A METHOD TO CALCULATE AGGREGATE EXCESS LOSS DISTRIBUTIONS

A Method to Calculate Aggregate Excess Loss Distributions

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Abstract

The purpose of the paper is to develop a method of calculating the aggregate loss distribution of excess claims based on a formula described in the book Risk Theory by Beard, Pentikainenen, and Pesonen. This formula requires that the claim frequency distribution satisfy a certain recursive relationship.

The first part of the paper shows that a claim frequency distributions of excess claims derived from a claim frequency distribution satisfying the recursive relationship also has that recursive property.

The second part describes a simple Pascal program that implements the calculation of aggregate loss distributions using these formulas.

Introduction

Like many actuarial departments, we have been using a variety of tools to determine the distribution of aggregate losses assuming we know something about the underlying frequency and severity parameters.

We have an analytical model based on the Fortran program described in the the article by Glenn Meyers and Phil Heckman. Through the work of our own staff we also have stochastic simulations. While these models have proven to be very useful, they do not work in all situations. The analytic models often don't behave well when there is a large probability spike in the severity distribution or a fractional expected number of claims. The effectiveness of simulation models may be questioned because of concerns about the "randomness" of the random number generators, at least for some of the PC based versions.

In their book, *Risk Theory*, Beard etal describe a method of calculating aggregate loss distributions for Compound Poisson processes when the claim frequency can be expressed in a particular recursive form and the claim severity distribution is discrete on uniformly spaced points. It turns out that the family of claim distributions satisfying the recursive relationship includes both the Poisson and Negative Binomial distributions, both of which are familiar to actuaries.

It is also true that any reasonable claim size distribution can be approximated to any desired degree of accuracy by an equally spaced discrete probability distribution, at least over a finite interval. Though, to suitably approximate many of the standard claim size distributions, it can require a relatively large number of grid points. On the other hand, the probability distribution of excess claims usually exhibits a fairly simple pattern and thus can be approximated reasonably well by a relatively small number of points.

The advantage of this method is that the computation does not have any of the convergence problems of the analytic approaches based on Fourier Series either when the number of claims is small or there is a point mass at the upper end of the interval. Of course, you do need to pay attention to the quality of the claim severity approximation.

To implement this approach there were two things that needed to be done.

The first was to answer a purely statistical question - given the probability distribution of first dollar claims, what is known about the probability distribution of excess claims. In particular, if you assumed that the starting distribution was Poisson or Negative Binomial with a given set of parameters, could you assume that the distribution of claims in excess of some loss amount was still Poisson or Negative Binomial and determine the new parameters? More generally, if the form of the original distribution satisfied the recursive relationship, will the resulting excess claim distribution?

The second was to write a computer program to carry out the calculations. In the second section I will describe an elementary version of such a program written in Turbo Pascal for the Macintosh.

The Frequency of Excess Claims

In this section, I consider the recursive formula described in *Risk Theory*. (esp. Sections 2.9 and 3.8) The intent is to show that the distribution of excess claims derived from such a recursive claim distribution is also recursive. The basic procedure I follow is to consider the probability of no excess claims and the probability of n excess claims.

Let Π_n stand for the probability of n excess claims, and let π stand for the probability that a claim is an excess claim and let $\theta = (1-\pi)$. Recall that $({}_iC_j)$ stands for the binomial coefficient $(i(i-1)...(i-j+1))/j!$

Make whatever assumptions about independence are necessary and assume that the claim frequency distribution satisfies the recursive relationship

$$P_{n+1} = [a+b/(n+1)]\,P_n\,,$$

where P_n, for $n \geq 0$, is the probability that the number of claims equals n. Since $\Sigma\, P_n = 1$, the parameter a must be less than one, and in the following assume that the parameter a is positive.

Then the probability of no excess claims can be expressed as

$$\Pi_0 = P_0 + P_1\theta + P_2\theta^2 + \ldots + P_n\theta^n + \ldots$$

and by using the recursive relationship

$$\Pi_0 = P_0 + (a+b)P_0\theta + (a+b/2)(a+b)P_0\theta^2 + \ldots + (a+b/n)\ldots(a+b)P_0\theta^n + \ldots .$$

Factoring out P_0, we have

$$\Pi_0 = P_0\,\{1 + (a+b)\theta + (a+b/2)(a+b)\theta^2 + \ldots + (a+b/n)\ \ldots\ (a+b)\theta^n + \ldots\}.$$

By rearranging the terms, we obtain

$$\Pi_0 = P_0\,\{1 + [(b/a)+1](a\theta) + [(b/a)+1][(b/a)+2](a\theta)^2/2 + \ldots$$
$$+ [(b/a)+1]\ \ldots\ [(b/a)+n](a\theta)^n/n! + \ldots\}.$$

Since $a\theta$ is less than one, the series inside the brackets converges and is equal to

$$\{1 - a\theta\}^{-[(b/a)+1]},$$

or alternatively

$$\{1 - a\theta\}^{-[(a+b)/a]},$$

and

(1) $\quad \Pi_0 = P_0/\{1 - a\theta\}^{[(a+b)/a]}.$

For the probability of n excess claims

$$\Pi_n = P_n \pi^n + P_{n+1} ({}_{n+1}C_1)\theta \pi^n + P_{n+2} ({}_{n+2}C_2)\theta^2 \pi^n + \ldots$$
$$+P_{n+q} ({}_{n+q}C_q) \pi^n\theta^q \pi^n + \ldots .$$

Again using the recursive relationship, we have

$$\Pi_n = P_n \pi^n + [a+b/(n+1)] P_n ({}_{n+1}C_1)\theta \pi^n$$
$$+ [a+b/(n+2)][a+b/(n+1)] P_n ({}_{n+2}C_2) \theta^2 \pi^n + \ldots$$
$$+ [a+b/(n+q)] \ldots [a+b/(n+1)] P_n ({}_{n+q}C_q) \theta^q \pi^n + \ldots$$

Factoring out $P_n \pi^n$, we have

$$\Pi_n = P_n \pi^n \{1 + [a+b/(n+1)] ({}_{n+1}C_1)\theta + [a+b/(n+2)][a+b/(n+1)] ({}_{n+2}C_2)\theta^2$$
$$+ \ldots + [a+b/(n+q)] \ldots [a+b/(n+1)] ({}_{n+q}C_q)\theta^q + \ldots \}.$$

Rearranging terms this becomes

$$\Pi_n = P_n \pi^n \{1 + [(b/a)+n+1] (a\theta) + [(b/a)+n+1][(b/a)+n+2](a\theta)^2/2 + \ldots$$
$$+[(b/a)+n+1] \ldots [(b/a)+n+q](a\theta)^q/n! + \ldots \}.$$

As before, since $a\theta$ is less than one, the series converges and is equal to

$$\{1 - a\theta\}^{-[(b/a)+n+1]},$$

or alternatively

$$\{1 - a\theta\}^{-[(a+b)/a]} \{1 - a\theta\}^{-n}$$

Thus

$$\Pi_n = P_n \{1 - a\theta\}^{-[(a+b)/a]}[\pi /\{1 - a\theta\}]^n$$

and

$$\Pi_{n+1} = [\pi/(1 - a\theta)][a+b/(n+1)] \Pi_n.$$

Thus Π_n satisfies the recursive relationship

$$(2) \qquad \Pi_{n+1} = [A+B/(n+1)] \Pi_n$$

with

$$(3) \qquad \begin{aligned} A &= [\pi/(1 - a\theta)] a \\ B &= [\pi/(1 - a\theta)] b. \end{aligned}$$

For the case where $a = 0$, the derivations above can be modified slightly to yield

$$\Pi_0 = P_0 \, e^{b\theta}$$

(4)

$$\Pi_n = P_n \, \pi^n \, e^{b\theta} \; .$$

These can be seen to satisfy equation (2) with $A = 0$ and $B = \pi b$.

In summary, excess losses generated from the family of distributions characterized by the recursive formula are also in the family of such recursive distributions, with parameters scaled by $\pi/(1 - a\theta)$.

As noted above, this family of distributions includes the Poisson and Negative Binomial distributions.

For the Poisson distribution, with expected value λ, we have for $n \geq 0$

$$P_n = e^{-\lambda} \, (\lambda^n/n!).$$

In this case, the recursive parameters are $a = 0$ and $b = \lambda$, and by (4) the excess distribution satisfies

$$\Pi_0 = P_0 \, e^{b\theta} = e^{-\lambda} \, e^{\lambda\theta} = e^{-\lambda} e^{\lambda - \pi\lambda} = e^{-\pi\lambda}$$

$$\Pi_n = P_n \, \pi^n \, e^{b\theta} = e^{-\lambda} \, (\lambda^n/n!) \, \pi^n \, e^{\lambda - \pi\lambda} = e^{-\pi\lambda} \, [(\lambda\pi)^n/n!]$$

Thus, the excess distribution will be a Poisson distribution with parameter $\pi\lambda$, which is, of course, what you would expect.

For the Negative Binomial with parameters h and n, we have for $k \geq 0$

$$P_k = ({}_{h+k-1}C_k) \;\; [h/(n+h)]^h \; [n/(n+h)]^k \; ,$$

The recursion parameters for this distribution are

$$a = n/(n+h),$$

$$b = n(h-1)/(n+h) \; .$$

And the mean and variance are given by

$$\mu = n$$
$$\sigma^2 = n + n^2/h.$$

Note that $(a+b)/a = h$ and $\{1 - a\theta\} = \{1 - n(1-\pi)/(n+h)\} = (n\pi+h)/(n+h)$, so from equation (1) we have

$$\begin{aligned}\Pi_0 &= P_0/\{1 - a\theta\}[(a+b)/a] \\ &= [h/(n+h)]^h/[(n\pi+h)/(n+h)]^h \\ &= [h/(n\pi+h)]^h\end{aligned}$$

And using (3) we see that

$$\begin{aligned}A &= [\pi/(1 - a\theta)]\, a \\ &= [\pi/(1 - \{n/(n+h)\}\theta)][n/(n+h)] \\ &= [(\pi n)/(n+h)][(n+h)/(n+h-n\theta) \\ &= [(\pi n)/\{n+h-n(1-\pi)\}] \\ &= (\pi n)/[(\pi n)+ h] .\end{aligned}$$

And

$$\begin{aligned}B &= [\pi/(1 - a\theta)]\, b \\ &= [\pi/(1 - a\theta)][n(h-1)/(n+h)]. \\ &= [\pi/(1 - \{n/(n+h)\}\theta)][n(h-1)/(n+h)] \\ &= [(\pi n)(h-1)/(n+h)][(n+h)/(n+h-n\theta)] \\ &= (\pi n)(h-1)/[(\pi n) + h] .\end{aligned}$$

Thus the excess distribution is again a Negative Binomial distribution with parameters πn and h. In particular, the mean of the excess distribution is πn and the variance is $\pi n + (\pi n)^2/h$.

Implications for the Negative Binomial

In the case of the Negative Binomial it is instructive to look at the relationship between means and variances for the original distribution and the excess distribution.

The following table summarizes the statistics for the two distributions.

	Original	Excess
Mean	n	πn
Variance	$n + n^2/h$	$\pi n + (\pi n)^2/h$
Ratio	$1 + n/h$	$1 + \pi\ (n/h)$.

Thus the ratio of the mean and variance becomes closer to unity as the size of the excess claim increases. Since the probabilities of an excess claim are often on the order of 10^{-3} or less, for most distributions the variance to mean ratio will be very close to unity. In some sense the derived claim frequency becomes nearly Poisson.

This observation would seem to run counter to the "common sense" view that the further out you are in the tail, the more volatile the claim distribution becomes. Of course, since this is all predicated on having perfect knowledge of the claim severity distribution with no questions about trend or loss development maybe it isn't so surprising.

The Program

The program basically implements the formulas shown in section 3.8 of Risk Theory by Beard etal.

The **parameter file** contains the name of the distribution, the mean and variance of the distribution of first dollar claims - either Poisson or Negative Binomial; the parameters of the Pareto claim size distribution, B, Q, P, T, and S; the upper and lower limit of the excess interval under consideration; the number of grid points to use for the claim size approximation and the Stopping Probability.

Following is an example of the parameter file:

Test Data	{Label}
10000 15000	{Expected Number and Variance of Claim Process}
2000 1.25 0.5 500 1000	{B, Q, P, S, T of Pareto Severity Dist}
1000000 5000000	{Attachment Point, Limit}
20 0.99999	{# of Pts in Sev Dist, Cum Prob stopping value}

The **output file** displays a variety of summary information to assess the quality of the estimation as well as the aggregate distribution itself.

Following is a copy of the summary data based on the Test Data parameters.

```
    Test Data
The Expected Number and Variance of First Dollar Claims
    10000           15000
The Pareto Parameters are
B  2000  Q 1.25  P 0.5000  S   500  T  1000
The AttachPt is    1000000  The Limit is    5000000
The Number of GridPts is 20, The Prob Stopping Value is 0.99990000

Prob of an XS Claim is 0.000350  Prob of a Limits Claim is 0.000037
The Prob of a Limits Claim given an excess claim 0.106712
Parameter a equals   0.000175, Parameter b equals   3.500974

The Mean and Variance of the Claim Frequency Distribution is:
          3.501762          3.502375

The Mean and Standard Deviation of the Actual Severity Distribution is
          1446050           1607771

The Mean and Standard Deviation of the Discrete Severity Distribution is:
          1477067           1583653

The Mean and Standard Deviation of the Actual Aggregate Distribution is:
          5063723           4046662
```

The Discrete Claim Severity Distribution is
0.328 0.127 0.089 0.066 0.051 0.040 0.032 0.026 0.022 0.019
0.016 0.014 0.012 0.011 0.009 0.008 0.008 0.007 0.006 0.110

The Cumulative Probability is 0.99990328
The Total Number of AggPts calculated is 112

The Mean and Standard deviation of the ESTIMATED Aggregate Distribution is:
5169444 4045234

This data allows one to assess the quality of the modeling of the severity distribution - compare the means and standard deviations of the actual and discrete severity distributions. If the fit is inadequate, the number of gridpoints, which is the number or points used to approximate the severity distribution, should be increased.

There is also a corresponding comparison for the aggregate distribution. Using the well known formula for mean and variance of a Compound Poisson process in terms of corresponding statistics for the frequency and severity distributions, the program computes The Mean and Standard deviation of the ESTIMATED Aggregate Distribution. This can be compared to The Mean and Standard Deviation of the Actual Aggregate Distribution, which is computed from the statistics of the claims frequency and actual severity distributions. If this comparison is not satisfactory, an increase in the Stopping Probability is indicated.

If you are satisfied with these estimates you can use the information written to the output file. This data set displays summary statistics and the aggregate distribution itself.

Following is an example of the output data set.

Test Data
The Mean and Variance of the Claim Frequency Distribution is:
3.501762 3.502375

The Mean and Standard Deviation of the Discrete Severity Distribution is:
1477067 1583653

The Mean and Standard Deviation of the ESTIMATED Aggregate Distribution is:
5169444 4045234

loss amount	probability	cum probability
0	0.030153453	0.030153453
250000	0.034619700	0.064773154
500000	0.033230037	0.098003191
750000	0.032365086	0.030153453
1000000	0.031735804	0.162104080
1250000	0.031000556	0.193104636
1500000	0.030124671	0.223229307
1750000	0.029144564	0.252373871
2000000	0.028098459	0.280472330
2250000	0.027016611	0.307488942
2500000	0.025921784	0.333410726
...	...	...

The first numbers are a restatement of the key statistics of the frequency, severity and aggregate distribution followed by the probabilities of the aggregate distribution.

In its present form, the program has limitations due mainly to my limited programming experience. The constraint is a limit on the amount of memory that can be specified in a single program unit. There are ways around it with more sophisticated memory management techniques, but that would have taken me too far afield.

Practically, the limitation means that the program is limited to evaluating the aggregate distribution at about 1000 points. If you describe the claim distribution with 25 points this means that you are limited to 40 claims. Not much of a constraint if the expected number of claims is 0.25.

However, since my main goal was to show that the calculations would work, the program appears to have accomplished those goals and could now be transferred to a less constrained environment or more talented programmer.

If anyone is interested in a program listing I would be glad to provide one with the understanding that it was not intended to be a finished product, is not warranted to be free of defects and comes with no technical support.

Reference

Beard, R.E., Pentikainenen, T. and Pesonen, E., *Risk Theory: The Stochastic Basis of Insurance*, (Third Edition), 1984.

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