Calibration Of A Jump Diffusion

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1 Introduction

This paper outlines an application of a weighted Monte Carlo method to a jump diffusion model in the presence of clustering and runs suggestive of contagion. The paper was originally submitted as a master's thesis in the Mathematics in Finance program at the Courant Institute of Mathematical Sciences, New York University, on March 15, 2003. The author wishes to make the material available to a wider audience. Explanatory material has been added to make the paper easier to read. The mathematics is unchanged.

Although the motivation for this application is actuarial in nature, the method is not limited to insurance accidents. In fact, the method has broad application to financial analysis. The first equation below may be seen as a theoretical bridge between the two fields of Actuarial Science and Finance; the underlying processes have a common construct.

In its most general form, a sample path corresponding to a stochastic process which is *differential*, *homogeneous*, and *increasing* may be decomposed as a linear part plus a pure jump function [12][10]. The process is referred to as a homogenous differential process with increasing paths. The process will be defined in Section 2. For now, we state only the equation:

$$p_t = p(t) = mt + \int_0^\infty \ell \wp([0,t] \times d\ell); t \geq 0 \tag{1}$$

The term ℓ corresponds to the size of a jump. The function $\wp$ counts the number of jumps. In this paper, the counts will be Poisson distributed. The term $[0,t] \times d\ell$ corresponds to a set over which the counts are taken. The expression $\wp([0,t] \times d\ell)$ is the number of jumps occuring up to time t of severity between ℓ and $\ell + d\ell$. In other words, the integrand corresponds to the well-known actuarial phrase "frequency times severity".

The linear part of the decomposition is mt. If the linear term is dropped, i.e. $mt = 0$, then the resulting pure jump function resembles an insurance aggregate loss. Frequency of accidents times severity of those accidents are summed over a given population to obtain the total loss amount [8]. If the linear part is replaced by a Brownian motion with or without drift, a financial model results. It's important to note here that in the financial model, there is a stochastic differential equation where the Brownian motion is in the exponential. There is no exponential term in the actuarial model; a jump is a jump.

Avellaneda has calibrated a variety of financial instruments [1]. Throughout this reference, a pricing model refers to a model for pricing less liquid instruments relatively to more liquid instruments (the benchmarks). Calibration of the Monte Carlo model is performed by assigning probability weights to the simulated paths. The weights are derived by minimizing the Kullback-Leibler relative entropy of the posterior measure to the prior (empirical) measure.

Recall the definition of entropy and don't feel bad if you look it up in Wikipedia. Entropy is a measure of how evenly energy is distributed in a system. Entropy is a measure of order versus disorder or randomness. Relative entropy measures the entropy of one state as compared to the entropy of a second state. A very rough analogy can be found in the measurement of temperature. Temperature is measured with thermometers, which may be calibrated to a variety of temperature scales such as degrees Fahrenheit, Celsius, or Kelvin. Relative entropy would be very roughly similar to a comparison of any two of these three temperature scales.

2 Modelling A Jump Diffusion

Applications of jump diffusions include option pricing, credit risk, and actuarial science. Applications in option pricing and actuarial science are outlined in the history of jump diffusion models below. For a financial model of credit risk, a suggested reference is [15].

A stochastic process with sample paths $p(t)$, $p(0) = 0$ is said to be *differential* if its increments $p[t_1, t_2) = p(t_2) - p(t_1)$ over disjoint intervals $[t_1, t_2)$ are independent, *homogenous* if the law of $p[t_1 + s, t_2 + s)$is independent of $s(\geq 0)$, and *increasing* if $p(t_1) \leq p(t_2)$ for $t_1 \leq t_2$.

A sample path may be decomposed into a linear part plus an integral of Poisson processes:

$$p_t = p(t) = mt + \int_0^\infty \ell \wp([0, t] \times d\ell); t \geq 0 \tag{2}$$

$\wp(dt \times d\ell)$ being Poisson distributed with mean $dt \times \nu' d\ell$ where $d\nu = \nu' d\ell$, ν' being the density function of the measure $d\nu$. The measure ν shouldn't be too large in the sense that the integral is finite:

$$\int_0^1 \ell d\nu + \int_1^\infty d\nu < \infty \tag{3}$$

Then:

$$\mathcal{P}[\wp(B) = n] = \frac{\beta^n}{n!} e^{-\beta}; for\ n \geq 0, B \subset ([0, +\infty) \times (-\infty, +\infty)), \tag{4}$$

$$\beta = \int_B dt \nu' d\ell \tag{5}$$

The process p_t is *differential* because the counts $\wp(B)$ attached to disjoint $B \subset [0, +\infty) \times (0, +\infty)$ are independent, and *additive* in the sense that $\wp(\bigcup_{n \geq 1}) = \sum_{n \geq 1} \wp(B_n)$ for disjoint B_1, B_2, etc. $\subset ([0, +\infty) \times (0, +\infty))$. $\wp([t_1, t_2) \times [\ell_1, \ell_2))$ is just *the number of jumps of* $p(t); t_1 \leq t < t_2$ of magnitude $\ell_1 \leq \ell < \ell_2$.

Natural extensions of the basic model would include stochastic volatility, mean reversion, and multiple jump processes.

Stochastic volatility accommodates volatility clustering, an important feature of the data. Mean reversion may account for perturbations induced by diffusion vs. perturbations created by jumps. Multiple jump processes may be used to distinguish between types of jumps or sizes of jumps. Additionally, the jump sizes may be modelled by various distributions.

Each of these scenarios will be described in turn.

3 Jump Diffusion Processes In The Literature

3.1 The Initial Pricing Model

The first pure jump model in the financial literature is attributed to Cox, Ingersoll and Ross [4].The model is illustrated as follows in the usual discrete pricing diagram.

$$S \begin{smallmatrix} \nearrow Su \\ \searrow S\exp^{(-w\Delta t)} \end{smallmatrix}$$

The upward and downward probabilities for stock price movement in time Δt are $\lambda\Delta t$ and $1-\lambda\Delta t$ respectively. The asset price declines at rate w except for occasional jumps occurring as a Poisson process with rate λ. The size of the jumps are modelled as u times the current asset price S.

Criticism of the model notes that jumps here can only be positive which is unrealistic in the financial markets except for the probability of ruin. Note that the exponential term in the diagram above cannot yield a negative value. However, this criticism would not hold for insurance losses. For instance, the value of a property lost in a fire cannot be negative, one does not lose negative time on the job due to an injury, and so forth.

Arguably, a reserve for future indemnity benefits, as an example, may be posted and subsequently netted down due to the death of the claimant. In such a case, however, the life expectancy of the claimant would have been quantified and posted as the initial reserve. The resulting downward movement could be seen as parameter risk. In other words, if the reserve had been estimated with greater accuracy, the downward movement would not have occurred. Further such arguments could be made to show that a pure jump process is useful in modelling insurance losses.

Another criticism of the financial model is that the process leads to a distribution of stock price values with a fat right tail and a thin left tail, the opposite to that observed for equities. Such a distribution, however, is common in insurance and especially in reinsurance. The time lags in discovering and reporting losses such as medical malpractice or products liability create a fat right tail. inflationary and social trends in jury awards may be very different ten years hence, leading to unexpected increases in the size of awards.

In Actuarial Science, pricing models for aggregate distributions of claim data occur in the cohort approach to collective risk theory. An analysis of collective risk theory and insurance models is beyond the scope of this paper. Aggregate loss distributions have been widely discussed in the actuarial literature. The interested reader is referred to basic, comprehensive treatments [2] [8] [6].

3.2 An Application To Option Pricing

Merton first suggested a modification to the standard option pricing model, a jump function added to the Brownian motion term [13]. The jump component represents the occasional discontinuous breaks observed in the financial markets.

Define:

μ_B the expected return from an asset associated with the Brownian motion
σ_B the volatility of the Brownian motion
λ the rate of occurrence of a jump
κ the average jump size (amplitude) as the change in asset price.

Then the model is written in the following form:

$$\frac{dS}{S} = (\mu_B - \lambda\kappa)dt + \sigma_B dW + \kappa dq \tag{6}$$

where κ is drawn from a normal distribution $\kappa \sim N(\mu_J, \sigma_J^2)$ for μ_J and σ_J the mean and standard deviation of the jump respectively, W is a Wiener process, and q is a Poisson process generating the jumps. The processes W and q are assumed to be independent.

If $\lambda\kappa$ is the contribution from the jumps then the remainder $\mu_B - \lambda\kappa$ is the expected growth rate provided by the geometric Brownian motion.

In a special case of Merton's model, the logarithm of the jump amplitude is normally distributed. The European call option price is then written as:

$$C = \Sigma_{n=0}^{\infty} \frac{e^{(-\lambda' T)}(\lambda'\tau)^n}{n!} f_n \tag{7}$$

where $\tau = T - t$, $\lambda' = \lambda(1+\kappa)$ and f_n is the Black-Scholes option price with parameters

$$\sigma_n^2 = \sigma^2 + \frac{n\sigma^2}{\tau} \tag{8}$$

for σ, the standard deviation of the normal distribution, and

$$r_n = r - \lambda\kappa + \frac{n(ln(1+\kappa))}{\tau} \tag{9}$$

for r, the interest rate. Terminating the infinite sum is not problematic since the factorial function grows rapidly.

Note that the model gives rise to fatter left and right tails than Black-Scholes and is consistent with implied volatilities in currency options but not in insurance losses.

The key assumption in Merton's model is that the jump component of the asset return models non-systematic risk. This assumption would be difficult to make in an actuarial model where the jumps may represent a mixture of both systematic risk and non-systematic risk. Insurance claim sizes tend to cluster due to insurance policy limits, trends in jury awards, and claims adjusters' case reserving practices. The jump component of risk may involve contract law, procedural law, and insurance goodwill. A more sophisticated model incorporating multiple jump processes may be required.

3.3 Stochastic Volatility

A stochastic volatility model may be of the following form.

$$\frac{dS}{S} = \mu_B dt + \sigma dW + kdq \tag{10}$$

where $k \sim N(\mu_J, \sigma_J^2)$ and

$$d(ln\sigma^2) = b(\mu_h - ln(h^2))dt + cdZ \tag{11}$$

The logarithm of the variance σ^2 follows a mean-reverting process with the Wiener error term dZ. This model is termed a stochastic volatility jump diffusion process (SVJD)[5]. The model has constant jump amplitude and a mean-reverting process for the volatility. In other words, the path of the volatility parameter is a mean-reverting process. Note the drift is also a mean-reverting process.

3.4 Multiple Jump Processes

In the general form of the sample path, denote the pure jump process by $\mathcal{J}_t = \int_0^\infty \ell \wp([0,t] \times d\ell)$ and replace mt by $d\mathcal{W}$ where W is a Wiener process with drift $d\mathcal{W} = \sigma d\mathcal{Z} + \mu dt$.

$d\mathcal{Z}$ is an independent Gaussian shock
σ is the variance
μ is the drift

$\mathcal{J}_t$ may be further decomposed as a multiple jump process $\mathcal{J}_t = \mathcal{J}_t^1 + \mathcal{J}_t^2$ where $\mathcal{J}_t^1$ has jump-amplitudes ≤ 1 and $\mathcal{J}_t^2$ has jump-amplitudes >1. The $\mathcal{J}_t^1$ term may be comprised of an mininite number of small jumps. In financial and actuarial applications, the $\mathcal{J}_t^1$ term would assume a finite number of jumps or insurance losses in a given period of time.

Then, referring back to equation (2):

$$p_t = mt + \int_0^\infty \ell \wp([0,t] \times d\ell) = d\mathcal{W} + \mathcal{J}_t^1 + \mathcal{J}_t^2 = \sigma d\mathcal{Z} + \mu dt + \mathcal{J}_t^1 + \mathcal{J}_t^2 \tag{12}$$

3.5 Jump Amplitudes

The jump amplitudes can give rise to a variety of models. The large jumps may be seen as rare events relative to the background noise of the diffusion. The jump amplitude may be time dependent.

Denote the positive measure by ν and the associated density function as ν', as before in the general form of the model. The measure ν is the product of the Poisson rate λ and the size of the jump.

The measure ν may not be a probability measure. In such a case, the jump diffusion model is not of the compound Poisson type. Further, $\int \nu'(dx) = \int d\nu$ may not be finite.

Processes with an infinite number of jumps may be modelled by jump amplitudes with densities given by:

1. $\nu'(x) = A \mid x \mid^{-1} e^{(-\eta_{\pm}|x|)}$ a variance gamma function
2. $\nu'(x) = A_{\pm} \mid x \mid^{-(1+\alpha)} e^{(-\eta_{\pm}|x|)}$ a tempered ("truncated") stable process
3. $\nu'(x) = \frac{Ae^{(-\lambda x)}}{\sinh(x)}$ a Meixner process

Note that in these cases, the singularity occurs near the origin as the denominator approaches zero. The small jumps may be truncated or the singularities may be omitted by dropping the $\mathcal{J}_t^1$ term.

In cases where $\lambda = \int \nu'(x)dx < +\infty$, the measure is finite and the measure ν can be normalized to define a probability measure μ which can be interpreted as the distribution of jump sizes:

$$\mu(dx) = \frac{\nu(dx)}{\lambda}$$

In these cases, it may be shown that one necessarily obtains a compound Poisson process as in formula (1). Processes constituted by stochastic variation in both the number of jumps and amplitude of the jumps are termed compound processes. The independence of the variables denoting the number of jumps and the jump amplitudes follows from the assumption of independent increments for the sample paths. The jump amplitudes x_i and x_j are independent of each other $\forall\ i \neq j$ by independence of increments. Each x_i has the same distribution by homogeneity.

Consider sample paths. Let $X(t) = \sum_{n=0}^{N(t)} x_n$ where x_n are independent identically distributed random variables, $N(t)$ is a Poisson process with rate λ. The sum of the jumps is compound Poisson. Without loss of generality, we may assume $X(0) = 0$. Let $d\mu = \frac{d\nu}{\lambda}$ be the normalization of the measure to a probability measure. Compute the characteristic function of the sample path:

$$\mathbb{E}(e^{ikX(t)}) = \mathbb{E}(e^{ik\sum_{n=0}^{N(t)} x_n}) = \mathbb{E}(e^{ikx_0}...e^{ikx_{N(t)}}) \tag{13}$$

By independence of N and x:

$$= \mathbb{E}((\int e^{ikx} d\mu(x))^{N(t)}) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} (\int e^{ikx} d\mu(x))^j \tag{14}$$

$$= e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} (\int e^{ikx} d\mu(x))^j = e^{-\lambda t} (e^{\lambda t \int e^{ikx} d\mu(x)})$$

$$= e^{\lambda t (\int e^{ikx} d\mu(x) - 1)} = e^{\lambda t \int e^{ikx} d\mu(x) - \lambda t \int d\mu(x)}$$

Substituting the normalized probability measure, one obtains:

$$e^{t \int (e^{ikx(t)} - 1) d\nu(x)} \tag{15}$$

In a financial or actuarial application, the number of jumps per unit of time is finite so the application may be described by a jump process of compound Poisson type.

4 Contagion

The next step in this exposition is a description of contagion as it affects the statistical properties of the number of jumps seen as a random process. We introduce contagion into the jump diffusion process by considering a "mixed" compound Poisson process. This type of process is often used in actuarial work when one accident effectively increases the probability of future accidents through a conditional probability.

We begin at the beginning with Polya's urn scheme and Polya's scheme of contagion [7]. Suppose an urn contains b black balls and r red balls. A ball is drawn at random. The ball drawn is always replaced and in addition, c balls of the same color are added to the urn. The absolute probability of the sequence *black, black* is by Bayes' theorem below. Let H denote the first drawing of *black* and let A denote the second drawing of *black*. The sequence *black, black* is denoted by AH. If the first ball drawn is black, the conditional probability of a black ball at the second drawing is $\frac{(b+c)}{(b+c+r)}$. The probability of the sequence AH is, by Baye's theorem:

$$P[AH] = P[A \mid H]P[H] = \frac{b}{(b+r)} \times \frac{(b+c)}{(b+c+r)} \tag{16}$$

If the first two drawings result in black, the urn contains $b+2c$ black balls and $b+r+2c$ balls in total. The conditional probability of a black ball at the third trial becomes $\frac{(b+2c)}{(b+2c+r)}$. The probability of any sequence can be calculated in this way. The ordering of the sequence is immaterial. Any sequence of n drawings resulting in n_1 black and n_2 red balls for $n_1+n_2=n$ has the same probability as the sequence first n_1 black balls and then n_2 red balls given by:

$$P_{n_1,n_2} = \frac{b(b+c)(b+2c)...(b+n_1c-c)r(r+c)...(r+n_2c-c)}{(b+r)(b+r+c)(b+r+2c)...(b+r+nc-c)} \tag{17}$$

The Polya process describes a model for contagion, where every accident increases the probability of future accidents. The applications of the process include contagious diseases, meteorology, lattices in crystal structure, industrial quality control, and insurance, where long runs are suggestive of contagion or accumulated chance effect. It can be shown that the limiting form of Polya's distribution of probabilities is the negative binomial distribution. The limiting form may be used in a mixed compound Poisson process as the distribution for the Poisson rate variable λ [7].

5 Calibration Of The Pricing Model

The calibration methods are taken from a pricing model developed for the financial markets due to Avellaneda [1]. One purpose of the model is to price less liquid instruments relative to more liquid instruments. Avellaneda has calibrated a variety of financial instruments. Further work was done by Cont [3]. Avellaneda's model for bid-ask spreads admits a jump diffusion, an enhancement proposed by this paper. The enhancement will be shown in the next section.

5.1 Theory Of The Model

Consider a simulation with sample paths denoted by $\omega_1, ..., \omega_\nu$. Define a uniformly weighted simulation to be one where each path has equal probability of occurrence. In a non-uniformly weighted simulation, we assign probabilities $p_1, ..., p_\nu$ to each path where the probabilities are not necessarily equal.

For a contingent claim that pays the holder h_i dollars if the path ω_i occurs, the value of the contingent claim in the non-uniformly and uniformly weighted scenario where $p_i = \frac{1}{\nu}\ \forall i$ is:

$$\Pi_h = \sum_{i=1}^{\nu} h_i p_i \tag{18}$$

and

$$\Pi_h = \frac{1}{\nu}\sum_{i=1}^{\nu} h_i \tag{19}$$

respectively.

A prior distribution is generated by simulating the paths of a stochastic process which are uniformly weighted. Probabilities $p_1, ..., p_\nu$ are then determined to simulate a posterior distribution comprised of non-uniformly weighted sample paths.

For two probability vectors $p_1, ..., p_\nu$ and $q_1, ..., q_\nu$, the relative entropy of p with respect to q is defined as:

$$D(p \mid q) = \sum_{i=1}^{\nu} p_i \log(\frac{p_i}{q_i}) \tag{20}$$

For the Monte Carlo simulation with equal weights, denote the uniform probability vector by $u = (1/\nu, \dots, 1/\nu)$. Then substitute $q_i = 1/\nu \equiv u_i$ into the above equation to derive the relative entropy distance immediately below.

The calibrated posterior probability measure is found by minimizing the Kullback-Leibler relative entropy of the prior and posterior measures. The relative entropy distance

$$D(p \mid u) = \log \nu + \sum_{i=1}^{\nu} p_i \log p_i \tag{21}$$

measures the deviation of the calibrated model from the prior data. Note that $D \geq 0$ with equality holding only if $p_i = \frac{1}{\nu}$.

For $p_i = \frac{1}{\nu}$:

$$D(p \mid u) = \log \nu + \frac{1}{\nu} \log \prod_{i=1}^{\nu} p_i = \log \nu + \frac{1}{\nu} \log(\frac{1}{\nu})^{\nu} = 0 \tag{22}$$

The relative entropy is directly related to the support of the measure. Suppose $p_i = \frac{1}{\nu^{\alpha_i}}$ for $i = 1, 2, ..., \nu$. Let N_α represent the number of paths with $\alpha_i = \alpha$.

Then, $\sum_\alpha N_\alpha = \nu$, $\sum_\alpha \frac{N_\alpha}{\nu^\alpha} = 1$, and:

$$D(p \mid u) = \log \nu + \sum_{\alpha} \frac{N_\alpha}{\nu^\alpha} \log \frac{1}{\nu^\alpha} \tag{23}$$

which reduces to:

$$\log \nu - \sum_{\alpha} \frac{\alpha N_\alpha}{\nu^\alpha} \log \nu = \log \nu (1 - \sum_{\alpha} \frac{N_\alpha}{\nu^\alpha} \alpha) = \log \nu (1 - \mathbb{E}^p(\alpha)) \tag{24}$$

A small relative entropy corresponds to a large expected value of α. A small α corresponds to a thin support, which implies that a large number of paths are discarded by the algorithm. A small α may also be seen as a mismatch of probabilities between the prior and posterior measures since the measure will be concentrated on a small number of paths in the posterior measure. One sees that it all depends on the measure. Therein lies the difficulty in calibrating a jump diffusion when the frequency of jumps is small.

To elucidate the theory, denote the set of sample paths as: $\omega^{(i)} = (x_1(\omega^{(i)}), ..., x_N(\omega^{(i)})$ for

$i = 1, 2, ..., \nu$

and the associated stochastic differential equation with Wiener process W

$$dX = \sigma(X,t)dW + \mu(X,t)dt \tag{25}$$

Denote the market prices of N benchmark instruments by $C_1, ..., C_N$ and the present value of the jth cashflow as $g_{1j}, g_{2j}, ..., g_{\nu j}$ for $j = 1, ..., N$. The price relations for the benchmark instruments, for $j = 1, ..., N$ are:

$$\sum_{i=1}^{\nu} p_i g_{ij} = C_j \tag{26}$$

5.2 The Calibration Algorithm

As before, denote the uniform probability vector by

$$u = (\frac{1}{\nu}, ..., \frac{1}{\nu})$$

In the case of the prior measure, we consider the following minimization problem.

Minimize:

$$D(p \mid u) = \log \nu + \sum_{i=1}^{\nu} p_i \log p_i \tag{27}$$

under linear constraints $C_j = \sum_{i=1}^{\nu} p_i g_{ij}$ for Lagrange multipliers $\lambda_1, ..., \lambda_N$:

$$\min_{\lambda}[\max_{p}\{-\log \nu - \sum_{i=1}^{\nu} p_i \log p_i + \sum_{j=1}^{N} \lambda_j (\sum_{i=1}^{\nu} p_i g_{ij} - C_j)\}] \tag{28}$$

Consider the max first. Differentiate with respect to p_i, for fixed i and equate the derivative to the Lagrange multiplier ϕ for the additional constraint $\sum p_i = 1$:

$$-\log p_i - 1 + \sum_{j=1}^{N} \lambda_j g_{ij} = \phi$$

Let $\phi = -\mu - 1$

Then

$$-\log p_i - 1 + \sum_{j=1}^{N} \lambda_j g_{ij} = -\mu - 1$$

$$\log p_i = \mu + \sum_{j=1}^{N} \lambda_j g_{ij}$$

Let $e^{\mu} = \frac{1}{Z}$. Then the maximum occurs at the value $p_i^* \ \forall i$,

$$p_i^* = \frac{e^{(\sum_{j=1}^{N} \lambda_j g_{ij})}}{Z} \tag{29}$$

The constraint

$$\sum_{i=1}^{\nu} p_i = 1 = \sum_{i=1}^{\nu} \frac{e^{(\sum_{j=1}^{N} \lambda_j g_{ij})}}{Z}$$

shows that Z is a normalizing constant.

Note:

$$\log p_i = \sum_{j=1}^{N} \lambda_j g_{ij} - \log Z \tag{30}$$

at the max p.

Thus, at the maximum,

$$\max_p \{-\log \nu - \sum_{i=1}^{\nu} p_i \log p_i + \sum_{j=1}^{N} \lambda_j (\sum_{i=1}^{\nu} p_i g_{ij} - C_j)\}$$

$$= -\log \nu - \sum_{i=1}^{\nu} p_i (\sum_{j=1}^{N} \lambda_j g_{ij} - \log Z) + \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{\nu} p_i g_{ij} - \sum_{j=1}^{N} \lambda_j C_j$$

$$= -\log \nu + \sum_{i=1}^{\nu} p_i \log Z - \sum_{j=1}^{N} \lambda_j C_j = -\log \nu + \log Z - \sum_{j=1}^{N} \lambda_j C_j$$

Now consider the minimum:

$$\min_{\lambda} [-\log \nu + \log Z - \sum_{j=1}^{N} \lambda_j C_j]$$

Differentiate with respect to λ_k and equate to zero:

$$\frac{1}{Z} \frac{\partial}{\partial \lambda_k} \sum_{i=1}^{\nu} e^{(\sum_{j=1}^{N} \lambda_j g_{ij})} - C_k = \frac{1}{Z} \sum_{i=1}^{\nu} g_{ik} e^{(\sum_{j=1}^{N} \lambda_j g_{ij})} - C_k = 0$$

Let

$$V(\lambda) = -\log \nu + \log Z(\lambda) - \sum_{j=1}^{N} \lambda_j C_j \tag{31}$$

By substituting (28) into (27), one sees that the optimization of (27) is equivalent to minimizing $V(\lambda)$. For the minimizing λ_k so determined, define the calibrated instrument:

$$\frac{\partial V(\lambda)}{\partial \lambda_k} = \sum_{i=1}^{\nu} p_i g_{ik} - C_k = \mathbb{E}^p(g_k(\omega)) - C_k \tag{32}$$

where $g_{ik} = g_k(\omega_i) = g_k(\omega)$ and $\mathbb{E}^p(g_k(\omega)) = \sum_{i=1}^{\nu} p_i g_k(\omega_i)$

6 Calibration Of The Jump Diffusion Model

Avellaneda models a bid-ask spread by minimizing the relative entropy and the sum of the weighted least-squares residuals:

$$\chi_w^2 = \frac{1}{2} \sum_{j=1}^{N} \frac{1}{w_j} (\mathbb{E}^p(g_j(w)) - C_j)^2 \tag{33}$$

where $w = (w_1, ..., w_N)$ is a vector of positive weights.

6.1 The Minimization

Minimize:

$$D(p \mid u) + \chi^2_w \tag{34}$$

Preliminaries:

Denote $\mathbb{E}^p\{g_i(w))\}$ by E_i. Then $\chi^2_w = \sum_{i=1}^N \frac{1}{w_i}(E_i - C_i)^2$.

Let

$$a_i = \frac{1}{\sqrt{w_i}}(E_i - C_i)$$

and

$$-b_i = \lambda_i \sqrt{w_i}$$

Utilizing the inequality $\frac{1}{2}a^2 + \frac{1}{2}b^2 \geq ab$ (right hand side is the inner product) and summing over i:

$$\frac{1}{2}\sum_{i=1}^N \frac{1}{w_i}(E_i - C_i)^2 + \frac{1}{2}\sum_{i=1}^N w_i \lambda_i^2 \geq -\sum_{i=1}^N \lambda_i (E_i - C_i)$$

i.e.

$$\chi^2_w \geq -\sum_{i=1}^N \lambda_i (\mathbb{E}\{g_i(w)\} - C_i) - \frac{1}{2}\sum_{i=1}^N w_i \lambda_i^2$$

It follows that

$$\min_p [D(p \mid u) + \chi^2_w] \geq \max_\lambda \{\min_p [D(p \mid u) - \sum_{j=1}^N \lambda_j (\mathbb{E}^p\{g_j(w)\} - C_j)] - \frac{1}{2}\sum_{j=1}^N w_j \lambda_j^2\} \tag{35}$$

The next equality holds by the following logic: $max(x) = -min(-x)$.

$$= -\min_\lambda [-\min_p [D(p \mid u) - \sum_{j=1}^N \lambda_j (\mathbb{E}'\{g_j(w)\} - C_j)] - \frac{1}{2}\sum_{j=1}^N w_j \lambda_j^2]$$

and since the last two terms are independent of p

$$= -\min_\lambda [\max_p [-D(p \mid u) + \sum_{j=1}^N \lambda_j \mathbb{E}^p\{g_j(w)\}] - \sum_{j=1}^N \lambda_j C_j + \frac{1}{2}\sum_{j=1}^N w_j \lambda_j^2]$$

It can be shown that the inequality is in fact an equality.

Now let D be the entropy rather than relative entropy $D = \sum_{i=1}^{\nu} p_i \log p_i$, where the term $\log \nu$ is dropped without loss of generality.

Then

$$\max_p[-D + \sum_{j=1}^{N} \lambda_j \mathbb{E}^p(g_j)] = \max_p[-D + \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{\nu} p_i g_{ij}]$$

$$= \max_p[-\sum_{i=1}^{\nu} p_i \log p_i + \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{\nu} p_i g_{ij}]$$

$$= -\sum_{i=1}^{\nu} p_i \sum_{j=1}^{N} \lambda_j g_{ij} + \sum_{i=1}^{\nu} p_i \log Z + \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{\nu} p_i g_{ij}$$

where the last equality holds at the maximum $p = p^*$. This line now reduces to:

$$\sum_{i=1}^{\nu} p_i \log Z = \log Z = \log Z$$

since $\sum_{i=1}^{\nu} p_i = 1$) and Z does not depend on p.

Therefore

$$\min_p[D(p \mid u) + \chi_w^2] = -\min_\lambda[\log Z - \sum_{j=1}^{N} \lambda_j C_j + \frac{1}{2} \sum_{j=1}^{N} w_j \lambda_j^2] \tag{36}$$

$$= -\min_\lambda[V(\lambda) + \frac{1}{2} \sum_{j=1}^{N} w_j \lambda_j^2] \tag{37}$$

Here $V(\lambda) = \log Z(\lambda) - \sum_j \lambda_j C_j$ is the function used in the case of exact fitting.

Differentiating with respect to λ_k,

$$\frac{\partial V(\lambda)}{\partial \lambda_k} + w_k \lambda_k = \sum_{i=1}^{\nu} p_i g_{ik} - C_k + w_k \lambda_k$$

$$= \mathbb{E}^p\{g_k(w)\} - C_k + w_k\lambda_k = 0$$

and we have the optimal λ_k:

$$\lambda_k^* = -\frac{1}{w_k}[\mathbb{E}^{p^*}\{g_k(w)\} - C_k] \tag{38}$$

Note, the minimization over p is the same as in the case of exact fitting since χ_w^2 does not depend on p, and so leads to the same values of p_i^*:

$$p_i^* = \frac{1}{Z(\lambda_i^*)} e^{(\sum_{j=1}^N \lambda_j^* g_{ij})} \tag{39}$$

6.2 The Calibration Algorithm

The minimizing function in the case of least-squares fitting is

$$\log(Z(\lambda) - \sum_{j=1}^N \lambda_j(\mathbb{E}^{p^*}\{g_j(w)\} - C_j) + \frac{1}{2}\sum_{j=1}^N w_j\lambda_j^2 \tag{40}$$

This may be seen where, in the minimization of $W(\lambda)$, the term $\sum_{j=1}^N \lambda_j(\mathbb{E}^{p^*}\{g_j(w)\} - C_j)$ is substituted for $-\sum_{j=1}^N \lambda_j C_j$. The substitution occurs since our assumption $C_k = \mathbb{E}^p\{g_k(w)\}$ no longer holds.

The term $\mathbb{E}^{p^*}\{g_j(w)\} - C_j$ is precisely the modelled bid-ask spread, the bid-ask spread being a small constant-valued jump. This term, however, may be any constant value. C_k is a constant, the instantaneous price observed in the market. As an expected value, $\mathbb{E}^{p^*}\{g_j(w)\}$ is not a stochastic term. In fact, an expected value is a constant.

If we replace the bid-ask spread with a larger jump term, the minimization is essentially unchanged. The mispriced asset value represented by the bid-ask spread is replaced by a larger mispriced value representing a shock. This paper proposes that if the shock occurs as a compound Poisson process, one may replace the bid-ask spread by the expected value of the compound Poisson jump.

The exhibits of the next section illustrate the concepts. See [9], [11], and [14] for background material, basic concepts, and formulas.

7 An Example: Exhibits

U.S. Treasury Yield Curve Rates as of 12/16/02
Term Structure of Interest Rates

Maturity	Zero Price	Zero Yields (continuous)
1	**0.985**	0.015
2	**0.962**	0.0194
3	**0.932**	0.0235
5	**0.855**	0.0314
7	**0.769**	0.0375
10	**0.660**	0.0415

`

Zero Yields for various maturities obtained from the U.S. Treasury website.

Zero Price calculated using continuous compounding and fit to a parabola with R squared = 99.75% to obtain Zero Price via function.

Fiting Parameters:
y = c + b*(x-d)^2

1.483766	c
-0.00053	b
-29.3765	d

P = exp (-r * t)

Valuing Coupon Bonds	
0 (nowyear)	0
s (zeroyear)	10
Bond Face Value (L)	1.00
Bond Coupon (cL)	0.05

Bond value (using ytm)	Bond yield to maturity
1.072	4.03%

ln (1+IRR (FV cash flows))

Check On Bond value **1.072**

Maturity	Zero Price (via function)	PV Bond Cash Flows	FV Bond Cash Flows	Zero Yields (via function)	Forward Rates (via function)	PV Bond Cash Flows (continuous)
0			-1.072			
1	0.992	0.050	0.050	0.84%	0.84%	0.048
2	0.959	0.048	0.050	2.11%	3.38%	0.046
3	0.925	0.046	0.050	2.61%	3.61%	0.044
4	0.890	0.044	0.050	2.93%	3.87%	0.043
5	0.853	0.043	0.050	3.17%	4.15%	0.041
6	0.816	0.041	0.050	3.38%	4.46%	0.039
7	0.778	0.039	0.050	3.59%	4.80%	0.038
8	0.739	0.037	0.050	3.79%	5.19%	0.036
9	0.698	0.035	0.050	3.99%	5.63%	0.035
10	0.657	0.690	1.050	4.20%	6.12%	0.702

time in years

fitted bond prices
these are PV

price * FV cashflow

coupons + face value
FV (full value) cash
flows are undiscounted

- ln (p_i) / t_i

- ln (p_i+1 / p_i)

CF * exp (-t_i * IRR)

Zero Yields and Forward Prices both based on Zero Price.

Exhibit 7.1

---- Descriptive Statistics for Variables ----

Variable	Minimum value	Maximum value	Mean value	Standard dev.
x	1	10	4.666667	3.386247
y	0.66	0.985	0.8605	0.1261091

---- Calculated Parameter Values ----

Parameter	Initial guess	Final estimate	Standard error	t	Prob(t)
c	1	1.48376593	0.5056422	2.93	0.06078
b	1	-0.000533394524	0.0004189297	-1.27	0.29264
d	1	-29.3764531	27.38353	-1.07	0.36199

---- Analysis of Variance ----

Source	DF	Sum of Squares	Mean Square	F value	Prob(F)
Regression	2	0.07931547	0.03965773	588.88	0.00013
Error	3	0.0002020314	6.73438E-005		
Total	5	0.0795175			

Fitting Program Output:

```
1: Title "Parabola";
2: Variables x,y;
3: Parameters c,b,d;
4: Function y = c + b*(x-d)^2;
5: plot;
6: data;
```

Beginning computation...

---- Final Results ----

NLREG version 5.3
Copyright (c) 1992-2002 Phillip H. Sherrod.

Parabola
Number of observations = 6
Maximum allowed number of iterations = 500
Convergence tolerance factor = 1.000000E-010
Stopped due to: Relative function convergence.
Number of iterations performed = 37
Final sum of squared deviations = 2.0203139E-004
Final sum of deviations = -3.3234526E-012
Standard error of estimate = 0.00820633
Average deviation = 0.00518032
Maximum deviation for any observation = 0.00895358
Proportion of variance explained (R^2) = 0.9975 (99.75%)
Adjusted coefficient of multiple determination (Ra^2) = 0.9958 (99.58%)
Durbin-Watson test for autocorrelation = 2.021
Analysis completed 29-Dec-2002 16:38. Runtime = 0.04 seconds.

Exhibit 7.2

Vasicek Model : see Hull (4th Edition) pp 567-9

RN model dr = a(b-r) dt + sr dz

a	**0.1779**	**Zero-coupon bond price**	
b	**0.0866**	P(0,s)	0.6006
r	**1.50%**	via fn	0.6006
0 (nowyr)	0.00		
s (zeroyr)	**10.00**	**Zero yield**	
zero life	10.00	R(0,s)	5.10%
σ_r	**2.00%**	via fn	5.10%
B(0,s)	4.6722	**Zero yield (infinite maturity)**	
A(0,s)	0.6442	R(∞)	8.02%
		Volatility of zero yield	
		σ_R(0,s)	0.93%

Cox, Ingersoll and Ross Model : see Hull (4th edition) pg 570

RN model dr = a(b-r) dt + s sqrt(r) dz

a	**0.2339**	**Zero-coupon bond price**	
b	**0.0808**	P(0,s)	0.5752
r	**1.50%**	via fn	0.5752
0 (nowyr)	0.00		
s (zeroyr)	**10.00**	**Zero yield**	
zero life	10.00	R(0,s)	5.53%
σ	**0.0200**	via fn	5.53%
γ	0.2356	**Zero yield (infinite maturity)**	
exp(γ(s-0))-1	9.5491	R(¥)	8.05%
B(0,s)	3.8547		
A(0,s)	0.6094	**Volatility of zero yield**	
		σP(0,σ)	0.09%

Vasicek Term Structure

Maturity	Vasicek Zero Yield	Zero Yield Volatility	Zero Price	Vasicek Forward Rate
	5.10%	0.93%		
0	1.50%	2.00%		
1	2.09%	1.83%	0.979	2.09%
2	2.61%	1.68%	0.949	3.13%
3	3.07%	1.55%	0.912	3.98%
4	3.47%	1.43%	0.870	4.68%
5	3.83%	1.32%	0.826	5.25%
6	4.14%	1.23%	0.780	5.72%
7	4.42%	1.14%	0.734	6.11%
8	4.67%	1.07%	0.688	6.43%
9	4.90%	1.00%	0.644	6.69%
10	5.10%	0.93%	0.601	6.91%

CIR Term Structure

Maturity	CIR Zero Yield	Zero Yield Volatility	Zero Price	CIR Forward Rate
	5.53%	0.09%		
0	1.50%	0.24%		
1	2.21%	0.22%	0.978	2.21%
2	2.82%	0.20%	0.945	3.44%
3	3.35%	0.18%	0.904	4.40%
4	3.80%	0.16%	0.859	5.17%
5	4.20%	0.14%	0.811	5.77%
6	4.54%	0.13%	0.762	6.25%
7	4.84%	0.12%	0.713	6.63%
8	5.10%	0.11%	0.665	6.93%
9	5.33%	0.10%	0.619	7.16%
10	5.53%	0.09%	0.575	7.35%

Exhibit 7.3

Using Monte Carlo Simulation to Value Zero Yields

Random Numbers (Excel)

Vasicek stochastic DE	
1st term	0.00127
2nd term	0.02000
CIR stochastic DE	
1st term	0.00154
2nd term	0.00245

r + dr	**Vasicek**	**CIR**
MC value	**0.0120**	**0.0160**
MC stdev	**0.0214**	**0.0026**

Vasicek Model : see Hull (4th Edition) p567-9

RN model dr = a(b-r) dt + sr dz

a	0.1779	**Zero-coupon bond price**	
b	0.0866	P(0,s)	0.6006
r	1.50%	MC Value	0.6089
0 (nowyr)	0.00	**Zero yield**	
s (zeroyr)	10.00	R(0,s)	5.10%
zero life	10.00	MC value	4.96%
σ_r	2.00%		
dt	0.10	**Volatility of zero yield**	
B(0,s)	4.6722	$\sigma_R(0,s)$	0.93%
A(0,s)	0.6442	MC value	1.00%

Cox, Ingersoll and Ross Model : see Hull (4th edition) p570

RN model dr = a(b-r) dt + s sqrt(r) dz

a	0.2339	**Zero-coupon bond price**	
b	0.0808	P(0,s)	0.5752
r	1.50%	MC Value	0.5729
0 (nowyr)	0.00	**Zero yield**	
s (zeroyr)	10.00	R(0,s)	5.53%
zero life	10.00	MC Value	5.57%
σ_r	2.00%		
dt	0.10		
γ	0.2356	**Volatility of zero yield**	
exp(γ(s-0))-	9.5491	$\sigma_R(0,s)$	0.09%
B(0,s)	3.8547	MC Value	0.01%
A(0,s)	0.6094		

Exhibit 7.4

Using Monte Carlo Simulation to Value Zero Yields

Quasi Random Numbers

Vasicek stochastic DE	
1st term	0.00127
2nd term	0.02000
CIR stochastic DE	
1st term	0.00154
2nd term	0.00245

r + dr	**Vasicek**	**CIR**
MC value	0.0154	0.0164
MC stdev	0.0194	0.0024

Vasicek Model : see Hull (4th Edition) pp 567-9

RN model dr = a(b-r) dt + sr dz

a	0.1779	**Zero-coupon bond price**	
b	0.0866	P(0,s)	0.6006
r	1.50%	MC Value	0.5993
0 (nowyr)	0.00	**Zero yield**	
s (zeroyr)	10.00	R(0,s)	5.10%
zero life	10.00	MC value	5.12%
σ_r	2.00%		
dt	0.10	**Volatility of zero yield**	
B(0,s)	4.6722	$\sigma_R(0,s)$	0.93%
A(0,s)	0.6442	MC value	0.91%

Cox, Ingersoll and Ross Model : see Hull (4th edition) pg 570

RN model dr = a(b-r) dt + s sqrt(r) dz

a	0.2339	**Zero-coupon bond price**	
b	0.0808	P(0,s)	0.5752
r	1.50%	MC Value	0.5720
0 (nowyr)	0.00	**Zero yield**	
s (zeroyr)	10.00	R(0,s)	5.53%
zero life	10.00	MC Value	5.59%
σ_r	2.00%		
dt	0.10		
γ	0.2356	**Volatility of zero yield**	
exp(γ(s-0))-	9.5491	$\sigma_R(0,s)$	0.09%
B(0,s)	3.8547	MC Value	0.01%
A(0,s)	0.6094		

Exhibit 7.5

Using Monte Carlo Simulation to Value Bond Prices
Comparison of Results
MRE (Minimum Relative Entropy) Prior Distribution

Zero Yield

time in years	Vasicek equation	Vasicek MC1	Vasicek MC2	CIR equation	CIR MC1	CIR MC2
0	1.50%	1.64%	1.42%	1.50%	1.45%	1.49%
1	2.09%	1.93%	2.03%	2.21%	2.23%	2.22%
2	2.61%	2.39%	2.57%	2.82%	2.84%	2.84%
3	3.07%	3.12%	3.03%	3.35%	3.41%	3.38%
4	3.47%	3.61%	3.45%	3.80%	3.83%	3.84%
5	3.83%	3.93%	3.81%	4.20%	4.26%	4.24%
6	4.14%	4.17%	4.14%	4.54%	4.57%	4.58%
7	4.42%	4.47%	4.43%	4.84%	4.89%	4.89%
8	4.67%	4.79%	4.68%	5.10%	5.16%	5.15%
9	4.90%	5.03%	4.91%	5.33%	5.39%	5.38%
10	5.10%	5.27%	5.12%	5.53%	5.59%	5.59%
Bond Price	**0.54706**	**0.54012**	**0.54478**	**0.52858**	**0.52478**	**0.52521**

Using Monte Carlo Simulation to Value Bond Prices
Comparison of Results
MRE (Minimum Relative Entropy) Prior Distribution

Forward Rate

time in years	Vasicek equation	Vasicek MC1	Vasicek MC2	CIR equation	CIR MC1	CIR MC2
0	2.09%	1.93%	2.03%	2.21%	2.23%	2.22%
1	2.09%	1.93%	2.03%	2.21%	2.23%	2.22%
2	3.13%	2.85%	3.10%	3.44%	3.45%	3.46%
3	3.98%	4.59%	3.97%	4.40%	4.57%	4.45%
4	4.68%	5.08%	4.69%	5.17%	5.08%	5.22%
5	5.25%	5.22%	5.27%	5.77%	5.97%	5.84%
6	5.72%	5.37%	5.76%	6.25%	6.12%	6.32%
7	6.11%	6.27%	6.15%	6.63%	6.82%	6.70%
8	6.43%	7.05%	6.48%	6.93%	7.05%	7.00%
9	6.69%	6.95%	6.75%	7.16%	7.23%	7.24%
10	6.91%	7.35%	6.98%	7.35%	7.39%	7.42%

Exhibit 7.6

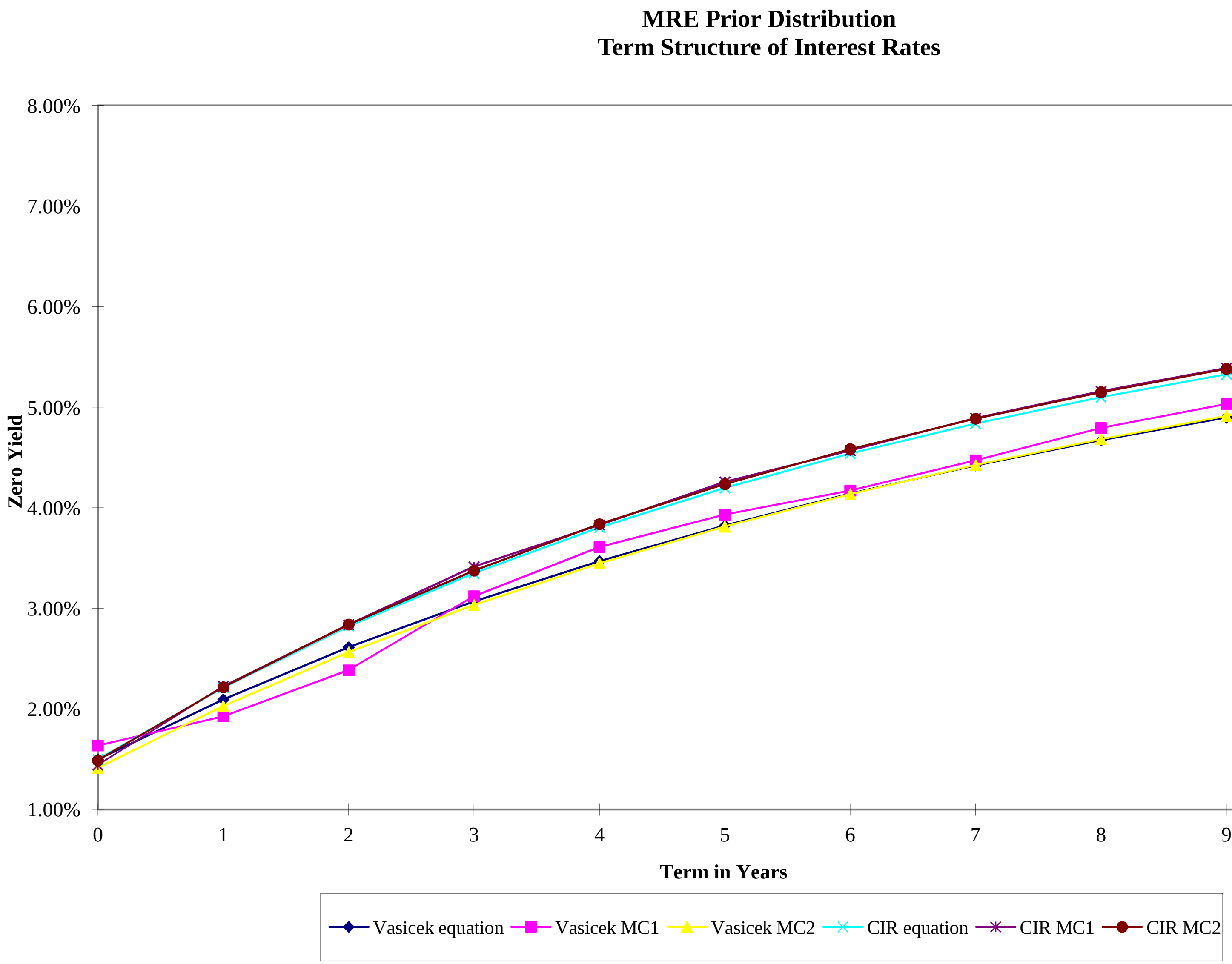
MRE Prior Distribution
Term Structure of Interest Rates
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
Zero Yield
0
1
2
3
4
5
6
7
8
9
10
Term in Years
Vasicek equation
Vasicek MC1
Vasicek MC2
CIR equation
CIR MC1
CIR MC2

Exhibit 7.7

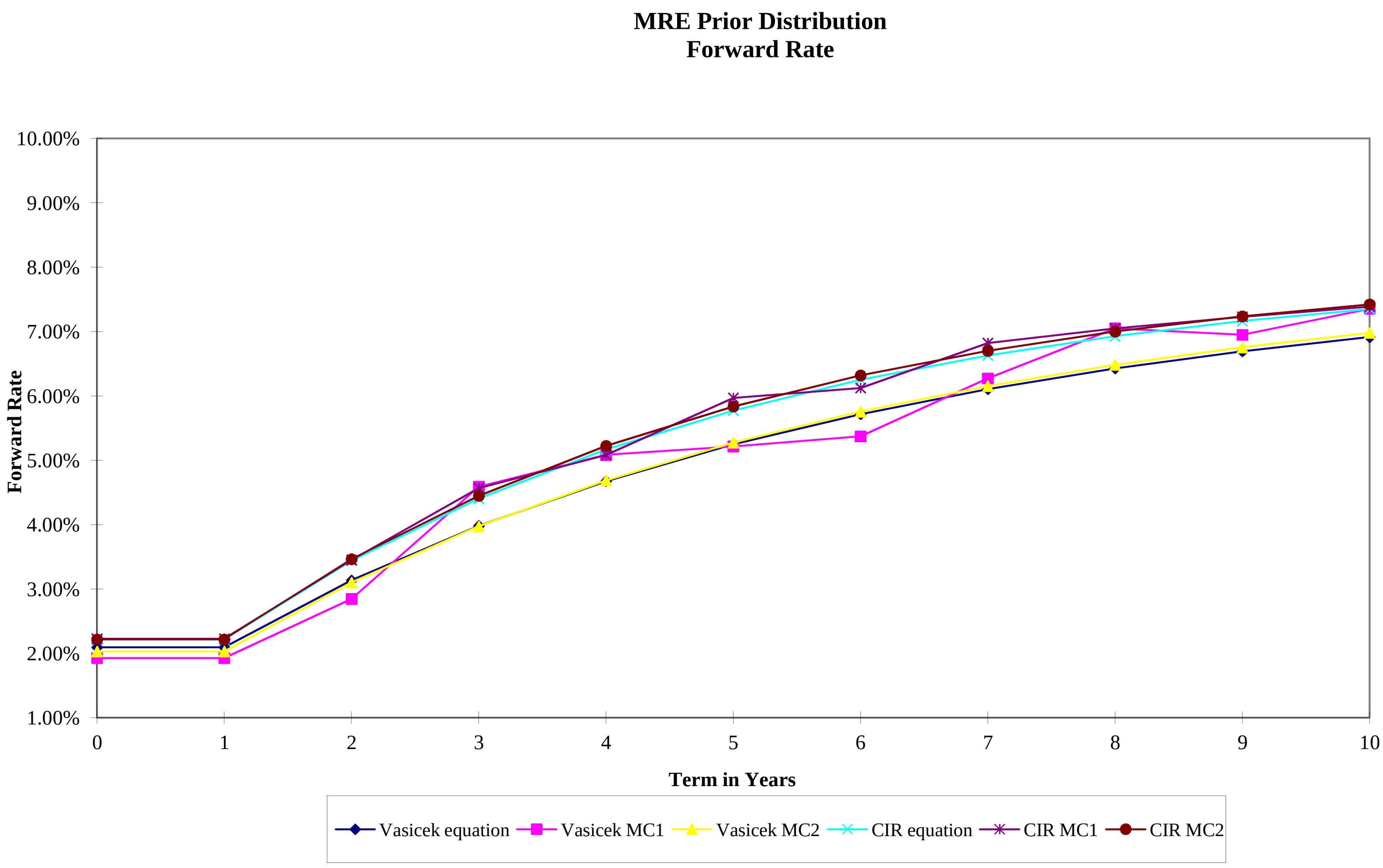
MRE Prior Distribution
Forward Rate
10.00%
9.00%
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
Forward Rate
0
1
2
3
4
5
6
7
8
9
10
Term in Years
Vasicek equation
Vasicek MC1
Vasicek MC2
CIR equation
CIR MC1
CIR MC2

Exhibit 7.8

Initial Bond Portfolio Value	0.522890

Final Bond Portfolio Value	0.506413

Benchmark instruments N=5, nu=11

price C_1	PV cash flow g_1j	price C_2	PV cash flow g_2j	price C_3	PV cash flow g_3j	price C_4	PV cash flow g_4j	price C_5	PV cash flow g_5j
0.693	0.059	0.608	0.050	0.523	0.040	0.438	0.030	0.353	0.020
	0.058		0.048		0.038		0.029		0.019
	0.055		0.046		0.037		0.028		0.018
	0.053		0.044		0.036		0.027		0.018
	0.051		0.043		0.034		0.026		0.017
	0.049		0.041		0.033		0.024		0.016
	0.047		0.039		0.031		0.023		0.016
	0.044		0.037		0.030		0.022		0.015
	0.042		0.035		0.028		0.021		0.014
	0.039		0.033		0.026		0.020		0.013
	0.194		0.192		0.191		0.189		0.187

The Vector Of Benchmark Prices C_j

0.692669	0.607779	0.522890	0.438000	0.353111

The Vector Of Final Values p_i

0.173543	0.171769	0.193366	0.209694	0.251628

The Matrix Of Present Valued Cash Flows g_ij

0.059495	0.049579	0.039663	0.029748	0.019832
0.057519	0.047932	0.038346	0.028759	0.019173
0.055479	0.046232	0.036986	0.027739	0.018493
0.053374	0.044479	0.035583	0.026687	0.017791
0.051206	0.042672	0.034137	0.025603	0.017069
0.048974	0.040811	0.032649	0.024487	0.016325
0.046677	0.038898	0.031118	0.023339	0.015559
0.044317	0.036931	0.029545	0.022158	0.014772
0.041892	0.034910	0.027928	0.020946	0.013964
0.039404	0.032837	0.026269	0.019702	0.013135
0.194332	0.192498	0.190665	0.188832	0.186999

The Matrix Of Lagrange Multipliers lambda*_j

-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000
-0.026586	-0.047202	0.171634	0.389973	1.000000

Exhibit 7.9

MRE: Find LaGrange Multipliers
Maximum allowed number of iterations = 500
Convergence tolerance factor = 1.000000E-010

Number of iterations performed = 5
Final function value = 4.9005938E-017
Analysis completed 2-Jan-2003 07:54. Runtime = 0.02 seconds.

---- Calculated Parameter Values ----

Parameter	Initial guess	Final estimate
L1	1	-0.26586057
L2	1	-0.0472018362
L3	1	0.171634464
L4	1	0.389972719
L5	1	1

Exhibit 7.10

Expected Value Of The Jump	0.255222

Severity	*Frequency*
0.10	6
0.15	263
0.20	741
0.25	765
0.30	503
0.35	265
0.40	111
0.45	31
0.50	13
0.55	2
0.60	0

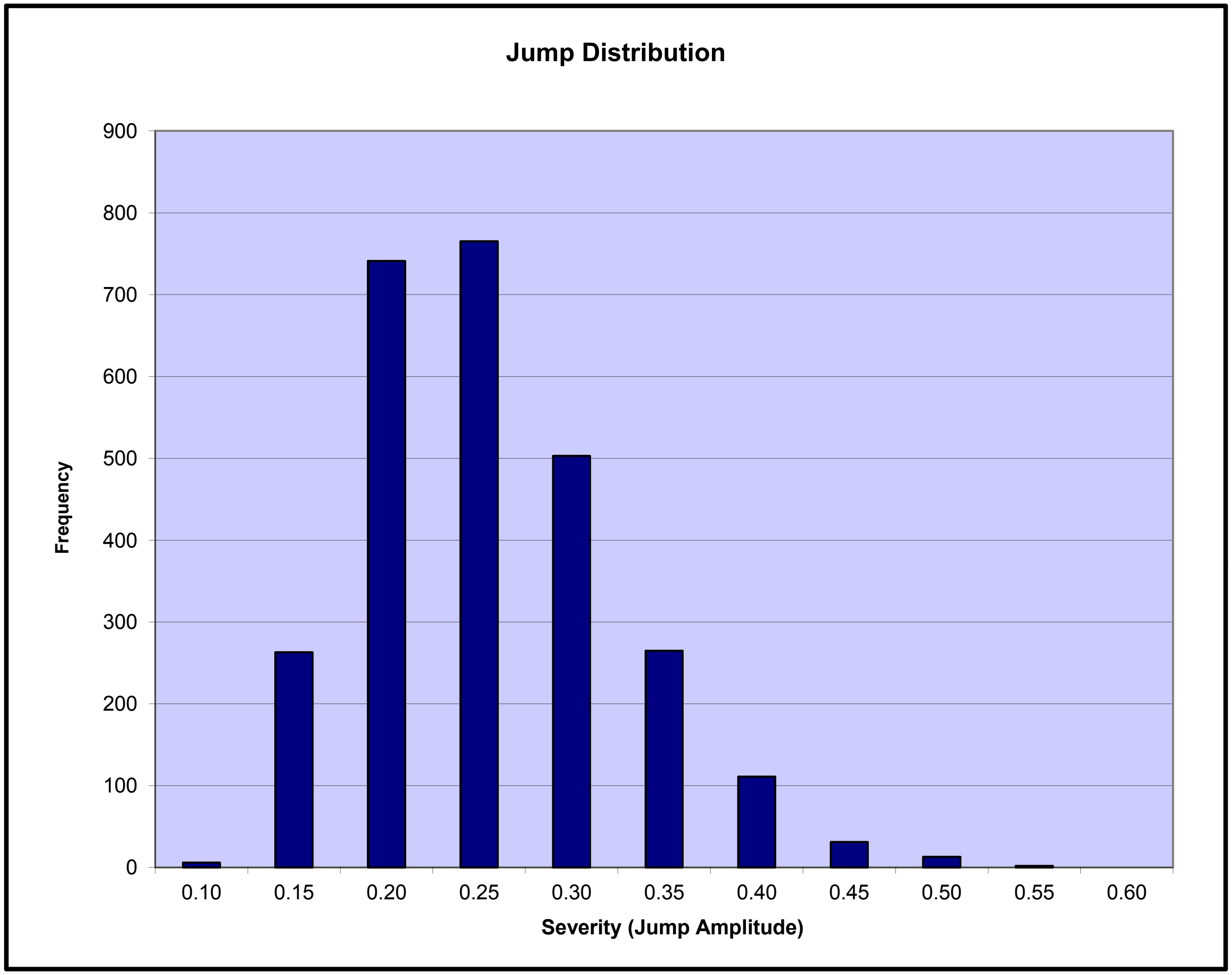

Exhibit 7.11

Initial Bond Portfolio Value	0.522890
Final Bond Portfolio Value	0.504591

Benchmark instruments N=5, nu=11

price C_1	PV cash flow g_1j	price C_2	PV cash flow g_2j	price C_3	PV cash flow g_3j	price C_4	PV cash flow g_4j	price C_5	PV cash flow g_5j
0.693	0.059	0.608	0.050	0.523	0.040	0.438	0.030	0.353	0.020
	0.058		0.048		0.038		0.029		0.019
	0.055		0.046		0.037		0.028		0.018
	0.053		0.044		0.036		0.027		0.018
	0.051		0.043		0.034		0.026		0.017
	0.049		0.041		0.033		0.024		0.016
	0.047		0.039		0.031		0.023		0.016
	0.044		0.037		0.030		0.022		0.015
	0.042		0.035		0.028		0.021		0.014
	0.039		0.033		0.026		0.020		0.013
	0.194		0.192		0.191		0.189		0.187

The Vector Of Benchmark Prices C_j

0.692669	0.607779	0.522890	0.438000	0.353111

The Matrix Of Present Valued Cash Flows g_ij

0.059495	0.049579	0.039663	0.029748	0.019832
0.057519	0.047932	0.038346	0.028759	0.019173
0.055479	0.046232	0.036986	0.027739	0.018493
0.053374	0.044479	0.035583	0.026687	0.017791
0.051206	0.042672	0.034137	0.025603	0.017069
0.048974	0.040811	0.032649	0.024487	0.016325
0.046677	0.038898	0.031118	0.023339	0.015559
0.044317	0.036931	0.029545	0.022158	0.014772
0.041892	0.034910	0.027928	0.020946	0.013964
0.039404	0.032837	0.026269	0.019702	0.013135
0.194332	0.192498	0.190665	0.188832	0.186999

The Vector Of Final Values p_i

0.159167	0.177378	0.197673	0.220289	0.245493

The Matrix Of Lagrange Multipliers lambda*_j

-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111
-1.276111	-1.276111	-1.276111	-1.276111	-1.276111

Exhibit 7.12

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