion can also be determined in the same way that Miccolis used to determine the coefficient of variance for calculating risk load. This method was to set the coefficient to result in a 5% risk load for a policy with the basic limit. Utility theory is useful to improve the understanding of risk loads, their meaning, and their implications. It will probably be a useful tool to help insurance company actuaries develop pricing rules. Appendix II gives some of the considerations relevant for the decision of whether to use utility. However, for a rating bureau, utility theory is not a complete theory of insurance pricing. A current influence upon the developing ideas of industry risk-load is the changing ideas of the role of the service bureau. In response, the bureau can give the risk-load as a function of the risk aversion level so that the subjective aspect of risk-load, selection of the company's risk aversion level, is left to the insurer, while the objective part, determined from data, is recognized as an essential bureau function. The graph of the risk adjusted cost function, as a function of the risk aversion level, for all risk aversion levels is a simple way to do this. In fact, it can be shown that the RAC function, as a function of the risk aversion level, uniquely encodes all of the probability information contained in the loss distribution.

This property of the complete family of exponential utility functions is known from the theory of transforms. The transform is the same function as the expected utility. Therefore the risk profile curve, which is the graph of RAC as a function of the risk aversion level, is as objective a measure of risk as is possible. Increased limits pricing is an essential topic today in light of the increasing popularity of large risk retention by the buyer. The increased risk retained by the insurer is something which the industry must maintain a careful awareness of. The risk of writing a policy is a strongly increasing function of the limits of coverage. The understanding provided by the theory of utility is useful for both insurers and regulators.

REFERENCES

1. Cozzolino, J. M., "A Method for the Evaluation of Retained Risk", Journal of Risk and Insurance, September, 1978, Volume XLV, No. 3.

2. Miccolis, R. S., "On the Theory of Increased Limits and Excess of Loss Pricing", PCAS LXIV, 1977.

3. Pratt, J. W., "Risk Aversion in the Small and in the Large", Econometrica, Vol. 32, No. 1, (Jan.-Apr. 1964).

4. Samuelson, P., "Risk and Uncertainty: A Fallacy of Large Numbers," Scientia, 6th Series, 57th year (April-May), 1963.

APPENDIX I: PROOF OF EQUATION 15

The starting points are equation 11, which defines the loss to the insurer who insures the excess layer from x to y, already denoted L(l,x,y), and the equation which gives the RAC for that layer. Our goal is to express it as a function of RAC(x,z) and RAC(z,y), which are the risk-loaded premiums of the two contiguous layers into which the (x,y) layer might be broken. Let us begin by applying the definitions to the sublayers. These are:

$$RAC(x,z) = \frac{F}{r} * \left\{ \int_0^x f(l)dl + \int_x^z e^{r*(l-x)} f(l)dl + e^{r*(z-x)} \int_z^\infty f(l)dl - 1 \right\} \tag{21}$$

and,

$$RAC(z,y) = \frac{F}{r} * \left\{ \int_0^z f(l)dl + \int_z^y e^{r*(l-z)} f(l)dl + e^{r*(y-z)} \int_y^\infty f(l)dl - 1 \right\} \tag{22}$$

The loss functions corresponding to these two layers are L(l,x,z) and L(l,z,y). These can be expressed in the same form as equation 11.

$$L(l,x,z) = \begin{cases} = 0 & \text{for } 0 < l < x \\ = (l-x) & \text{for } x < l < z \\ = (z-x) & \text{for } z < l < \infty \end{cases}$$

$$L(l,z,y) = \begin{cases} = 0 & \text{for } 0 < l < z \\ = (l-z) & \text{for } z < l < y \\ = (y-z) & \text{for } y < l < \infty \end{cases} \tag{23}$$

The next step in preparation is to write equation 12 which expresses the layer losses from each layer as the terms in each layer.

This will express the L(l,x,y) in terms of four loss size intervals rather than three.

Layer (x,y) Loss		Layer (x,z) Loss		Layer (z,y) Loss	
0	=	0	+	0	for $0 < l < x$
$(l-x)$	=	$(l-x)$	+	0	for $x < l < z$
$(l-x)$	=	$(z-x)$	+	$(l-z)$	for $z < l < y$
$(y-x)$	=	$(z-x)$	+	$(y-z)$	for $y < l < \infty$

The first step in the proof of equation 15 is to begin with equation 14 and to split the middle integral, which goes from x to y into two integrals corresponding to the layers of the table above. The result is:

$$RAC(x,y) = \frac{F}{r} * \left\{ \int_0^x f(l)dl + \int_x^z e^{r*(l-x)} f(l)dl - 1 \right\}$$

$$+ \frac{F}{r} * \left\{ \left[e^{r*(z-x)} \right] \left[\int_z^y e^{r*(l-z)} f(l)dl + e^{r*(y-z)} \int_y^\infty f(l)dl \right] \right\}$$

The expression above for RAC(X,Y) has two curly brackets on its right hand side. The following expression can be added into the first bracket and balanced by subtraction within the second bracket.

$$e^{r*(z-x)} \int_z^\infty f(l)dl$$

The resulting equation appears as:

$$RAC(x,y) = \frac{F}{r} * \left\{ \int_0^x f(l)dl + \int_x^z e^{r*(l-x)} f(l)dl + \right.$$

$$\left. + \; e^{r*(z-x)} \int_z^\infty f(l)dl \quad -1 \right\} +$$

$$+ \frac{F}{r} * \left\{ \left[e^{r*(z-x)} \right] \right.$$

$$\left. \left[- \int_z^\infty f(l)dl + \int_z^y e^{r*(l-z)} f(l)dl + e^{r*(y-z)} \int_y^\infty f(l)dl \right] \right\}$$

The new negative term in the second bracket can be replaced by its equivalent value shown in the next equation:

$$-\int_z^{\infty} f(l)\,dl = -1 + \int_0^z f(l)\,dl$$

The equation for RAC(x,y) now looks like the following:

$$RAC(x,y) = \frac{F}{r} * \left\{ \int_0^x f(l)\,dl + \int_x^z e^{r*(l-x)} f(l)\,dl + e^{r*(z-x)} \int_z^{\infty} f(l)\,dl \quad -1 \right\} +$$

$$+ \frac{F}{r} * \left\{ \left[e^{r*(z-x)} \right] \left[\int_0^z f(l)\,dl + \int_z^y e^{r*(l-z)} f(l)\,dl + e^{r*(y-z)} \int_y^{\infty} f(l)\,dl - 1 \right] \right\}$$

At this point it is easy to recognize that we have equation 15:

$$RAC(x,y) = RAC(x,z) + RAC(z,y) * e^{r*(z-x)}$$

PERSPECTIVES IN THE APPLICATION OF UTILITY THEORY

There are two utility theories. The one used by economists to rationalize the purchase of possible market baskets of goods has nothing to do with risk. Many people, never exposed to the utility theory of risk, erroneously assume that they learned something about it in their required course in economics. Where would you have studied this relatively new risk theory? The theory of Von Neumann and Morgenstern is the one we are concerned with. It is a theory based upon three consistency axioms for choice among lotteries. The properties of the utility curve are derived from the axioms. There are several books which contain this theory, including references 1, 2, 8, 9, 10, 14, and 17 for example. There are several more under the subject name "Statistical Decision Theory". The book by Morris DeGroot, entitled "Optimal Statistical Decisions" is an excellent example and a fine pressentation of the derivation from the axioms.

Another perspective is that other applications areas exist in addition to insurance. Oil and gas exploration is another highly risky business. Some of those practitioners also apply utility theory and there is an extensive literature on risk. Operations research people often tended to be the users and trustees of the knowledge of utility theory in general, but the study of risk is rapidly growing, including new disciplines called risk management and risk analysis.

Another perspective is that the theory of utility has developed considerably over the years and there is now a general realization that the exponential family of utility functions is the simplest to apply. It is unique in its "portfolio property" which is additivity of the values of independent random variables. Without this, the complication of evaluating hundreds or thousands of "lotteries" would be insurmountable. With this type of function, the expected utility is essentially the moment generating function, which we know from probability theory, so the mathematics is already in place. Science usually begins with the simplest model, when the choice is available, proceeding to more complicated models only when experience reveals the need to do so. That is how we ought to proceed.

The family of exponential utilities is a one parameter family. The parameter is called the local risk aversion function, so named by John Pratt, who explored the properties of many functional forms of utility functions. The fact that this function is a constant for the exponential is often called the "wealth independence" property. It is reasonable to expect that every decision maker has their own individual risk aversion level, the parameter of the exponential. We can make the measurement of risk more objective by computing and showing the spectrum of certainty equivalent values for each possible risk aversion level from zero to infinity. This graph has been called the "Risk Profile Curve". Lotteries can be compared against each other by comparing their risk profile curves. Reference [5] gives the details of "Risk Profile Dominance". In practice we find that real decision makers want to know how they "ought to" behave regarding risk. Utility theory was not meant to answer that question. One widely accepted idea is the greater the wealth the smaller the risk aversion. When constructing a theory which involves a whole population of companies or individuals, we often find a Pareto distribution of wealth levels. A simple model for the population of risk aversions is that each individual's risk aversion level is the reciprocal of their wealth level.

In my experience, I was fairly successful in advising oil exploration companies when I recommended that their risk aversion level be taken as the reciprocal of their exploration budget. In many cases, individuals in positions of responsibility are found to be too risk averse compared to this guideline and the advice is often welcome news. Personality traits can influence this but probably ought not to.

Applying exponential utility theory is simple because you only need one number, the risk aversion level, to get started. For the application to risk loads, for example, we can determine the risk aversion level which gives the risk load of 5% for the basic limits policy. This will be illustrated in Appendix 3 which contains the example. In general, since only one parameter need be determined, one past decision is sufficient too determine the past risk aversion level. An alternative to uing the parameter r is provided by Van Slyke [18]. He recommends a risk tolerance type of parameter and calls it capacity, intending it to be measure of capacity. A model such as this would be very useful if it found general acceptance.

The interested reader ought to examine one general reference, such as [2] or [10], and the two papers by Pratt [13] and Samuelson [15].

The idea that the local risk aversion ought to be declining with wealth is appealing to many people and was first expressed by Kenneth Arrow. Reference [19] quickly assumes declining risk aversion with wealth. An alternative, but similar, hypothesis is that of population heterogeneity; all individuals in the population have different risk aversion levels.

The weatlthier individuals have the smaller risk aversion levels. Their risk aversion levels all remain constant. I think that this possibility needs empirical testing rather than debate because these two hypotheses are difficult to distinguish between. The simpler of the two models is that of population heterogeneity. This model as assumed by Lintner [12] in his early derivation of the capital asset pricing model.

An objection to exponential utility theory was expressed by Richard Woll in his review of the discussion paper by Cozzolino and Kleinman in the Discussion Paper Program of 1982. Woll states that "While the characteristic of constant risk aversion is extremely useful, ...,it provides no limit on the number of independent risks which a particular insurer might be willing to write, given no external constraints." In his next paragraph Woll states that "This implication of exponential utility flies in the face of the historical and intuitive notion that there is always some limit to the amount of business one is willing to write with a given amount of capital." This objection to exponential utility is less real than imagined because it is not the role of risk aversion to limit the amount of business written. The realistic nature of the limitation is the limitation of the capacity available to any insurer. Some think that capacity is not real, perhaps just a construct of the regulators. But another natural cause of limitation for any business is the finite nature of the stock of good opportunities available in the whole world. As an insurer trys to obtain more risks of a given type, the quality declines; the additional risks are not of the same quality. In that case of declining quality, risk aversion will also play a role; the lower quality of the marginal risk makes it more risky and so it fails to meet a constant cutoff.

In all businesses, there are very real limitations of the number of good prospects available. Much of the effort expended in many businesses is that of finding the opportunities worthy of investment.

If the utility function really had decreasing local risk aversion, then as the insurer takes more risks whose expected profit is positive, the insurer's risk aversion would decline and become less of a restriction on the acceptance of marginal risks. Perhaps this proves that the desired restriction is not the role of utility but rather the result of some other general aspect of business.

One last consideration; don't wait to find the "perfectly correct" utility curve before beginning to apply this methodology. Utility curves, like probability distributions, are never perfectly correct, although we can distinguish that some are better than others. As for wealth independence, if you think that the utility curve is changing over time, then you can reestimate the risk aversion periodically, perhaps annually, as is done for other financial parameters of business firms. Slow changes are easily handled this way.

UTILITY REFERENCES

1. Borch, Karl H., "The Economics of Uncertainty", Princeton Studies in Mathematical Economics, Princeton University Press, Princeton, NJ, 1968.

2. Bowers, N. L., Jr., H. U. Gerber, J. C. Hickman, D. Jones, C. J. Nesbit, "Actuarial Mathematics", The Society of Actuaries, Itasca IL, 1986.

3. Brockett, P. L., and Linda L. Golden, "A Class of Utility Functions Functions Containing all the Common Utility Functions", Management Science, Vol.33, No.8, August, 1988.

4. The CAS Committee on Theory of Risk, "Risk Theoretic Issues in Loss Reserving", Presented at the CAS Annual Meeting, November 1984.

5. Cozzolino, J. M., "A New Method for the Evaluation of Retained Risk", Journal of Risk and Insurance, Vol. XLV, No. 3, Sept. 1978.

6. Cozzolino, J. M., "The Evaluation of Loss Control Options", Journal of Risk and Insurance, Vol. XLV, No. 3, Sept. 1978.

7. Cozzolino, J. M. and Naomi B. Kleinman, "A Capacity Management Model based on Utility Theory", Discussion Paper Program, 1982.

8. Freifelder, Leonard R., "A Decision Theoretic Approach to Insurance Ratemaking", Huebner Foundation for Insurance Education, Monograph No. 4, Philadelphia, 1976.

9. Gerber, Hans U., "An Introduction to Mathematical Risk Theory", Huebner Foundation for Insurance Education, Monograph No. 8, Philadelphia, 1979.

10. Gupta, S.K., and J.M. Cozzolino, Chapter 11, "Decisions Under Uncertainty with Risk Aversion", in "Fundamentals of Operations Research for Management", Holden-Day, 1975, pp. 231-264.

11. Hewitt, Charles C., Jr., "Decision Making Through Utility Theory", appears in the RANDOM SAMPLER Section of The Actuarial Review, May 1984, pp. 3 and 11.

12. Lintner, J., "Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", Review of Economic Studies, February, 1965.

13. Pratt, J. W., "Risk Aversion in the Small and in the Large", Econometrica, Vol. 32, No. 1-2, Jan-April, 1964.

14. Raiffa, Howard, "Decision Analysis: Introductory Lectures on Choices under Uncertainty", Addison-Wesley, Reading, MA, 1968.

15. Samuelson "Risk and Uncertainty: A fallacy of large numbers", Scientia 6th Series, 57th Year, May-April 1963 (4 pages).

16. Steeneck, L. R., "Reinsuring the Captive/Specialty Company", PCAS, Vol. LXX, 1983.

17. Von Neumann, John, and Oskar Morgenstern, "Theory of Games and Economic Behavior", Princeton University Press, Princeton, NJ 1944.

18. Van Slyke, O.E., Discussion of "Reinsuring the Captive/Specialty Company" PCAS, Vol. LXXII, 1985.

19. Venter, Gary G., "Utility with Decreasing Risk Aversion", PCAS, Vol. LXX, 1983.

APPENDIX III

The Miccolis paper, "On the Theory of Increased Limits and Excess of Loss Pricing," is very complete but did not illustrate all of the capabilities it contains. The formula for the covariance between the excess loss of two adjacent layers was given (equation 39 of that paper) but it was not illustrated. The capability of computing the variance-based risk-loaded premiums for excess layers was illustrated.

The formula (equation 43 of that paper) was stated as a formula for the amount of risk reduction due to layering. In addition, it can be used to compute the correct variance-based, risk-loaded premium for an excess layer; it is also a formula for the correction of a premium determination by subtraction of ground-up rates. It seems useful to illustrate those here because the main purpose is to illustrate RAC-based risk-loaded premiums. The presence of variance-based risk-loads in the same paper is useful to allow comparisons.

Example A will be the Miccolis example of a lognormal severity distribution with the parameters μ equal to 8.9146, and σ equal to 1.7826. The mean frequency is given as 0.1 losses per year. This is a long-tailed distribution, appropriate for the medical malpractice loss of one doctor. The annual expected loss of this frequency and severity combination is $3,644. The distribution of annual aggregate loss has a variance of 2.4181E09 squared dollars, and the standard deviation is $49,174.

In example A, the formulas supplied by Miccolis for partial integral, partial mean, and partial noncentral second moment were used to produce the layer results shown in the following tables:

Table A-III-1

Layer Definitions Layer #	Start	Top	Frequency in Layer
1	$ 0	$ 25,000	0.075172
2	25,000	50,000	0.010569
3	50,000	100,000	0.007011
4	100,000	300,000	0.005343
5	300,000	500,000	0.000992
6	500,000	1,000,000	0.000614
7	1,000,000	1,300,000	0.000110
8	1,300,000	1,500,000	0.000043
9	1,500,000	2,000,000	0.000061
10	2,000,000	3,000,000	0.000047
11	3,000,000	4,000,000	0.000017
12	4,000,000	5,000,000	0.000008
13	5,000,000	7,500,000	0.000008
14	7,500,000	10,000,000	0.000003
15	10,000,000	15,000,000	0.000002
	Total Frequency =		0.10000

Table A-III-2

Layer #	Exp. Loss Freq in layer	above layer	Exp. Loss below & inc.	Exp. Loss zero to top	Exp. Loss Excess Layer
1	492.25	0.02483	492.25	1,112.92	1,112.92
2	373.81	0.01426	866.06	1,578.95	466.03
3	491.68	0.00725	1,357.74	2,082.39	503.44
4	881.85	0.00190	2,239.58	2,810.61	728.22
5	378.00	0.00091	2,617.59	3,073.40	262.78
6	418.45	0.00030	3,036.04	3,333.67	260.27
7	124.56	0.00019	3,160.60	3,404.62	70.95
8	59.97	0.00014	3,220.57	3,437.58	32.97
9	104.17	0.00008	3,324.74	3,492.80	55.21
10	112.64	0.00004	3,437.38	3,549.00	56.21
11	58.90	0.00002	3,496.28	3,576.42	27.42
12	35.62	0.00001	3,531.90	3,592.00	15.58
13	46.56	0.00000	3,578.46	3,610.32	18.32
14	21.99	0.00000	3,600.45	3,617.23	6.91
15	20.04	0.00000	3,620.49	3,620.49	3.26
Sum =	3,620.49				

Table A-III-3

(All variance figures have been divided by 1000)

Layer #	Variance Freq. in Layer	above layer	Variance below & incl.	Var of layer zero to top
1	6,229	0.024827	6,229	21,746
2	13,753	0.014258	19,982	55,627
3	35,873	0.007247	55,855	128,320
4	160,588	0.001903	216,442	387,751
5	147,215	0.000912	363,657	591,562
6	296,676	0.000298	660,334	957,964
7	141,951	0.000188	802,285	1,119,508
8	83,722	0.000145	886,007	1,211,530
9	180,154	0.000084	1,066,161	1,402,290
10	274,697	0.000037	1,340,858	1,675,739
11	203,442	0.000020	1,544,300	1,864,866
12	158,942	0.000012	1,703,242	2,003,755
13	282,726	0.000004	1,985,968	2,224,928
14	189,530	0.000002	2,175,498	2,343,264
15	242,625	0.000000	2,418,124	2,418,124
Total Var=	2,418,124			
Std. Dev. =	49,174.42			

The variance scaling was reversed before computing the standard deviation, in this and all similar tables.

The righthand column of the table above shows the variances of ground up layers (from zero to the tops of the numbered layers). The next thing of interest would be to show the variance of the excess loss for each numbered layer. If subtraction of ground-up layer variances were correct, the results would be just the differences of the successive numbers in that column, after the entry for the bottom layer. Subtraction results are shown in the next table after a correction determined by the Miccolis formula. The correction term is shown separately in the next column. The variance of the excess loss in the top layer is also useful in computing the correlation, shown in the next column, between the ground-up layer which excludes the top layer shown in the layer column and the top layer itself. The first entry here is for row 2 of the table; that represents the correlation between excess loss in layer one and excess loss in layer two. The third row is the correlation between excess loss in the combined first two layers and excess loss in the third layer counting up from the bottom.

Table A-III-4

Layer	Var of layer zero to top	Var/Excess Layer	Var Reduction	Correl of O,T w/top layer
1	21,746	21,746		
2	55,627	10,579	23,302	0.733945
3	128,320	22,349	50,344	0.691373
4	387,751	113,786	145,645	0.590111
5	591,562	46,140	157,671	0.583871
6	957,964	106,129	260,273	0.516184
7	1,119,508	19,651	141,894	0.515368
8	1,211,530	6,307	85,715	0.508690
9	1,402,290	25,115	165,644	0.473712
10	1,675,739	48,628	224,821	0.429720
11	1,864,866	24,613	164,514	0.404548
12	2,003,755	14,273	124,616	0.381568
13	2,224,928	38,012	183,161	0.331595
14	2,343,264	14,698	103,637	0.286408
15	2,418,124	9,674	65,187	0.216404

A risk charge of 5% of the expected value pure premium was used by Miccolis to as a standard to determine the coefficient of variance in the pricing formula. The coefficient was determined to be 2.559E-06. The following table gives the resulting premiums for all ground-up layers and all excess layers.

Table A-III-5

Lambda = 2.56E-06

Layer#	Exp. Loss zero to top	Exp. Loss Excess Layer	Premium zero to top	Premium Excess layer
1	$1,113	$1113	$1,169	$1,169
2	1,579	466	1,721	493
3	2,082	503	2,411	561
4	2,811	728	3,803	1,019
5	3,073	263	4,587	381
6	3,334	260	5,785	532
7	3,405	71	6,269	121
8	3,438	33	6,538	49
9	3,493	55	7,081	119
10	3,549	56	7,837	181
11	3,576	27	8,349	90
12	3,592	16	8,720	52
13	3,610	18	9,304	116
14	3,617	7	9,614	45
15	3,620	3	9,808	28

Table A-III-6

Percentage Risk Loads

Layer Definitions Layer #	Start	Top	Percent for zero to top	Percent for excess layer
1	$ 0	$ 25,000	4.76	4.76
2	25,000	50,000	8.27	5.49
3	50,000	100,000	13.62	10.20
4	100,000	300,000	26.09	28.56
5	300,000	500,000	33.00	31.00
6	500,000	1,000,000	42.37	51.06
7	1,000,000	1,300,000	45.70	41.48
8	1,300,000	1,500,000	47.42	32.87
9	1,500,000	2,000,000	50.68	53.79
10	2,000,000	3,000,000	54.72	68.89
11	3,000,000	4,000,000	57.16	69.67
12	4,000,000	5,000,000	58.81	70.10
13	5,000,000	7,500,000	61.20	84.15
14	7,500,000	10,000,000	62.37	84.48
15	10,000,000	15,000,000	63.09	88.37

This verifies the often stated opinion that the risk load is a larger fraction of the excess layer premium than it is of the primary premium. While it is also quite large for very high limits primary policies, those ground-up coverages in the top half of the list are not often written as single policies because of their high risk loads, which can be avoided by the common combination of primary plus excess covers. Excess layers, on the other hand, can be kept small to hold down their expected losses, but their percentage risk loads are still high because of their risk structure. A sufficiently thin excess layer approaches the risk characteristics of the Poisson distribution which has a long tail for the cases of small frequency.

The next example is similar to the first example; it has the same Poisson frequency but the severity is piece wise constant with constant density within each layer. The layers have the same frequency within the layer for each layer as for the lognormal, but the mean and variance within each layer will be somewhat different than for the lognormal. The main motive for this difference is to facilitate the calculation of the RAC within each layer. Based upon the lognormal, the RAC is difficult to compute because the moment generating function for the lognormal can only be expressed as a series expansion. The lognormal has all moments but the series is difficult to express in any simple form. In addition, the motive also exists to illustrate how easy the RAC is to compute when each layer is approximated as a rectangular density function.

EXAMPLE B

Table A-III-1 remains the same in the B example as in the A table, because the layer frequencies have been kept the same. But the layer mean (the mean of all aggregate loss from losses whose size is within that layer) is just the layer frequency multiplied by the average of the upper and lower endpoints of the layer. Because of the Poisson frequency within each layer, the variance of aggregate loss (the variance of the sum of all losses whose size is within that layer) is given as the frequency in that layer multiplied by the second non-central moment of the layer severity. The formulas for these and for the RAC within a layer are given at the end of this appendix.

The next five tables repeat the last five tables so the reader can see the size of the differences between the two models. The next two tables have some differences from their example A counterparts.

Table A-III-7

Layer #	Exp. Los in layer	Freq above in layer	Exp. Loss below & incl.	E. Loss for zero to top	Exp. Loss excess layer
1	$ 939.65	939.6542	$ 939.65	$1,560.32	$1,560.32
2	396.33	396.3317	1,335.99	2,048.88	488.56
3	525.85	525.8496	1,861.84	2,586.49	537.61
4	1,068.61	1,068.6149	2,930.45	3,501.48	914.99
5	396.72	396.7247	3,327.18	3,782.98	281.50
6	460.49	460.4910	3,787.67	4,085.30	302.31
7	126.41	126.4126	3,914.08	4,158.10	72.80
8	60.24	60.2403	3,974.32	4,191.33	33.24
9	106.13	106.1288	4,080.45	4,248.51	57.18
10	117.06	117.0581	4,197.51	4,309.13	60.62
11	60.11	60.1075	4,257.61	4,337.76	28.62
12	36.07	36.0670	4,293.68	4,353.78	16.03
13	48.58	48.5772	4,342.26	4,374.12	20.34
14	22.49	22.4920	4,364.75	4,381.53	7.41
15	20.97	20.9707	4,385.75	4,385.72	4.19
Sum =	$4,385.72				

Table A-III-8

(All variance figures have been divided by 1000)

Layer #	Variance Freq in Layer	Above Layer	Variance below & incl.	Var/Layer zero to top
1	15,661	0.024827	15,661	31,178
2	15,413	0.014258	31,074	66,718
3	40,899	0.007247	71,973	144,438
4	231,533	0.001903	303,506	474,815
5	161,996	0.000912	465,502	693,407
6	358,160	0.000298	823,662	1,121,292
7	146,199	0.000188	969,861	1,287,084
8	84,480	0.000145	1,054,341	1,379,865
9	186,989	0.000084	1,241,330	1,577,459
10	296,547	0.000037	1,537,877	1,872,758
11	211,807	0.000020	1,749,684	2,070,251
12	162,969	0.000012	1,912,654	2,213,166
13	307,655	0.000004	2,220,309	2,459,268
14	198,144	0.000002	2,418,453	2,586,218
15	265,628	0.000000	2,684,081	2,684,081
Sum =	2,684,081	(Scaled by E-04)		
Std. Dev. =	$ 51,808			

Table A-III-9

Layer	Var of Layer zero to top	Var of Excess Layer	Var Reduct By Layering	Correll of O,T with top layer
1	31,178	31,178		
2	66,718	11,113	24,428	0.615219
3	144,438	23,959	53,761	0.644774
4	474,815	147,378	182,999	0.610913
5	693,407	49,689	168,903	0.543396
6	1,121,292	125,573	302,312	0.508375
7	1,287,084	20,191	145,601	0.481853
8	1,379,865	6,361	86,420	0.476027
9	1,577,459	26,062	171,532	0.451003
10	1,872,758	52,817	242,482	0.419143
11	2,070,251	25,760	171,733	0.390379
12	2,213,166	14,692	128,224	0.367209
13	2,459,268	42,743	203,359	0.330303
14	2,586,218	15,841	111,109	0.281305
15	2,684,081	13,980	83,883	0.220475

The variance-based risk-loaded premiums for the B example are given in the next tables, with their percentages of risk load. The λ value used is that which gives the 5% risk load for the basic policy whose limit is $25,000.

Table A-III-10

Layer #	E. Loss for zero to top	Exp. Loss Excess Layer	Premium zero to top	Premium Excess Layer
1	$1,560	$1,560	$1,640	$1,640
2	2,049	489	2,220	517
3	2,586	538	2,956	599
4	3,501	915	4,717	1,292
5	3,783	282	5,557	409
6	4,085	302	6,955	624
7	4,158	73	7,452	124
8	4,191	33	7,722	50
9	4,249	57	8,285	124
10	4,309	61	9,102	196
11	4,338	29	9,636	95
12	4,354	16	10,017	54
13	4,374	20	10,667	130
14	4,382	7	11,000	48
15	4,386	4	11,254	40

Table A-III-11

Layer Definitions Layer #	Start	Top	% Risk-Load zero to top	% Risk-Load Excess Layer
1	$ 0	25,000	4.86	4.86
2	25,000	50,000	7.69	5.50
3	50,000	100,000	12.50	10.24
4	100,000	300,000	25.76	29.19
5	300,000	500,000	31.93	31.11
6	500,000	1,000,000	41.26	51.53
7	1,000,000	1,300,000	44.20	41.51
8	1,300,000	1,500,000	45.73	32.87
9	1,500,000	2,000,000	48.72	53.84
10	2,000,000	3,000,000	52.65	69.04
11	3,000,000	4,000,000	54.98	69.73
12	4,000,000	5,000,000	56.54	70.11
13	5,000,000	7,500,000	59.00	84.32
14	7,500,000	10,000,000	60.17	84.55
15	10,000,000	15,000,000	61.03	89.51

The differences between the examples A and B are now evident and are apparently minor, based on comparison of the two sets of 5 tables for each. The next series of tables will focus upon the differences between variance risk load and RAC, and upon the properties of RAC as a risk-loaded premium, all entirely based upon the B example.

The first idea to illustrate is that the risk aversion level can be selected on the same basis as the λ coefficient of variance was selected. The result is that the risk aversion level is $r = 4.93E\text{-}06$, also a very small number. The reciprocal of the risk aversion level will also be noted since this is sometimes called risk tolerance. For the stated risk aversion level the risk tolerance is $202,840 indicating a very small insurer. The set of premiums calculated by RAC for the ground-up policies is given in the next table.

Table A-III-12

Risk Aversion = 4.93E-06, Risk Tolerance = $202,840

Layer #	Policy Limit	E. Loss for zero to top	RAC of layer zero to top	% Risk-Load zero to top	Var Prem zero to top
1	$ 25,000	$1,560	$1,640	5.1	$1,640
2	50,000	2,049	2,225	8.6	2,220
3	100,000	2,586	2,995	15.8	2,956
4	300,000	3,501	5,307	51.6	4,717
5	500,000	3,783	7,292	92.8	5,557

The reason that the results are not given for the higher policy limits is that the premium becomes very large for the higher limits at this large risk aversion level. If an insurer is so risk averse that it requires a 5% risk load at a policy limit of $25,000, it is too risk averse to write policy limits of $500,000 or more. That conclusion seems reasonable in light of the fact that most small primary companies do not write high limits policies.

Another risk aversion level to consider is that which makes the premium for top policy limits as determined by RAC equal to that determined by variance with the same λ we have been using, 2.559E-06. This is .5682E-06 and it corresponds to a risk tolerance of $1,759,944.

Table A-III-13

Risk Aversion = 5.68E-07, Risk Tolerance = $1,759,944

Layer #	Policy Limit	E. Loss for zero to top	RAC of layer zero to top	% Risk-Load zero to top	Var Prem zero to top
1	25,000	1,560	1,569	0.6	1,640
2	50,000	2,049	2,068	0.9	2,220
3	100,000	2,586	2,628	1.6	2,956
4	300,000	3,501	3,642	3.9	4,717
5	500,000	3,783	3,994	5.3	5,557
6	1,000,000	4,085	4,447	8.1	6,955
7	1,300,000	4,158	4,586	9.3	7,452
8	1,500,000	4,191	4,660	10.1	7,722
9	2,000,000	4,249	4,813	11.7	8,285
10	3,000,000	4,309	5,058	14.8	9,102
11	4,000,000	4,338	5,264	17.6	9,636
12	5,000,000	4,354	5,468	20.4	10,017
13	7,500,000	4,374	6,154	28.9	10,667
14	10,000,000	4,382	7,200	39.1	11,000
15	15,0000,00	4,386	11,254	61.0	11,254

The next table shows the premiums for all excess layers. Also shown for perspective are the expected loss, the risk-load as a fraction of the premium, and the variance-based premium.This is at the same risk aversion level last used.

Table A-III-14

Risk Aversion = 5.682E-07, Risk Tolerance = $1,759,944

Start	Top	Exp. Loss Excess Layer	RAC of Excess Layer	Var Premium Excess Layer	Layer RI Load as
$0	25,000	1,560	1,569	1,640	0.5
25,000	50,000	489	492	517	0.6
50,000	100,000	538	544	599	1.2
100,000	300,000	915	958	1,292	4.5
300,000	500,000	282	296	409	4.9
500,000	1,000,000	302	341	624	11.4
1,000,000	1,300,000	73	79	124	7.6
1,300,000	1,500,000	33	35	50	5.3
1,500,000	2,000,000	57	65	124	12.4
2,000,000	3,000,000	61	79	196	22.9
3,000,000	4,000,000	29	37	95	23.5
4,000,000	5,000,000	16	21	54	23.9
5,000,000	7,500,000	20	40	130	49.1
7,500,000	10,000,000	7	15	48	49.7
10,000,000	15,000,000	4	14	40	69.6

Table A-III-15

Risk Aversion = 5.682E-07, Risk Tolerance = $1,759,944

Start	Top	Exp. Loss Excess Layer	RAC of Excess Layer	Var Premium Excess Layer	Layer RI Load as
$ 0	$ 25,000	$1,560	$1,640	$1,640	4.8
25,000	50,000	489	517	517	5.5
50,000	100,000	538	602	599	10.6
100,000	300,000	915	1,412	1,292	35.2
300,000	500,000	282	452	409	37.7
500,000	1,000,000	302	1,069	624	71.7
1,000,000	1,300,000	73	158	124	53.8
1,300,000	1,500,000	33	55	50	40.0
1,500,000	2,000,000	57	225	124	74.5
2,000,000	3,000,000	61	1,292	196	95.3
3,000,000	4,000,000	29	652	95	95.6
4,000,000	5,000,000	16	379	54	95.7
5,000,000	7,500,000	20	222,912	130	99.9
7,5000,000	10,000,000	7	86,181	48	99.9

In spite of the small size and high risk aversion represented in the table above, this insurer is able to write most of the excess layers evaluated. The premiums are excessively large for the top six layers. Apparently, risk sharing works very well, but there are enough larger insurers, with smaller risk aversion to write these excess layers at lower cost.

The "Risk Profile Curve is a graph of the RAC as a function of the risk aversion level. Here it is a graph as a function of the risk tolerance which is the reciprocal of the risk aversion. Risk tolerance is an amount of money and so may appear more meaningful."

Figure two is the risk profile curve for the top excess layer,which starts at 7.5 million dollars and runs to 15 million dollars. At low risk tolerance, the risk loaded premium is very large but then it declines, approaching the expected loss pure premium which is just $4.00.

FIGURE 1

Risk Profile Curve for Basic Policy

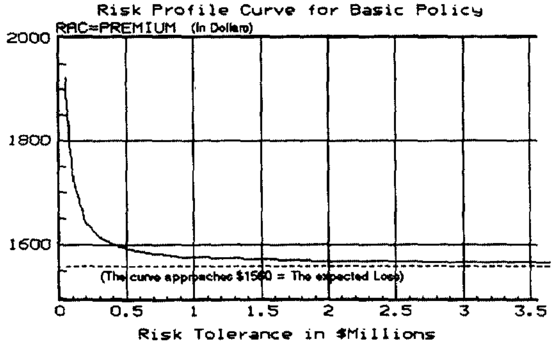
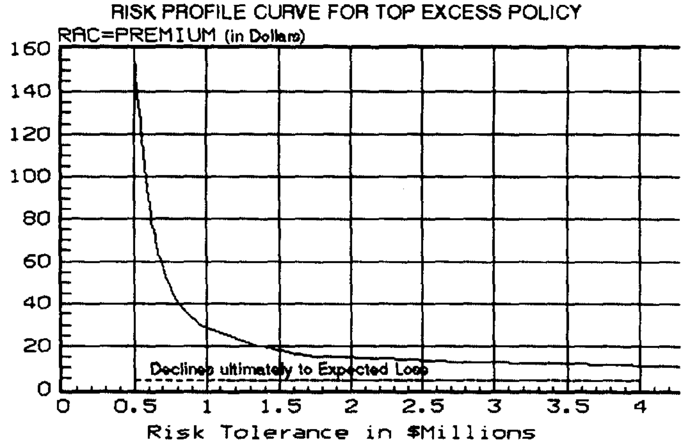


FIGURE 2

RISK PROFILE CURVE FOR TOP EXCESS POLICY



The Equations

When using a piecewise constant density for severity, where each layer has a constant density, the formulas needed for the results presented are given below. The set of three equations is for aggregate loss for all losses whose size is between the lower end point L_{i-1} and the upper end point, L_i of layer $_i$. The expected amount of aggregate loss, given the frequency F_i, in this layer is:

$$EL(L_{i-1}, L_i) = F_i \frac{L_{i-1}+L_i}{2}$$

In the special case when the layer has zero width, the expected loss is Fi x Li itself. The Variance in the layer, also based upon the rectangular severity is given by the next equation:

$$VAR(L_{i-1}, L_i) = F_i \frac{L_{i-1}^2 + L_{i-1} . L_i + L_i^2}{3}$$

When the layer endpoints are the same, the VAR is just

$$VAR(L_{i-1}, L_i) = F_i L_i^2$$

The RAC in the layer, a function of risk aversion level r, has the following formula:

$$RAC(L_{i-1}, L_i) = (\frac{F_i}{r}) . \left[\frac{\exp(r.L_i) - \exp(r.L_{i-1})}{r.(L_i - L_{i-1})} - 1 \right]$$

The special case when the layer has zero width has the special formula as follows:

$$RAC(L_{i-1}, L_i) = (\frac{F_i}{r}) . [\exp(r.L_i) - 1]$$

The special cases of zero width usually occur when there is a policy limit. Then all the layers above are effectively collapsed into a degenerate layer at that limit and the frequency of the degenerate layer is the frequency above that layer. This is very conveniently organized into a spreadsheet format.