THE ACTUARIAL PARADIGM: A NONTECHNICAL EXPOSITION

Lee Smith

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OVERVIEW

In his insightful and ground-breaking essay entitled The Structure of Scientific Revolutions, Thomas S. Kuhn discussed the notion of scientific advancement. In particular, he introduces the idea of a paradigm, a body of knowledge and beliefs which provides direction for research. The development and acceptance of a paradigm is a key factor in transforming a group of practitioners into a profession.

The beginning of a scientific revolution is the discovery of an anomaly relative to expectations which the existing paradigm creates. A "crisis" occurs when standard analysis is not producing desired results. A search for an alternative paradigm commences, and its acceptance completes the revolution.

Developments in the liability insurance area in recent years have caused casualty actuaries to reevaluate the tools they use in solving problems in their domain. Simultaneously, the range of issues determined to be within the purview of the actuarial profession has expanded rapidly. As a result, the generally agreed upon body of knowledge of casualty actuaries have been undergoing something of a revolution.

The intent of this paper is to survey the elements which are impacting on the issues casualty actuaries face today and to discuss the implications of that on our ability to develop and maintain a paradigm which can respond to the challenges. Such a paradigm must help define the profession as well as provide a basis for problem solving. To the extent that the scope of actuarial activity is such that some aspects of that activity are not amenable to the notion of a paradigm, those aspects will be considered as separate from the primary activity of professional actuaries.

As actuaries approach the 100th anniversary of their profession in North America, they will pause to reflect on the status of their profession. The challenges faced today provide perspective on the larger issues to be faced in coming years. Socioeconomic changes of the type that have caused the liability insurance area to be revolutionized will continue to provide actuaries with opportunities to creatively resolve some of society's most perplexing problems.

ELEMENTS OF THE CURRENT ACTUARIAL PARADIGM

To evaluate the current core of actuarial science, it may be helpful to develop a notion of what an actuary is. Conventional wisdom uses phrases like "insurance mathematician," and "ratemaking and reserving specialist." A more objective way of defining an actuary's role might be to consider the body of knowledge encompassed by the professional examinations which an actuary must take. The following brief listing of key topics covered in the 1989 Syllabus of Examinations of the Casualty Actuarial Society provides a valuable framework:

- Calculus and Linear Algebra
- Probability and Statistics
- Operations Research
- Numerical Methods
- Interest and Life Contingencies
- Credibility Theory and Loss Distributions
- Economics
- Theory of Risk and Insurance
- Ratemaking
- Reserves

- Accounting
- Law
- Financial Operations
- Forecasting

A review of some of the topics covered under these broad headings provides additional perspective:

- Simulation
- Bayesian Credibility
- Loss Distributions
- Risk Management
- Exposure Bases
- Statistical Plans
- Litigation
- Risk Classification
- Increased Limits Pricing
- Experience Rating

From these elements an actuarial paradigm could arise.

Plan of Paper

This paper is structured so as to introduce some key actuarial concepts, survey elements from related subjects which are considered a part of actuarial science, and to discuss a possible paradigmatic structure for actuarial science. Related subjects discussed here include law, mathematics, probability, statistics, economics and accounting.

The first area of knowledge reviewed is that of tort law. In that review strands from other areas of law are reviewed to the extent they have impact on actuarial science. Whether law is a component of an actuarial paradigm is doubtful, but the law greatly impacts the values and phenomena being analyzed by actuaries.

The next area considered is that of accounting. Like law, accounting probably contributes little to the actuarial core of knowledge, but yet it must be understood for actuarial science to realize its full potential. Ultimately the results of actuarial calculations are filtered through accounting rules and conventions.

Economics is the third area surveyed. While, like law and accounting, economics is somewhat peripheral to actuarial science, it is proposed herein that economics does contribute directly to the actuarial paradigm. This is best seen in the context of pricing which requires economic elements, but is also apparent in terms of generally overlapping methodologies.

Physics is given attention because it provides the best example of the application of mathematical principles to the analysis of complex phenomena. Like actuaries, physicists are attempting to uncover patterns and rules which will help them better explain and predict real world phenomena. The models physicists have developed and the mathematics they have employed will provide direction to actuaries in search of their core models.

Philosophy is briefly covered to provide a sense of general importance to the work of the actuary. As a professional in search of procedures by which to analyze and quantify socioeconomic phenomena, actuaries are involved in activity with epistemological, ethical, and even metaphysical implications. While those may not be apparent in the day to day routine of actuarial work, an awareness of their existence provides useful perspective for the actuary in his/her long-term planning.

Mathematics is the subject which is identified as most likely to provide the key to development of a paradigm to cover key aspects of actuarial science. A brief survey of some of the broad areas of mathematics reveals the richness of that subject and the power it can bring to actuarial sciences. The subject matter of probability and statistics is covered as a separate area.

Utility theory and interest theory are briefly reviewed to provide perspective on how these specialized topics impact on actuarial science. Selected general actuarial topics are then discussed to illustrate some of the specific areas of current interest to casualty actuaries. Appendices are included to provide additional perspective on the elements to consider in developing an actuarial paradigm.

Risk Theory

Risk theory is the study of the random variation of insurance claims. The probability distribution of the amount of claims expected to arise from an insurance portfolio over a period of time is to be estimated. This distribution may be derived from distributions of claim counts and average claim sizes expected to arise from portfolio components in the time frame.

Estimation of losses arising from insurance portfolios is a key aspect of actuarial work. Risk theory provides a theoretical basis for loss estimation. As such, it must be considered to be a key aspect of the actuarial paradigm.

Complicating Elements

Analysis of losses arising from liability insurance coverages involves the consideration of a variety of factors. As opposed to the individual risk-theoretic approach which disaggregates the portfolio and the collective approach which assumes losses follow a particular distribution, the loss analysis performed by most actuaries in the course of their standard activity is based on aggregated data from an unknown mathematical structure. Historical losses for a group of occurrences or policies are evaluated and relationships derived from that data are used to estimate losses for the category under review.

For a typical group of policies covering liability insurance exposure, losses can be influenced by such things as policyholder characteristics, jurisdictional attributes, contract language, social and economic conditions, weather, and chance. Both the occurrence of a claim and its amount will be subject to wide variability depending upon the mix of factors involved.

The actuarial paradigm of loss estimation, then, will be grounded in the theory of mathematics and statistics. Certain aspects of the problems facing actuaries, on the other hand, will involve the evaluation of a number of factors for which the theoretical core of actuarial science has no component. It is in balancing this need for a theoretical core with the reality of the scope of the actuary's work that the challenge of providing a framework will arise.

Other Issues

Actuaries respond to a number of issues requiring tools not directly related to those used in the estimation of losses from a portfolio of insurance policies. Included among these issues are expense levels, profit margins, interest income, exposure bases, rate level indices, and credibility. In addition, many of the factors impacting on losses are such that incorporation into a standardized loss estimation model is prohibitively complex.

The implication is that the body of knowledge deemed to encompass the subject of actuarial science will consist of a number of paradigms as well as some elements that fall outside the paradigm structure. Any discussion of the paradigmatic structure of actuarial science will need to define the scope of issues and elements covered by the paradigms.

LAW

One of the factors which influences the aspects of liability insurance within the actuary's domain is the legal environment. The general area of law which has had the greatest impact in the liability insurance area is tort law. The evolution and interpretation of cases and statutes related to torts has been a significant factor in the insurance crises which have occurred in recent years.

Tort law, which provides the underlying structure from which liability losses emanate, is concerned with allocating losses which result from human activity. Unlike contract theory, which attempts to assess damages based on a very narrow range of criteria, tort theory attempts to assess the consequences of the tort actions. A balance is sought between the utility of a certain kind of conduct and the harm which may arise from that conduct.

Originally, liability under tort was grounded in actual intent or in a cause and effect relationship between the conduct and the damage. Today, the concept of social fault has arisen whereby consequences alone may determine liability. Tort liability generally falls into one of three classes--based on intent; based on negligence; or strict.

Negligence is roughly described as conduct which falls below the standard established by law. The standard of care requirement is that which a prudent person would follow under the circumstances. A professional is expected to have the same skill and learning as a typical member of his/her profession and to apply that skill and learning with the same care as is generally exercised.

In recent years a number of creative bases for allocating damages has arisen. This creativity creates problems for actuaries attempting to quantify potential losses. In a 1980 case, for example, in which the plaintiff could not identify the manufacturer of a drug which had injured his mother, the concept of "market share liability" arose whereby several companies could be held responsible.

In a 1975 case, a landlord was held liable for a criminal act by an unrelated party in his building. A 1969 case removed immunity from tort liability which a charitable institution had enjoyed, and a 1973 case removed the defense of governmental immunity from a local Board of Education.

In an article in the December 12, 1988 issue of the Wall Street Journal a brand new legal damage theory was described. Named "hedonistic damages," this approach advocates urges the award of large sums to the estates of victims to provide for the intrinsic value of life.

A range of the implied value of human life is calculated from statistics on amounts people will pay to reduce the risk of death. The range is from $66,000 which is the value of prompt coronary care to $12 million which is the value of safer airline travel.

Product liability has become a significant component of tort law. The notion of recovering from manufacturers for negligence was established in a 1916 case involving Buick Motor Co. A 1960 case involving Chrysler established that even a signed disclaimer by the plaintiff did not eliminate the liability. A 1979 Act of Congress established a wide range of remedies for consumers of products with implied warranties. All of this activity impacts on the problem solving activity of actuaries.

In addition to tort law, there are several other areas of law that impact on the work actuaries perform. Property law is concerned with the notion of ownership. Property law attempts to establish principles by which to evaluate situations involving property rights. Many colorful and historically revealing notions underlie the theoretical structure of property law.

A system of estates exists in property law which distinguishes types of ownership. The estate types include fee simple, fee tail, life and leasehold. A fee simple is the most commonly known type of ownership representing an estate that has no end point and has no limitations on inheritability. Defeasible fees are estates that may end if certain conditions arise.

A future interest is an interest in property which can inure in the future. The Statute of Uses created a type of future interest in a transferee called an executory interest. These interests may spring out of the transferor or shift to a transferee. The Rule against Perpetuities arose to curb indestructible executory interests.

Estates may be held concurrently. Commonly known forms of concurrent ownership include Tenancy in Common, joint tenancy and tenancy by the entirety. Co-tenants have various rights and obligations relative to the co-tenancy.

In addition to providing perspective to the actuaries with regard to principles underlying insurance contract language, the study of property law provides insights into the structure of society. It also provides a fascinating analytical framework in the analysis of estate interests which has many features of a deterministic mathematical model. Along with contract law, property law provides the systematic underpinnings of much of the social structure which exists in modern democracies.

Contract law may be looked at as an abstraction. All particularities of person and subject matter are removed from pure contract theory. It is what is left after accounting for all the specialized areas of law.

For a contract to exist there must be mutual assent and consideration. Mutual assent often manifests itself in terms of offer and acceptance. Contract law attempts to uphold the reasonable expectations of parties relying on promises.

Determining whether a contract arises involves evaluating the consideration involved and deciding whether there was a meeting of minds. The two essential features of consideration are the item bargained for and its legal value. The requirements of a valid offer are intent, definitiveness and communication. Modes of acceptance are different for unilateral and bilateral contracts.

Once it is determined that a contact has been formed, there are various remedies if a party breaches. Among these are damages, specific performance, recission and restitution. Damages may be compensatory, punitive, or nominal.

Damages in contract are to be distinguished from those in tort. The contract breaker is not held liable for all consequences of his/her actions. As articulated in the 1854 Hadley v. Baxendale case, damages for breach of contract are "such as may fairly and reasonably be considered either arising naturally...or such as may reasonably be supposed to have been in the contemplation of both parties..." Doctrines such as promissory estoppel are causing changes in this scheme, however, and contract damages may ultimately be subsumed into the mainstream of tort law.

Constitutional law is another area of law which provides actuaries with perspective on the legal system which impacts on the phenomena they are modeling. Equal protection is a limitation on government regulation to guard against arbitrary discrimination. Discrimination by race and sex has been greatly scrutinized by recent court decisions. The notion of establishing reasonable classifications to be permitted has analogies in actuarial science.

Another area of constitutional law of interest to actuaries is that of state regulation. Under the tenth amendment the states have all power not specifically given to the Federal Government. Where there is a conflict, the Federal Government controls. Federal power with respect to the regulation of commerce is found in the Commerce Clause.

Actuarial science is faced with the problem of estimating losses which will arise from evolving legal theories. To the extent these legal theories are dynamic in the sense that future damages will not be similar to past damages, predictability is affected. The actuarial paradigm will be effective only to the extent it can quantify the effect of the dynamism.

In a 1972 Harvard Law Review article, George Fletcher set forth two paradigms for tort liability which would replace the fault and nonfault categories. The first paradigm, reciprocity, provides all individuals the same degree of security from risk and grants compensation whenever a disproportionate distribution of risk injures someone more than his/her fair share. The paradigm of reasonableness established the notion of a "reasonable man" as the basis of determining what ought to be done in a given set of circumstances.

Lawyers, like mathematicians, deduce conclusions from axioms about undefined terms. Chains of deductive legal reasoning give meaning to these terms only at the point where conclusions are drawn. These meanings are not necessarily consistent from one application to the next.

Should tort and other law become paradigmatic and should the paradigms lend themselves to a degree of predictability, the contingencies to be estimated by actuaries in the liability insurance area may become more quantifiable. Until then, the actuary must continue to use models which are constructed to estimate losses arising from a very subjective legal system. Appendix A provides a glossary of legal terms which are of relevance to the subject of actuarial science.

ACCOUNTING

Actuaries produce results which become elements of income statements and balance sheets of entities with insurance types of exposure. Understanding the basic accounting principles which affect the way actuarial results are interpreted is an essential aspect of the actuary's work. To provide perspective on the accounting implications of the actuarial paradigm, a review of key accounting principles will be useful.

Presenting a fair picture of the financial status of a business organization is the goal to which the various accounting rules relate. Some accounting principles which relate to actuarial calculations are:

- Recording the value of assets at historical cost.
- Recognizing revenue at the completion of a transaction.
- Matching costs to related revenues by period.
- Basing financial statement elements on objective evidence.
- Being conservative where evidence is unclear or conflicting.
- Determining materiality subjectively.
- Disclosing anything which could affect the financial position of the entity.
- Documenting changes in procedures.

The basic accounting equation which provides a framework for all financial accounting is Assets = Liabilities + Equity. The balance sheet shows the components of the equation at a point in time. Another key statement is the income statement which displays revenues and expenses. A statement which has become more prominent in recent years is the Statement of Changes in Financial Position which provides useful information on liquidity changes.

Valuation of assets is a topic of increasing interest in recent years. Traditionally assets have been valued based on historical or acquisition costs. Inflation levels of recent years, however, have caused concern regarding the relevance of such an approach. Most alternatives, however, require that subjective judgments be made. Different methods may provide a better picture in different situations.

Valuation of liabilities for many entities is not a significant problem. The notion is that a liability is a known amount expected to be paid in the future. Liabilities of financial institutions like insurance companies, however, are not so easy to estimate.

A firm's leverage position measures the degree to which it is obligated for fixed costs. As regards financial leverage, the interest rate on borrowed money should be less than the expected rate of return on the venture supported by the borrowing. Break-even analysis involves dividing fixed costs by the margin per unit to determine the volume of sales needed to cover fixed costs. Decisions as to desired degree of leverage must consider stability of earnings (risk).

Amortization involves spreading costs over time. Depreciation is a form of amortization which involves changing the value of an asset over time as its value changes.

Various methods have arisen to provide a basis for depreciation. The Tax Acts of 1981 and 1982 established the idea of accelerated cost recovery which allowed depreciation costs to be accelerated.

An effect of using accelerated depreciation for tax purposes relative to financial reporting services is to produce deferred taxes. The difference between tax expense reported and tax expense paid is shown as a liability. This deferred tax amount may never be paid unless the firm declines in size and is profitable as it declines.

The basic inventory equation is: Ending Inventory = Beginning Inventory plus Purchases less Sales. A question arises as to how to value cost of goods sold. Two well known methods are first-in, first-out (FIFO) and last-in, first-out (LIFO).

The analysis of long-term projects involves capital budgeting. Approaches to capital budgeting include:

1. Payback Method

2. Net Present Value Method

3. Internal Rate of Return Method

The idea is to evaluate alternative investment opportunities so as to choose the mix that will provide the best results.

The words "cost" and "expense" have specific meaning in accounting usage. The cost refers to the amount paid for an item at acquisition. Expenses are costs which are matched to revenues which have resulted from the expenditures. Before a cost becomes an expense it is an asset.

Financial analysis is often enhanced by the calculation of certain analytical measures. Certain ratios have provided helpful insights into various financial aspects of a company. Among these are:

1. Liquidity ratios: Ratios of current assets to liabilities.

2. Competition ratios: Relationships of key values for the entity with the same values for similar entities.

3. Efficiency ratios: Measures which show speed of turnover of key items.

4. Solvency ratios: Measures which show how operations and financial status support ongoing operations.

5. Profitability ratios: Measures of profit relative to various benchmarks.

Since actuaries are involved in the calculation of a number of balance sheet and income statement components, they must be aware of the accounting principles which must be met in recording actuarial items. Improper interpretation of the end product of an actuarial calculation will offset the value created by the mathematical ingenuity.

ECONOMICS

As was the case with the legal environment, the economic environment which impacts on the values the actuary is attempting to estimate is very dynamic. The impact of national economic policies and microeconomic forces on the liability losses the actuary is attempting to estimate can be dramatic. The actuary must understand these economic forces to be in a position to estimate the losses which will be dependent on those forces.

Because the economic paradigm which would help evaluate economic variables is in a state of revolution, the actuary gets little guidance from economic science. Macroeconomic variables like employment, inflation, and interest rates which can greatly impact insurance costs are becoming less and less predictable. As a result, loss estimation models must provide for the assimilation of a wide variety of economic scenarios.

The microeconomics of the insurance industry also adds to the complexity of actuarial modeling. Evolving marketing and underwriting strategies including changing of policy coverages, revised reinsurance programs, and changing target markets make historical data inadequate to the task of evaluating future loss patterns. Consumer demand and production theories provide an analytical framework for the evaluation of these topics.

The work of actuaries is inextricably bound up with that of economists. The development of rate levels and allocation of overall rate level to groupings of customers is an economic function. Elements of economic theory influence actuarial models. A full discussion of the underpinnings of actuarial science must account for the various economic elements.

Economics is concerned with the allocation of scarce resources. It evaluates the problems of what to produce, how to produce it, and how to distribute the results. The starting point in a market economy is generally an analysis of supply of and demand for products.

Microeconomics

A simple demand schedule relates units of a good which will be purchased with prices at which the good would be offered. For a normal good, the higher the price the lower the quantity demanded. Among the reasons for this pattern is the fact that consumers will shift spending to the point at which the total amount available to spend is allocated so as to produce the most utility. An increase in the price of a particular good makes other goods relatively more attractive.

A simple supply schedule relates units of a good which will be offered for sale with possible price levels. Because the presumption is that profit is a primary reason for production, and that costs of production are given, higher prices lead to greater profits and more output. Assumptions must be made as to the time frames in which output adjustments can be made.

The equilibrium price and quantity for a good occurs at the point where the supply and demand curves intersect. An equilibrium position is considered stable if a deviation from equilibrium brings into operation market forces which move the curves back to the same equilibrium position. Ceteris paribus, a shift in either the supply or demand curve, will cause the equilibrium point to move.

A concept of importance to actuaries in their pricing work is that of elasticity. Elasticity is a measure of the change in demand which results from a change in price level. When an actuary is attempting to estimate the policyholder mix to be written under a set of proposed rates, he/she must estimate both the volume and mix changes which will occur. A form of elasticity measure is needed.

Utility theory comes in many forms, but in basic demand theory it relates to the degree of satisfaction the consumer receives from consuming a good. To maximize the total satisfaction which can be derived from expenditure of a given amount of income, the utility of the last dollar spent on the commodities purchased must be the same. In other words, the quotient of the marginal utility and price of every good would be equal.

Graphically, indifference curves can be used to illustrate the notion of maximizing utility when two goods are involved. Indifference curves show combinations of two goods which yield equal utility. The marginal rate of substitution between the goods is the slope of the curve and represents the amount of one good which will be given up to gain a unit of the other.

The budget constraint line shows the various combinations of the goods which can be purchased given available expenditure by the consumer and the prices of the good. The point at which the budget line is tangent to the highest indifference curve is the point of maximum utility or equilibrium. To increase utility, the consumer must enter into exchange with another who has a different marginal rate of substitution.

These notions are important to the actuary in pricing work because they provide guidance as to reasons for market reactions to alternative price levels. Also important is the supply side of the market. As demand theory is the primary economic paradigm for demand, production theory provides the supply paradigm.

A production function shows levels of production that are possible in a given time frame given the available input factors. The law of diminishing returns is reflected in a marginal product curve which eventually declines. Each input factor provides for various production stages which reflect the relative average product level.

A graphical device in production theory somewhat analogous to the indifference curve in demand theory is the isoquant. An isoquant shows combinations of two input factors which produce a particular output level. The marginal rate of technical substitution of one factor for another is the amount of one a firm can release by increasing the other by a unit without changing the level of production. It is equal to the ratio of the marginal products of the factors.

An isocost shows combinations of input factors which a firm can purchase given its budget for such factors and factor prices. The equilibrium for a producer occurs where the ratio of the marginal products of the factors equals the ratio of their prices. At that point, the marginal product resulting from the last dollar spent on each input is the same.

Factor inputs in the insurance industry do not follow the patterns established in basic economic textbooks. Similarly, costs of production do not arise as a conventional textbook illustrates. There are enough analogies, however, that economic theory provides a sound context for a discussion of insurance pricing.

Cost curves show the minimum cost associated with production of various output levels. Total costs are composed of fixed and variable elements. In a given time frame, the shapes of the various cost curves follow patterns as follows relative to quantity produced:

1. Average fixed costs decline as quantity rises.

2. Average variable costs decline at first and then rise.

3. Marginal costs decline at first and then rise..

4. Total average costs decline at first and then rise.

Pricing an insurance product involves determining levels of fixed and variable costs in a given time frame. The primary cost associated with a casualty insurance product is loss potential. Loss potential is variable in the sense that increasing quantities sold at some point are correlated with relaxed underwriting standards.

Many expense elements of the cost associated with insurance are fixed in the time frame involved. As a result, insurance cost curves for a given product could follow patterns not unlike those in economic theory, ceteris paribus.

An issue of much interest in the insurance industry is that of the level of competition. Many aspects of the market model which theoretically results in optimal societal welfare depend on the assumption that pure competition prevails in the market of interest. Price and output in noncompetitive markets are not considered optimal.

Other market types often considered are monopoly, monopolistic competition, and oligopoly. Within each of these are significant variations. The germane question is not whether markets are perfectly competitive, but whether they are workably competitive.

For the market model to be fully operative, perfect competition must exist in the factor as well as the product markets. To maximize profit, a company will produce the optimal output level given the optimal combination of factor inputs. This level occurs where the ratio of marginal product to the price of each input equals the reciprocal of the marginal cost of a product and its price. Where perfect competition does not exist, marginal revenue differs from price, and equilibrium occurs at a nonoptimal point.

This fairly abstract theory of market equilibrium actually leads to a notion of general equilibrium for a society and a measure of welfare therein. A condition known as Pareto optimality can arise at which the marginal rates of transformation and the marginal rates of substitution associated with the goods in a society are equal. The fact that this can occur only where perfect competition exists is the basic argument in favor of pure competition. A complication not handled in this notion is that of externality which belies the optimality of results from a market economy.

Macroeconomics

To this point the focus of the discussion of economic theory has been primarily in the area generally thought of as microeconomics. The subject area of macroeconomics also has much to offer the actuary in terms of perspective on underpinnings of actuarial science. In addition to macroeconomic variables such as unemployment and interest rates which may enter directly into actuarial algorithms, basic macroeconomic theory has elements of importance to actuarial science.

It is generally becoming recognized that for macroeconomics to have a needed degree of scientific rigor it must develop linkages to microeconomic foundations. While this will severely strain existing macroeconomic modeling patterns, it will lead to models which are more demonstrably related to real world phenomena. The elements of microeconomic theory lend themselves to modeling techniques on aggregates which can have similar degrees of success.

The problem of general equilibrium in a market economy was tackled by Wald in 1936. An exchange economy consisting of individuals, commodities, and stocks of goods could be "solved" given a set of exchange equations. The question arises, however, as to whether such an abstract system has any real economic content.

Keynes theory of macroeconomics was not reconcilable with general equilibrium theory. Samuelson introduced a dynamic adjustment mechanism. Von Neumann introduced mathematical techniques including mini-max and fixed point notions which advanced the mathematical aspects of the analysis.

A model associated with Arrow, Debreu, and McKenzie has provided a new foundation for work on the microeconomic foundations of macroeconomic theory. They introduced production sets, preference structures, and a more vigorous treatment of the elements of a market economy.

Hicks introduced the notion of temporary equilibrium. Lange introduced some of the uncertainty, probability, and risk concerns which arose in the Keynesian system. Patinkin dealt with the idea that various financial assets have the function of providing a store of purchasing power.

The transition from the micro to the macro level involves determination of aggregation rules which allow choice-theoretic household demands to aggregate to a consumption function. In Microfoundations, Weintraub illustrates that general equilibrium theory is merely an example of general systems theory the latter of which applies to systems in general.

General systems theory (GST) finds structural similarities among fields of science. If economics is a science, GST is applicable. Making micro and macroeconomics compatible means defining them so a systematic structure can emerge. Rather than a single model providing for the economic paradigm, economics has become a body of developing knowledge responding to particular problems.

In this context, the rather abstract ideas of macroeconomics take on more meaning. Macroeconomics is generally thought of as dealing with behavior of aggregated variables like income, price, and growth levels. The simple two-sector model considers relationships among income, savings, investment, and consumption. Government and international sectors may be added.

The simple model provides a basis for determining the effects on national income of changes in levels of basic variables. This basis is the multiplier effect, the idea being that a change in the level of a variable, say investment, has a multiplicative effect on national income because of dependence effects.

Equilibrium in commodity markets is often analyzed using a graphical device known as our IS curve. In a market, equilibrium occurs where the volume of output equals planned spending. The IS schedule shows equilibrium income levels at various interest rates.

Equilibrium in money markets is often analyzed in terms of an LM curve. Money demand is a function of general types of demand. Equilibrium occurs where supply and demand of money are equal, and the LM curve gives combinations of interest and income consistent with equilibrium in the money markets.

When equilibrium occurs in the money and product markets at the same income and interest rate level, a form of general equilibrium arises. Monetary and fiscal policy can be judged according to their effect on this equilibrium point. The effectiveness of the various policies in this model are a function of the relative slopes of the curves.

Economic theory is obviously rich with tools which can be of value to the actuary. Like the other underpinnings of actuarial science, however, economics makes no claims of uncovering truth for the objects of its study. In an essay entitled <u>An Essay on the Nature and Significance of Economic Science</u>, Lord Robbins discusses the elements of economic reasoning.

The first step involves defining economics. "Economics is the science which studies human behavior as a relationship between ends and scarce means which have alternative uses." The nature of economic generalizations is discussed so as to identify the postulates underlying economic theory and challenge their realism.

Generalizations of economics are not only based upon assumptions regarding relative valuations but also on assumptions regarding human psychology. Unlike the natural sciences, transition from qualitative to quantitative aspects of social sciences is tenuous. Ultimately, economics deals with ethical and moral questions regarding optimal forms of society.

Actuarial Science and Economics

To the extent the actuarial paradigm does not reflect the dynamic economic environment which is impacting the problem which the paradigm is attempting to resolve, it may eventually face a crisis of the type discussed in Kuhn's book. Our concepts, models, and shared beliefs will ultimately be evaluated from the perspective of their success in solving problems which will have economic roots. Our ability to integrate the useful parts of economic theory into our body of knowledge will impact on our success in providing useful results.

A specific issue facing actuaries in the area of ratemaking which has a strong economic element is that of appropriate profit load. While economists have not resolved the issue of equilibrium or optimal profit levels in a dynamic industry in a market economy, there are some generally agreed upon notions. The appropriate profit level is that which attracts the amount of capital needed to produce the optimal level of output.

Profitability in the insurance industry has been a topic of significant interest for some time. The National Association of Insurance Commissioners (NAIC) conducted studies in the late 1960s, the results of which were published in their 1970 Proceedings. Six alternative measures of return were suggested for monitoring, but no concensus was reached on a particular approach.

In utility regulation where much work has been done in determining an appropriate profit standard, cost of capital has evolved as a key measure. Capital is separated into long-term debt, preferred stock and common stock, and the appropriate return on each component is determined. The degree of risk is a factor in making these determinations.

In the area of insurance, risk loads are becoming more of interest. Given the large uncertainties in the reserve setting process, for example, the issue of providing a margin beyond expected value to cover potential adverse developments is arising. The sense that standard reserve setting practices may have a downward bias in the type of dynamic environment insurers have recently faced gives additional impetus to this idea. Appendix B provides a glossary of economic terms which are of importance to actuarial science.

PHYSICS

Physics, like economics, provides examples of mathematical modeling which are of interest to the actuarial profession. Two key developments which have reshaped our world view in the twentieth century were Planck's Quantum and Einstein's Relativity theories. Quantum mechanics deals with probabilities for phenomena for which no conceptual framework can be designed. By

Heisenberg's uncertainty principle, we cannot know both the position and momentum of a particle. We cannot observe reality without changing it. Mathematics applied to real world phenomena has produced some startling results.

Subatomic particles can only be dealt with statistically. Activity of individual units cannot be predicted, but broad patterns can be. Just as Einstein uncovered a contant quantity at the macrolevel, Planck discovered a constant at the microlevel. Planck's constant relates the frequency of a light wave with its energy. Mathematical calculations have produced results which human reason could not have imagined.

Light was shown to be both wave-like and particle-like. This duality dealt a fatal blow to standard ways of considering causalty. Experiments showed that the paths of individual photons could not be determined. The notion of a probability wave arose which provided a bridge between the ancient metaphysical duality notions of the ideal and the real. Mathematics has provided the tools by which physicists have been able to unlock seemingly unknowable mysteries.

The physical world by this view is an interconnecting group of relationships rather than a structure made up of components. The notion of a wave function as developed by Schrodinger allows for the development of a probability function which allows for application of deterministic rules. Quantum theory as thus evolved provides a bridge to the real world of three spatial dimensions and time from a potential world of infinite dimension.

Concrete reality is sacrificed by this approach. Metaphysical issues related to uncovering ultimate reality remain. The basic duality of mind and matter becomes as much an epistemological as a metaphysical issue. Proof in the mathematical sense that a particular world view holds up becomes impossible. Mathematics has paradoxically led to a finding that mathematics is limited in power.

Heisenberg determined that since activity in the subatomic realm can never be known, observation should take precedence over mathematical modeling. He devised a means of organizing data which provided an alternative to wave equations in calculating transition probabilities for the effect of an experiment. Heisenberg's uncertainty principle arose from his discovery that nature presents itself in such a way that the accuracy of developments will always be limited. "What we observe is not nature itself, but nature exposed to our method of questioning."

Einstein's special theory of relativity merges space with time and energy with mass. Lorentz transformations were a key mathematical tool by allowing for relating observations from different frames of reference. Combined with the discovery of the constancy of the speed of light, this tool led Einstein to the development of the notion of a space-time continuum. Abstract mathematics led to a new world view.

Einstein's general theory of relativity attempted to generalize his prior results to apply to all frames of reference. The mathematical equations of the general theory describe the changing structure of gravitational fields. All reality could be described by space-time and motion.

Quantum Field Theory is an ad hoc theory which merges aspects of quantum mechanics and relativity. The complementarity of key aspects of the physical universe is reflected in quantum field theory which lacks mathematical and conceptual consistency. With fields hypothesized as providing the substance of the universe, matter becomes a transitory phenomenon.

Bells' theorem illustrated the dilemma posed by quantum theory. It showed that if the statistical predictions of quantum theory are correct, our normal way of thinking about the world is not. The principle of local causes is inconsistent with the statistical predictions of quantum theory.

Because actuaries, like physicists, attempt to build mathematical models to better understand and predict physical phenomena, they too must attempt to determine the extent to which their models fit reality. Actuaries study phenomena with significant subjective components for which mathematical modeling is difficult to apply. Because relativity is such a widely recognized and useful paradigm of physical science, a review of some of the mathematical elements of it illustrates how mathematics can support an area of applied mathematics like actuarial science.

The important point of this discussion as regards actuarial science relates to the complexity of the mathematics involved in analyzing real world phenomena. Whether the scope of the search for the core of our paradigm would include lead to such lengths is not known. Before we can determine the proper level of abstraction, however, it is important to consider the alternatives.

To conclude this digression on the use of mathematics in one of the more interesting areas of physical science, a review of the general theory of relativity and its mathematics is warranted. The special theory had a mathematical requirement that algebraic statements of fundamental laws be invariant under Lorentz transformations of coordinates. Einstein used tensor calculus to provide for invariance under general transformations.

General relativity is analytically formulated in terms of a G-tensor. Formulating the general theory involved finding field equations specifying the G-tensor in terms of the distributions of mass and energy in space-time. Every feature of the universe may eventually be reduced to a geometrical property of the space-time manifold.

A fundamental metric tensor is determined by the gravitational field the effect of which is expressed by taking the components of the Ricci tensor equal to zero. Given this mathematical structure, many physical phenomena can be modeled with great precision. The subject matters of vector and tensor analysis provide a rich structure of mathematics for this purpose. Their use in the area of modern physics provides an important illustration of the use of mathematics in an applied area.

PHILOSOPHY

Philosophy concerns itself with the ultimate questions of existence. While this may seem somewhat removed from the concerns of actuarial science, it must be noted that the subject matter underlying actuarial science and the impact of actuarial activity have implications which are ultimately philosophical in nature. Consider the welfare effects of classification of risks, for example, or the epistemological implications of the determinations as to how actuarial models are to be developed.

Metaphysics, the branch of philosophy which deals with reality can no longer be separated from physical science. As we saw in the survey of physics, reality is no longer considered a deterministic notion. Complementarity is the rule in physical as well as social science.

Epistemology is the branch of philosophy which deals with what we can know and how we can gain knowledge. As we have already seen, physical science has determined that the mere act of gathering knowledge impacts the content of the knowledge. As was the case with metaphysics, recent developments in physics have totally changed the way epistemological issues are posed.

Other branches of philosophy deal with large questions related to particular concerns. Ethics concerns itself with questions of morality and obligation. Government concerns itself with the ordering of human affairs so as to realize some agreed upon principles. Logic is concerned with principles of valid reasoning. The concern of aesthetics is beauty.

The work of actuaries has governmental, ethical, metaphysical, epistemological and even aesthetic elements. The nature of government impacts the nature and use of actuarial calculations. The economic implications of actuarial calculations have ethical aspects. The relation of actuarial models to reality is a metaphysical issue. Epistemological questions arise as actuaries determine what they can know and how they can know it. The aesthetic aspect of actuarial science has been commented upon by many nonactuaries.

The Socratic method of finding truth uses the technique of counter-example to develop questions which help clarify definitions. This method is widely used in law school to allow law students to refine their ability to argue a position according to a particular kind of logic and set of rules. Recent developments in physics would suggest that this method of pursuing truth continues to have great credibility.

The notion of a dualistic metaphysics was formed by Plato. He identified physical objects which are in a constant state of flux and ideas or essences which are permanent and unchangeable. His famous allegory of the cave provides a metaphor for the distinction between true and superficial knowledge.

Plato's theory of knowledge involves the construction of a hierarchy of knowledge. A basic distinction is made between knowledge and opinion. The four levels of knowledge are imagination, belief, understanding and reason. Understanding of the type reached by the mathematician is inferior to reason which uncovers true knowledge through the use of dialectic.

Aristotle's metaphysics was not dualistic, but considered matter and form inseparable. The universe has its own form. His ethics involved seeking for the natural role. The Aristotelian synthesis ruled philosophical thinking for over a thousand years.

Descartes provided a philosophical basis for modern science. He developed the modern duality between mind and matter. Newton followed with a physics which explained the universe.

New schools of thought arose to challenge the mechanistic view of the universe. Empiricists like John Locke felt knowledge arose from sense perception, and that metaphysical systems based on reason were of questionable value. David Hume carried this thought to its logical conclusion that only sense perceptions and feelings exist.

Kant developed a theory of knowledge partly in response to the radical conclusions of Hume. For Kant, reality conformed to the concepts of the human mind. The laws of nature became dependent upon categories in the mind.

Hegel provided a synthesizing philosophy, bringing in elements of romanticism, rationalism and Kantianism. Hegelian dialectic reaches higher levels of truth by synthesizing opposite notions. Ultimate reality becomes manifest in the nation state the values of which all its components would accept.

Existentialism is a modern philosophical school. Existentialism stresses the subjectivity of the individual and the need for the individual to accept responsibility for his/her own actions. As opposed to a systematic structure, existentialism tends to present its themes most effectively in literature of authors like Camus and Sartre.

Actuaries bring a world view to their work and that world view affects their approaches and expectations. Currently, standards of practice are being discussed which will provide some additional structure to the approach actuaries will take toward their work. The search for a structure from within with which to evaluate the work of an actuary relative to a paradigmatic structure will be an ongoing one.

MATHEMATICS

While actuaries must be knowledgeable about such diverse areas as economics, law, accounting, government, business, and finance, their unique identity is found within the context of the subject of mathematics. It is through mathematical tools that actuaries filter the various factors influencing the variables they are attempting to estimate, and it is these tools that provide the solutions the actuary presents. Our body of knowledge and the center of gravity of our discipline is ultimately an aspect of mathematical science.

As an applied field impacted by a wide variety of subjective factors, actuarial science, like physics, must ultimately define its scope. Physical science abstracts away many complicating elements of the real world in order to apply structured models to its problems. Actuarial science must do likewise, at least for aspects of its work which are determined to be covered by the paradigm.

To identify the mathematical core of the actuarial paradigm, then, we must evaluate the subject matter of mathematics and extract the elements most relevant to the actuarial model. From these elements, we will be in a position to construct a theoretical foundation for actuarial science. That theoretical foundation may or may not have application in a large percentage of the practical situations faced by an actuary in the course of normal business.

Foundations

In his book, The Philosophy of Mathematics, Korner discusses three modern schools of the philosophy of mathematics. They are the logicist school, the formalist school, and the intuitionist school.

The logicist school, associated with thinkers like Frege and Russell, attempts to confine pure mathematics to a manageable number of concepts and principles. Cantor's logic of classes, however, led to contradictions requiring that any logicist system have one postulate which is pragmatically grounded. Mathematical theories are existential in a sense in which logic is not.

A leading proponent of the formalist school is Hilbert. Hilbert attempts to reconcile finite mathematics with transfinite. The ideal is a consistent, complete, mechanical formalism of all mathematics.

The intuitionist school rejects the notion that mathematics requires the support of an extended logic or rigorous formalization. While the subject matter of the metamathematics of the formalists was perceptual objects, intuitionists deal with nonperceptual conceptions. Brouwer formulated the fundamental theses of the intuitionist philosophy.

Intuitionists consider mathematics to be independent of language and logic. A sequence of abstract entities, the natural numbers, is the starting point. Godel's proof of the inadequacy of the original metamathematics does not affect the intuitionist approach.

Mathematics, then, which provides a foundation for actuarial science, does not itself have a universal set of accepted results. When Godel proved that the consistency of mathematics could not be proven from existing principles, the hope of finding infallible objective laws and standards evaporated. This has not rendered mathematics impotent, however.

The work of Copernicus and Kepler, for example, while no longer the last word on astronomy, still influences our thinking about mathematical laws which underlie physical phenomena. Newton's laws of motion still explain mathematically most laws of nature not withstanding the fact that relativity and quantum theory have replaced the implied mechanistic determination of Newtonian physics. Actuarial science, then, while not responding adequately to all questions for which answers are desired, does provide a basis for structuring a response.

Mathematical reality cannot be unambiguously incorporated into axiomatic systems. Nonetheless, developments in such diverse areas of pure mathematics as non-Euclidean geometry, topology, group theory, tensor analysis, and matrix theory continue to yield rich rewards in many practical applications. Actuarial science can draw from the subject matter of mathematics in developing its paradigmatic structure.

Mathematics provides the basic tools of actuarial science. Because actuarial science covers a broad range of problems, it uses tools from a broad range of mathematical subjects. This section will survey broad areas of mathematical science which include elements of importance to actuaries in developing a core body of knowledge.

Calculus--Calculus deals with the real number system and the theory of limits. The limiting process is explored using differentiation and integration. Its study is preparatory to a more rigorous study of the real number system and the concept of a limit.

The derivative measures the instantaneous rate of change of a function. Marginal concepts which arise regularly in the social sciences can be expressed as the derivative of a respective total function. Marginal costs and revenues, for example, which are important concepts to actuaries, may be found by taking the derivative of associated total cost and revenue functions.

Integration involves finding the function for which the rate of change is known. An actuary could be faced with the need to determine the capital base into which a known investment is flowing. Capital level would be the integral with respect to time of net investment. Integration may be used to derive a total function from a marginal one, as well as to find probabilities from a given density function.

Special functions studied within the calculus include power, exponential, and logarithmic functions. These functions are widely used by actuaries. Exponential functions provide a basic tool for interest theory. In economics, production functions which measure output relative to input combinations often assume a power curve.

Multivariable functions create another layer of problems for applied mathematicians. The profit of an insurance company, for example, might be considered a function of the costs and revenues associated with several products. Partial derivatives of the proper degree would be used to determine optimum output levels of each product. Maximization or minimization under constraint can often be accomplished by the use of a Lagrangian function.

Differential equations may be used to find functions where the underlying growth rate is known. A formula for the total value of a sum of money compounding continuously for a defined period at a defined interest rate can be derived using differential equations. Differential equations are utilized in a large number of applications in the physical sciences.

Linear Algebra--Linear algebra involves the solution of simultaneous linear equations. This may be done in a variety of ways, often associated with the notions of elimination or determinant. In a more abstract sense, the idea is to find the kernel of a linear mapping and to characterize the subspace spanned by a set of vectors.

Matrices have arisen as a tool for expressing a complicated system of equations succinctly. A matrix is a rectangular array of numbers, and matrix algebra provides a shorthand method of solving equation systems. Determinants are values derived from a matrix using prescribed rules.

A matrix which provides perspective on the power of the tools of matrix algebra is the Leontief matrix which is used in economic input-output analysis. The level of total output for an economy, including intermediate as well as final output, can be solved by matrix algebra. A vector representing final demand for each product and another representing input values in the production of products provide the basis for the solution of the level of output of each product.

More pertinent to the immediate work of many actuaries is the value of linear algebra in evaluating variables more microeconomic in nature. Linear algebra provides a basis for the solution of a system of equations involving several variables. Linear statistical models are widely used for regression, analysis of variance, and design of experiments.

Certain determinants and matrices have special value to applied mathematicians. A Jacobian determinant allows testing for linear and nonlinear dependence of a system of equations. A Hessian is a determinant composed of all second order partial derivatives of a multivariable system and provides a second order test for optimization. Bordered Hessian determinants are valuable in solving problems involving constrained optimization.

Linear programming has as its objective the determination of an optimal allocation subject to constraints. One method often used to solve such problems is the simplex algorithm. The algorithm moves from one basic feasible solution to another until an optimal solution is found.

Profit maximization is a category of problem to which this approach may be applied. Profit is a function of output levels of products, the production of which is subject to a number of constraints. Constraint equations are established in matrix form, and a simplex tableau is set up by which to determine optimal allocations.

As was the case with calculus, we see that linear algebra has many tools which would be valuable to people solving problems of the type actuaries face. Because calculus and linear algebra are prominent mathematical topics in the actuarial syllabus, they have been given significant attention. Because actuarial science draws on many other areas of mathematics, however, those must also be reviewed to uncover any threads which may be important in the determination of an actuarial paradigm.

Vector and Tensor Analysis--A vector is a quantity having both magnitude and direction. Vector algebra includes the operations of addition, subtraction, and multiplication. The laws of association, distribution, and commutation apply. Vector functions are differentiable, integrable, and have many physical applications.

Vector analysis provides a basis for tensor analysis. Tensor analysis, which is of great consequence in such areas as general relativity theory and differential geometry, is a result of the study of consequences of the fact that physical laws are independent of any particular coordinate system.

A tensor is a system of numbers or functions whose components obey a certain law of transformation when the variables undergo a linear transformation. Tensors may be of any order. An aggregate of tensors at every point of a region of space is a tensor field.

The covariant derivative of a vector field is a mixed tensor. The divergence, curl, gradient, and LaPlacian notions all involve the covariant derivative. Intrinsic and tensor derivatives of a tensor reduce at a given point to ordinarily derivatives of the tensor.

Tensor calculus is a key aspect of the analysis of particle dynamics. It is also applicable to the theory of electricity and magnetism. Green's Theorem and Stoke's Theorem use tensor calculus to provide for transformations of volume and surface integrals to line integrals. This leads to special relativity.

The representation of the motion of a particle under the action of a force system in space-time is a curve. The equations of motion of the particle may be investigated using tensor calculus. The relativistic form of the equations of motion of a continuous medium may be expressed in terms of world tensors. Electrodynamical equations are also expressed in terms of world tensors.

The metric for special relativity affects the formulas for the LaPlacian and divergence. Maxwell's equations in vacuum may be extended to space-time via dual tensors under Lorentz Transformations. A tensor field may be defined on a manifold.

Algebraic Systems--Algebraic systems are sets of objects together with relationships and operations defined for the sets. Two systems are considered isomorphic if there exists one to one correspondence between the sets and if relations and operations defined on the sets are preserved in the correspondence. Well-known systems include the real numbers, natural numbers, integers, complex numbers, and rational numbers.

More abstract systems include groups, rings, integral domains, and fields, each of which have distinguishing characteristics. Notions introduced previously, including vector spaces, matrices, linear algebra, and polynomials are all algebraic systems. Boolean algebras involve binary operations and are widely used in computer science.

Topology--As was the case with tensor analysis, topology covers the analysis of concepts which are of importance to applied mathematicians like actuaries. The notions of countability, equivalence, boundaries, and completeness all arise within the area of topology. Technically, a topology is a class of subsets satisfying certain axioms.

Topology involves the investigation of properties like compactness and connectedness which have fairly concrete application. Metric spaces are topological spaces and include Hilbert spaces, which exhibit phenomena not occurring in Euclidean space. Threads from the subject matter of topology are found in many of the theoretical structures which have arisen in the area of actuarial science.

Numerical Analysis--Numerical analysis is another subject covered by the actuarial syllabus and intimately related to the work of actuaries. Numerical analysis involves developing and evaluating methods for arriving at numerical results. As such, it cuts across many other areas of mathematics.

Use of a polynominal to approximate a function is a common approach used in numerical analysis. A particular polynominal, called a collocation polynominal, can be found which coincides with the function being estimated at various points. As is the case with all approximations, minimization of error is a key criteria for the selection.

Other polynominals are also of value in solving numerical problems. Osculating polynominals match derivatives, as well as primary values with a given function at specified points. Least-squares polynominals minimize the squares of errors utilizing the idea of orthogonal projection in a vector space. Mini-max approximation polynominals minimize the maximum error (equal-error property).

Polynominal approximation is also a basis for a variety of integration solutions. Series approximation is another aspect of numerical analysis, as is the solution of differential equations. Finding roots of systems of equations utilizes numerical analysis techniques.

Solving systems of linear equations, as discussed previously, is a key area of applied mathematics. Reducing such a system using matrices and vectors with an algorithm from numerical analysis is a common activity of applied mathematicians. Gaussian elimination is a key algorithm used for this purpose.

Special problems often arise in applying mathematical principles to physical problems and numerical analysis techniques tend to provide useful approaches. A system of linear equations may be overdetermined in the sense that the coefficient matrix may have more rows than columns. Boundary conditions may make solutions of differential equations problemmatical. Numerical analysis techniques provide a way of handling such problems.

Monte Carlo methods solve certain types of problems by the use of random numbers. They may be used to provide early approximations for more cumbersome algorithms. Simulation and sampling are two such methods which provide for approximations to the description real phenomena.

Problems involving approximation often involve the selection of an element from a set which is close to an element not in the set. A decision as to how the distance between the elements is to be measured is needed. A metric space provides an abstract concept for such a problem.

Concepts from topology may be defined in metric space. Compactedness, closedness, denseness and boundedness are examples. Normed linear spaces are particular kinds of metric spaces, and Banach and Hilbert spaces are particular normed linear spaces.

The inner product of two vectors in linear space obeys established postulates. Vectors in an inner product space may be orthogonal, and if an orthogonal set of vectors is of unit norm, the set is orthonormal. Orthonormal sets of vectors allow for explicit solutions to problems involving the location of a point on a subspace of minimum distance from an external point, a problem faced by actuaries in regression analysis.

In the Cartesian Plane, a convex set is one where any of its point pairs can be joined by a line which is contained in the set. Convexity is a property which, when it exists, provides for better modeling. Convex programming has arisen as a way of locating the minimum points of a convex function defined on a convex set.

The Weierstrass theorem provides that there exists a sequence of polynomials which will converge to a continuous function uniformly on a closed bounded interval. A set of vectors in a space satisfies the Haar condition if every set of them is linearly independent. A Markoff system is a system of continuous functions on an interval which satisfies the Haar condition in its initial segments.

Least squares approximation involves quadratic norms, and uses orthogonal polynomials, Fourier series, and harmonic analysis. Orthogonal polynomials are utilized in the process of numerical integration. A transformation of the Fourier Series provides for uniform approximation of continuous functions.

Tchebycheff approximation may be appropriate for approximation problems where parameters to be determined have not occurred linearly. Existence, unicity, characterization and computation of the approximations are issues of primary concern. The minimax algorithm may apply in rational approximation.

Integral Theory--Because the functions actuaries work with are not always well behaved, standard integrals do not always produce optimal results. In addition to techniques from numerical analysis, the actuary may utilize fairly recent developments in integration theory. These developments have proven especially valuable in the continuous advancement of measure theory.

Classical integration theory has proven inadequate to many problems which have arisen in physics and other applied sciences. The standard integral does not provide for integration over sets with a large number of parameters, applies only to a fairly small number of well defined functions, and requires that the domain of integration be homogeneous. The modern theory of integration, which traces back to Lebesgue, has provided for a much wider range of application.

Measure Theory

Because actuaries use probability theory (to be discussed later) in solving problems in their domain, and because probability theory is based on a rich and extensive mathematical structure, that structure must be an aspect of the actuarial paradigm. Probability has the formal properties characterizing a class of functions of sets known as measures. Measure theory which has a rich mathematical foundation, then, is a component of actuarial science.

Miscellaneous Topics

LaPlace transforms are useful in the solution of linear differential equations. Situations arise where a function can be expressed as the product of two functions each of which is a transform of a known function. One of the properties of LaPlace transforms is that the product is an integral called the convolution of the functions.

Orthogonality is a notion that often arises in actuarial science. Two functions are said to be orthogonal over a region if the integral of their product vanishes over the region. Sets of characteristic functions corresponding to boundary value problems are orthogonal with respect to a weighting function. A set of normalized orthogonal functions is said to be orthonormal.

In addition to ordinary integrals defined along a line segment, there is a corresponding integral defined along a curve which is given the name line integral. Similarly, corresponding to an ordinary double integral defined on a region in a plane, there are surface integrals defined on a surface. Line and surface integrals have found many applications in theoretical work done in the physical and social sciences.

In Riemann integration, the functions and regions of integration are assumed to be bounded. Stieltjes integrals involve two functions defined on a closed interval. Stieltjes integrals play an important part in the formulation of such key notions in probability and statistics as first moment, variance, and mathematical expectation.

Point-set theory provides additional grounding for the study of continuous functions, integration, and series. The Bolzano-Weierstross Theorem states that there is at least one point of accumulation in a bounded infinite set. Cauchy's Convergence Theorem provides conditions for convergence of a sequence. The Heine-Borel Theorem states that a finite number of open sets may be chosen from a collection which covers a bounded and closed point set in such a way that the set is covered by the new collections.

The gamma function is an improper integral which has some useful properties. It provides a convenient way of interpolating between factorial values. There is a connection between the gamma and beta functions.

Fourier Series play a key role in the solution of many problems in the physical sciences. Lebesgue integrals are often used in dealing with Fourier Series where discontinuous and unbounded functions are involved. Bessel's inequality arises as a result of the orthogonality relations among the terms in a Fourier Series.

Because functions in mathematics are exact, but those in the physical and social sciences are not, the determination of a function by applied scientists will be different from that for a mathematician. Methods known as relaxation methods have arisen in the applied sciences which allow for computational facility. These methods allow for evaluation of approximate values of the required function for a finite set of values.

Modern developments in quantum physics indicate that statistical laws are the fundamental laws of nature. Probability calculus of the type utilized by actuaries is based on empirical assumptions about the stability of relative frequencies of events. The pure and applied aspects of mathematics are inextricably bound, and actuaries are in a position to utilize developments in mathematical science to push the frontiers of their profession.

Mathematics builds on itself. Algebra builds upon arithmetic. Geometry builds on both. Calculus builds on all three. Topology builds on geometry, set theory, and algebra. Noncantorian set theory takes the axioms of restricted set theory and adds a form of the negation of the axiom of choice such as the negation of the continuum hypothesis. Nonstandard analysis takes the standard universe of mathematics and adds nonstandard models.

To introduce the notion of complex numbers, it is instructive to consider vectors. Rotations in space lead to a new algebra of quaternions. Quantities which require more than three numbers for their identification may be handled by tensor algebra.

Rates of change of derived magnitudes with respect to length are called gradients. Because there is a tendency in nature toward uniformity, gradients enter into may physical laws. The divergence of a vector also has many physical applications and is the limit of a surface integral. The curl is a vector often used in physical science and, like the divergence, is usually expressed in terms of derivatives of functions.

Many of the fundamental equations of physics are formulated in terms of partial differential equations. These include the field equations which include a group of terms called the LaPlacian of the function. The LaPlacian measures the difference between local and average values of the function in an infintessimal neighborhood.

Summary--This section provides a sketch of certain areas of the subject matter of mathematics which may be useful to the actuary in the course of practical or theoretical work. The actuary must borrow extensively from the subject matter of mathematics and, in moving toward the definition of paradigms, must identify elements which may be part of the paradigm structure.

While there are many mathematical tools which cannot be covered in a concise but broad survey of the type which is attempted here, this survey may at least provide some perspective on the breadth of mathematical tools available to the actuary. Appendix C provides a glossary of mathematical terms which are related to actuarial considerations.

PROBABILITY AND STATISTICS

Actuarial science is best described as an area of applied mathematics. Like physics, which attempts to analyze and explain physical phenomena using mathematical models, actuarial science attempts to solve problems arising in the real world using mathematical techniques. Because the areas of inquiry of actuarial science (particularly in the liability insurance area) involve phenomena for which the modeling involves a wide range of possible variables, probabilistic as opposed to deterministic models tend to be favored.

Probability theory originally developed in response to some very practical problems involving gambling and commerce. The basic idea was that if there are 'n' possible ways in which an event 'E' can happen, and 'm' of those ways are favorable, the probability of a favorable outcome is 'm/n,' if all the ways are equally probable of happening. In a more general sense, probability is the limit of the frequency ratio as the number of trials of a repetitive event increases without limit.

As in the case with mathematics, probability theory has not led to truth or certainty for the class of problems in its area. No system of axioms adequately reflects behavior of real world phenomena. Whatever randomness exists in the phenomena being analyzed by applied mathematicians using available tools is too varied to be handled axiomatically.

Overview

A lack of rigorous foundation for pure mathematics and probability theory provides the backdrop for a discussion of another key tool of actuaries, statistical science. Statistics involves collecting, organizing, summarizing and analyzing data and making decisions based on that analysis.

Information about a population may be inferred by analyzing samples drawn therefrom. Statistical inference involves the estimation of population parameters from sample statistics. R. A. Fisher suggested three criteria for such estimation as consistent, efficient, and sufficient.

When a relationship is thought to exist between two or more variables, a mathematical form describing that relationship can often be found. When a data set is analyzed by finding equations or curves which fit the data, the term curve fitting is used. The method of least squares involves finding the curve which minimizes the sum of the squares of the distances of the data points from the curve.

Curve fitting is a technique used extensively by actuaries. To model the phenomena they analyze, it is often useful and necessary to define an expected relationship between components of the phenomena. Once that relationship is defined, various assertions can be made about the expected behavior of the phenomena and the implications of that to the problem being addressed by the actuary.

The Bayesian approach to inference regards probability as a degree of reasonable belief rather than a frequency. Bayes' Theorem provides for posteriori probabilities based on prior probabilities and the impact of new data on the likelihood function. The need for the use of unknown prior probability creates an additional problem for this branch of probability theory.

One attempt to deal with this problem was provided by Wald. The notion was to modify the problem as Lebesgue had modified the ordinary integral to solve a class of problems which had previously been open. Wald came up with the Theory of Statistical Decision Functions, which unified a number of prior theories.

Classical probability theory, then, which provides a key foundation for actuarial science, has some fairly serious limitations. A finite set of possible outcomes is required. The outcomes must be equally likely.

A more general model which incorporates features of real world problems of the type faced by actuaries which is abstract at the core but which provides for adaption to concrete problems will be even less rigorous.

In addition to uncertainty related to the mathematical models themselves, actuaries face uncertainties related to the data to which the models apply. Oskar Morgenstern provided a study of the properties of economic data which discussed problems related to the "accuracy of economic observations." A key point made was that the acceptability of a measurement is inextricably tied to the use to which it is put.

Among the sources of inaccuracies involved in the use of economic statistics, Morgenstern listed the lack of designed experiments, observation errors, deliberate falsification, lack of definition, and interdependence. He suggests, at a minimum, that economic statistics be published only in conjunction with an estimate of their error.

The Subject Matter

The subject matter of statistics involves the collection, analysis, presentation, and utilization of numerical information. The purpose is to make inferences and reach decisions in the face of uncertainty.

A common way to organize data in performing statistical analysis is to establish a frequency distribution. Such a distribution breaks data into groups and totals the number of observations in each group. A cumulative frequency distribution shows for each class the total number of observations in all classes up to and including the class itself.

The mean or average value of the elements in the population is the most common measure of central tendency. The standard deviation is the most widely known measure of dispersion.

A random variable is one whose values are associated with some probability of being observed. The set of all possible values of a random variable is called a probability distribution. A number of probability distributions figure prominently in actuarial science.

The normal distribution is a probability distribution commonly used in statistical science. Given the mean and standard deviation, the normal distribution is completely determined. The normal distribution is a continuous probability distribution.

The Poisson distribution is a discrete probability distribution. It is useful in determining the probability of a certain number of events in a given time frame. The assumption is that the events are independent and that the mean per unit of time is constant. Actuaries have found these assumptions tolerable.

Probability Theory

Modern Probability Theory, which builds upon a mathematical foundation, considers outcomes as points in a sample space. An event is a set of points in the space. A random variable is a function defined at each point of the sample space.

Because the models used by actuaries to approximate the phenomena they are evaluating have characteristics which are mathematically complex, an understanding of them requires solid grounding in the core elements of probability theory. One such element is the Moment Generating Function (MGF) of the random variable in question. Since the MGF is a complete characteristic, conclusions reached in terms of MGFs can be translated into corresponding properties of the probability distribution.

Since many of the distributions utilized by actuaries are related, the shorthand provided by the use of MGFs is enhanced. The geometric distribution is a special case of the negative binomial. The exponential distribution is a special case of the gamma. The gamma distribution is the continuous analogue of the negative binomial.

By developing a theoretical distribution of losses expected to occur for a model portfolio of policies, actuaries have a starting point for the various types of loss analysis they perform, Even if they are working with an aggregated database and attempting to estimate losses from statistics not conformable to the assumptions in the theoretical model, the results of the analysis can still be judged in the context of how well they fit what the corresponding idealized model would have produced.

Two models used in developing a loss distribution for a group of policies relate to the probability of the occurrence of the insured event and the amount of loss which arises once a loss occurs. Models also exist by which to determine the aggregate loss distribution directly. Like all models, those which attempt to reflect insurance contingencies of the type faced by actuaries must find a balance between computational facility and accuracy of representation of the phenomena being modeled.

One probability distribution which is often selected to model the occurrence of insured events is the Poisson. The Poisson Model can be used to calculate the probability of the occurrence of specified number of losses from groups of exposure units in a given time frame. When the probability of occurrence is not constant over time, or not uniform among members of the group, a compound Poisson or negative binomial distribution is often used.

Given the distribution representing loss occurrence, a distribution of amount of loss is needed to derive an aggregate loss distribution for a group of exposures. Size of loss distributions are obviously very sensitive to the nature of underlying insurance coverage. Among the distributions often discussed as providing reasonable approximations to size of loss distributions for common insurance coverages are the Lognormal, Pareto, and gamma.

Once distributions of loss occurrence and amount of loss are determined, aggregate claim distributions can be developed. The notion of convolution sometimes arises in this regard. Convolutions allow for the development of cumulative distribution functions given occurrence and amount distributions.

The notion of convolution arises in the mathematical theory of generalized functions as well. This subject is based on the theory of topological linear spaces. Convolution is the operation that maps an ordered pair of elements into the product element. Recent research in the area of integral transformations has involved investigation of the conventional convolution transformation. Extension to the N-dimensional case has been explored. .

Because compound distributions adequate to the task of estimating aggregate claims in the collective model become so complex as to make calculation impractical for many applications, approximation methods have been developed. Among these are the normal approximation graduation by orthogonal polynomials, the Esscher approximation, the Cornish-Fisher Expansion, and the Edgeworth Series.

In addition to direct determination of distribution functions, indirect methods are available. Simulation is an approach widely used when the problem to be solved has an analytical structure which is too complex to be modeled effectively. The simulation model describes the process in terms of individual events.

The first step in performing a simulation is to develop a model which adequately represents the phenomena being analyzed. Elements of the model which are unknown may be simulated using random numbers to obtain random observations from a selected probability distribution. Conventional methods are available by which random numbers may be generated by computers.

Monte Carlo techniques are often utilized to increase the efficiency of the simulation and increasing the precision of sample estimators. Stratified sampling is a particularly effective technique for a number of actuarial applications. The statistical theory underlying simulations is the same as that which applies to physical experimentation.

The essence of a Monte Carlo study is the specification of sets of parameter values for postulated distributions. Drawings from the distributions provide samples to which estimation techniques may be applied. The sampling distributions of the estimates are then analyzed in relation to expectations.

The key actuarial tools for developing loss estimates and variability estimates for exposure related to a group of insurance policies are fairly well established. The problems faced by actuaries in their everyday work, however, do not always fit the model specifications very well. As a result, techniques used in practice often have at best a tenuous relationship to the mathematical underpinnings.

Consider a typical actuarial problem, the need to establish reserves for a line of insurance for a company which has written the line in question for a period of 10 years. The standard actuarial approach is to consider payment and reserve patterns over the relevant history to determine the ultimate value of losses expected to arise for each segment. Building a model which considers development and trend factors as well as the basic frequency and severity components at each point in time would be a significant mathematical challenge.

Once a loss distribution for an insurance portfolio is established, the theory of risk and ruin theory come into play. An insurance company is "ruined" if the aggregate amount of claims in a period equals or exceeds the funds available to provide for such claims. Current models are limited in the sense that not all the risk elements facing an insurer are incorporated, but they provide perspective on the very important solvency issue.

In addition to the complexity associated with the establishment of mathematical models to complex physical phenomena, actuaries have developed the notion of credibility to apply to the particular situations they encounter. Because information available for a category of risk being evaluated by an actuary may be sparse, appeal to the law of large numbers is not always sufficient. Credibility procedures allow an actuary to incorporate consideration of the impact of the relative volume of data available on the results of the calculations performed.

Credibility standards can be established based on an assumed distribution of claim counts. Adding the variability in the size of claim distribution allows for the calculation of more refined standards. Given the credibility standard, Bayesian or parameter-free methods of utilizing partial credibilities may be considered.

Statistics

Given the uncertainty involved in the mathematical modeling of physical phenomena, various statistical methods have arisen by which to make inferences based upon available information. Sampling theory studies relationships between populations and samples drawn from the population. Given the known characteristics of the population, information about random samples drawn from the population is determined. Statistical inference uses principles of sampling theory to infer information about populations from samples drawn from the population.

One of the more common uses of statistical estimation theory is the development of confidence intervals. By making assumptions about the underlying distribution, confidence levels for population parameters can be estimated based on samples drawn therefrom. Hypothesis testing involves evaluation of a particular hypothesis in the light of sample information and statistical decision rules.

Where samples are small and it cannot be assumed sampling distributions are approximately normal, distributions often utilized are the "students" and chi-square distributions. To complete statistics for such distributions, sample observations and population parameters must be obtained. The number of degrees of freedom of such a statistic is the number of independent observations in the sample less the number of population parameters which must be estimated.

Tests have been developed based on the chi-square distribution which have much significance for actuaries. Chi-square tests for hypotheses regarding goodness of fit and independence of variables have been widely utilized. Another technique widely used by actuaries involves analysis of variance.

Analysis of variance is used to test mean differences. The one way analysis of variance model may be represented by a linear equation, as may the two-way model. Testing hypotheses regarding populations from which sample means emerge is a common aspect of many actuarial problems.

Regression analysis is another area of statistics widely used by actuaries. Regression analysis attempts to predict the value of one variable from the values of associated variables. Among the assumptions of the linear regression model are the randomness of the dependent variable, the linear relationship of the dependent and independent variable, and homoscedasticity.

The least squares regression line is that for which the sum of the squared deviations between the estimated and actual values of the dependent variable is minimized. The means of the probability distributions of the dependent variable at each value of the independent variable have a systematic (linear) relationship to each other value. Inferences can be made regarding the parameters of the regression line.

Correlation analysis measures the degree of relationship between the variables. The correlation coefficient is a common value used in evaluating the relationship because it is included in a test statistic which approximates the distribution and is therefore amenable to statistical testing. In particular, the significance of the coefficient can be tested.

The notion of a probability weighted average is a critical one in probability theory. Integrals which are generalized sums provide a basis for expression of these sums. The abstract integral is a key tool in the evaluation of mathematical expectation.

The mean value is the coordinate of the center of mass of the probability distribution. The standard deviation is the radius of gyration of the mass distribution, and the variance is the second moment or moment of inertia about the center. Moment generating and characteristic functions involve LaPlace and Fourier transforms from Classical Integration Theory.

Random processes are infinite families of random variables. The covariance and convolution functions which play roles in such areas of importance to actuarial science as extrapolation and smoothing of time series are important to the analytical treatment of many random processes. Random (stochastic) processes generalize the notion of random variables allowing for the extension of probability models to dynamic systems.

Empirical work with random processes involves finding a function which is typical over an interval of sufficient length. The mathematical problem is to express conditions such that one can use physical information to determine whether a useful probability model can be developed.

In addition to standard regression analysis techniques, a number of variations have arisen to deal with specific problems. Qualitative explanatory variables can be introduced by the use of dummy variables. Where the dependent variable is influenced by past as well as current values of the independent variable, the distributed lag model is available.

Problems which often arise when economic data is being analyzed include multicollinearity, heteroscedasticity, auto correlation, and measurement errors. Also, the dependent variable in one system may be an explanatory variable in another in which case the system is a simultaneous equation system.

The notion of Bayesian analysis plays a role with regard to much actuarial work. Decision making based on probabilities, economic consequences, and expected utility involves the use of Bayesian techniques. Criteria for the use of and value of sample information also fall in this area. Bayesian procedures permit decision makers to enter opinions in a formal way.

By pointing out the problem in arguing from the particular to the general in statistical inference, Bayes created a challenge for future statisticians. Fisher developed a postulate requiring the selection from among all possible values of a population parameter that value which maximizes the likelihood that the sample obtained would arise. Wald, concerned about continued assumptions about prior probabilities in Fisher's approach, developed an alternative standard involving minimizing the maximum risk of loss.

Given the complexity and uncertainty involved in the mathematical theory of probability and statistics, actuaries could find it difficult to pick models applicable to problems at hand. This has not generally been the case for a variety of reasons. One is that actuaries are not always aware of or concerned about the mathematical rigor of the model being used. Another is that many of the problems faced by actuaries are best handled by approaches which are fairly simple in mathematical structure.

Operations Research and Forecasting

Because actuaries are often involved in business decisions, in addition to being concerned with reducing numerical chaos, they are concerned about reducing the chaotic aspects of management decision making. An area of applied mathematics which has grown dramatically in recent years to deal with such problems is known as operations research. The various techniques falling in this area are involved in a quest for organizing activities so as to provide for desired results.

Like the other areas of applied mathematics surveyed in this paper, aspects of operations research arise in the contexts of other areas of mathematics and have been commented upon elsewhere herein. Linear programming has been introduced as a tool by which resource allocation may be judged. It provides an algorithm by which to optimize a function subject to constraints.

Network analysis allows for a graphic representation of the activities involved in completing a project. Game theory provides a basis for decision making under uncertainty. Queueing theory allows for systematic analysis of service time. Markov chains can be developed with discount factors allowing for optimization of a functional equation with unbounded horizon.

Forecasting is a particular area of statistical science which is a significant aspect of the actuary's work. A stable mathematical or statistical data structure allows for specification of a model from which forecasts may be made. Models used in forecasting have analogues in other areas of statistics, and a brief review will provide valuable perspective.

Perhaps the simplest model is the constant mean model which assumes values of the dependent variable vary from a mean value only by a random error element. The linear trend model assumes the dependent variable is a function of time. Regression models consider variables in addition to time.

In stochastic models the random element plays the dominant role in determining the model structure. Seasonal models incorporate the idea of periodic variation effects. Probabilistic models estimate future probabilities of events.

Game Theory is an area in which significant theoretical progress is currently being made. A game is characterized by a set of rules having a certain formal structure governing the behavior of certain components. Game Theory allows the reduction of any game to a simple or normal form. The aim of each player in a game is to maximize expected utility.

Perfect-information games are such that the players know the choices and outcomes of all prior moves and they have a pure value. The matrix of a finite game of perfect information has a saddle point. Each player has a good pure strategy. Every finite game has a value and each player has at least one good mixed strategy.

Any finite game can be solved by solving a finite number of linear equation systems. Each statistical game postulates a sample space which describes all outcomes of an experiment. A decision function provides a rule which associates with each outcome a point in the strategy space.

Statistical games utilize the notions of loss and risk functions. A priori and a posteriori probability distributions are also involved in the development of game strategies. Preference patterns are described by utility functions. Choice principles include Minimax, Bayes, and Maximin.

Game Theory provides an example of a subject designed to produce very pragmatic results using abstract mathematical concepts. The basic theorem of Game Theory is an assertion about convex sets in Cartesian n-Space. Convex sets and functions involve topological notions of boundedness, closedness, denseness, continuity, and uniformity.

Statistical science is clearly rich with tools for use in handling actuarial problems. As is the case with more abstract mathematical subjects, the tools afforded by statistical science are subject to challenge both as to internal consistency and external application. The continuing challenge of actuarial science is to explore ways to optimize the use of these tools in areas where they can be profitably applied.

MATHEMATICS AND ACTUARIAL SCIENCE

Given this uncertainty at the abstract core of actuarial science, what can we say about the results of actuarial calculations which blend the uncertainty of mathematical models with the subjectivity of the social and economic forces impacting the topic of evaluation? The discipline of statistics, as discussed above, helps bridge the gap from the mathematical models to the real world.

The statistical view of nature is that while deterministic laws may be developed which summarize phenomena which occur in the real world, these laws merely provide probabilistic, as opposed to deterministic, information. Real world phenomena do not consistently follow deterministic patterns and therefore these phenomena can only be evaluated statistically.

In actuarial science, this view manifests itself in a variety of ways. To make calculations manageable, actuaries often assume that the phenomena they are evaluating follow specified patterns. As an example, it is often assumed that the claims arising in a particular time interval for a particular insurance product will follow a Poisson distribution.

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Computer models are now being developed, the purpose of which is to gain insight into the complexities of the real world by simulating natural phenomena. Because the mathematical equations used by scientists to describe simple phenomena become overwhelmed when attempting to describe the countless interactions involved in real world phenomena, the new simulations attempt to minimize the mathematical structure imposed initially so as to allow the phenomena themselves to determine the evolution of the structure. In this way, a better fit is expected to arise.

The core of the actuarial paradigm, then, must be found in the subject matter of mathematics. Given the core, various aspects of the related field of statistics would be brought in to expand the range of problems which can be solved. Finally, adaptation to the various subjective areas such as law, economics, business, operations, and the physical world itself, is necessary. Appendix D provides a glossary of econometric terms which would be of interest to actuaries.

UTILITY THEORY

Utility theory provides insights into problems involving decision making under uncertainty. Actuaries are often involved in calculating expected values related to various types of transactions. Decisions are not always based on the expected outcome of an event. The value associated with various outcomes may be related to, but not equal to, the expected value.

A utility function relates level of wealth to utility given defined levels of uncertainty in the decision problem. For a utility function to exist, certain conditions must be met. Different forms of utility functions reflect different attitudes as to risk aversion. The greater the risk aversion, the more the decision maker will pay to eliminate the risk.

Utility theory requires that decision makers can express preferences for outcomes which affect wealth levels. Given a choice between alternative outcomes, the decision makers' choices determine the utility function. The simplest function, a linear one, assumes risk neutrality, i.e., that the level of risk does not affect the value placed on an outcome by the decision maker. Risk neutral decision makers make decisions based on expected values.

A form of utility function which is felt to provide a basis for analyzing the behavior of many decision makers under risk conditions is the exponential one. It assumes a constant aversion to risk in the sense that changes in wealth do not result in changes in behavior. Exponential utility functions provide many computational advantages. Other types of utility functions reflect other risk attitude structures.

Another way of viewing a utility function is as an indifference curve showing combinations of risk and return to which an investor is indifferent. The higher the risk involved the higher return needed to provided the same utility level.

In 1979 the Casualty Actuarial Society had a Call Paper Program on the subject Total Return Due A Property-Casualty Insurance Company. Several papers from that program discussed the notion of a Capital Asset Pricing Model (CAPM). CAPM provides a framework for determining the relationship between risk, return, and reward for taking risk.

Modern Portfolio Theory which incorporates the ideas of risk and utility into investment strategy develops the notion of an efficient portfolio. Efficient portfolios maximize return for a given amount of risk or minimize risk for a given level of return. Identification of efficient portfolios requires information regarding each investment's expected return, variance of return, and covariance of return with that of other investments under consideration.

CAPM assumes all investors are risk averse and that for efficient portfolios the standard deviation of return is an appropriate measure of risk. A linear relationship between risk and return is hypothesized. For individual investments, risk is measured by covariance between the investment's return and that of the market as well as the relative volatility of the return on the investment. Return on an investment becomes a function of the riskless rate of return, the market rate of return, and the sensitivity of the investment's return to that of the market.

The measure of systematic risk (beta) can enter into valuation theory by impacting the discount rate used to value the income stream anticipated from the asset being valued. The value of the various utility-based models to casualty actuaries will be related to success in measuring the needed parameters.

THE THEORY OF INTEREST

In addition to estimating the loss and expense elements arising from the provision of insurance coverage, the actuary is involved in the estimation of cash flows which such coverage produces. Such estimation includes a number of elements. Key among those are the timing of the various cash payments made and the interest rate which properly reflects the valuation of cash deposits held by one or the other party.

In liability insurance, payments made under an insurance contract and their timing are influenced by the same kind of factors as influence the total amount of losses arising for a portfolio of policies. The data available to an actuary might include payments which have arisen historically under portfolios considered to be similar. From that data, the actuary must attempt to estimate future cash flows and assess a relative value on those given an interest rate which allows for appropriate evaluation of the cash flows.

As we have seen in the discussion regarding the syllabus of material to be studied by actuaries, the theory of interest is a key area of study. The theory of interest uses some of the tools of growth and decay analysis which apply to the physical sciences to analyze the impact of timing differences in cash flows given a discount or interest rate. Given the period of investment, the interest rate, and an amount at a point in time, the actuary can use existing tools to resolve any problem involving the time value of money.

AN ACTUARIAL PARADIGM

Actuarial science is ripe for, and probably in the midst of, a scientific revolution. This is to some degree a result of the fact that it is attempting to deal with a wide range of issues which are impacted by a wide variety of factors.

Standard approaches have often been unsuccessful in solving problems they were expected to address, and adjustments to these approaches have been evolving rapidly. This paper will suggest that the actuarial profession is at a crossroad which can lead to a new paradigm structure of significantly greater breadth than one we generally recognize today.

As is the case with economics, any description of a paradigm of actuarial science must recognize that a number of components exist. Some of the better known components of the economic paradigm are covered by the categories microeconomics, macroeconomics and welfare economics. In actuarial science the categories are not so well defined.

While actuarial science deals with areas of law, accounting, economics, and management, it is fundamentally a mathematical discipline. While a broad number of issues are dealt with by actuaries, the most fundamental is the estimation of liabilities arising from contingencies covered by insurance contracts. Once these liabilities are estimated, the values can be put into various actuarial models for the development of rate level, rate structure, reserves, and other "actuarial" items.

Life and casualty actuaries evaluate different types of contingencies. Losses associated with many life insurance products are felt to be predictable enough to fit some fairly well defined patterns which are quantified in various types of mortality tables. Losses associated with property/casualty insurance products, particularly those covering some of the more significant exposure to liability losses, are not felt to be likely to fall into any easily predetermined patterns.

Casualty losses for a particular exposure for a particular entity must be estimated by considering the unique characteristics of the exposure. Variables affecting casualty losses for a category at a point in time are numerous and include the nature of the coverage itself, the operating and socioeconomic environment, chance, and the insured population.

The actuarial paradigm may be defined as the methodology or methodologies generally agreed upon by members of the actuarial community to solve the problems being addressed by that community. In the evolving area of actuarial methodology relating to estimation of losses arising from liability insurance, there is often a dichotomy between methodology used in practice and methodology developed in theory.

The model often used in practice when adequate historical data is available utilizes data organized in the form of development tables. These tables organize claim count and claim amount information for a grouping of policies or incidents in such a way that changes in these values over time can be analyzed.

A theoretical approach to the problem of loss estimation has been discussed under the general title of risk theory. The idea is that the occurrence of an insured event is a random variable, and given that occurrence, the amount of loss which will be generated will fit a probability curve.

A BRIEF SURVEY OF CURRENT ACTUARIAL TOPICS

We have now reviewed some of the factors impacting on the problems actuaries are asked to resolve. We have also identified mathematics as the primary tool which the actuary applies to resolution of those problems. Of interest now is a discussion of some topics often associated with actuarial science.

Ratemaking--An area generally associated with actuarial science is that of development of rates or premiums to be charged for defined insurance coverages. In addition to estimating losses expected to arise for the portfolio of risks to be written at the rates being calculated, the actuary is faced with the issues of determining a price which covers other costs associated with the insurance product. Those costs include the cost of capital required to provide the product and the appropriate profit level associated therewith.

The ratemaking activity of the actuary, therefore, includes many elements of modification to whatever general paradigm is established for actuarial science. Those include estimating future social and economic factors which will influence needed rate level, determining a profit load which provides adequate return to capital, estimating how elasticity of demand and other factors will affect the mix of policyholders expected to purchase the coverage being priced, and estimating nonloss costs associated with provision of the product.

<u>Reserving</u>--Estimating loss reserves for defined categories of business may be the area of actuarial activity most directly related to an underlying paradigm. Unlike ratemaking, the estimation of loss reserves can be done from a particular data framework using a defined mathematical model. Peripheral elements like expense components and profit loads need not be considered in estimating loss reserves for a defined category.

Where deviation from a standard model may arise in the reserve estimation process is in the specification of the category for which reserves are being estimated and in the nature of the data which is available. The type of exposure involved, including policy limits, retention, policyholder mix, coverage mix, and volume of claim potential may result in modifications to the core model.

Other--In addition to standard pricing and reserving work, actuaries are called upon to provide analysis related to a wide variety of other issues. For purposes of paradigm evaluation, these other issues are considered peripheral or subsumed. A review of some of them, however, provides perspective on the texture and richness of actuarial activity which emanates from an abstract core.

One area receiving a good deal of actuarial attention currently is that of valuation. Actuaries are expanding the role they play in the determination of value of an insurer's assets and liabilities. This led to much more attention to cash flows associated with these assets and liabilities, as well as more direct recognition of possible effects of economic and operating conditions.

Additional knowledge regarding accounting rules also becomes important, as does evaluation of the uncertainties attached to the various components of the analysis. An understanding of the components of asset categories and their characteristics also proves to be important.

A second "secondary" area of actuarial science falls under the heading risk classification. While losses for a portfolio may be estimable given certain assumptions, that portfolio may consist of heterogeneous components. Risk classification attempts to evaluate differences in average loss costs associated with different groupings of risks.

Methodologies for identifying and evaluating risk differences are varied. Each individual exposure unit has unique characteristics, so any classification scheme must abstract away certain elements of difference. Once the scheme is determined, the problem of developing statistically sound differentials given the general credibility and homogeneity characteristics of the database remains.

Actuaries thus deal with a wide range of issues which may not fit neatly into a core model. Topics such as reinsurance, excess limits, individual risk rating, utility theory, interest theory, and graduation which the actuary faces all may be dealt with in the context of a core model or a component thereof, but the realism of the assumptions which must be made to achieve results is always a significant challenge.

THE NATURE OF ACTUARIAL WORK

While the mathematical models which apply to actuarial science are rich and robust, the work done by casualty actuaries is often difficult to transform into problems to which the models are well adapted. The data available by which to perform a typical ratemaking calculation, for example, would consist of a multitude of elements for the category under consideration. These would include:

- Paid and incurred losses for the coverage for a period of years.
- Various expense elements for the coverage for a period of years.
- Exposure measures for the coverage for a period of years.
- Premium measures for the coverage for a period of years.

The loss data can be structured in a variety of ways depending upon desired accounting period and time lag structure. Expense elements can be broken in a number of ways. Effects of such items as residual markets, excess limits, coverage variations, individual risk rating, and elasticity of demand can be incorporated in various ways.

Determination of target profit level and the way of reflecting it in the price level is another complicating element. Determination of trend factor and weights given to various historical periods add to the complexity, as does reflection of credibility.

Once overall rate level is determined, allocation to various risk classifications is normally done. The classifications themselves must be determined first. Once a satisfactory classification scheme has been developed, actuarial methodologies for spreading loss costs must be determined.

The introduction of claims made coverages in recent years has added to the challenges faced by actuaries in rate and reserve analysis as well as in the other areas in which they work. By breaking the standard occurrence coverage into seemingly endless potential components, the implementation of claim-made policies has provided additional challenges in the areas of credibility, discounting, data structure, policy limits, risk classification, expense and profit loads, trend analysis, payment and reporting patterns, and reinsurance analysis.

Actuarial work, then, like the work of other applied scientists, encompasses a myriad of processes, elements, and interrelationships to which mathematical modeling can provide only rough approximations. The challenge is to improve on the mathematical foundations of actuarial science while simultaneously developing optimal approaches to real world problems. Appendix E provides a glossary of insurance terms which relate to aspects of the actuary's role in the business world.

CONSIDERATIONS IN DEVELOPING AN ACTUARIAL PARADIGM

The first step in determining whether actuarial science is a body of knowledge for which there is or can be developed a core model or paradigm from which research can emanate is to define the group of problems this model will attempt to address. In the area of risk theory actuaries have developed a model which can guide research related to loss estimation in a way consistent with Kuhn's notion of the concept of a paradigm. It is in adapting that model to the practical situations which actuaries face that the question of the nature of the body of knowledge known as actuarial science arises.

In his book, Kuhn indicated that "by focusing attention upon a small range of relatively esoteric problems, the paradigm forces scientists to investigate some part of nature in a detail and depth that would otherwise be unimaginable." If actuaries could function as pure scientists, they could take well defined models from probability theory, interest theory, utility theory, and number theory, and develop core models for the resolution of well defined problems. Actuarial science as currently defined, however, is an applied discipline, and results therefrom must resolve real, as opposed to idealized, problems.

The real problems to be solved fit the theoretical models available to actuaries only loosely. Once the cumulative loss distribution for a portfolio of exposures has been determined from risk theory, for example, fitting that portfolio to an actual portfolio which exists in the dynamic insurance world becomes difficult. Even if the portfolio presented a fairly good fit at a point in time, adjusting it to conditions expected to occur at a future point in time presents serious problems.

Even if actuarial paradigms can be developed for several of the abstract problems facing actuaries, the degree of certainty produced from those paradigms should not be overestimated. We have seen that even in that most abstract of disciplines (mathematics) for which truths were at one time felt secure no certainty exists. As we move away from mathematics toward the body of knowledge underlying actuarial science, the uncertainty levels increase.

Godel proved that the mathematics which had developed up to his time was fatally flawed. The concept of a universally accepted, infallible body of reason was found to be an illusion. Mathematicians have retreated from the search for the solution to large problems to narrow topics where proofs are felt to be safe.

Physics and the other sciences have undergone similar revolutions. Quantum mechanics has replaced classical physics in explaining the phenomenon of the world in which we live. The mechanistic determination of Newtonian physics has given way to the probabilistic perspective of quantum mechanics. Heisenberg's uncertainty principle shows that we cannot know both the position and the momentum of a particle. The metaphysical implication is that the world philosophers have been trying to understand and explain is not fully comprehensible.

The epistemological quandary faced by physicists is magnified for persons dealing with less abstract problems. The complementarity which haunts physicists is overwhelming for social scientists. Shakespeare's plays often poignantly illustrate the complementarity of values faced by ordinary and extraordinary people in their everyday existence.

Actuarial science ultimately must fit into some larger scheme of things. As an applied social science, actuarial science is involved with a very diverse group of phenomena. To adequately address the problems it faces, the profession must have a perspective which encompasses those phenomena. Appendix F provides some statistical data which illustrates the range of topics actuaries handle.

CONCLUSION

This paper was conceived from the notion that, as it approaches its 100th anniversary, the actuarial profession in North America is faced with profound choices. This is particularly the case in the area of liability insurance. The tools available to the actuary and the results expected from the actuary may not be ideally matched at any point in time, but the challenges provided by the continuing need to model complex phenomena produce great opportunities for the profession.

This paper has surveyed a number of subjects which are felt to provide perspective on the body of knowledge from which actuaries draw. Mathematics has been emphasized as the area from which the core of the actuarial paradigm is expected to be found. Other subjects, such as economics and law, are surveyed because they impact on the structure of the modeling which must be done and because they provide perspective on the use to which actuarial analysis is put.

The nature of a synopsis presented in a paper such as this is such that none of the bodies of knowledge which are surveyed are rigorously treated. The need addressed in this paper is not the need for rigorous treatment of a narrow area of actuarial science, but the need for a consolidation of existing knowledge.

As quantifiers of significant social phenomena, actuaries face many of the issues facing any applied mathematician. Because physicists have developed very complex mathematical models by which to evaluate the phenomena in their domain, some attention was paid to some of the more significant models developed in the physical sciences. Those models may provide actuaries with insights into more refined theoretical approaches which can be developed with respect to the type of problems they face.

It is suggested that for actuarial science to be a body of knowledge as described by Kuhn, the scope of problems to be resolved by its paradigm must be defined. Loss estimation, a key aspect of the actuary's work, is best viewed as mathematically based. Similarly, interest theory and utility theory, which underlie certain actuarial problems, have a mathematical foundation. To the extent legal, regulatory, operational, accounting, economic, or other phenomena interact with the subject matter of actuarial science, they can be integrated without changing the basic paradigms.

No claim to originality is made. The intent is to provide a glimpse of the universe of information which an actuary must sift through to determine the nature of his/her subject. Hopefully, it provides many of the elements which will be needed in developing a paradigm for our profession.

APPENDIX A: SELECTED GLOSSARY OF LEGAL TERMS OF INTEREST TO ACTUARIES

a fortiori	All the more; for a stronger reason.
Abatement	A making less; a suspension of action.
Accretion	Addition to property by natural causes.
Additur	A remedy by which a new trial is denied if the defendant agrees to an increased judgment.
Adhesion Contract	A standard form prepared by one party.
Adverse Possession	The holding of land under a claim of right inconsistent with that of the true owner.
Aleatory Contract	An agreement in which performance of a party is dependent upon the occurrence of an uncertain event.
Assumpsit	An action for damage for breach of contract.
Bailment	Agreement created by delivery of property by the owner.
Casualty	An unforeseen circumstance occasioning loss.
Certiorari	An action whereby a cause is moved to a superior court.
Collateral	Property which is subject to a security interest.
Color of Title	Apparent ownership of land based upon a written instrument.

Consideration	The matter of inducement of a contract.
Contingency	An unforeseeable, but possible event.
Curtesy	The estate of the husband in wife's fee simple or fee tail estates after her death.
Defeasance	A condition for the determination of an estate or interest based upon a performance or occurrence.
Detinue	A personal action for recovery of goods or value.
Discovery	Method by which parties to a lawsuit gather facts.
Easement	A right which the owner of real property has with regard to another property.
Equality	The condition of persons when none has an unfair advantage.
Estate	The condition and circumstance in which a person stands with regard to those around him and her property.
Estate in Fee Simple	An estate free of restrictions, conditions, or limitations.
Estate in Reversion	An estate remaining in the grantor.
Estoppel	A declaration by which a person is precluded from bringing controverting evidence.
Executory	Not completed.

Fair Cash Value	The instrinsic value of shares of stock in a corporation.
Fair Market Value	The amount an article would bring if sold in the market under normal conditions.
Feoffment	The transfer of possession of a freehold estate.
Fiduciary	One who has a duty to act in another's best interest.
Gift Intervivos	An irrevocable gift from a live donor.
Hornbook	A one-volume work containing elementary principles of an area of law.
Hypothecation	The deposit of securities to secure repayment of a loan.
Implead	To bring a new party into a lawsuit.
Insurable Interest	A concern in the subject of insurance entitling the possessor to obtain insurance.
Insurance	The act of providing against a possible loss by entering into a contract with a party willing to make good the loss should it occur.
Interlocutory	Incident to a suit still pending.
Interpleader	A procedure whereby persons with claims against another may be joined as parties to a lawsuit.
Intestacy	The condition of a person who dies without a valid will.

Joint and Several Liability	A situation where one or more liable parties, or all of them, may be sued by the plaintiff.
Jurisdiction	The authority of a court to hear and decide an action or lawsuit.
Leading Question	An inquiry which suggests an answer to a witness.
Letter of Credit	An irrevocable credit issued by a financial institution.
Liability	A present or potential duty.
Livery of Seisen	An act by which real property is transferred.
Malpractice	Improper performance by a professional of duties incumbent on account of professional status.
Mortality Tables	Statistical charts showing life expectancy.
Negligence	Failure to use ordinary care or to follow the standard established by law under the circumstances.
Nuisance	Activities which harm others under the facts of the suit.
Pleadings	Opposing written statements of parties to a lawsuit.
Premium	The consideration paid for issuance of an insurance policy.

Product Liability	Law dealing with responsibility of manufacturers to buyers concerning quality of merchandise and consequences resulting from substandard quality.
Remainder	Residue of interest in real property.
Remittitur	Disposal of a lawsuit by ordering a new trial unless a party agrees to damages different from that from trial court.
Replevin	Lawsuit to recover possession of specific chattels.
Reversion	Residue of estate left in grantor.
Stare Decisis	To follow precedent.
Subrogation	The substitution of a person to the rights of another concerning a claim the former has paid.
Surety	A person who makes himself responsible for the obligations of another.
Title	A valid claim of right.
Tort	A legally recognized private wrong other than a breach of contract.
Trover	A special form of trespass.
Usury	Interest in excess of the legal maximum.
Voir Dire	A preliminary examination of a witness to determine competence.
Writ	A written court order or judicial process.

APPENDIX B: SELECTED GLOSSARY OF ECONOMIC TERMS

Adjustment Lag	The time taken for a variable to adjust to charges in its determinants.
Agglomeration Economics	Cost savings resulting from enterprises locating near one another.
Aggregation Problem	The problem of deriving predictable macroeconomic behavior from that of underlying units.
Automatic Stabilizers	Relationships which reduce the volatility of cyclical fluctuations in the economy.
Average Cost Pricing	A pricing rule whereby the firm adds a mark-up onto average variable costs to cover average total costs including a fair net profit margin.
Average Revenue Product	Average product of an input factor multiplied by average revenue (price).
Averch-Johnson Effect	Profit maximizing response of a regulated firm faced with a determined rate of return.
Axioms of Preference	Axioms individuals are assumed to obey in consumer demand theory, including transitivity, completeness, selection, dominance, continuity, and convexity.

Balance Budget Multiplier	The ratio of the change in real income to a change in government expenditure which is matched by a change in tax revenue.
Balanced Growth	A dynamic condition where all real variables are growing at the same rate.
Cobb Douglas Production Function	A homogeneous production function.
Collective Good	A commodity with the characteristic of nonexcludability.
Comparative Advantage	The idea behind the notion that specialization and free trade lead to a higher level of general welfare.
Comparative Statics	The comparison of equilibrium positions.
Coefficient of Concentration	A statistical measure of the degree to which an economic activity is concentrated.
CES Production Function	A linearly homogeneous production function.
Consumer Equilibrium	The point at which the consumer maximizes utility.
Consumer's Surplus	A measure of consumer benefit resulting from the ability to buy a good at the market price.
Consumption Function	The relationship between consumption and income.

Contract Curve	The locus of points where the marginal rate of substitution between goods is the same for the individuals.
Cross Elasticity of Demand	The responsiveness of the quantity demanded of one good to a change in the price of another.
Cyclical Unemployment	Short run unemployment caused by deficient demand.
Devaluation	A fall in the exchange rate of a currency.
Distributive Judgment	A value judgment as to appropriate distributional effects of economic policies.
Economics of Scale	Reduction in long-run average cost resulting from expanded output.
Elasticity	A measure of the percentage change in the value of one variable with respect to a change in another.
Externalities	Side effects of production or consumption for which the market is ineffective in providing for.
Fair Rate of Return	That which allows for continued capital attraction.
Fiscal Policy	The use of taxation and government expenditure to affect the level of economic activity.
General Equilibrium	A condition whereby all markets in an economy are simultaneously in equilibrium.

Gini Coefficient	A measure of distribution equality.
Harrod-Domar Growth Model	A one sector growth model.
Herfindahl Index	A measure of market concentration.
Indifference Curve	The locus of combinations of goods to which the individual is indifferent.
Input-Output	A method of analysis in which the economy is represented by a set of linear production functions.
IS-LM Diagram	The diagram detailing simultaneous equilibrium in money and product markets.
Isoquant	Combinations of inputs which produce a particular output.
Iso Profit Curves	The focus of combinations of independent variables of the profit function which yield equal profit.
Laffer Curve	An illustration of the thesis that there exists an optimal tax rate for maximizing government tax revenue.
Laspeyres Price Index	A base year weighted price index.
Lerner Index	An indicator of monopoly power.

Leverage	The relationship between long-term debt and capital.
Loss Function	A disutility function to be minimized.
Marginal Efficiency of Capital	The rate of discount at which the present value of net returns from an asset equal its cost.
Marginal Rate of Substitution	The amount of one good required to compensate a consumer for the loss of another.
Opportunity Cost	The value of foregone alternatives.
Paasche Price Index	A current year weighted price index.
Pareto Optimum	A condition in the economy where to make someone better off someone else must be made worse off.
Permanent Income Hypothesis	The idea that consumption depends on lifetime income including asset depletion.
Wealth Effect	Increase in aggregate expenditure due to fall in prices and interest rates.
Welfare Economics	The normative area of economics.

APPENDIX C: SELECT GLOSSARY OF MATHEMATICAL TERMS

Abelian Group	A group in which the binary operation obeys the commutative law.
Accumulation Point	The limit of a sequence of points of a set.
Algorithm	A standardized procedure for the solution of a particular type of problem.
Applied Mathematics	The area of mathematics concerned with solution of physical problems by supporting empiricism with logically deduced solutions.
Boolean Algebra	A ring with an identity element which reproduces each element.
Calculus of Variations	The study involving the finding of a function for which a given expression attains an extreme value.
Cauchy's Test	Ratio test for convergence or divergence of an infinite series.
Center of Mass	The point at which the entire mass may be concentrated in theory to produce the same effect as when originally distributed.
Cramer's Rule	A rule using determinant notation for the solution of simultaneous equations.

DioPhantine Analysis	The determination of integral solutions of certain algebraic equations.
Equilibrium	The state of a body when the resultant of all forces acting on it is zero.
Group	A mathematical system consisting of a set of elements subject to a binary operation satisfying certain axioms.
Hyperbolic Space	A non-Euclidean space based on the postulate that through a point external to a given line several lines parallel to the given line exist.
Integral Domain	A set of elements subject to two binary operations satisfying certain axioms.
Invariant Property	That property of a function which remains unaltered under a transformation.
Isomorphism	Two mathematical systems in one-to-one correspondence.
Klein Bottle	A bottle with a single surface and properties of a three-dimensional strip.
Linear Algebra	The study of the algebraic properties of vector spaces.
Linear Transformation	A transformation produced by the use of algebraic linear equations.

Manifold	A class with subclasses.
Mathematical Programming	The process of finding an optimum value of a function where the variables are subject to constraints which often take the form of equations.
Mathematics	A system of organized thinking of an analytic and synthetic nature.
Measure	The size of something expressed in standard units.
Neighborhood	Part of a line which contains a given point.
Number Field	A set of numbers which is arithmetically closed.
Number Theory	The study of integers and the relationships between them.
Order of Group	The number of elements in a finite group.
Orthogonal	Right-angled.
Paradox	An apparent contradiction between two reasonable conclusions.
Parametric Equations	Equations in which the parameters are expressed as functions.
Postulate	As assumption upon which a logical argument is based.

Probability	A numerical measure of the likeliness of an event occurring.
Quadrature	The process by which a square equal in area to that of a given surface is found.
Range	The set of all possible values.
Relativity	The theory that all natural processes take place in a four dimensional space-time continuum.
Resolution	The process of determining two vectors equivalent to a given one.
Resultant	The vector quantity equivalent to two or more vector qualities.
Sampling	The process of obtaining a sample of population for the purpose of inference.
Set Theory	The Boolean algebra of sets or classes.
Simplex	A topological terms for the simplest geometric figure in a space.
Space	A set of elements which satisfy a set of postulates.
Standard Form	An accepted way of writing an expression.
Statistics	The study of the methodology of collecting and analyzing data.

Surd	A numerical expression containing an irrational number.
Surface	A set of points forming a two-dimensional space.
Symmetric Relation	A relation which is identical to its own inverse.
System	A set of elements which have a common property.
Theory	The principles involved in the development of a central concept.
Topological Property	A property of a geometric figure which remains invariant under a topological transformation.
Transcendental	Nonalgebraic.
Transfinite Number	An infinite cardinal or ordinal number.
Vector Quantity	A quantity with magnitude and direction.
Wave Length	The distance between successive points on a wave which represent the same phase of disturbance.

APPENDIX D: SELECTED GLOSSARY OF ECONOMETRIC TERMS

Analysis of Variance	The breakdown of total variation in a dependent variable into the proportions accounted for by variation in explanatory variables.
Arima Forecasts	Forecasts generated from models which capture time series characteristics of a variable by relating current to lagged values.
Asymptotic Distribution	The probability distribution to which a statistic tends as the sample size increases indefinitely.
Autoregression	The regression of a variable on its own lagged values.
Bayesian Techniques	Methods of statistical analysis in which prior information is formally combined with sample data to produce estimates or test hypotheses.
BLUE	The best linear unbiased estimator has the smallest variance of all unbiased linear estimators.
Binary Variable	A variable which can take two values and is normally used to provide for qualitative factors.
Blus Residuals	Best linear unbiased estimators with a scalar covariance.
Central Limit Theorem	A theorem which states that the sum and mean of a set of random variables will follow a normal distribution if the sample is large enough.

Coefficient Determination	A statistic which summarizes the explanatory power of an equation.
Coefficient of Variation	A measure of the degree to which a variable is distributed around its mean value.
Contingency Table	A device for measuring the degree of association between variables.
Correlation	The degree to which two variables are linearly related.
Correlogram	A plot of the correlation coefficient between the current value of a variable and lagged values agains the length of the lag.
Covariance	A measure of the degree to which two variables are linearly related.
Critical Value	The value in a probability distribution above or below which a specified percentage of probability lies.
Cross Sectional Analysis	Analysis of a set of observations taken at a point in time.
Degrees of Freedom	The number of pieces of information which can vary independently of each other.

Deseasonalization	The process of removing seasonal influences from data.
Detrending	The process by which a time trend is removed from data.
Distributed Lags	The result of formulating mathematical relationships for coefficients of lagged values.
Disturbance Term	The error term in a regression equation.
Dummy Variable	A binary variable which accounts for shifts in econometric relationships.
Durbin-Watson Statistic	A statistic which diagnoses serial correlation in error terms in a regression.
Estimation	The quantitative determination of the parameters of economic models through statistical manipulation of data.
Fourier Analysis	A method by which time series data can be transformed to the frequency domain.
Full Information Maximum Likelihood	A technique for estimating systems of simultaneous equations.
General Linear Model	A model which specifies the dependent variable as a linear function of independent variables.

Heteroscedasticity	A situation where the variance of the error term in a regression equation is not constant between observations.
Identification Problem	A condition arising in the estimation of parameters of simultaneous equations where the equation being estimated cannot be uniquely identified.
Indirect Least Squares	A procedure for estimating parameters of simultaneous equations which avoids bias.
Joint Probability Distributions	Probability distributions which give the probability with which two or more variables take certain values.
Kalman Filtering	An optimal method of predicting endogenous variables and updating parameter estimates.
Klein-Goldberger Model	An econometric model of the economy.
Lagrangian Technique	A method for solving constrained optimization problems.
Linear Programming	A technique for formulation and analysis of constrained optimization of problems.
Logit Analysis	A linear probability model where values of the dependent variable are constrained within probability limits.

Markov Process	A process which relates the current value of a variable to previous values and an error term.
Maximum Likelihood	An estimation technique involving maximization of the likelihood function of the observations with respect to the parameters being estimated.
Moments	Summary statistics which characterize a distribution.
Multicollinearity	A condition in which two or more explanatory variables in a regression are highly correlated.
Orthogonal	A condition arising from uncorrelated variables.
Parameter	A quantity which remains constant in a given context.
Random Walk	A time series model in which the current value of a variable is equal to the most recent value plus a random element.
Rank Correlation	A measure of the rank relationship between variables.
Recursive Model	Current and lagged values of one variable determine the value of another values of which determines the value of the former.
Residual	The difference between an actual data point and its estimated value.

Simulation	The generation of a range of estimates based on alternative assumptions.
Simultaneous Equation Bias	A bias resulting from feedback effects between equations in the model.
Singular Matrix	A matrix whose determinant is zero because of linear dependence.
Spectral Analysis	A technique by which cyclical properties of a variable can be established from time series data.
Stochastic Process	A time related series subject to random variation.
Systems Estimator	An estimator used to gain estimates of all the parameters in a simultaneous equation system.
Two Stage Least Squares	A procedure for estimation of structural form parameters of simultaneous equation systems.
Von Neumann Ratio	A test statistic for detecting serial correlation of residuals in regression analysis.

APPENDIX E: SELECTED GLOSSARY OF ACTUARIALLY ORIENTED INSURANCE TERMS

Actuary	A person concerned with the application of mathematical theories to the practical problems of insurance related fields.
Classification	The underwriting or rating group into which a particular risk is placed.
Combined Ratio	The sum of the loss and expense ratios.
Deductible	The amount of loss paid by the insured.
Deficiency Reserve	Supplemental reserve required when the gross premium for a class of life insurance policies is less than the net premium.
Defined Contribution Plan	A pension plan where an individual account is maintained for each participant.
Deviation Rate	A premium change that differs from the scheduled rate.
Discrimination	Treating a given class of risks differently from other similar risks.
Earned Premium	That part of a policy premium for protection already provided.

Excess Limit	A coverage limit above the basic one established.
Experience Rating	Ratemaking based on specific experience of the risk.
Facultative Reinsurance	A type of reinsurance available on a per risk basis.
Governing Classification	The main operation of any employer for workers compensation insurance.
Graduated Life Table	An actuarial table derived from a mortality experience curve.
Gross Net Premiums	Gross premiums less return premiums.
Insurance	A contractual relationship whereby one party for consideration agrees to reimburse another for loss from designated contingencies.
Lapse Ratio	The ratio of policies surrendered to those in force.
Law of Large Numbers	Theory of probability that is the basis of insurance.
Liability Insurance	Coverage whereby the insured is protected against claims from other parties.
Loss Conversion Factor	A factor applied to losses to provide for expenses.
Loss Reserve	The portion of assets available to pay probable claims.

Manual Rate	Cost of a unit of insurance.
Maximum Probable Loss	The longest loss expected for a given risk.
Merit Rating	A system for measuring differences of a specific risk from a standard risk.
Mortality Table	A statistical table showing the death rate at each age.
Net Retention	The amount of insurance a ceding company keeps.
Portfolio Reinsurance	A type of reinsurance whereby the reinsurer assumes all obligations of a certain type.
Prospective Reserve	A reserve base on the present value of future claims less the present value of future net premiums.
Pure Loss Cost	The ratio of reinsurance losses incurred to the ceding company's subject premium.
Pure Premium	The amount of money needed to pay the loss portion of the insurance coverage.
Quota Share Treaty	A reinsurance arrangement whereby each company accepts a stated proportion of premium and losses for covered risks.
Reinsurance	An agreement between insurance companies to spread the risk of loss.

Retrocession	A cession of reinsurance by a reinsurer.
Retrospective Rating	A rating method by which the final premium is based on experience of the risk for the covered period.
Self-Insurance	The systematic provision of a fund to cover losses the entity may suffer.
Standard Premium	Premium before adjustment for discounts or retrospective adjustments.
Tontine	A reverse form of life insurance.
Tort	A wrongful act not involving a breach of contract.

APPENDIX F: SELECTED STATISTICS OF INTEREST TO ACTUARIES

I. Federal Government Receipts and Expenditures

Year	Receipts	Expenditures	Deficit
1977	$357 Billion	$ 402 Billion	$ 45 Billion
1987	854 Billion	1,002 Billion	148 Billion

II. Federal Reserve Board Discount Rates; Annual CPI Change

Date	Discount Rate	Year	CPI Change
May 5, 1981	14	1980	13.5%
September 1, 1988	6.5	1987	3.6

III. Unemployment Insurance Data for 1987

Claims	Benefits	Average Weekly Benefit
$17 million	$14 billion	$140

IV. Productivity and Unit Labor Costs

Year	Output Per Hour	Unit Labor Costs
1975	92.9	91.7
1987	132.4	139.7

V. Foreign Trade

Year	Exports	Imports	Deficit
1980	$220 Billion	$241 Billion	$ 21 Billion
1987	253 Billion	424 Billion	171 Billion

VI. Pollution Abatement and Control Expenditures

Year	Amount
1975	$30 Billion
1985	$74 Billion

VII. Probabilities

Full House (Poker)	12 (Dice)	13-0-0-0 (Bridge)	Heads (Fair Coin)
1/694	1/36	1/159 Billion	1/2

VIII. World Population; U.S. Population

Year	World Population	U.S. Population
1960	3 Billion	179 Million
1987	5 Billion	243 Million

IX. Social Security Statistics

	OASI Trust Fund		DI Trust Fund		HI Trust Fund	
Year	Receipts	Expenditures	Receipts	Expenditures	Receipts	Expenditures
1970	$ 32 Billion	$ 30 Billion	$ 5 Billion	$ 3 Billion	$ 6 Billion	$ 5 Billion
1987	211 Billion	188 Billion	20 Billion	21 Billion	63 Billion	51 Billion

X. Births and Deaths in the U.S.

Year	Births	Total Deaths	Motor Vehicle Deaths	Home Accident Deaths
1970	3.7 Million	1.9 Million	54,633	27,000
1987	3.8 Million	2.1 Million	48,700	20,500

XI. Life Insurance Statistics

Year	Life Insurance Purchases	Insurance In Force	Life Expectancy
1970	$ 193 Billion	$1,402 Billion	70.8
1986	1,309 Billion	6,720 Billion	74.9

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