EXCESS LOSS DISTRIBUTIONS OVER AN UNDERLYING ANNUAL AGGREGATE

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Excess Loss Distributions over an Underlying Annual Aggregate

by Joseph R. Schumi

Abstract

The purpose of the paper is to develop a method of calculating the aggregate loss distribution for a policy covering excess claims over occurrence limit plus claims arising from the primary losses over an underlying annual aggregate.

Usually, when working with losses from more than one source you would determine the aggregate distributions of each component and convolute the result to get the overall distribution. The problem is that the two distributions - the excess over occurrence limits and the excess over the retained annual aggregate are not independent.

Using results developed in my earlier note, I develop the conditional probability distribution of the number of non-excess claims based on the number of excess claims. It is argued that, in the probability subspace defined by a particular number of excess claims, the random variables describing the distributions of excess and retained losses are independent and thus so are the distributions of the excess losses and the excess of the retained losses over an annual aggregate. Thus the distribution of their sum can be determined by convoluting the respective distributions.

The conditional results for zero, one, two etc, excess claims are then summed using the probabilities of that number of excess claims.

Finally I outline a computer implementation of the process. I have created a simple demonstration version in Turbo Pascal for the Macintosh. It is limited in that I used a simple loss distribution to limit the number of points required for the calculations.

While not developed explicitly in this paper, this approach could also be used to determine increased limits factors as a function of the expected number of claims when an underlying aggregate is involved.

Introduction

In many situations we are asked to analyze the loss distribution of an excess policy which includes coverage for retained losses that exceed some aggregate accumulation. While it is possible to determine the distributions of both the retained losses and the excess losses, it is not readily apparent how to combine the distributions because the distributions are not independent, since a claim could contribute to each distribution.

The procedure described in this note decomposes the problem in such a way as to make the two distributions independent.

The key to this decomposition is to determine the excess and retained loss distributions for a given and fixed number of excess claims. The first step is to determine the claim frequency distribution of non-excess claims conditioned on the number of excess claims. This problem is solved under the assumption of a negative binomial claim distribution using Bayes Theorem.

The next step is to determine the aggregate loss distributions for the excess and retained losses. For a given number of excess claims, say N_x, the distribution of excess losses is given by the N_x-fold convolution of the excess loss distribution, which will be denoted $f_{xL}(X)^{N_x}$. The distribution of losses below the underlying per occurrence loss limit, L, is given by the Compound Distribution of the non-excess claims plus N_xL, the amount contributed by the excess claims. From this, the distribution of underlying losses over an annual aggregate, Agg, can be determined. I will call this $f_{Agg}(X|N_x)$.

The key observation is that in the subspace defined by the number of excess claims, the aggregate distribution of the excess claims and the aggregate distribution of the retained losses are independent.

The argument is as follows: The excess distribution is determined by the N_x-fold convolution of the excess claim distribution. The distribution of the aggregate retained losses is made up of two components. The first is the aggregate distribution of the non-excess claims. Under the usual assumptions the individual claims are independent, thus the size of the non-excess claims is not influenced by the size of the excess claims and vise versa. The other component is the retained portion of the excess claims. In the entire probability space, this isn't independent of the number of excess claims, but for a given number of excess claims it is a fixed amount. Thus, in each subspace defined by the number of excess claims, the distribution of the retained amounts is independent of the distribution of amount of the excess losses.

Finally, since the excess losses and the retained losses are independent, so are the excess losses and the excess of the retained losses over the annual aggregate.

This is discussed in more detail in Appendix A.

Thus for any number of excess claims, say N_x, the total loss distribution is given by the convolution of $f_{xL}(X)^{N_x}$ and $f_{Agg}(X|N_x)$.

Finally the total distribution, $f_{xL,Agg}(X)$ is obtained by summing the conditional distributions $f_{xL}(X)^{N_x} * f_{Agg}(X|N_x)$ weighted by the probability distribution of N_x, that is

$$f_{xL,Agg}(X) = \sum f_{xL}(X)^{k} * f_{Agg}(X|k)\ P(N_x = k).$$

The Conditional Distribution of the Number of Claims

Let π stand for the probability that a claim is an excess claim. Recall that $({}_nC_k)$ stands for the binomial coefficient $(n)!/((n-k)!(k)!)$

If the basic claim process is Negative Binomial with parameters h and n, then for $k \geq 0$ we have that the probability of k claims is given by

$$P_k = ({}_{h+k-1}C_k)\ [h/(n+h)]^h\ [n/(n+h)]^k .$$

In this case, the excess claims are also a Negative Binomial with parameters h and πn and again for $k \geq 0$, the probability of k claims is given by

$$P_k = ({}_{h+k-1}C_k)\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^k.$$

Consider the probability that the total number of claims is N and that the number of excess claims is N_x, call this $P(N_x \cap N)$. On the one hand,

$P(N_x \cap N) =$ P(nr of excess claims = N_x| nr of claims = N)P(nr of claims = N)

or in algebraic terms

$$P(N_x \cap N) = {}_NC_{N_x}\ \pi^{N_x}\ (1-\pi)^{(N-N_x)}({}_{h+N-1}C_N)\ [h/(n+h)]^h\ [n/(n+h)]^N.$$

On the other hand,

$P(N_x \cap N) =$ P(nr of claims = N|nr of excess claims = N_x)P(nr of excess claims= N_x)

$$P(N_x \cap N) = P(N|N_x)\ ({}_{h+N_x-1}C_{N_x})\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^{N_x}.$$

Combining the equations we have

$$P(N|N_x) = \frac{{}_NC_{N_x}\ \pi^{N_x}\ (1-\pi)^{(N-N_x)}\ ({}_{h+N-1}C_N)\ [h/(n+h)]^h\ [n/(n+h)]^N}{({}_{h+N_x-1}C_{N_x})\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^{N_x}.}$$

Let $\eta = N-N_x$, the number of non-excess claims.

Considering the combinatorial terms, we have

$$\frac{{}_{N}C_{N_x}\,({}_{h+N-1}C_{N})}{({}_{h+N_x-1}C_{N_x})}$$

which equals

$$\frac{\dfrac{(N!)}{(N-N_x)!(N_x)!}\cdot\dfrac{(h+N-1)!}{(h-1)!(N)!}}{\dfrac{(h+N_x-1)!}{(h-1)!(N_x)!}}$$

and by multiplying through we have

$$\frac{(N!)(h+N-1)!\ \ (h-1)!(N_x)!}{(N-N_x)!(N_x)!\ \ (h-1)!(N)!(h+N_x-1)!}$$

which reduces to

$$\frac{(h+N-1)!}{(N-N_x)!(h+N_x-1)!}\ .$$

Which can be rewritten as

$$\frac{(h+N_x+\eta-1)!}{(h+N_x-1)!(\eta)!}$$

and which equals $({}_{h+N_x+\eta-1}C_{\eta})$.

The rest of the equation is

$$\frac{\pi^{N_x}\,(1-\pi)^{(N-N_x)}\,[h/(n+h)]^{h}\,[n/(n+h)]^{N}}{[h/(\pi n+h)]^{h}\,[\pi n/(\pi n+h)]^{N_x}}$$

Rearranging terms

$$\frac{\pi^{N_x}\ \ [n/(n+h)]^{N_x}\ \ \ [h/(n+h)]^{h}\ \ \ [n/(n+h)]^{(N-N_x)}(1-\pi)^{(N-N_x)}}{[\pi n/(\pi n+h)]^{N_x}\ \ \ [h/(\pi n+h)]^{h}}$$

Summarizing

$$[(\pi n+h)/(n+h)]^{N_x} \quad [(\pi n+h)/(n+h)]^{h} \quad [n(1-\pi)/(n+h)]^{(N-N_x)}$$

And finally

$$[(\pi n+h)/(n+h)]^{(h+N_x)} \, [n(1-\pi)/(n+h)]^{(N-N_x)}$$

or

$$[(\pi n+h)/(n+h)]^{(h+N_x)}[n(1-\pi)/(n+h)]^{\eta}$$

Now $(\pi n+h)/(n+h)$ can be expressed as $(h+N_x)/(h+N_x+X)$ where $X = (h+N_x)\, n(1-\pi)\, /(h+\pi n)$. Thus $n(1-\pi)/(n+h) = X/(h+N_x+X)$

and the full expression becomes

$$({}_{h+N_x+\eta-1}C_{\eta}\,)[(h+N_x)/(h+N_x+X)]^{(h+N_x)}[X/(h+N_x+X)]^{\eta}.$$

This is the conditional probability that the total number of claims is N and also the conditional probability that the number of non-excess claims is $\eta = N-N_x$.

Thus the distribution of non-excess claims is a Negative Binomial with $h' = h+N_x$ and $n' = (h+N_x)\ n(1-\pi)\ /(h+\pi n)$.

Recall that this is a distribution with mean n', i.e. $(h+N_x)\ n(1-\pi)\ /(h+\pi n)$. In particular, if $N_x = \pi n$, the expected number of excess claims, then the expected number of non-excess claims is $n(1-\pi)$ and the total expected number of claims is n.

The Conditional Loss Distributions

Assuming that we know the loss severity distribution we can now determine the conditional excess and primary aggregate loss distributions.

For a given number of excess claims N_x, the aggregate distribution of the excess claims is the N_x-fold convolution of the loss severity distribution truncated below at the loss retention L. While this does not generally have a closed-form solution, it can be easily determined numerically for for any given N_x.

For the primary layer, the loss distribution of the non-excess claims is a compound distribution with claim frequency distribution as determined above and with a conditional loss severity distribution derived from the original severity distribution restricted to losses up to the occurrence limit. This distribution does not have a point mass at its upper limit for the losses over the occurrence retention. Finally, this distribution is shifted to the right by $N_x L$ to account for the retained portion of the excess losses.

There are several numerical tools available to determine the aggregate distribution, the choice of which depends on the parameters of the frequency and severity.

Now, since under the conditional assumption of a known fixed number of excess claims the distribution of the excess losses and the retained losses are independent, the distribution of the sum of the retained and excess losses is given by the convolution of their respective distributions.

The Total Loss Distribution

Finally having determined $f_{xL,Agg}(X|N_x)$ [$= f_{xL}(X)^{N_x} * f_{Agg}(X|N_x)$] for each N_x, we can determine $f_{xL,Agg}(X)$ by multiplying each $f_{xL,Agg}(X|N_x)$ by $P(N_x)$, where $P(N_x)$ is the probability that the number of excess claims is N_x. The number of terms to be calculated is determined by specifying a stopping probability parameter. The stopping probability is compared to

$F_{xL,Agg}(X) = \sum_{all\ X} f_{xL,Agg}(X)$ after each step. If $F_{xL,Agg}(X)$ is less than the stopping probability, the process is repeated for N_x+1.

Implementation

I created a simple program to carry out these calculations. In the calculations in this program, I based the severity distribution on a simple Pareto distribution of the form $(B^q/(q-1)/(X+B)^{q+1}$, for $X \geq 0$ and computed the aggregate distributions using the method I described in my previous note.

In general, the various distributions could be determined using any of tools at one's disposal; all that is needed is a device to obtain the aggregate distributions of the primary and excess losses, translate it and convolute it. In particular, if the expected number of primary claims is large, the primary aggregate distribution would probably best be obtained using one of the Fourier transform methods.

In the example, the starting assumptions were that we expected 10 claims with a variance of 11, that the primary claims represented the first five points of the severity distribution and that the primary aggregate retention was ten loss units. This was set up in the parameter file read by the program. The parameters of the frequency distributions for the excess claims and non-excess claims for a given number of excess claims were derived as described above. In particular, h = 100 and n = 10 for the primary distribution and h = 100 and n = 0.688 for the excess distribution.

In the main part of the program I loop on the number of excess claims.

I use the method of calculating aggregate distributions discussed in my previous note to determine the aggregate non-excess losses using the normalized probabilities for first five points from the loss severity distribution as the non-excess severity distribution. The non-excess distribution is translated by $500N_x$ to include the retained part of the excess claims in the aggregate primary loss distribution.

Finally, the excess of the annual aggregate retention distribution is determined by adding the probabilities of all of the points below the aggregate retention and assigning this probability to zero and assigning the probabilities of the points above the aggregate retention to the corresponding point shifted downward by the aggregate retention.

The excess convolutions are performed using the normalized probabilities of the upper 15 points of the original severity distribution.

The two distributions are then convoluted to yield the total distribution for the current number of excess claims.

This distribution is then added to the previous distributions by weighting the current distribution by the probability of the current number of excess claims.

$$f_{xL,Agg}(X) \text{ [this step]} = f_{xL,Agg}(X) \text{ [last step]} + f_{xL,Agg}(X|k)\ P(N_x = k).$$

The cumulative distribution obtained from this distribution is evaluated at the maximum aggregate loss value, X_{max}. This value is compared to the stopping value. That is, if

$$F(X_{max}) \text{ [this step]} < \text{aggregate stopping parameter}$$

then increase the value of the excess claim count by one and repeat the calculations. In some situations the memory constraints of a PC may be such that it is not possible to make X_{max} large enough for $F(X_{max})$ to exceed the stopping value.

In each step, the aggregate distribution calculations have been stopped when the probability first exceeds a given stopping probability parameter. The overall calculated aggregate probability cannot exceed this value by more than a slight amount. A slight amount because each step will exceed it by some amount. Thus, in general, the overall stopping value must be less than or equal to the value used for the individual steps.

Another possible problem is that as the number of excess claims increases, the convoluted excess distribution will require additional points to satisfy its stopping parameter. It is possible that these subcalculations might be truncated because of array size constraints which would cause the probability to be understated and making the aggregate stopping value unattainable. While this is easy to deal with given sufficient computing resources, the outcome of the individual steps should be monitored.

Closing Comments

As a check of the calculations, I performed these calculations using an annual aggregate of zero, that is, the insurance company assumes all of the losses. The resulting distribution, within the precision controlled by the stopping probabilities, turned out be the same as the aggregate distribution calculated directly from the frequency parameters and the total severity distribution. I consider this to be a check both of the program and the algorithm. Selected data from these runs is included as Appendix B

Also attached as Appendix C is a copy of part of the output file of the program. In each step I included a number of statistics that allowed me to determine if the calculations were correct. For example, the mean and variance of the convoluted distribution should be the sum of those statistics for the input distributions, the mean and variance of the aggregate distribution should relate to the mean and variance of the frequency and severity distributions by the well known formulas, etc.

Appendix A

Following is a sketch of the proof of the independence of the the primary retained losses over the annual aggregate and the excess losses in the probability subspace defined by the number of excess claims.

Recall the definition of and some facts about independent random variables.

Definition. A set of random variables is independent if every finite subset is independent.

Theorem A. Random variables are independent if and only if their joint distribution function factors into a product of their individual distributions.

Theorem B. Any Borel measurable functions of independent random variables are again independent random variables.

Theorem C. Any Borel measurable functions of disjoint sets of independent random variables are independent random variables.

The total probability space can be thought of as a set of Cartesian products of the interval $I = (0, \infty)$, where the number of terms in the product correspond to the number of claims. This is the total probability space can be expressed as $\Omega = \bigcup I^N$ where I^0 is a single point. Let Ω_{N_x} stand for the subspace of N_x excess claims. In any I^N, under the usual assumptions, the N claims are independent random variables. Thus the joint distribution $g(x_1, \ldots, x_N) = f(x_1) \ldots f(x_N)$.

Now define two new random variables. $X_{ex} = L$ if $X \leq L$ and $X_{cx} = X$ if $X > L$ and $X_{pri} = X$ if $X \leq L$ and $X_{pri} = L$ if $X > L$, where L is the occurrence limit. These are Borel measurable functions with respect to the sigma-algebra generated by the original random variables. Thus, if these random variables are substituted for any of the original random variables, the resulting set of random variables is still independent.

Now restrict attention to the subspace I^N and assume we are in the subspace of I^N where the first N_x claims are excess claims and the remaining $N-N_x$ are non-excess claims. In this subspace, the probability density of these claims is identical with the joint probability distribution given by substituting X_{cx} for the first N_x claims and X_{pri} for the remaining $N-N_x$ claims, since it is zero outside of this space. Since this distribution factors on the entire space, it factors on the subspace. Obviously any other configuration of excess claims would yield the same result.

Next consider, the two random variables defined, respectively, as the sums of the excess and primary claims. By Theorem C, these are independent random variables. Finally, in the intersection of subspace I^N and Ω_{N_x}, the random variable defined as the sum of the excess claims less $N_x L$ and the random variable defined to be the maximum of zero and the sum of the non-excess claims plus $N_x L$ less the annual aggregate are again Borel measurable and independent.

Appendix B

Following are two copies of the output data sets. The first is from a run that calculates the total aggregate distribution corresponding to the underlying claim frequency and severity distribution in the normal fashion.

The second performs the calculation as described in the paper by separately calculating the primary and excess components, performing the convolution of these terms for each excess loss and finally calculating the weighted sum with the annual aggregate retention set to zero.

Note that the means and standard deviations are equal and the probabilities at the loss amounts shown are very close.

First Method

The mean and std dev of the aggregate distribution is:

2082 1270

loss amount	probability	Cumm Prob	Pure Prem Ratio
0	0.000073	0.000073	1.000000
100	0.000520	0.000593	0.951974
200	0.001932	0.002524	0.903974
300	0.004962	0.007486	0.856066
400	0.009942	0.017429	0.808396
500	0.016618	0.034046	0.761204
600	0.024190	0.058236	0.714811
700	0.031618	0.089854	0.669579
800	0.037971	0.127825	0.625865
900	0.042648	0.170473	0.583975
1000	0.045443	0.215916	0.544134
2000	0.025905	0.590415	0.258203
3000	0.018752	0.793156	0.106955
4000	0.007832	0.919984	0.039573

Second Method

The mean and std dev of the aggregate distribution is:

2082 1270

loss amount	probability	Cumm Prob	Pure Prem Ratio
0	0.000073	0.000073	1.000000
100	0.000520	0.000593	0.951973
200	0.001932	0.002524	0.903970
300	0.004962	0.007486	0.856060
400	0.009942	0.017428	0.808388
500	0.016617	0.034046	0.761195
600	0.024189	0.058235	0.714799
700	0.031618	0.089853	0.669565
800	0.037970	0.127823	0.625850
900	0.042647	0.170471	0.583958
1000	0.045443	0.215913	0.544116
2000	0.025905	0.590410	0.258186
3000	0.018752	0.793154	0.106957
4000	0.007830	0.919947	0.039601

Appendix C

The following is a sample of the output data set from the computer program. The first section echoes the input parameters - a label for the run (Test Data),the mean (10.0) and variance (11.0) of the first dollar claims process, the unit loss value (100), the number of points in the primary distribution (5), the number of points in the aggregated retention (10) and the total number of points in the severity distribution (20). The last line shows the probabilities controlling the aggregate loss calculation, the main loop and the printing of the excess pure premium table and the maximum number of points to be used in the loss arrays.

So in this example, the primary layer includes losses up to 500 with an annual aggregate of 1000. The excess layer is losses from 500 to 2000.

```
   Test Data
  10.00     11.00
   100.0        5         10         20
0.99990   0.99990   0.99990       200

The Mean of the Primary Severity Distribution is        128
The Stdev of the Primary Severity Distribution is        77
The Mean of the Excess Severity  Distribution is       794
The Std Dev of the Excess Severity Distribution is       560

The number of excess claims is   0 with Prob 0.50383
the expected number of non-excess claims is  9.249
the variance number of non-excess claims is 10.104
the nr of points in the primary agg dist is 36
the mean of the primary aggregate is       1184
the stdev of the primary aggregate is       469

the nr of points in the primary excess agg dist is 26
the mean of the primary excess aggregate is        312
the stdev of the primary excess aggregate is       369
the nr of excess claims is 0 and the nr of xsPts is 1
the xs number of points is 1 with total prob 1.0000000
the mean of the excess distr is        0
the stdev  of the excess distr is        0

the total number of points in the int dist is 27
the mean of the intermediate distr is       312
the stdev of the intermediate distr is       369

writing out the Total Probability
0.1858302   0.0480659   0.0457648   0.0418939   0.0370323
0.0317229   0.0264131   0.0214297   0.0169782   0.0131599
  ...

finished with case Nx equals 0
total probability equals 0.5038258
the mean of the cumm distr is       157
```

Since the number of points in the primary aggregate distribution did not exceed 200, another input parameter, the step was completed because the probability stopping parameter was exceeded. This information is reported for one, two etc. excess claims.

The next section is from the step where N_X equals five.

The number of excess claims is 5 with Prob 0.00069
the expected number of non-excess claims is 9.711
the variance number of non-excess claims is 10.609
the nr of points in the primary agg dist is 37
the mean of the primary aggregate is 1243
the stdev of the primary aggregate is 480

the nr of points in the primary excess agg dist is 52
the mean of the primary excess aggregate is 2743
the stdev of the primary excess aggregate is 480

the nr of excess claims is 5 and the nr of xsPts is 76
the xs number of points is 76 with total prob 1.0000002
the mean of the excess distr is 3972
the stdev of the excess distr is 1251

the total number of points in the int dist is 128
the mean of the intermediate distr is 6716
the stdev of the intermediate distr is 1340

(Note: the mean of the intermediate distribution is the sum of the means of the primary excess aggregate and the excess distribution, and you can verify by squaring the standard deviations to see that the variances add.)

writing out the Total Probability
0.1858302 0.0497242 0.0493546 0.0477042 0.0452040
0.0422148 0.0390128 0.0357931 0.0326833 0.0297591
...

finished with case Nx equals 5
total probability equals 0.9999080
the mean of the cumm distr is 1156

Since the loop has reached the stopping probability of 0.9999, the program exits the main loop, calculates the excess pure premium ratios for the aggregate distribution, writes it to the output file and stops.