Movement Analysis

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Abstract.
Consistent with the requirements of Actuarial Standards of Practice (ASOPs) 36 and 41 (paragraphs 4.5 and 3.5, respectively), this paper derives simple but mathematically sound formulas for explaining differences in estimates of ultimate from one period to the next. Specifically, the change in ultimate is decomposed into the movement due to loss experience relative to the movement due to changes in assumptions or methods. The approach outlined below is for use in common reserving situations where the Bornhuetter-Ferguson (BF) or Chain-Ladder (CL) methods are used, but can also be easily extended in other circumstances.

Keywords.
Reserving; Communication.

1. INTRODUCTION

In booking the unpaid claims reserve, good governance as well as Actuarial Standards of Practice (ASOPs)[1,2] require that the actuary clearly communicates any material differences in the estimate of ultimate relative to earlier projections. This is so that management has the necessary tools to assess, challenge or validate the actuary's recommendation and make their own determination as to the final carried amount.

In order to do so effectively, the actuary needs to be transparent as to why the estimate of ultimate changed. Did it move as a result of loss experience emerging more or less favorably than expected or did it move because of changes in the underlying methods or assumptions? And in the case of the latter, what impact did these changes have on the final result and why were these changes warranted?

To address these questions, this paper derives simple but mathematically sound formulas for explaining differences in estimates of ultimate loss from one period to the next. Here, the change in ultimate is decomposed into the movement due to loss experience relative to the movement due to changes in assumptions or methods. While most actuaries will already perform this type of analysis in some form (typically via successive substitution of new data and new assumptions into new methods), the "movement analysis" outlined below offers a consistent approach for communicating as well as quantifying change which will work in many practical situations.

[1] Explanation of Material Differences – If a later actuarial communication produced by the same actuary, which opines on the same issue, includes materially different results or expresses a different opinion from the former communication, then the later communication should make it clear that the earlier results or opinion are no longer valid and explain why they have changed. [excerpted from ASOP 41: 3.5]

[2] Changes in Opining Actuary's Assumptions, Procedures, or Methods – If a change occurs in the opining actuary's assumptions, procedures, or methods from those previously employed in providing an opinion on the entity's reserves, and if the actuary believes that the change is likely to have a material effect on the results of the actuary's reserve analysis, then the actuary should disclose the nature of the change. [excepted from ASOP 36: 4.5]

1.1 Outline

The Executive Summary in Section 2 presents the movement analysis in its complete form with all the formulas needed to implement this analysis within a practical setting presented in Table 2.

Section 3 proceeds to develop these formulas iteratively by isolating and quantifying the impact of loss experience as well as certain methodological or assumption changes on estimates of ultimate where the Bornhuetter-Ferguson (Section 3.1) or Chain-Ladder (Section 3.2) methods are used. For readability, the actual derivations of the key formula are contained within a Technical Appendix to this paper. While these sections are not exhaustive as to situations that might arise in practical reserving settings, they can easily enough be extended to other circumstances as will be discussed in Section 3.3. This section should prove useful for understanding the how and why of this analysis intuitively.

Finally, to illustrate this analysis, an example is included in Section 4. Also provided is a workbook which includes the necessary formula to implement this analysis in Excel.

1.2 Notation

The following notation is used within this paper:

- q_k is the percentage of loss developed at time k;
- C_k is the actual loss at time k;
- u is the initial expected loss ratio (IELR);
- P is the premium; and
- U_k is the estimate of ultimate loss at time k.

Using this notation, the Bornhuetter-Ferguson (BF) and Chain-Ladder (CL) estimates of ultimate loss at time k can be written as:

Table 1. BF and CL projections of ultimate loss.

Method	Formulation
BF method	$U_k = C_k + uP(1 - q_k)$
CL method	$U_k = C_k / q_k$

Further, "hats" are used to indicate updated assumptions. For instance, where q_k should be taken to be the original assumption of the percentage of loss developed at time k, $\hat{q}_k$ would be the revised assumption as to the percentage of loss developed at time k.

2. EXECUTIVE SUMMARY

In Section 3, the movement analysis is derived by iteratively considering each of the following:

- The movement in ultimate as a result of loss experience emerging differently from expectations;
- The movement in ultimate as a result of premiums emerging differently than originally anticipated;
- The movement in ultimate as a result of changes to key assumptions including (i) development patterns and (ii) IELRs; and
- The movement in ultimate by switching between the CL and BF methods.

That said, the table below provides the complete set of equations for producing the movement analysis. As the exact form depends on what the current and prior methods are, the table is split across this dimension with dots "•" used to indicate where the result is invariant to the method. These equations are also programmed into the attached Excel workbook.

Table 2. Movement Analysis.

Movement in ultimate due to:	**Method** Prior	**Method** Current	**Formulation** $(U_{k+1} - U_k)$
Loss experience	BF	•	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
	CL	•	$= \left((C_{k+1} - C_k) - \frac{C_k}{q_k}(q_{k+1} - q_k)\right) \times \left(\frac{1}{q_{k+1}}\right)$
Change in method	BF	CL	$+ C_{k+1}\left(\frac{1}{q_{k+1}} - 1\right) - uP(1 - q_{k+1})$
	CL	BF	$+ uP(1 - q_{k+1}) - C_{k+1}\left(\frac{1}{q_{k+1}} - 1\right)$
Change in premium	•	BF	$+ (\hat{P} - P)u(1 - q_{k+1})$
Change in development pattern	•	BF	$+ u\hat{P}[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})]$
	•	CL	$+ C_{k+1}\left[\left(\frac{1}{\hat{q}_{k+1}} - 1\right) - \left(\frac{1}{q_{k+1}} - 1\right)\right]$
Change in IELR	•	BF	$+ (\hat{u} - u)\hat{P}(1 - \hat{q}_{k+1})$

3. MOVEMENT ANALYSIS

3.1 The Bornhuetter-Ferguson (BF) Method

Consider the situation where the estimate of ultimate is set equal to the BF method. Here, the estimate of ultimate can change for any of four reasons:

- Loss experience that emerges more or less favorably than expected;
- Premium amounts that are restated;
- Changes in the development pattern; or
- Changes in the IELR.

The following considers each of these in turn.

3.1.1 Movement in ultimate due to loss experience

Assuming that no assumptions are updated, the change in ultimate $U_{k+1} - U_k$ can be written as:

Table 3. Decomposition of movement in ultimate for the BF method.

Movement in ultimate due to:	Formulation $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$

The above should be recognizable as the actual vs. expected (AvE) statistic when using the BF method. $C_{k+1} - C_k$ represents actual emergence and $uP(p_{k+1} - p_k)$ represents expected emergence. Indeed, the change in the BF ultimate without any changes in assumptions reduces to the AvE statistic with claims emergence that is more or less favorable than expected flowing entirely through to the change in ultimate.

3.1.2 Movement in ultimate due to change in development pattern

Suppose as a result of loss experience emerging differently from expectations, the development pattern is revised. Using "hats" to indicate updated assumptions, the change in ultimate is written as:

Table 4. Decomposition of movement in ultimate for the BF method.

Movement in ultimate due to:	Formulation $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
Change in development pattern	$+ uP[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})]$

While this derivation is less straightforward than above, observe that the change in ultimate is decomposed into the AvE statistic from Table 3 and a remainder. In this instance, the remainder is just the difference in the estimated reserve at time $k+1$ implied by the current and prior selected development patterns, or the movement in ultimate due to the change in pattern.

It should be noted that while the subscripts k and $k+1$ might indicate points at which the development pattern is selected, this method will work equally well in instances where the selected development pattern is interpolated. For example, if loss development factors are selected over periods from 3-15 months, 15-27 months and so forth, it is no problem to interpolate the pattern as at 6, 9 and 12 months in order to apply the movement analysis to the most recent year over the subsequent three quarters.

3.1.3 Movement in ultimate due to change in IELR

Going one step further, should an adjustment be made to the IELR as well as the development pattern, the change in ultimate is written as:

Table 5. Decomposition of movement in ultimate for the BF method.

Movement in ultimate due to:	**Formulation** $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
Change in development pattern	$+ uP[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})]$
Change in IELR	$+ (\hat{u} - u)P(1 - \hat{q}_{k+1})$

Again, the change in ultimate can be decomposed into the movement in ultimate due to loss experience, the movement in ultimate due to change in development pattern and a remainder. Here, the remainder is just the difference in the estimated reserve at time $k+1$ implied by change in IELR (using the current development pattern), or the movement in ultimate due to the change in IELR.

As an aside, note that the order in which the development pattern and IELRs are considered matters. This is obvious from the above as the change in IELR is based on the current development pattern. The above order seems reasonable as it might be practice to select the development pattern prior to the IELR; however, it is easy enough to consider these changes in reverse order as:

Table 6. Decomposition of movement in ultimate for the BF method (alternate formulation).

Movement in ultimate due to:	**Formulation** $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
Change in IELR	$+ (\hat{u} - u)P(1 - q_{k+1})$
Change in development pattern	$+ \hat{u}P[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})]$

3.1.4 Movement in ultimate due to change in premium

The next natural extension is to consider the impact that changes in premiums will have on the estimate of ultimate. Similar to the prior subsection, a decision needs to be made as to the order in which to consider changes in premium relative to other changes. Although there is an argument to consider it prior to loss experience, the below considers it after making an allowance for deviations in loss experience relative to expectation but prior to changes in assumptions. This is so that the AvE statistic will tie to any prospective estimates of loss emergence computed at prior periods.

Table 7. Decomposition of movement in ultimate for the BF method.

Movement in ultimate due to:	Formulation $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
Change in premium	$+ (\hat{P} - P)u(1 - q_{k+1})$
Change in development pattern	$+ u\hat{P}[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})]$
Change in IELR	$+ (\hat{u} - u)\hat{P}(1 - \hat{q}_{k+1})$

The above equation provides a near-complete decomposition of the movement in ultimate into each of the key drivers of change when using the BF method, with Tables 3-6 only representing partial solutions. In the next sections, we extend these formulas to consider situations when using the CL method, moving between the CL and BF methods and netting down estimates of gross ultimate loss for the impact of reinsurance.

3.2 The Chain-Ladder (CL) Method

The formulas from the prior section can be extended in situations where the CL, rather than BF, method is used as follows:

Table 8. Decomposition of movement in ultimate for the CL method.

Movement in ultimate due to:	Formulation $(U_{k+1} - U_k)$
Loss experience	$= \left((C_{k+1} - C_k) - {C_k}/{q_k}\,(q_{k+1} - q_k) \right) \times \left({1}/{q_{k+1}} \right)$
Change in development pattern	$+ \left({C_{k+1}}/{\hat{q}_{k+1}} - {C_{k+1}}/{q_{k+1}} \right)$

In some regards, while this analysis is simpler as there is only one assumption to consider (the development pattern), it is important to note that the AvE statistic is expressed slightly differently than in the previous section. In contrast to the BF method, deviations between actual and expected loss experience under the CL method do not correspond one-to-one to movements in ultimate; rather they are leveraged by the expected percentage developed at the future period. This makes sense because CL estimates of future losses depend on historical loss experience, whereas BF estimates of future losses are invariant to historical loss experience. The table below outlines these differences.

Table 9. AvE Statistic vs. Movement in Ultimate due to AvE Statistic.

Method	AvE statistic	Movement in ultimate due to loss experience
BF method	$(C_{k+1} - C_k) - uP(q_{k+1} - q_k)$	$(C_{k+1} - C_k) - uP(q_{k+1} - q_k)$
CL method	$(C_{k+1} - C_k) - {C_k}/{q_k}\,(q_{k+1} - q_k)$	$\left[(C_{k+1} - C_k) - {C_k}/{q_k}\,(q_{k+1} - q_k) \right] \times \left[{1}/{q_{k+1}} \right]$

3.3 Extensions

There are a number of extensions to the above analysis, some of which are considered below.

3.3.1 Movement in ultimate due to change in reinsurance recovery rate

While the above analysis could equally apply to gross or net projections, a common approach to netting down gross projections is to assume a recovery rate on the reserves (i.e., the percentage of gross reserves that might be recovered from reinsurers). Using r and $\hat{r}$ to refer to the current and proposed recovery rate with C_k referring to net of reinsurance losses (but P, u and q all still gross of reinsurance), the movement analysis when using the BF method is as follows:

Table 10. Decomposition of movement in ultimate for the BF method.

Movement in ultimate due to:	Formulation $(U_{k+1} - U_k)$
Loss experience	$= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)(1 - r)$
Change in premium	$+ (\hat{P} - P)u(1 - q_{k+1})(1 - r)$
Change in development pattern	$+ u\hat{P}[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})](1 - r)$
Change in IELR	$+ (\hat{u} - u)\hat{P}(1 - \hat{q}_{k+1})(1 - r)$
Change in recovery rate	$+ \hat{u}\hat{P}(1 - \hat{q}_{k+1})[(1 - \hat{r}) - (1 - r)]$

Note that the first four formulas in the above are very similar to those shown in Table 7, but multiplied by $1 - r$ and with gross losses replaced by net losses. The movement in ultimate due to change in recovery rate is then just the gross reserve multiplied by the change in recovery rate.

3.3.2 Movement in ultimate due to change in method

Consider the situation of switching between the BF and CL methods, perhaps because losses are believed to be sufficiently developed so that historical loss experience, rather than initial expectations, is more predictive of future emergence. Again, the question of in which order to consider these changes arises. In this situation, as different projection methods utilize different sets of data and assumptions, it makes sense to consider the change in method after any changes due to loss experience, but before changes in premium or assumptions. And when switching from the BF to CL method, this seems logical as the CL method uses neither premiums nor IELRs and thus these items are irrelevant to the change in ultimate.

With that in mind, the change in ultimate is decomposed as:

Table 11. Decomposition of movement in ultimate including change in method (BF to CL).

Movement in ultimate due to:	**Formulation** $\left(U_{k+1} - U_k\right)$
Loss experience	$= \left(C_{k+1} - C_k\right) - uP\left(q_{k+1} - q_k\right)$
Change in method	$+ C_{k+1}\left({}^{1}\!/\!{}_{q_{k+1}} - 1\right) - uP\left(1 - q_{k+1}\right)$
Change in development pattern	$+ C_{k+1}\left[\left({}^{1}\!/\!{}_{\hat{q}_{k+1}} - 1\right) - \left({}^{1}\!/\!{}_{q_{k+1}} - 1\right)\right]$

If moving from the CL to BF method, the movement in ultimate due to loss experience in the above would be set equal to the leveraged AvE statistic described in the previous section, the order of terms in the "change in method" would be reversed and the movement in ultimate due to change in premium, development pattern or IELR would all revert to those shown in Table 7. This is shown below:

Table 12. Decomposition of movement in ultimate including change in method (CL to BF).

Movement in ultimate due to:	**Formulation** $\left(U_{k+1} - U_k\right)$
Loss experience	$= \left(\left(C_{k+1} - C_k\right) - {}^{C_k}\!/\!{}_{q_k}\left(q_{k+1} - q_k\right)\right) \times \left({}^{1}\!/\!{}_{q_{k+1}}\right)$
Change in method	$+ uP\left(1 - q_{k+1}\right) - C_{k+1}\left({}^{1}\!/\!{}_{q_{k+1}} - 1\right)$
Change in premium	$+ \left(\hat{P} - P\right)u\left(1 - q_{k+1}\right)$
Change in development pattern	$+ u\hat{P}\left[\left(1 - \hat{q}_{k+1}\right) - \left(1 - q_{k+1}\right)\right]$
Change in IELR	$+ \left(\hat{u} - u\right)\hat{P}\left(1 - \hat{q}_{k+1}\right)$

Tables 11 and 12 above provide complete decompositions of the movement in ultimate into each of the key drivers of change. Note that these tables are combined into a complete analysis as presented in the Executive Summary.

3.3.3 Other

There are a number of other common scenarios for which the above can easily be extended including changes in data (i.e., relying on paid vs. incurred data), adjustments to the data, changes in currency, weighting between projection methods and so forth. That said, in practice the results might never be this clean. There could be other adjustments or idiosyncrasies involved (i.e., actuarial judgment) in the selection of ultimate loss that do not easily fall into one or another bucket and thus would be captured in a remaining catch-all residual which should ideally be minimal and explainable.

4. PRACTICAL EXAMPLE

To illustrate the application of this analysis, consider the following example. Tables A and B show exhibits illustrating the projection of ultimate loss at two subsequent year-ends. Here, Items (2), (3) and (5) are assumptions with Items (1) and (4) assumed to come from the data. The estimate of ultimate is then computed as (4) / (3) for the CL method or (4) + (1) x (2) x [1 – (3)] for the BF method. The ultimate loss ratio (ULR) is also shown.

Table A. Estimate of ultimate as at 31 December 2014.

Year	Premium	IELR	Pattern	Loss	Method	Selected Ultimate	ULR
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
2011			100%				
2012	10,000	65%	95%	5,916	CL	6,227	62%
2013	10,000	65%	85%	5,108	BF	6,083	61%
2014	10,000	65%	35%	3,337	BF	7,562	76%
Total	**30,000**					**19,872**	**66%**

Table B. Estimate of ultimate as at 31 December 2015.

Year	Premium	IELR	Pattern	Loss	Method	Selected Ultimate	ULR
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
2012	10,000	60%	99%	6,098	CL	6,160	62%
2013	10,000	65%	98%	6,321	CL	6,450	65%
2014	9,000	70%	80%	4,961	BF	6,221	69%
Total	**29,000**					**18,831**	**65%**

Table C then computes the movement analysis by applying the relevant formulas from Table 2. For example, the movement in ultimate due to loss experience for 2014 is solved as:

$$\begin{aligned} &= (C_{k+1} - C_k) - uP(q_{k+1} - q_k) \\ &= (4{,}961 - 3{,}337) - 65\% \times 10{,}000 \times (85\% - 35\%) \\ &= -1{,}626 \end{aligned}$$

While the remaining implementation can be found in the attached Excel workbook, note that there is no residual as the analysis described above fully decomposes the change in ultimate into each of the key drivers.

Table C. Movement Analysis.

Year	Change in Ultimate: Prior	Current	Change	Movement in ultimate due to change in: Experience	Method	Premium	Pattern	IELR	Residual
2012	6,227	6,160	(68)	(129)	0	0	62	0	0
2013	6,083	6,450	367	563	8	0	(204)	0	0
2014	7,562	6,221	(1,341)	(1,626)	0	(98)	293	90	0
Total	**19,872**	**18,831**	**(1,042)**	**(1,192)**	**8**	**(98)**	**150**	**90**	**0**

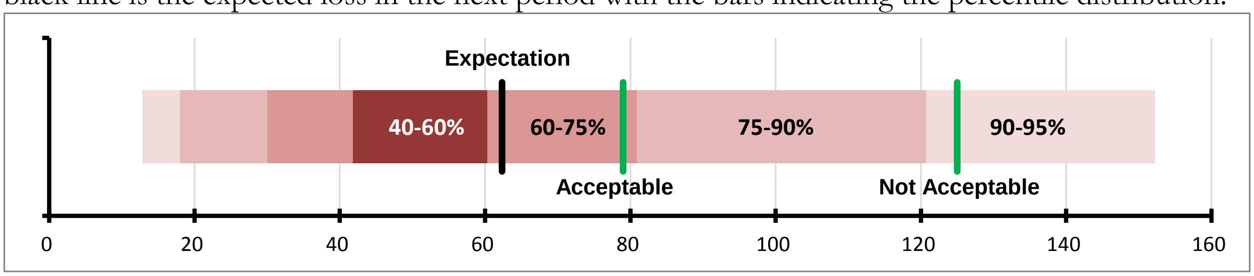
5. CONCLUSION

As actuaries become increasingly influential, there is an additional responsibility to move from opaqueness to transparency. By clearly communicating and quantifying the impacts of certain decisions, we can ensure that management has the appropriate information to assess, challenge or validate our recommendations and make their own determination as to the final carried amount.

The above presented several simple formulas for doing this on a deterministic and retrospective basis in a number of situations that commonly arise in actuarial practice.

That said, there are two useful and practical extensions of the above that are worth highlighting. The first involves moving toward reserve reports that not only isolate the key drivers of change between prior estimates, but also provide prospective estimates as to how losses are expected to emerge in future periods. This should enhance management information as emergence can then be monitored on a regular basis (rather than waiting until the next formal reserve review) and deviations from expectations can be flagged and explored in more detail. In regard to the latter, the other useful extension is to report not just expected emergence, but also to provide a range around that expectation so a determination can be made as to whether or not divergences from expectations are statistically significant.

As an example, consider the below figure which illustrates what this analysis might look like. The black line is the expected loss in the next period with the bars indicating the percentile distribution.



In this instance, emergence of 79 might be acceptable as it falls within the 75th percentile, but emergence around 125 might not be acceptable as it falls above the 90th percentile. In the former instance, the actuary might leave the key assumptions unchanged, but in the latter instance the actuary may wish to modify one or more assumptions as the deviation in claims experience relative to expectation appears to be statistically significant. This is more or less akin to hypothesis testing.

This should be especially doable in Europe as the formal implementation of Solvency II draws near where insurance risk is measured on a one-year basis and thus emergence profiles of loss as well as the distribution around those estimates should be readily available.

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A. TECHNICAL APPENDIX

The following derives some of the key formulas expressed in this paper with the remainder fairly straightforward but tedious to derive and thus omitted here for presentation purposes.

Table (3): $U_{k+1} - U_k = [C_{k+1} + uP(1-q_{k+1})] - [C_k + uP(1-q_k)]$

$$\begin{aligned}
&= [C_{k+1} + uP(1-q_{k+1})] - [C_k + uP(1-q_k)] \\
&= C_{k+1} + uP - uPq_{k+1} - C_k - uP + uPq_k \\
&= (C_{k+1} - C_k) - uP(q_{k+1} - q_k)
\end{aligned}$$

Table (4): $U_{k+1} - U_k = [C_{k+1} + uP(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)]$

$$\begin{aligned}
&= [C_{k+1} + uP(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)] \\
&= C_{k+1} + uP - uP\hat{q}_{k+1} - C_k - uP + uPq_k \\
&= (C_{k+1} - C_k) - uP\hat{q}_{k+1} + uPq_k + [uPq_{k+1} - uPq_{k+1}] \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + uP(q_{k+1} - \hat{q}_{k+1}) \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + uP[(1-\hat{q}_{k+1}) - (1-q_{k+1})]
\end{aligned}$$

Table (5): $U_{k+1} - U_k = [C_{k+1} + \hat{u}P(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)]$

$$\begin{aligned}
&= [C_{k+1} + \hat{u}P(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)] \\
&= C_{k+1} + \hat{u}P - \hat{u}P\hat{q}_{k+1} - C_k - uP + uPq_k \\
&= (C_{k+1} - C_k) + \hat{u}P - \hat{u}P\hat{q}_{k+1} - uP + uPq_k + [uPq_{k+1} - uPq_{k+1}] + [uP\hat{q}_{k+1} - uP\hat{q}_{k+1}] \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + uP(q_{k+1} - \hat{q}_{k+1}) + (\hat{u}P - uP) - (\hat{u}P\hat{q}_{k+1} - uP\hat{q}_{k+1}) \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + uP[(1-\hat{q}_{k+1}) - (1-q_{k+1})] + (\hat{u} - u)P(1-\hat{q}_{k+1})
\end{aligned}$$

Table (6): $U_{k+1} - U_k = [C_{k+1} + \hat{u}P(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)]$ (alternative formulation of Table 5)

$$\begin{aligned}
&= [C_{k+1} + \hat{u}P(1-\hat{q}_{k+1})] - [C_k + uP(1-q_k)] \\
&= C_{k+1} + \hat{u}P - \hat{u}P\hat{q}_{k+1} - C_k - uP + uPq_k \\
&= (C_{k+1} - C_k) + \hat{u}P - \hat{u}P\hat{q}_{k+1} - uP + uPq_k + [uPq_{k+1} - uPq_{k+1}] + [\hat{u}Pq_{k+1} - \hat{u}Pq_{k+1}] \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + (\hat{u}P - uP) - (\hat{u}Pq_{k+1} - uPq_{k+1}) + \hat{u}P(q_{k+1} - \hat{q}_{k+1}) \\
&= [(C_{k+1} - C_k) - uP(q_{k+1} - q_k)] + (\hat{u} - u)P(1-q_{k+1}) + \hat{u}P[(1-\hat{q}_{k+1}) - (1-q_{k+1})]
\end{aligned}$$

Table (7): $U_{k+1} - U_k = \left[C_{k+1} + \hat{u}\hat{P}(1-\hat{q}_{k+1})\right] - \left[C_k + uP(1-q_k)\right]$

$$
\begin{aligned}
&= \left[C_{k+1} + \hat{u}\hat{P}(1-\hat{q}_{k+1})\right] - \left[C_k + uP(1-q_k)\right] \\
&= C_{k+1} + \hat{u}\hat{P} - \hat{u}\hat{P}\hat{q}_{k+1} - C_k - uP + uPq_k \\
&= (C_{k+1} - C_k) + \hat{u}\hat{P} - \hat{u}\hat{P}\hat{q}_{k+1} - uP + uPq_k + \left[u\hat{P}q_{k+1} - u\hat{P}q_{k+1}\right] + \left[u\hat{P}\hat{q}_{k+1} - u\hat{P}\hat{q}_{k+1}\right] + \left[u\hat{P} - u\hat{P}\right] + \left[uPq_{k+1} - uPq_{k+1}\right] \\
&= \left[(C_{k+1} - C_k) - uP(q_{k+1} - q_k)\right] + \left(u\hat{P} - uP\right) - \left(u\hat{P}q_{k+1} - uPq_{k+1}\right) + u\hat{P}(q_{k+1} - \hat{q}_{k+1}) + \left(\hat{u}\hat{P} - u\hat{P}\right) - \left(\hat{u}\hat{P}\hat{q}_{k+1} - u\hat{P}\hat{q}_{k+1}\right) \\
&= \left[(C_{k+1} - C_k) - uP(q_{k+1} - q_k)\right] + \left(\hat{P} - P\right)u(1 - q_{k+1}) + u\hat{P}\left[(1 - \hat{q}_{k+1}) - (1 - q_{k+1})\right] + (\hat{u} - u)\hat{P}(1 - \hat{q}_{k+1})
\end{aligned}
$$

Table (8): $U_{k+1} - U_k = \frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_{k+1}}{q_k}$

$$
\begin{aligned}
&= \frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_k}{q_k} \\
&= \frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_k}{q_k} + \left[\frac{C_k}{q_{k+1}} - \frac{C_k}{q_{k+1}}\right] + \left[\frac{C_{k+1}}{q_{k+1}} - \frac{C_{k+1}}{q_{k+1}}\right] \\
&= \left[\frac{C_{k+1}}{q_{k+1}} - \frac{C_k}{q_k} + \frac{C_k}{q_{k+1}} - \frac{C_k}{q_{k+1}}\right] + \left(\frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_{k+1}}{q_{k+1}}\right) \\
&= \left[C_{k+1} - C_k \frac{q_{k+1}}{q_k} + C_k - C_k\right] \times \left(\frac{1}{q_{k+1}}\right) + \left(\frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_{k+1}}{q_{k+1}}\right) \\
&= \left[(C_{k+1} - C_k) - \frac{C_k}{q_k}(q_{k+1} - q_k)\right] \times \left(\frac{1}{q_{k+1}}\right) + \left(\frac{C_{k+1}}{\hat{q}_{k+1}} - \frac{C_{k+1}}{q_{k+1}}\right)
\end{aligned}
$$

Table (10): $U_{k+1} - U_k = \left[C_{k+1} + \hat{u}\hat{P}(1-\hat{q}_{k+1})(1-\hat{r})\right] - \left[C_k + uP(1-q_k)(1-r)\right]$

Omitted.

Table (11): $U_{k+1} - U_k = \left[\frac{C_{k+1}}{\hat{q}_{k+1}}\right] - \left[C_k + uP(1-q_k)\right]$

$$
\begin{aligned}
&= \left[\frac{C_{k+1}}{\hat{q}_{k+1}}\right] - \left[C_k + uP(1-q_k)\right] \\
&= \frac{C_{k+1}}{\hat{q}_{k+1}} - C_k - uP + uPq_k + \left[C_{k+1} - C_{k+1}\right] + \left[uPq_{k+1} - uPq_{k+1}\right] + \left[\frac{C_{k+1}}{q_{k+1}} - \frac{C_{k+1}}{q_{k+1}}\right] \\
&= \left[(C_{k+1} - C_k) - uP(q_{k+1} - q_k)\right] + \left(\frac{C_{k+1}}{q_{k+1}} - C_{k+1}\right) - uP + uPq_{k+1} + C_{k+1}\left(\frac{1}{\hat{q}_{k+1}} - \frac{1}{q_{k+1}}\right) \\
&= \left[(C_{k+1} - C_k) - uP(q_{k+1} - q_k)\right] + \left[C_{k+1}\left(\frac{1}{q_{k+1}} - 1\right) - uP(1 - q_{k+1})\right] + C_{k+1}\left[\left(\frac{1}{\hat{q}_{k+1}} - 1\right) - \left(\frac{1}{q_{k+1}} - 1\right)\right]
\end{aligned}
$$

Table (12): $U_{k+1} - U_k = \left[C_{k+1} + \hat{u}\hat{P}(1-\hat{q}_{k+1})\right] - \frac{C_k}{q_k}$

Omitted.