ON MAXIMIZING PROFIT THROUGH PRICING

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Abstract

Several important practical problems and techniques which are familiar to both actuaries and underwriters are discussed informally and then analyzed mathematically. The main subjects are the degree to which profitability can be improved by estimating the expected losses of risks more accurately, and the usefulness of various methods of improving accuracy. The effect of raising or lowering the general level of insurance rates is also analyzed.

The lognormal distribution, the bivariate normal distribution, and some mathematical results relating to prior and posterior distributions are used to produce methods of estimating the value of accuracy, the effects of adverse selection, the value of a "second opinion," and the relation between rate increases or decreases and loss ratios. Mathematical guidelines for using the "stop and go" and "follow the lead" methods of evaluating risks are derived.

Several suggestions for research are mentioned at the end, as well as many other places in the paper. Some of the ideas that are used can also be applied to other types of pricing besides property and casualty insurance, such as life insurance and real estate. Two applications to loss reserving are given.

I. INTRODUCTION

Obviously, if an insurance company charges too much premium for a risk, it is likely to lose that business, and if it charges too little, it is likely to lose money. There hasn't been any published formal study, however, on the subject of how much it is worth to increase the accuracy of pricing, or on certain commonly used methods of increasing accuracy, or on the effects of raising or lowering prices. There have been many formal papers, however, on improving accuracy through better classification and experience rating systems. Certain insurance carriers have used some of these papers with great success.

Insurance company managers are interested in the speed with which pricing is done, but it seems that they often underestimate the value of accuracy as well as the effects on accuracy of various methods that can be used to improve it. Methods of quantifying both of these things will be discussed. The problem of quantifying them relates to the questions of how much money should be spent on data, classification and experience rating systems, and underwriters' and actuaries' salaries. The effect of adverse selection on pricing adequacy and the effect on profitabilty of raising or lowering prices will be examined. Also, applications to loss reserving are presented.

The problem of determining the value of accuracy in pricing was mentioned in [1] by the Committee on Risk Classification of the American Academy of Actuaries:

> "Economic incentive also requires the risk classification system to be efficient. The additional expense of obtaining more refinement should not be greater than the reduction in expected costs for the lower cost risk classification."

In the above quotation, an upper bound is indicated for the value of accuracy. There is much more to the subject that is worth analyzing.

II. THE VALUE OF ACCURACY

A. Long-Term Considerations in the Selection Process

The immediate expected profitability of a risk depends on the relation between the risk's expected losses and its experience-modified premium (if there is an experience modification), but it should be noted that another criterion in selecting risks is the relation between the expected losses and the unmodified premium. The future experience rating credits and debits will not fully reflect the experience unless the risk has 100% credibility. Therefore, in the long run, a risk with lower expected losses than the provision for losses in its unmodified premium will also tend to have expected losses which are less than the provision for losses in the modified premium. The reverse is true for risks with higher expected losses. These long-term considerations involve an estimate of how long the risk will continue renewing its coverage with the company if it is selected.

B. Estimating the Next Year's Expected Loss Ratio for a Risk

When an insurer offers to accept a risk for a certain rate, its offer is based implicitly or explicitly on an estimate of the expected value of the risk's "losses". (More precisely, losses per exposure unit. The amount of future exposure may be unknown. But simply the word "losses" will be used in this paper.) The situation can be represented as follows:

1. A prior, or "a priori", distribution (see [2]) exists which indicates the probability of the expected value of the losses being in any given interval. The provision for losses in the standard (experience-rated) premium is sometimes used as the mean of the prior distribution. In some situations there is no "standard" premium. A prior distribution may be thought of as a probability distribution based on some amount of information and analysis and prior to further information and analysis.

2. Some amount of effort is made to estimate how the expected value of the risk's losses compares with the mean of the prior distribution.

The mean of the prior distribution may be somewhat higher or lower than the provision for losses in the standard premium. For one thing, risks whose expected losses are lower than the provision for losses are more likely to have already found an insurer.

The method that the insurer uses to further evaluate the risk and to estimate the expected losses may be, for example, the use of schedule rating, underwriting judgement, its own classification or experience rating system, or additional experience beyond what was used in the experience rating. (There may not be an experience rating.)

For each possible value x of the expected losses, there is a probability distribution of the estimates given that a risk's losses have expected value x.

C. Use of the Lognormal

It will be assumed that the probability distributions of the expected losses of risks, and of the estimates of expected losses of a risk, are lognormal. Since this assumption is believed to be a useful approximation of reality, rather than merely arbitrary, some remarks about the lognormal are in order.

It's natural to use the lognormal to approximate a probability density function f(x) if it seems that f(x) should increase as x gets closer to the median and that, for all y, the probability of x being greater than y times the median is approximately the same as the probability of the median being greater than y times x. These are properties of the lognormal and seem to be properties of the distributions of expected losses and estimates mentioned above. The fact that the lognormal has been found to approximate the distributions of many random variables studied in nature has been related to Liapounov's central limit theorem ([3], p. 276) for the sum of independent random variables. Liapounov proved that if an infinite series of random variables X_1, $X_2, \ldots$ are independent, and μ_i and σ_i are the mean and standard deviation of X_i, then if:

$$E(|X_i-\mu_i|^3) < \infty \text{ for } i = 1, 2, \ldots$$

and, $$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E(|X_i-\mu_i|^3)}{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{\frac{3}{2}}} = 0$$

then, as $n \to \infty$, the distribution of $\sum_{i=1}^{n} X_i$ approaches a normal distribution. The relation of this to the lognormal is that if Y is the product of random variables Y_1, $Y_2, \ldots$, then log Y is the sum of random variables log Y_1, log $Y_2, \ldots$ Along these lines, consider the following remark of Arthur L. Bailey [4]:

> "The only condition necessary to produce a Normal Logarithmic Distribution is that the amount of an observed value be the product of a large number of factors, each of which is independent of the size of any other factor."

Dropkin [5] produced a close fit to automobile claim frequency data using the Gamma distribution for expected frequencies and the Poisson for the frequency distribution for a given expected frequency. It can be verified that the distribution of expected frequencies he derived is not far from a lognormal distribution. A lognormal distribution of expected average severities is used by Hewitt in [6].

Statistical studies could be made in order to fit lognormal or other distributions to the expected losses and estimated expected losses (for various methods of estimating) mentioned above. However, even rough estimates of these distributions, based on experience and opinion, may, with the help of the mathematical relationships which will be discussed, produce additional insight into the relation between pricing accuracy and profitability. These mathematical relationships can also help in evaluating certain methods that can be used to improve accuracy.

Distributions of expected losses and of estimated expected losses are difficult to approximate, but they are more within the range of experience, and also easier to study statistically, than the relation between pricing accuracy and profitability, or the effects on profitability of various pricing methods. Distributions of expected losses are implicit in experience rating systems.

Clearly it is impractical for an insurance company to experiment with its profitability in order to study the effects of pricing accuracy directly.

D. Use of the Bivariate Normal

Since we will assume that both the prior distribution of expected losses and the distribution of estimates of expected losses, for given expected losses, are lognormal, the distribution of estimate/expected losses over all possible values of expected losses must also be lognormal.

The "estimating method" may not be independent of the method used to estimate the median m of the prior distribution of expected losses, so it seems appropriate to use a bivariate normal distribution for the joint probability distribution of log (m/expected losses) and log (estimate/expected losses).

For those unfamiliar with the bivariate normal distribution, the following quote is given from page 302 of [3]:

"For many populations the joint distribution of the scores of the individuals in the population on two related tests will be approximately a bivariate normal distribution."

The above clearly is relevant if we consider the method of estimating the expected losses and the method of producing the median m of the prior distribution as "two related tests", and the two estimates of expected losses as "scores".

E. Implications of the Model

The situation facing the person (or method) estimating the expected value of the losses of a risk is modeled as follows:

1. The prior distribution of the possible expected values of the losses of the risk is lognormal with median m.

2. Given any expected value of the losses, the probability distribution of the possible estimates that may be made is lognormal.

3. The distribution X_2 of log (estimate/expected losses) over all possible values of expected losses, and the distribuion X_1 of log (m/expected losses), are bivariate normal, and X_2 has mean 0. (Note that this implies that

the distribution X_1' of log (expected losses/m) and the distribution X_2' of log (expected losses/estimate) are bivariate normal, and that X_2' has mean 0.)

It should be noted that although it is assumed in 3 above that the distribution of log (expected losses/estimate) has mean 0, this does not mean that it is assumed that the distribution will actually have a mean of 0 for any given method of estimating. Zero is selected as the mean of the distribution of possible means, and the variance of the distribution of possible means increases the variance of the above distribution. If, for a method of estimating, it is believed that some non-zero number is the mean of the distribution of possible means, the method can be adjusted so that it is believed that zero is the mean.

The theorem and corollary below will be used repeatedly. See the Appendix for the proofs.

<u>THEOREM</u>. Assume that the distribution X_1 of log (expected losses/m), where m is the median of the prior distribution, and the distribution X_2 of log (expected losses/estimate), are bivariate normal. Assume that X_2 has mean 0, that the standard deviations of X_1 and X_2 are σ_1 and σ_2, respectively, and that the correlation between X_1 and X_2 is ρ. Assume that $\sigma_1 > 0, \sigma_2 > 0$ and $-1 < \rho < 1$. Then, given that the estimate of a risk's expected losses is x, the probability distribution of the risk's expected losses has

mean m $(\exp(\nu + \frac{\sigma^2}{2}))$ and variance $m^2(\exp(2\nu+\sigma^2))(\exp(\sigma^2)-1)$ where

$$\nu = ((1-\frac{\rho\sigma_2}{\sigma_1})\sigma_1^2 \log \frac{x}{m})/((1-\rho^2)\sigma_2^2 + (1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2)$$

$$\sigma^2 = ((1-\rho^2)\sigma_2^2\sigma_1^2/((1-\rho^2)\sigma_2^2 + (1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2)$$

<u>COROLLARY.</u> Suppose that all the conditions of the theorem are satisfied. Then, given that the estimate of a risk's expected losses is equal or less than E, the mean of the probability distribution of the expected losses of the risk is

$$m\cdot\exp((\sigma^2+a^2)/2)\ \frac{\phi(\frac{1}{v}\log(E/m)-a)}{\phi(\frac{1}{v}\log(E/m))}$$

where $\phi(x)$ is the standard normal distribution function and

$$\sigma^2 = (1-\rho^2)\sigma_2^2\sigma_1^2/((1-\rho^2)\sigma_2^2 + (1-\rho\sigma_2/\sigma_1)^2\sigma_1^2)$$
$$a = (1-\rho\sigma_2/\sigma_1)^2\sigma_1^2/((1-\rho^2)\sigma_2^2 + (1-\rho\sigma_2/\sigma_1)^2\sigma_1^2)^{\frac{1}{2}}$$
$$v = ((1-\rho\sigma_2/\sigma_1)^2\sigma_1^2 + (1-\rho^2)\sigma_2^2)^{\frac{1}{2}}$$

The probability that the estimate of a risk's expected losses is equal or less than E is $\phi(1/v \log(E/m))$.

<u>Example 1 - Fixed Rates, No Adverse Selection</u>

Suppose that an insurance company has decided, before evaluating a certain risk, that they will accept it for a certain premium P (fixed rates) if its expected losses are equal or less than E. There are many considerations that this decision could be based on, such as the volume of business the company desires, the long term prospects of the risk, etc. Suppose that the prior distribution of expected losses for the risk is lognormal and that the median of the prior distribution of expected losses is m. Suppose it is known that the risk is willing to accept an offer of insurance at premium P. (Adverse selection will not operate after an offer is made.[2])

The accuracy of the method of estimating the risk's expected losses affects the mean and variance of the distribution of expected losses of those risks which are accepted.

Given the conditions of the above theorem and using numbers to illustrate, assume E = .82m (i.e. log(E/M) = -.2). Assume that the probability of log (expected losses/m) being between -.5 and .5 (i.e. .606m $\leq$ expected losses $\leq$ 1.649m) is .683 (i.e. $\sigma_1 = .5$), and $\rho = .5$. Then the mean of the probability distribution of the expected losses of the risks selected is given by the above corollary and

$$s^2 = (.1875\,\sigma_2^2)/(\sigma_2^2 - .5\sigma_2 + .25)$$

$$a = (.25(1-\sigma_2)^2)/(\sigma_2^2 - .5\sigma_2 + .25))^{\frac{1}{2}}$$

$$v = ((1-\sigma_2)^2(.25) + .75\sigma_2^2)^{\frac{1}{2}}$$

and the probability of the risk being accepted is also given by the corollary.

2 See the next example.

Therefore, if for example σ_2 =.333, then the probability of the risk being accepted is .325 and the mean of the probability distribution of the expected losses of the risks selected is .802m. If σ_2 =.667, then the probability of the risk being accepted is .369 and the mean of the probability distribution of the expected losses of the risks selected is 1.072m.

It will not be proven here, but it would seem that an increase in accuracy decreases the variance of the expected losses of the risks chosen, as well as the mean. The problem of working out the mathematical relationship, using the equations of this paper, is suggested for further research.

<u>Example 2 - Variable Rates, Adverse Selection</u>

The more accurate a company is in estimating expected losses, the less room it leaves for adverse selection. For example, a perfect estimate for every risk would make adverse selection impossible. Therefore, a lower loading for adverse selection is necessary when a company is making an offer to a risk it is better able to price. Incidentally, this is a reason for an underwriter to be wary of making offers to types of risks he is not expert in.

Suppose insurance company A decides what profit margin it wants and offers to insure a certain risk based on that profit margin and an estimate E of the risk's expected losses. Suppose insurance company B is competing for the risk and that X_1 is the probability distribution of log(expected losses/m) where m is the median of the distribution of expected losses for the risk, and X_2 is the probability distribution of log(expected losses/estimate of company B). Suppose that X_1 and X_2 satisfy all the conditions of the theorem. Suppose that company B will make a better offer than company A if, and only if, their estimate of the expected losses is equal or less than E. According to the corollary the probability p of this happening is

$$\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)$$

and the mean μ_1 of the probability distribution of the expected losses of the risk, given that company B makes a better offer, is

$$m \cdot \exp((\sigma^2 + a^2)/2)\ \frac{\Phi(\frac{1}{v}\log(\frac{E}{m}) - a)}{\Phi(\frac{1}{v}\log(\frac{E}{m}))}$$

Let μ_2 = the mean of the probability distribution of expected losses for the risk prior to company B deciding whether to make an offer. The expected value C of the losses, given that company B does not make a better offer, satisfies the equation $(1-p)C + p\mu_1 = \mu_2$.
Therefore $C = \frac{\mu_2 - p\mu_1}{1-p}$. Since $\mu_1 < \mu_2$, $C > \frac{\mu_2 - p\mu_2}{1-p} = \mu_2$.

If the risk ends up on company A's books in this scenario, the expected losses equal C , not μ_2. The above equations for C, μ_1 and p enable one to solve for C in terms of μ_2. If company B is the only competing company, and if the probability that the risk will get a competing bid from company B is p', then the expected losses of the risk, given that it becomes insured by company A, is $(1-p')\mu_2 + p'C$.

In a more realistic and complicated example, in which the risk may get competing bids from several companies, the model of this paper could still be used. It would be necessary to estimate the probability of company A getting bids from each subset of a set of competing companies, the accuracy of the competing companies, and the correlation between the estimates of each pair of companies. This complicated application of the ideas in this paper is left as a suggestion for further research.

III. ON IMPROVING ACCURACY

A. Are Two Heads Better Than One?

It is well known that a more accurate estimator deserves more weight than a less accurate one. Also, for given degrees of accuracy for estimators 1 and 2, there is more to be gained by using both estimators if the correlation between the estimators is lower. Little is gained if they are very highly correlated.

There is a direct application of the theorem in this paper to the question of how to weight two estimates of the expected losses of a risk.

Suppose that the prior distribution referred to in the theorem is the probability distribution of expected losses given the first estimate. Also, suppose that the estimate x referred to in the theorem is the second of the two estimates. Then the probability distribution of the risk's expected losses, given the two estimates, has the mean and variance referred to in the theorem, i.e. the mean is $m(\exp(\mu+\frac{\sigma^2}{2}))$ and the variance is $m^2(\exp(2\mu+\sigma^2))(\exp(\sigma^2)-1)$, where μ and σ are as in the theorem. Since for each value of x, $m(\exp(\mu+\frac{\sigma^2}{2}))$ is the mean of the probability distribution given the two estimates, it is also true that it is the estimate which minimizes the variance of the errors. Therefore, the estimate $m(\exp(\mu+\frac{\sigma^2}{2}))$ is a more accurate estimator than either of the two original estimators.

It would certainly seem that estimators 1 and 2 can be used to produce a more accurate estimate than the $m(\exp(\mu+\frac{\sigma^2}{2}))$ referred to above. If 1 and 2 are people rather than methods, for example, they can discuss their estimates with each other and they may discover mistakes, oversights, or poor judgments that were used. So only a lower bound for the benefits of the "two heads" method has been presented in this section.

The problem of weighting n estimates will not be dealt with in detail here. It will be mentioned, however, that the above method of using two estimates can be extended to n estimates by supposing that the prior distribution referred to in the theorem is the probability distribution of expected losses given the first n-1 estimates. Some estimate for the correlation between this prior distribution and the nth estimate must be used.

The above analysis of whether "two heads" are better than one also applies to two estimates of loss reserves. The probability distribution of ultimate losses for any set of accident years can well be estimated by a lognormal, it would seem.

B. Following the Lead

Forms of "following the lead" are sometimes used by both primary and reinsurance companies. Various "follow the lead" strategies are used by primary companies when they get information about what was or is charged by their competitors for individual risks or classes of risks.

A reinsurance company which operates in the reinsurance broker market will sometimes be told by a broker, when given a reinsurance treaty to consider, that a certain reinsurance company has "taken the lead" by agreeing to accept some share of the treaty at certain terms. The broker may also point out that certain other reinsurance companies have "followed the lead" by agreeing to take various shares. It may be that only the "lead" company has actually priced the treaty, and that the other companies are simply following it.

Given the fact that a certain reinsurance company has been the only reinsurance company to price a certain treaty, the fact that it found it acceptable could be used by another reinsurance company to produce a "prior probability distribution" of the expected losses of the risk. The mean of this distribution would be the premium minus the reinsurer's expenses and expected profit margin, and the variance would be based on the estimate of the accuracy of its pricing. However, if it is possible that the rate was rejected by several other reinsurers before being accepted by the "lead" reinsurer, a much different prior distribution would be estimated. The significance of the fact that a certain company has taken the lead, and that certain other companies have followed, is dependent on estimates of:

1. How many companies have accepted the rate, and how many have rejected it? How many are merely "following the lead" without pricing the treaty themselves?

2. How accurate are the companies which have rejected or accepted the rate, and what are their expected profit margins?

3. What are the correlations between the various pairs of companies' estimates?

No attempt will be made to indicate how an estimate of the prior distribution should be made based on the above, however, this discussion is intended to show that due to the unknowns inherent in the "follow the lead" method, it is important to use it only as a supplement to other methods of estimation.

C. Stop and Go Pricing

Instead of deciding beforehand how much time to spend estimating whether a risk is acceptable at some price (which is presumed fixed), an insurer could use the following stop and go strategy. A quick estimate can be made, and then, if the price seems much too high or much too low, a decision is made. If the price seems on the borderline, further time is spent attempting to estimate more accurately. For example, a "second opinion" may be used.

Consider an example of what can easily happen when stop and go pricing isn't used. Suppose that an actuary is deciding whether the reinsurance company he works for should take a share of a certain excess loss treaty in the broker market at a rate of 6%, and that there is no possibility of getting a higher rate. If the actuary makes a very lengthy and detailed analysis, and then estimates that the treaty will have a loss ratio of 300%, he is likely to wish that he had made a quicker estimate first and then decided whether to continue his analysis.

A mathematical analysis is as follows. For each amount of time t, the best estimate of a risk's expected losses that a company can make in time t satisfies the following. The expected profit for each risk considered is p(t)(profit(t))-expense(t), where the terms are defined as follows. First, p(t) is the probability that the risk will be selected after time t as having expected losses equal or less than the "break even" point. Profit(t) is the expected profit on risks selected after time t (without subtracting the expense of estimating expected losses). Expense(t) is the expense of spending the amount of time t estimating the expected losses.

Suppose that t_2 is the amount of time which maximizes p(t)(profit(t))-expense(t) and that at time t_1, such that $t_1<t_2$, during the process of estimating which takes time t_2, a "preliminary" estimate can be made. It may be possible to improve upon the value $p(t_2)(\text{profit}(t_2))$-expense$(t_2)$ by using a "stop and go" method. This method is to stop the process of estimating at time t_1 if the risk appears to be one which should clearly be accepted or clearly rejected, and to go on with the process of estimating until time t_2 if the risk is on the borderline.

Assume that at time t_1 the risk's expected losses are estimated and that the probability distribution of expected losses, given that estimate, has median m and is such that the distribution X_1 of log(expected losses/m) is normal with mean 0 and standard deviation σ_1, and that the distribution X_2 of log(expected losses/estimate at time t_2) will be normal with mean 0 and standard deviation σ_2. Assume that $\sigma_1 > 0, \sigma_2 > 0$ and X_1 and X_2 are bivariate normal with $-1 < \rho < 1$.

If the criterion for selecting the risk is that the estimate of the expected losses is $\leq$ E, and the estimate at time t_1 is greater than E, then the probability that the decision will be changed if the estimating process continues from time t_1 to t_2 is, as shown in Example 1 of section 2E,

$$\phi(\tfrac{1}{v}\log(\tfrac{E}{m}))$$

and the mean expected losses μ, given that the decision is changed at time t_2, is

$$m \cdot \exp((\sigma^2 + a^2)/2)\,\frac{\phi(\tfrac{1}{v}\log(\tfrac{E}{m}) - a)}{\phi(\tfrac{1}{v}\log(\tfrac{E}{m}))}$$

Therefore, if E is a break-even point for the future profitability of a risk chosen at time t_2, then the expected gain from continuing the estimating process until time t_2, is (p)(E-μ)-(expense(t_2)-expense(t_1)).

If the estimate at time t_1 is less than E, then the probability p that the decision will be changes at time t_2 is

$$1 - \phi(\tfrac{1}{v}\log(\tfrac{E}{m}))$$

and the mean expected losses μ, given that the decision is changed at time t_2 is

$$m \cdot \exp\left(\frac{\sigma^2 + a^2}{2}\right)\frac{1 - \phi(\tfrac{1}{v}\log(\tfrac{E}{m}) - a)}{1 - \phi(\tfrac{1}{v}\log(\tfrac{E}{m}))}$$

and the expected gain from continuing the estimating process until t_2 is $(p)(\mu-E)-(\text{expense}(t_2)-\text{expense}(t_1))$. Whether the estimate at time t_1 is greater or less than E, p is very small if there is a great difference between E and m, so there are cases in which it is not worth continuing the process of estimating until time t_2. It is worth continuing precisely when the above expected gain is greater than zero. So, in many pricing situations, the optimal decision-making process is not simply to choose t so as to maximize $p(t)(\text{profit}(t))-\text{expense}(t)$. Additional benefits can result from a reasonable stop and go strategy. The problem of optimizing the strategy seems to be a practical and interesting one, but will not be considered further here.

D. The Value of the Accuracy of a Single Factor

Sometimes a single factor affects so many underwriting decisions that it is worth a very great deal of effort to be accurate. In such a case it certainly is useful to be able to estimate how much expense should be put into estimating the factor. For example, the estimate of incurred but not reported losses can have an overall effect on the individual estimates of expected losses of a company's risks. If individual estimates don't seem consistent with overall estimates of expected losses, a change in individual estimates may be indicated. Other factors which may have a great effect are trend and development factors, classification systems and experience rating systems.

The accuracy of a company's estimates of a risk's expected losses depends on the accuracy of each factor used. The effects of the accuracy of a single factor on expected profitability can therefore be estimated by using methods based on those of this paper.

Example - The Value of Accuracy in Loss Reserving

Suppose a company makes individual estimates of expected losses of risks in such a way that the sum of the individual estimates equals an overall estimate which is based on the estimated ultimate losses for the preceding accident year. Suppose that the actual expected losses for the preceding accident year are t times as great as the estimate and that for each risk, the estimate would be multiplied by t if the estimated ultimate losses for the preceding accident year were correct. For any estimate e of the expected losses of a risk, let m_e be the corresponding median of the probability distribution of expected losses. Then since, log(expected losses/m_e) = log(expected losses/($t \cdot m_e$)) + log t, the variance of the probability distribution of log(expected losses/m_e) equals the variance of the probability distribution of log(expected losses/$t \cdot m_e$) plus the variance σ_t of the probability distribution of log t. Thus in this example, the effect of the accuracy of the estimated losses for the preceding accident year on the accuracy of individual estimates is given by the above equation. The variance σ_t depends on the accuracy of the loss reserve estimation.

IV. THE EFFECT OF RATE INCREASES ON LOSS RATIOS

When rates are increased, expected loss ratios may not be lowered as much as estimated, due to the fact that adverse selection tends to be increased.

It isn't even necessarily true that a company will decrease its expected loss ratio at all by demanding higher rates. (Also, of course, its volume may decrease greatly.) By demanding higher rates, a company may allow the competition to take most of the better risks to the point that the company will have a higher expected loss ratio in spite of its higher rates. For instance, an automobile insurer which raises its rates enough may find it is becoming an assigned risk underwriter. A mathematical illustration will be given.

Suppose company A and company B are the only two competitors for a type of business, that they each charge the same rate, write the same amount of premium and have the same expected loss ratio. Suppose that there is no inflation or loss trend, and that company A raises its rate by X%. If company B is sufficiently accurate at estimating expected losses, and if enough risks apply to it for its lower rate, then company B may be able to write 50% of the premium for that type of business with expected losses per exposure unit which are more than X% below the average. In this case the expected loss ratio of company A will go up in spite of its X% rate increase.

At the end of example 2 of second IIE, a method of estimating the effect of adverse selection is discussed. By estimating the effect of adverse selecton both with and without a rate increase the effect of the rate increase on expected loss ratios can be estimated.

V. SUGGESTIONS FOR RESEARCH

Several unsolved problems have been mentioned. Some other problems which seem to be fruitful areas for further research are as follows.

Given a certain line of business, market, competitive environment, classification system, estimating method, etc.:

1. What methods (e.g. new classification system, more than one opinion, stop and go pricing, etc.) should be used to improve accuracy in the most cost efficient way?

2. How much time and expense should be put into improving accuracy in order to maximize profit?

3. What is the optimal level of pricing, given certain volume constraints?

VI. CONCLUSION

The problem of maximizing profit through the level and accuracy of pricing can, to some extent, be analyzed mathematically. This analysis requires some estimates relating to the current market, the available set of risks, and various methods of pricing. There are many important practical applications of the mathematical analysis.

REFERENCES

[1] American Academy of Actuaries Committee on Risk Classification "Risk Classification Statement of Principles," June, 1980.

[2] Mayerson, A.L., "A Bayesian View of Credibility," PCAS LI, 1964, p.85.

[3] DeGroot, M.H., Probability and Statistics, (2nd ed.) Addison-Wesley Publishing Co., 1986.

[4] Bailey, A.L., "Sampling Theory in Casualty Insurance," PCAS XXIX, 1942 p.50 and PCAS XXX, 1943, p.31.

5] Dropkin, L.B., "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records," PCAS XLVI, 1959, p.165.

[6] Hewitt, C.C., "Credibility for Severity," PCAS LVII, 1970, p.148.

APPENDIX

PROOFS OF THEOREM AND COROLLARY

Lemma 1. (See p. 303 of [3].) Suppose that the random variables X_1 and X_2 have a bivariate normal distribution, and that the correlation of X_1 and X_2 is ρ. Suppose also that $E(X_1)=\mu_1, E(X_2)=\mu_2, Var(X_1)=\sigma_1^2, Var(X_2)=\sigma_2^2, \sigma_1>0, \sigma_2>0, -1<\rho<1.$ Then the conditional distribution of X_2 given that $X_1=x_1$ is a normal distribution for which the mean is $\mu_2+\rho\sigma_2\left(\frac{x_1-\mu_1}{\sigma_1}\right)$ and the variance is $(1-\rho^2)\sigma_2^2$.

Lemma 2. (See p. 324 of [3].) Suppose that an element is chosen at random from a normal distribution for which the value of the mean θ is unknown ($-\infty<\theta<\infty$) and the value of the variance σ^2 is known ($\sigma^2>0$). Suppose also that the prior distribution of θ is a normal distribution with given values of the mean μ and the variance v^2. Then the posterior distribution of θ, given that the element chosen equals x_1, is a normal distribution for which the mean μ_1 and the variance v_1^2 are as follows;

$$\mu_1=\frac{\sigma^2\mu+v^2x_1}{\sigma^2+v^2}$$

$$v_1^2=\frac{\sigma^2v^2}{\sigma^2+v^2}$$

Proof of Theorem. Given that log(expected losses/m) = x_1, then, since X_1 and X_2 are bivariate normal, and X_1 must have mean 0 since it has median 0 and is normal, the probability distribution of log(expected losses/estimate) is normal with mean $(\rho\sigma_2x_1)/\sigma_1$ and variance $(1-\rho^2)\sigma_2^2$ by Lemma 1.

Therefore the distribution of log(estimate/m), given that log(expected losses/m)=x_1, is normal with mean $x_1(1-(\rho\sigma_2)/\sigma_1)$ and variance $(1-\rho^2)\sigma_2^2$. The set of all such distributions, as x_1 ranges between $-\infty$ and ∞, is therefore a set of normal distributions, each of which has variance $(1-\rho^2)\sigma_2^2$, and whose means are normally distributed with mean 0 and variance $(1-(\rho\sigma_2)\sigma_1)^2\sigma_1^2$. If the estimate of expected losses is x, then log(estimate/m)= $\log\frac{x}{m}$ and it follows from Lemma 2 that the posterior distribution of the means of the above set of normal distributions has mean

$$\frac{(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}{(1-\rho^2)\sigma_2^2+(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}\log\frac{x}{m}$$

and variance

$$\frac{(1-\rho^2)\sigma_2^2(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}{(1-\rho^2)\sigma_2^2+(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}$$

As mentioned above, if log(expected losses/m)=x_1, then the distribution of log(estimate/m) has mean $x_1(1-(\rho\sigma_2)/\sigma_1)$. So the posterior distribution of log(expected losses/m) has a mean μ which is $1/(1-(\rho\sigma_2)/\sigma_1)$ times the above mean and a variance σ^2 which is $1/(1-(\rho\sigma_2)/\sigma_1)^2$ times the above variance, so

$$\mu=\frac{(1-(\rho\sigma_2)/\sigma_1)\sigma_1^2}{(1-\rho^2)\sigma_2^2+(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}\log\frac{x}{m}$$

$$\sigma^2=\frac{(1-\rho^2)\sigma_2^2\sigma_1^2}{(1-\rho^2)\sigma_2^2+(1-(\rho\sigma_2)/\sigma_1)^2\sigma_1^2}$$

It is a well known theorem of statistics that if the logs of a distribution are normally distributed with mean μ and variance σ^2, the mean of the distribution is $\exp(\mu+\frac{\sigma^2}{2})$ and the variance is $(\exp(2\mu+\sigma^2))(\exp(\sigma^2)-1)$. This gives the mean and variance of the distribution of (expected losses/m) and our theorem follows immediately. Q.E.D.

Proof of Corollary. It can be seen from the proof of the theorem that the probability distribution of log(estimate/m) is normal with variance $v^2=(1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2+(1-\rho^2)\sigma_2^2$. (In the proof, given log(expected losses/m)=x_1, the distribution of log(estimate/m) has mean $x_1(1-\frac{\rho\sigma_2}{\sigma_1})$ and variance $(1-\rho^2)\sigma_2^2$ and the distribution of X_1 has variance σ_1^2.) The probability p that the risk will be accepted is

$$\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)$$

where $\Phi(x)$ is the standard normal distribution function.

It can also be seen from the statement of the theorem that given that $(1/v)\log(\text{estimate}/m)=t$, the probability distribution of log(expected losses/m) has mean

$$\mu=\frac{vt(1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2}{(1-\rho^2)\sigma_2^2+(1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2}=\frac{t(1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2}{((1-\rho^2)\sigma_2^2+(1-\frac{\rho\sigma_2}{\sigma_1})^2\sigma_1^2)^{\frac{1}{2}}}$$

and variance

$$\sigma^2 = \frac{(1-\rho^2)\sigma_2^2\sigma_1^2}{(1-\rho^2)\sigma_2^2 + \left(1-\frac{\rho\sigma_2}{\sigma_1}\right)^2\sigma_1^2}$$

The mean of the probability distribution of the expected losses of the risk, given that it is accepted (i.e. estimate $\leq$ E), is therefore

$$\frac{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{v}\log\frac{E}{m}} m\cdot\exp\left(-\frac{t^2}{2}\right)\exp\left(\nu+\frac{\sigma^2}{2}\right)dt}{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)}$$

There is an a such that ν = at, so for that a the above integral equals

$$\frac{\int_{-\infty}^{\frac{1}{v}\log\frac{E}{m}} m\cdot\exp\left(-\frac{t^2}{2}\right)\exp\left(at+\frac{\sigma^2}{2}\right)dt}{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)} = \frac{\int_{-\infty}^{\frac{1}{v}\log\frac{E}{m}} m\cdot\exp\left(\frac{\sigma^2+a^2}{2}\right)\exp\left(-\frac{(t-a)^2}{2}\right)dt}{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)}$$

$$= \frac{m\cdot\exp((\sigma^2+a^2)/2)\int_{-\infty}^{\frac{1}{v}\log\left(\frac{E}{m}\right)-a}\exp\left(-\frac{t^2}{2}\right)dt}{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)}$$

$$= m\cdot\exp((\sigma^2+a^2)/2)\,\frac{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)-a\right)}{\Phi\left(\frac{1}{v}\log\left(\frac{E}{m}\right)\right)}$$

Q.E.D.