STATEMENT OF ACTUARIAL OPINION – INSTRUCTIONS FOR 1991

NAIC
(with a letter and attachment
from R. Michael Lamb)

Statement of Actuarial Opinion
Instructions for 1991 Blank (Due March 1, 1992)

The National Association of Insurance Commissioners (NAIC) adopted a revision to the instructions for the 1991 Annual Statement Blank due March 1, 1992 regarding the scope and content of the Statement of Actuarial Opinion on casualty loss reserves.

The next seven pages is Instruction 12 as adopted. The ten pages following those are a letter and attachment from R. Michael Lamb, Chairman of the NAIC Casualty Actuarial (Technical) Task Force to the Chairman of the NAIC Blanks Task Force dated June 26, 1990. That material annotates the changes.

Due to the significance of the scope of these changes, we thought this material would be useful to you.

12. (1) STATEMENT OF ACTUARIAL OPINION

There is to be included or attached to Page 1 of the Annual Statement, the statement of a qualified actuary, entitled "Statement of Actuarial Opinion," setting forth his or her opinion relating to loss and loss adjustment expense reserves.

(2) DEFINITIONS

"Qualified actuary" is a person who is either:

(a) A member in good standing of the Casualty Actuarial Society, or

(b) A member in good standing of the American Academy of Actuaries who has been approved as qualified for signing casualty loss reserve opinions by the Casualty Practice Council of the American Academy of Actuaries, or

(c) A person who otherwise has competency in loss reserve evaluation as demonstrated to the satisfaction of the insurance regulatory official of the domiciliary state. In such case, at least 90 days prior to the filing of its annual statement, the insurer must request approval that the person be deemed qualified and that request must be approved or denied. The request must include the NAIC Biographical form and a list of all loss reserve opinions and/or certifications issued in the last 3 years by this person.

Notwithstanding the above, a domiciliary commissioner may, by bulletin or regulation, specify who may sign an opinion. Also, a domiciliary commissioner may require particular qualifications, including independence, for specific insurers.

"Insurer" means an insurer authorized to write property and/or casualty insurance under the laws of any state and includes but is not limited to fire and marine companies, general casualty companies, local mutual aid societies, statewide mutual assessment companies, mutual insurance companies other than farm mutual insurance companies and county mutual insurance companies, Lloyd's plans, reciprocal and interinsurance exchanges, captive insurance companies, risk retention groups, stipulated premium insurance companies, and non-profit legal services corporations.

"Annual Statement" means the annual financial statement required to be filed by insurers with the commissioner.

(3) CONTENT

The opinion shall be in the format of and contain the information required by this Section 12 of the Annual Statement Instructions: Property and Casualty.

(4) EXEMPTIONS

A certified copy of the approved exemption must be filed with the annual statement in all jurisdictions in which the company is authorized.

Automatic Exemption

(a) An insurer otherwise subject to the requirement that has less than $1,000,000 total direct plus assumed written premiums during a calendar year or that has less than a total of 1,000 policyholders and certificate holders at the end of a calendar year, in lieu of the certification required for the calendar year, may submit an affidavit under oath of an officer of the insurer that specifies that amount of direct plus assumed premiums written and the total number of policyholders and certificate holders.

(b) An insurer who intends to file for an exemption under this section must submit a letter of intent to its domiciliary commissioner no later than December 1 of the calendar year for which the exemption is to be claimed. The commissioner may deny the exemption prior to December 31 of the same year if he deems the exemption inappropriate.

Exemption for Insurers under Supervision or Conservatorship

Unless ordered by the domiciliary commissioner, an insurer that is under supervision or conservatorship pursuant to statutory provision is exempt from the filing requirements contained herein.

Exemption for Nature of Business

An insurer otherwise subject to the requirement and not eligible for an exemption as enumerated above may apply to its domiciliary commissioner for an exemption based on the nature of business written. This exemption is available to those companies writing property lines only.

Financial Hardship Exemption

(a) An insurer otherwise subject to this requirement and not eligible for an exemption as enumerated above may apply to the commissioner for a financial hardship exemption.

(b) Financial hardship is presumed to exist if the projected reasonable cost of the certification would exceed the lesser of:

(i) One percent of the insurer's capital and surplus reflected in the insurer's latest quarterly statement for the calendar year for which the exemption is sought; or

(ii) Three percent of the insurer's projected net direct plus assumed premiums written during the calendar year for which the exemption is sought as reflected in the insurer's latest quarterly statement filed with its domiciliary commissioner.

(5) Such a statement of opinion must consist of a paragraph identifying the actuary; a scope paragraph identifying the subjects on which an opinion is to be expressed and describing the scope of the actuary's work (see sections 8-11 below); and an opinion paragraph expressing his or her opinion with respect to such subjects (see sections 12-14 below). One or more additional paragraphs may be needed in individual cases if the actuary considers it necessary to state a qualification of his or her opinion or to explain some aspect of the annual statement which is not already sufficiently explained in the annual statement.

(6) The opening paragraph should generally indicate the actuary's relationship to the company. For a company actuary the opening paragraph of the actuarial opinion should contain the sentence:

> "I, (name and title of actuary), am an officer (employee) of (named insurer) and a member of the American Academy of Actuaries and meet its qualification standards. (and/or) I am a Fellow/Associate of the Casualty Actuarial Society."

For a consulting actuary, the opening paragraph of the actuarial opinion should contain the sentence:

> "I, (name and title of actuary, am associated with the firm of (name of firm). I am a member of the American Academy of Actuaries and meet its qualification standards. (and/or) I am a Fellow/Associate of the Casualty Actuarial Society. I have been retained by the (name of insurer) with regard to loss and loss adjustment expense reserves."

For a person other than a member of the American Academy of Actuaries or a member of the Casualty Actuarial Society, the opening paragraph of the opinion should contain the sentence:

> "I, (name and title), am an officer (employee) of (name of insurer), and I have demonstrated competency in loss reserving to the satisfaction of (regulatory official of domiciliary state)."
>
> or
>
> "I, (name and title of consultant), am associated with the firm of (name of firm). I have demonstrated competency in loss reserving to the satisfaction of (regulatory official of domiciliary state) and have been retained by the (name of insurer) with regard to loss and loss adjustment expense reserves."

(7) The following are examples, for illustrative purposes, of language which in typical circumstances would be included in the remainder of the statement of actuarial opinion. The illustrative language should be modified as needed to meet the circumstances of a particular case, and the actuary should in any case use language which clearly expresses his or her professional judgment.

(8) The scope paragraph should contain a sentence such as the following:

> "I have examined the actuarial assumptions and methods used in determining reserves listed below, as shown in the Annual Statement of the company as prepared for filing with state regulatory officials, as of December 31, 19__."

The paragraph should list those items and amounts with respect to which the actuary is expressing an opinion. The list should include but not necessarily be limited to:

(a) Reserve for unpaid losses (Page 3, Item 1)

(b) Reserve for unpaid loss adjustment expenses (Page 3, Item 2).

(c) Reserve for unpaid losses - Direct and Assumed (Schedule P, Part 1, Cols. 13 and 15).

(d) Reserve for unpaid loss adjustment expenses - Direct and Assumed (Schedule P, Part 1, Cols. 17 and 19).

(9) If the actuary has examined the underlying records and/or summaries, the scope paragraph should also include a sentence such as the following:

"My examination included such review of the actuarial assumptions and methods used and of the underlying basic records and/or summaries and such tests of the calculations as I considered necessary."

(10) If the actuary has not examined the underlying records and/or summaries, but has relied upon those prepared by the company, the scope paragraph should include a sentence such as one of the following:

(a) "I relied upon underlying records and/or summaries prepared by the responsible officers or employees of the company or group to which it belongs. In other respects, my examination included such review of the actuarial assumptions and methods used and such tests of the calculations as I considered necessary."

(b) "I relied upon (name of accounting firm) for the accuracy of the underlying records and/or summaries. In other respects, my examination included such review of the underlying actuarial assumptions and methods used and such tests of the calculations as I considered necessary."

(11) The actuary should comment in the scope section, as appropriate, on relevant topics such as the following to the extent they affect, or could affect, the loss reserves; discounting, salvage/subrogation, loss portfolio transfers, financial reinsurance, and reinsurance collectibility. If the company reserves will create exceptional values using the NAIC IRIS tests, the actuary should include an explanation.

(12) The opinion paragraph should include a sentence which covers at least the points listed in the following illustration:

"In my opinion, the amounts carried in the balance sheet on account of the items identified above

(a) are computed in accordance with accepted loss reserving standards and principles.

(b) make a reasonable provision for all unpaid loss and loss expense obligations of the Company under the terms of its policies and agreements.

(c) meet the requirements of the insurance laws of (state of domicile)."

(13) The actuary should describe the actuarial assumptions and/or methods which have been used. If there has been any material change in the actuarial assumptions and/or methods from those previously employed, that change should be described in the statement of actuarial opinion by inserting a phrase such as:

"A material change in actuarial assumptions (and/or methods) was made during the past year, but such change accords with accepted loss reserving standards."

A brief description of the change should follow.

The adoption of new issues or coverages requiring underlying actuarial assumptions which differ from actuarial assumptions used for prior issues or coverages is not a change in actuarial assumption within the meaning of this paragraph.

(14) If the actuary is unable to form an opinion, he or she should refuse to issue a statement of opinion. If the actuary's opinion is adverse or qualified, the actuary should issue an adverse or qualified actuarial opinion explicitly stating the reason(s) for such opinion.

(15) The statement must include assurance that workpapers supporting the actuarial opinion will be maintained at the company and available for examination for seven years. The wording for an actuary employed by the company should be similar to the following:

"Workpapers supporting the findings expressed in this statement of actuarial opinion will be retained for a period of seven years in the administrative offices of the company and available for regulatory examination."

The wording for a consulting actuary retained by the company should be similar to the following:

"Workpapers supporting the findings expressed in this statement of actuarial opinion have been provided to the company to be retained for a period of seven years at its administrative offices and available for regulatory examination."

(16) The statement should conclude with the signature of the actuary responsible for providing the opinion. The signature should appear in the following format:

_____
Signature of actuary
Printed name of actuary
Address of actuary
Telephone number of actuary

Department of Insurance and Finance

21 LABOR AND INDUSTRIES BUILDING • SALEM, OREGON 97310

June 26, 1990

Mr. Robert Solitro
Director of Examinations
New Hampshire Insurance Department
169 Manchester Street
Concord, NH 03301

Re: Statement of Actuarial Opinion
Annual Statement Instructions for Property/Casualty Companies
Proposals from the Casualty Actuarial Task Force for 1991

Dear Bob:

The NAIC Casualty Actuarial Task Force recommends some changes to the Instructions relating to the Statement of Actuarial Opinion for property/casualty companies. For the most part, these have to do with the content of the statement and are needed for consistency with the changes adopted by the Blanks Task Force for 1990. We also recommend some substantive changes, which I wish to describe.

Paragraph (8): We want to add reserves for direct and assumed losses and loss adjustment expenses to the list of items for the scope paragraph to which the actuary is to express an opinion. Reserves on the direct and assumed basis represent the total potential liability should reinsurance agreements fail. Technical impairment on a direct and assumed basis should be of regulatory concern even if ceded loss reserves provide sufficient surplus relief.

New Paragraph (11): We want to insert a new requirement for the scope section for comment on items which could affect the loss reserves, such as: discounting (if and when permitted), salvage/subrogation, loss portfolio transfers, financial reinsurance, and reinsurance collectibility. These items are particularly relevant to the difference between direct and net reserves. Both regulators and industry representatives have expressed concern about the potential impact of these items on apparent solvency.

In addition, our task force recommends a required explanation from the actuary if the company reserves will cause exceptional values on the IRIS tests. This explanation should assist the examiner teams which review the IRIS results each year.

Paragraph (12): In the opening sentence of the opinion paragraph, the "fairly stated" phrase needs to be dropped. This is an accounting concept not translated into actuarial principles beyond "accepted loss reserving standards and principles," which is sufficient language.

We further recommend substituting the phrase "reasonable" for "good and sufficient," which seems to imply guaranteed adequacy despite all contingencies known or unknown. Actuaries facing the older phrase have expressed considerable discomfort with it. The term "reasonable" is preferred by most practicing actuaries as referring to an appropriate value based on all factors which are known or can be known at the current time--in other words, the best state-of-the-art estimate.

Our task force discussed other phrases such as "adequate" and "sufficient," but did not choose to use any other than current actuarial practice. Some members noted that section (iii) specifies that the opinion items must "meet the requirements of the insurance laws of" the state of domicile, which usually include a term such as "sufficient."

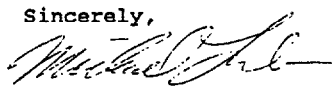
Paragraph (13): The actuary should describe the assumptions and methods used to determine the loss and expense reserves, rather than simply stating that any changes meet accepted standards. This will help us to evaluate the quality of efforts made to determine reserves and will help examiners interpret the workpapers.

New Paragraph (15): We recommend adding another paragraph or clause stating that workpapers supporting the opinion will be available at the company for examiners to review. A seven-year retention was selected to comfortably cover two triennial examinations.

New Paragraph (16): Finally, the signature line was just dangling at the end of the Instructions. We recommend a paragraph giving mention of it and also calling for a printed name, address, and phone number so we may easily contact the actuary directly.

With these revisions, we believe the Statement of Actuarial Opinion for property/casualty companies will be a useful tool for our efforts to monitor solvency.

Sincerely,



R. Michael Lamb, FCAS, MAAA
Casualty Actuary
Insurance Division
(503) 378-4271

RML:psm
7156u

Enclosure

12. (1) STATEMENT OF ACTUARIAL OPINION

There is to be included or attached to Page 1 of the Annual Statement, the statement of a qualified actuary, entitled "Statement of Actuarial Opinion," setting forth his or her opinion relating to loss and loss adjustment expense reserves.

(2) DEFINITIONS

"Qualified actuary" is a person who is either:

(a) A member in good standing of the Casualty Actuarial Society, or

(b) A member in good standing of the American Academy of Actuaries who has been approved as qualified for signing casualty loss reserve opinions by the Casualty Practice Council of the American Academy of Actuaries, or

(c) A person who otherwise has competency in loss reserve evaluation as demonstrated to the satisfaction of the insurance regulatory official of the domiciliary state. In such case, at least 90 days prior to the filing of its annual statement, the insurer must request approval that the person be deemed qualified and that request must be approved or denied. The request must include the NAIC Biographical form and a list of all loss reserve opinions and/or certifications issued in the last 3 years by this person.

Notwithstanding the above, a domiciliary commissioner may, by bulletin or regulation, specify who may sign an opinion. Also, a domiciliary commissioner may require particular qualifications, including independence, for specific insurers.

"Insurer" means an insurer authorized to write property and/or casualty insurance under the laws of any state and includes but is not limited to fire and marine companies, general casualty companies, local mutual aid societies, statewide mutual assessment companies, mutual insurance companies other than life, farm mutual insurance companies, county mutual insurance companies, Lloyd's plans, reciprocal and interinsurance exchanges, captive insurance companies, risk retention groups, stipulated premium insurance companies, and non-profit legal services corporations.

"Annual Statement" means the annual financial statement required to be filed by insurers with the commissioner.

(3) CONTENT

The opinion shall be in the format of and contain the information required by this Section 12 of the Annual Statement Instructions: Property and Casualty.

(4) EXEMPTIONS

A certified copy of the approved exemption must be filed with the annual statement in all jurisdictions in which the company is authorized.

Automatic Exemption

(a)[.] An insurer otherwise subject to the requirement that has less than $1,000,000 total direct plus assumed written premiums during a calendar year or that has less than a total of 1,000 policyholders and certificate holders at the end of a calendar year, in lieu of the certification required for the calendar year, may submit an affidavit under oath of an officer of the insurer that specifies that amount of direct plus assumed premiums written and the total number of policyholders and certificate holders.

(b)[.] An insurer who intends to file for an exemption under this section must submit a letter of intent to its domiciliary commissioner no later than December 1 of the calendar year for which the exemption is to be claimed. The commissioner may deny the exemption prior to December 31 of the same year if he deems the exemption inappropriate.

Exemption for Insurers under Supervision or Conservatorship

Unless ordered by the domiciliary commissioner, an insurer that is under supervision or conservatorship pursuant to statutory provision is exempt from the filing requirements contained herein.

Exemption for Nature of Business

An insurer otherwise subject to the requirement and not eligible for an exemption as enumerated above may apply to its domiciliary commissioner for an exemption based on the nature of business written. This exemption is available to those companies writing property lines only.

Financial Hardship Exemption

(a)[.] An insurer otherwise subject to this requirement and not eligible for an exemption as enumerated above may apply to the commissioner for a financial hardship exemption.

[(b)][.] Financial hardship is presumed to exist if the projected reasonable cost of the certification would exceed the less[o]er of:

[(i)][.] One percent of the insurer's capital and surplus reflected in the insurer's annual statement [filed with the board] for the calendar year for which the exemption is sought; or

[(ii)][.] Three percent of the insurer's net direct plus assumed premiums written during the calendar year for which the exemption is sought as reflected in the insurer's annual statement filed with its domiciliary commissioner.

(5) Such a statement of opinion must consist of a paragraph identifying the actuary; a scope paragraph identifying the subjects on which an opinion is to be expressed and describing the scope of the actuary's work (see sections 8-[10]11 below); and an opinion paragraph expressing his or her opinion with respect to such subjects (see sections [11-13]12-14 below). One or more additional paragraphs may be needed in individual cases if the actuary considers it necessary to state a qualification of his or her opinion or to explain some aspect of the annual statement[s] which is not already sufficiently explained in the annual statement[s].

(6) The opening paragraph should generally indicate the actuary's relationship to the company. For a company actuary the opening paragraph of the actuarial opinion should contain the sentence:

> "I, (name and title of actuary), am an officer (employee) of (named insurer) and a member of the American Academy of Actuaries and meet its qualification standards. (and/or) I am a Fellow/Associate of the Casualty Actuarial Society."

For a consult[ant]ing actuary, the opening paragraph of the actuarial opinion should contain the sentence:

> "I, (name and title of actuary [consultant]), am associated with the firm of (name of firm). I am a member of the American Academy of Actuaries and meet its qualification standards, (and/or) I am a Fellow/Associate of the Casualty Actuarial Society, I [and] have been retained by the (name of insurer) with regard to loss and loss adjustment expense reserves."

For a person other than a member of the American Academy of Actuaries or a member of the Casualty Actuarial Society, the opening paragraph of the opinion should contain the sentence:

'I, (name and title), am an officer (employee) of (name of insurer), and I have demonstrated competency in loss reserving[."] to the satisfaction of (regulatory official of domiciliary state)."

or

"I, (name and title of consultant), am associated with the firm of (name of firm). I have demonstrated competency in loss reserving to the satisfaction of (regulatory official of domiciliary state) and have been retained by the (name of insurer) with regard to loss and loss adjustment expense reserves."

(7) The following are examples, for illustrative purposes, of language which in typical circumstances would be included in the remainder of the statement of actuarial opinion. The illustrative language should be modified as needed to meet the circumstances of a particular case, and the actuary should in any case use language which clearly expresses his or her professional judgment.

(8) The scope paragraph should contain a sentence such as the following:

'I have examined the actuarial assumptions and methods used in determining reserves listed below, as shown in the Annual Statement of the company as prepared for filing with state regulatory officials, as of December 31, 19__."

The paragraph should list those items and amounts with respect to which the actuary is expressing an opinion. The list should include but not necessarily be limited to:

[(i)] (a) Reserve for unpaid losses (Page 3, Item 1)

[(ii)] (b) Reserve for unpaid loss adjustment expenses (Page 3, Item 2).

(c) Reserve for unpaid losses - Direct and Assumed (Schedule P, Part 1, Cols. 13 and 15)

(d) Reserve for unpaid loss adjustment expenses - Direct and Assumed (Schedule P, Part 1, Cols. 17 and 19.

(9) If the actuary has examined the underlying records and/or summaries, the scope paragraph should also include a sentence such as the following:

"My examination included such review of the actuarial assumptions and methods used and of the underlying basic

records and/or summaries and such tests of the [and] calculations as I considered necessary."

(10) If the actuary has not examined the underlying records and/or summaries, but has relied upon those prepared by the company, the scope paragraph should include a sentence such as one of the following:

[(i)] (a) "I relied upon underlying records and/or summaries prepared by the responsible officers or employees of the company or group to which it belongs. In other respects, my examination included such review of the actuarial assumptions and methods used and such tests of the calculations as I considered necessary."

[(ii)] (b) "I relied upon (name of accounting firm) for the accuracy of the underlying records and/or summaries. In other respects, my examination included such review of the underlying actuarial assumptions and methods used and such te[x]sts of the calculations as I considered necessary."

(11) The actuary should comment in the scope section, as appropriate, on relevant topics such as the following to the extent they affect, or could affect, the loss reserves: discounting, salvage/subrogation, loss portfolio transfers, financial reinsurance, and reinsurance collectibility. If the company reserves will create exceptional values using the NAIC IRIS tests, the actuary should include an explanation.

[(11)](12) The opinion paragraph should include a sentence which covers at least the points listed in the following illustration:

"In my opinion, the amounts carried in the balance sheet on account of the items identified above

[(i)] (a) are computed in accordance with accepted loss reserving standards and [are fairly stated in accordance with sound loss reserving] principles.

[(ii)] (b) make a reasonable provision for all unpaid loss and loss expense obligations of the Company under the terms of its policies and agreements. [are based on factors relevant to policy provisions.]

[(iii)] (c) meet the requirements of the insurance laws of (state of domicile)."

[(iv) make a good and sufficient provision for all unpaid loss and loss expense obligations of the Company under the terms of its policies and agreements."]

[(12)](13) The actuary should describe the actuarial assumptions and/or methods which have been used. If there has been any material change in the actuarial assumptions and/or methods from those previously employed, that change should be described in the statement of actuarial opinion by inserting a phrase such as:

> "A material change in actuarial assumptions (and/or methods) was made during the past year, but such change accords with accepted loss reserving standards."

A brief description of the change should follow.

The adoption of new issues or coverages requiring underlying actuarial assumptions which differ from actuarial assumptions used for prior issues or coverages is not a change in actuarial assumption within the meaning of this paragraph.

[(13)](14) If the actuary is unable to form an opinion, he or she should refuse to issue a statement of opinion. If the actuary's opinion is adverse or qualified, the actuary should issue an adverse or qualified actuarial opinion explicitly stating the reason(s) for such opinion.

(15) The statement must include assurance that workpapers supporting the actuarial opinion will be maintained at the company and available for examination for seven years. The wording for an actuary employed by the company should be similar to the following:

> "Workpapers supporting the findings expressed in this statement of actuarial opinion will be retained for a period of seven years in the administrative offices of the company and available for regulatory examination.

The wording for a consulting actuary retained by the company should be similar to the following:

> "Workpapers supporting the findings expressed in this statement of actuarial opinion have been provided to the company to be retained for a period of seven years at its administrative offices and available for regulatory examination.

(16) The statement should conclude with the signature of the actuary responsible for providing the opinion. The signature should appear in the following format:

_____

Signature of actuary
Printed name of actuary
Address of actuary
Telephone number of actuary

j:\jeo\misc\lossres

CONTROVERSIES IN THE FOUNDATION OF STATISTICS (REPRINT)

Bradley Efron

Controversies in the Foundations of Statistics

by Bradley Ephron

This lively and wide-ranging article explores the philosophical battles among Bayesians, classical statisticians (frequentists), and a third group, termed the Fisherians. At this writing, no clear winner has emerged, although the frequentists may currently have the upper hand.

The article gives examples of the approach to estimation of the mean of a distribution by each camp, and some problems with each approach. One section discusses Stein's estimator more rigorously than the Scientific American article by Ephron and Morris. Ephron speculates on the future of statistical theory.

This article will give you insight regarding the fundamental problems of statistics that affect your work (in particular, as regards credibility). The bases of some common actuarial methods are still controversial.

This article is presented as part of a program of reprinting important papers on the foundations of casualty actuarial science. It is reprinted with the generous permission of the Mathematical Association of America. It originally appeared in the American Mathematical Monthly, Volume 85, Number 4, April 1978, pages 231 to 246.

CONTROVERSIES IN THE FOUNDATIONS OF STATISTICS

BRADLEY EFRON

1. Introduction. Statistics seems to be a difficult subject for mathematicians, perhaps because its elusive and wide-ranging character mitigates against the traditional theorem-proof method of presentation. It may come as some comfort then that statistics is also a difficult subject for statisticians. We are now celebrating the approximate bicentennial of a controversy concerning the basic nature of statistics. The two main factions in this philosophical battle, the Bayesians and the frequentists, have

---

Bradley Efron received his Ph.D. in Statistics from Stanford in 1964 under the direction of Rupert Miller. He holds professorships at Stanford in both the Statistics Department and the Department of Preventive Medicine. His interests cover most of theoretical and applied statistics, with special emphasis on the application of geometrical methods to statistical problems. — *Editors*

alternated dominance several times, with the frequentists currently holding an uneasy upper hand. A smaller third party, perhaps best called the Fisherians, snipes away at both sides.

Statistics, by definition, is uninterested in the special case. Averages are the meat of statisticians, where "average" here is understood in the wide sense of any summary statement about a large population of objects. "The average I.Q. of a college freshman is 109" is one such statement, as is "the probability of a fair coin falling heads is 1/2." The controversies dividing the statistical world revolve on the following basic point: just *which* averages are most relevant in drawing inferences from data? Frequentists, Bayesians, and Fisherians have produced fundamentally different answers to this question.

This article will proceed by a series of examples, rather than an axiomatic or historical exposition of the various points of view. The examples are artificially simple for the sake of humane presentation, but readers should be assured that real data are susceptible to the same disagreements. A counter-warning is also apt: these disagreements haven't crippled statistics, either theoretical or applied, and have as a matter of fact contributed to its vitality. Important recent developments, in particular the empirical Bayes methods mentioned in Section 8, have sprung directly from the tension between the Bayesian and frequentist viewpoints.

2. The normal distribution. All of our examples will involve the normal distribution, which for various reasons plays a central role in theoretical and applied statistics. A normal, or Gaussian, random variable x is a quantity which possibly can take on any value on the real axis, but not with equal probability. The probability that x falls in the interval $[a, b]$ is given by the area under Gauss' famous bell-shaped curve,

$$\text{Prob}\{a \leqq x \leqq b\} = \int_a^b \phi_{\mu,\sigma}(x)dx, \tag{2.1}$$

where

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]. \tag{2.2}$$

For convenience we indicate such a random variable by

$$x \sim \mathcal{N}(\mu, \sigma^2), \tag{2.3}$$

with σ^2 instead of σ as the second argument by convention.

Figure 1 illustrates the normal distribution. The high point of $\phi_{\mu,\sigma}(x)$ is at $x = \mu$, the curve falling off quickly for $|x - \mu| > \sigma$. Most of the probability, 99.7%, is within ± 3 σ-units of the central value μ. We can write $x \sim \mathcal{N}(\mu, \sigma^2)$ as $x = \mu + \varepsilon$, where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$; adding the constant μ merely shifts $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ μ units to the right.

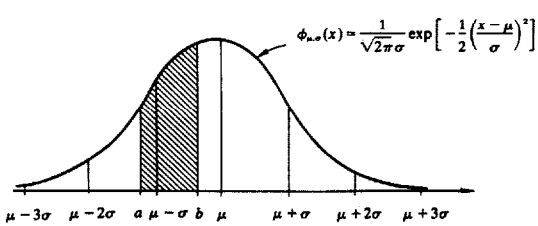


FIG. 1. The normal distribution. The random quantity $x \sim \mathcal{N}(\mu, \sigma^2)$ occurs in $[a, b]$ with probability equal to the shaded area. 68% of the probability is in the interval $[\mu - \sigma, \mu + \sigma]$, 95% in $[\mu - 2\sigma, \mu + 2\sigma]$, 99.7% in $[\mu - 3\sigma, \mu + 3\sigma]$.

The parameter μ is the "mean" or "expectation" of the random quantity x. Using "E" to indicate expectation,

$$\mu = E\{x\} = \int_{-\infty}^{\infty} x\phi_{\mu,\sigma}(x)dx. \tag{2.4}$$

The reader may wish to think of $E\{g(x)\}$ for an arbitrary function $g(x)$ as just another notation for the integral of $g(x)$ with respect to $\phi_{\mu,\sigma}(x)dx$,

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x)\phi_{\mu,\sigma}(x)dx. \tag{2.5}$$

Intuitively, $E\{g(x)\}$ is the weighted average of the possible values of $g(x)$, weighted according to the probabilities $\phi_{\mu,\sigma}(x)dx$ for the infinitesimal intervals $[x, x+dx]$. In other words, $E\{g(x)\}$ is a theoretical average of an infinite population of $g(x)$ values, where the x's occur in proportion to $\phi_{\mu,\sigma}(x)$.

It is easy to see, by symmetry, that μ is indeed the theoretical average of x itself when $x \sim \mathcal{N}(\mu, \sigma^2)$. A more difficult calculation (though easy enough for friends of the gamma function) gives the expectation of $g(x) = (x-\mu)^2$,

$$E\{(x-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2\phi_{\mu,\sigma}(x)dx = \sigma^2. \tag{2.6}$$

The parameter σ, called the "standard deviation," sets the scale for the variability of x about the central value μ, as Figure 1 shows. A $\mathcal{N}(1, 10^{-6})$ random variable will have almost no perceptible variability under repeated trials, 997 out of 1000 repetitions occurring in [.997, 1.003], since $\sigma = 10^{-3}$. A $\mathcal{N}(1, 10^{6})$ random variable is almost all noise and no signal, in the evocative language of communications theory.

The normal distribution has a very useful closure property that makes it as easy to deal with many observations as with a single one. Let $x_1, x_2, x_3, \ldots, x_n$ be n independent observations, each of which is $\mathcal{N}(\mu, \sigma^2)$, μ and σ being the same for all n repetitions. Independence means that the value of x_1, say, does not affect any of the other values: observing $x_1 > \mu$ does not increase or decrease the 34% probability that $x_2 \in [\mu, \mu+\sigma]$, etc. A familiar (non-normal) example of independent variables $x_1, x_2, x_3, \ldots$ is given by successive observations of a well-rolled die.

Let

$$\bar{x} = \sum_{i=1}^{n} x_i/n \tag{2.7}$$

be the observed average of the n independent $\mathcal{N}(\mu, \sigma^2)$ variables. It is easy to show that

$$\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n). \tag{2.8}$$

The distribution of $\bar{x}$ is the same as that for the individual x_i except that the scaling parameter has been reduced from σ to $\sigma/\sqrt{n}$. By taking n sufficiently large we can reduce the variability of $\bar{x}$ about μ to an arbitrarily small level, but of course in real problems n is limited and $\bar{x}$ retains an irreducible component of random variability.

In all of our examples σ will be assumed known to the statistician. The unknown parameter μ will be the object of interest, the goal being to make inferences about the value of μ on the basis of the data $x_1, x_2, x_3, \ldots, x_n$. In 1925 Sir Ronald Fisher made the fundamental observation that in this situation *the average $\bar{x}$ contains all possible information about μ*. For any inference problem about μ, knowing $\bar{x}$ is just as good as knowing the entire data set $x_1, x_2, x_3, \ldots, x_n$. In modern parlance, $\bar{x}$ is a "sufficient statistic" for the unknown parameter μ.

It is easy to verify sufficiency in this particular case. Given the observed value of $\bar{x}$, a standard

probability calculation shows that the random quantities $x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x}, \ldots, x_n - \bar{x}$ have a joint distribution which does not depend in any way on the unknown parameter μ. In other words, what's left over in the data after the statistician learns $\bar{x}$ is devoid of information about μ. (This deceptively simple principle eluded both Gauss and Laplace!)

3. Frequentist estimation of the mean. The statistician may wish to estimate the unobservable parameter μ on the basis of the observed data $x_1, x_2, x_3, \ldots, x_n$. "Estimate" usually means "make a guess $\hat{\mu}(x_1, x_2, x_3, \ldots, x_n)$ depending on $x_1, x_2, \ldots, x_n$, with the understanding that you will be penalized an amount which is a smooth increasing function of the error of estimation $|\hat{\mu} - \mu|$." The usual penalty function, which we shall also use here, is $(\hat{\mu} - \mu)^2$, the squared-error loss function originally introduced by Gauss.

Fisher's sufficiency principle says that we need only consider estimation rules which are a function of $\bar{x}$. The most obvious candidate is $\bar{x}$ itself,

$$\hat{\mu}(x_1, x_2, \ldots, x_n) = \bar{x}. \tag{3.1}$$

This estimation rule is "unbiased" for μ; no matter what the true value of μ is,

$$E\bar{x} = \mu. \tag{3.2}$$

Unbiasedness is by no means a necessary condition for a good estimation rule, as we shall see later, but it does have considerable intuitive appeal as a guarantee that the statistician is not trying to slant the estimation process in favor of any particular μ value.

The expected penalty for using $\hat{\mu} = \bar{x}$ is, according to (2.6) and (2.8),

$$E(\hat{\mu} - \mu)^2 = \sigma^2/n. \tag{3.3}$$

Gauss showed that among all unbiased estimation rules $\hat{\mu}(x_1, x_2, \ldots, x_n)$ which are linear in $x_1, x_2, x_3, \ldots, x_n$, the rule $\hat{\mu} = \bar{x}$ uniformly minimizes $E(\hat{\mu} - \mu)^2$ for every value of μ. In the early 1940's this result was extended to include any unbiased estimator at all, linear or nonlinear. The proof, which depends on ideas Fisher developed in the 1920's, was put forth separately by H. Cramér in Sweden and C. R. Rao in India.

If we agree to abide by the unbiasedness criterion and to use squared-error loss, $\bar{x}$ seems to be the best estimator for μ. It is helpful for the statistician to provide not only a "point estimator" for μ, $\bar{x}$ in this case, but also a range of plausible values of μ consistent with the data. From (2.8) and Figure 1 we see that

$$\text{Prob}\{|\bar{x} - \mu| \leqq 2\sigma/\sqrt{n}\} = .95, \tag{3.4}$$

which is equivalent to the statement

$$\text{Prob}\{\bar{x} - 2\sigma/\sqrt{n} \leqq \mu \leqq \bar{x} + 2\sigma/\sqrt{n}\} = .95. \tag{3.5}$$

The interval $[\bar{x} - 2\sigma/\sqrt{n}, x + 2\sigma/\sqrt{n}]$ is called a "95% confidence interval" for μ. The theory of confidence intervals was developed by J. Neyman in the early 1930's. As an example, suppose $n = 4$, $\sigma = 1$, and we observe $x_1 = 1.2$, $x_2 = 0.3$, $x_3 = 0.7$, $x_4 = 0.2$. Then $\bar{x} = 0.6$ and the 95% confidence interval for μ is $[-.04, 1.6]$.

All of this seems so innocuous and straightforward that the reader may wonder where the grounds for controversy lie. The fact is that all of the results presented so far are "frequentist" in nature. That is, they relate to theoretical averages with respect to the $\mathcal{N}(\mu, \sigma^2/n)$ distribution of $\bar{x}$, with μ assumed fixed at its true value, whatever that may be. Unbiasedness itself is a frequentist concept; the theoretical average of $\hat{\mu}$ with μ held fixed, $E\hat{\mu}$, equals μ. Results (3.3) and (3.5), and the Cramér–Rao theorem, are frequentist statements. For example, the proper interpretation of (3.5) is that the interval $[\bar{x} - 2\sigma/\sqrt{n}, \bar{x} + 2\sigma/\sqrt{n}]$ covers the true value of μ with frequency 95% in a long series of independent repetitions of $\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n)$.

Nobody doubts that these results are true. The question raised by Bayesians and Fisherians is whether frequentist averages are really relevant to the process of inference scientists use in reasoning from noisy data back to the underlying mathematical models. We turn next to the Bayesian point of view.

4. Bayesian estimation of the mean. So far we have considered μ to be a fixed, albeit unknown, quantity. Suppose though that μ itself is a random variable, known to have the normal distribution with mean m and standard deviation s,

$$\mu \sim \mathcal{N}(m, s^2), \tag{4.1}$$

m and s being constants known to the statistician. For example, if μ is the true I.Q. of a person randomly chosen from the population of the United States, (4.1) holds with $m = 100$ and $s = 15$ (approximately). About 68% of I.Q.'s are between 85 and 115, about 95% between 70 and 130, etc. Information like (4.1), a "prior distribution for μ" in the language of the Bayesians, changes the nature of the estimation process.

Standard I.Q. tests are constructed so that if we test our randomly chosen person to discover his particular μ value, the overall test score*, say $\bar{x}$, is an unbiased normally distributed estimator of μ as in Section 3,

$$\bar{x} \mid \mu \sim \mathcal{N}(\mu, \sigma^2/n), \tag{4.2}$$

with $\sigma/\sqrt{n}$ about 7.5. We can expect $\bar{x}$ to be within 7.5 I.Q. points of μ 68% of the time, etc. The notation "$\bar{x} \mid \mu$" emphasizes that the $\mathcal{N}(\mu, \sigma^2/n)$ distribution for $\bar{x}$ is *conditional* on the particular value taken by the random quantity μ. The reason for this change in notation will be made clearer soon.

Bayes' theorem, originally discovered by the remarkable Reverend Thomas Bayes around 1750, is a mathematical formula for combining (4.1) and (4.2) to obtain the conditional distribution of μ given $\bar{x}$. In this case the formula gives

$$\mu \mid \bar{x} \sim \mathcal{N}(m + C(\bar{x} - m), D), \tag{4.3}$$

where

$$C = \frac{n/\sigma^2}{1/s^2 + n/\sigma^2} \quad \text{and} \quad D = \frac{1}{1/s^2 + n/\sigma^2}. \tag{4.4}$$

For example, if $\bar{x} = 160$ (and $m = 100$, $s = 15$, $\sigma/\sqrt{n} = 7.5$) then

$$\mu \mid \bar{x} \sim \mathcal{N}(148, (6.7)^2). \tag{4.5}$$

Expression (4.5), or more generally (4.3), is the "posterior distribution for μ given the observed value of $\bar{x}$." It is possible to make such a statement in the Bayesian framework because we start out assuming that μ itself is random. In the Bayesian framework the averaging process is reversed; the data $\bar{x}$ is assumed fixed at its observed value while it is the parameter μ which varies. In (4.5) for example, the conditional average of μ given $\bar{x} = 160$ is seen to be 148. If we randomly selected an enormous number of people, gave them each an I.Q. test, and considered the subset of those who scored 160, this subset would have an average true I.Q. of 148; 68% of the true I.Q.'s would be in the interval $[148 - 6.7, 148 + 6.7]$, etc.

How should we estimate μ in the Bayesian situation? It seems natural to use the estimator $\mu^*(\bar{x})$ which minimizes the conditional expectation of $(\mu - \mu^*)^2$ *given the observed value of* $\bar{x}$. From (4.3) it is

* The symbols $\bar{x}$ for the test score and $\sigma/\sqrt{n}$ for its standard deviation are chosen to agree with our previous notation, even though real I.Q. scores aren't actually the average of n independent test items. Perfect normality, as expressed in (4.2), is an ideal only approximated by actual test scores.

easy to derive that this "Bayes estimator" is

$$\mu^*(\bar{x}) = m + C(\bar{x} - m), \tag{4.6}$$

the mean of the posterior distribution of μ given $\bar{x}$. Having observed $\bar{x} = 160$, the Bayes estimate is 148, not 160. Even though we are using an unbiased I.Q. test, so many more true I.Q.'s lie below 160 rather than above that it lowers the expected estimation error to bias the observed score toward 100. Figure 2 illustrates the situation.

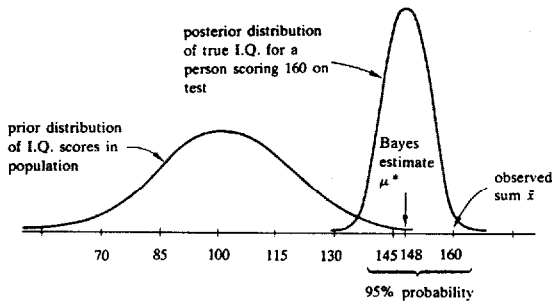


FIG. 2. I.Q. scores have a $\mathcal{N}(100, (15)^2)$ distribution in the population as a whole. A randomly selected person scoring 160 on a normal unbiased I.Q. test with standard deviation 7.5 points is estimated to have a true I.Q. of 148. The probability is 95% that the person's true I.Q. is in the interval [134.6, 161.4].

Confidence intervals have an obvious Bayesian analogue, from (4.3),

$$\text{Prob}\{\mu^*(\bar{x}) - 2\sqrt{D} \leqq \mu \leqq \mu^*(\bar{x}) + 2\sqrt{D} \mid \bar{x}\} = .95. \tag{4.7}$$

The notation $\text{Prob}\{\cdot \mid \bar{x}\}$ indicates probability conditional on the observed value of $\bar{x}$. In the I.Q. example, $\text{Prob}\{134.6 \leqq \mu \leqq 161.8 \mid \bar{x} = 160\} = .95$.

Nobody (well, almost nobody) disagrees with the use of Bayesian methods in situations like the I.Q. problem where there is a clearly defined and well-known prior distribution for μ. The Bayes theory, as we shall see, offers some striking advantages in clarity and consistency. These advantages are due to the fact that Bayesian averages involve only the data value $\bar{x}$ actually seen, rather than a collection of theoretically possible other $\bar{x}$ values.

Difficulties and controversies arise because Bayesian statisticians wish to use Bayesian methods when there is no obvious prior distribution for μ, or going even further, when it is clear that the unknown μ is a fixed constant with no random character at all. (For example, if μ is some physical constant, such as the speed of light, being experimentally estimated.) It is not perversity that motivates this Bayesian impulse, but rather a well-documented casebook of unpleasant inconsistencies in the frequentist approach.

As an example of the kind of difficulties frequentists experience, let us reconsider the I.Q. estimation problem, but without assuming knowledge of the prior distribution (4.1) for μ. In other words, assume only that we observe $\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n)$, $\sigma/\sqrt{n} = 7.5$, and wish to estimate μ. Having observed $\bar{x} = 160$, the results of Section 3 tell us to estimate μ by $\hat{\mu} = 160$, with 95% confidence interval $[\hat{\mu} - 2\sigma/\sqrt{n}, \hat{\mu} + 2\sigma/\sqrt{n}] = [145, 175]$.

Suppose now that the frequentist receives a letter from the company which administered the I.Q. test: "On the day the score of $\bar{x} = 160$ was reported, our test-grading machine was malfunctioning. Any score $\bar{x}$ below 100 was reported as 100. The machine functioned perfectly for scores $\bar{x}$ above 100."