Reinsurance Arrangements Minimizing the Total Required Capital

Yingjie Zhang, PhD, FCAS
The Cincinnati Insurance Companies
PO Box 145496, Cincinnati, OH 45250
E-mail: yingjie_zhang@cinfin.com

Abstract

Reinsurance reduces the required capital of the primary insurer but increases that of the reinsurer. Capital is costly. All capital costs, including that of the reinsurer, are ultimately borne by primary policyholders. Reducing the total capital of insurers and reinsurers lowers the total capital cost and the total primary policy premium. A reinsurance arrangement is considered optimal if it minimizes the total required capital. This optimal reinsurance is shown to be an attracting equilibrium under price competition. Evidence suggests that there is an inverse relationship between the total required capital and the correlation between the losses held by different insurers. Examples are constructed to support this observation.

Keywords

Required capital, capital cost, optimal reinsurance, subadditive risk measure, correlation between losses

Acknowledgements

I thank Neb Bojer, Evan Glisson, Art Zaremba and Ka Chun Yeung for reviewing the paper. They have helped me correct errors and improve presentation.

1 Introduction

A new type of optimal reinsurance is introduced in this paper. Reinsurance serves many purposes, one of which is to reduce the required capital by lessening the volatility of losses. From the shareholder point of view, capital is costly because of income taxes and agency costs. Shareholders pay income taxes two times on their capital investment, first at the corporate level and then at the personal level when they sell the stock. They would not owe the corporate tax if they invested directly in the securities market. Agency costs exist because of the separation of ownership and control. They include monitoring and bonding expenditures and other losses in profits due to a misalignment of managers' decisions and shareholders' welfare. Taxes and agency costs, altogether called capital costs, generally have an increasing relationship with the amount of capital (Jensen and Meckling 1976, Perold 2005, Chandra and Sherris 2006, Zhang 2008). Thus carrying less capital is desirable.

Reinsurance transfers losses from a ceding company to a reinsurer. Such losses are often highly volatile. So this transfer of losses increases the capital requirement of a reinsurer while reducing that of a ceding company. Consequently, capital costs of the reinsurer increase and those of the ceding company decrease. The total capital cost, the sum of that of both companies, may go either way. Capital costs are funded by premium. Primary policy premiums include charges to cover primary insurers' capital costs; reinsurance premiums include charges to cover reinsurers' capital costs. But reinsurance premiums are funded through premiums of primary policies. Therefore, the total capital costs of primary insurers and reinsurers are ultimately borne by primary policyholders. If a treaty reduces a ceding company's capital costs more than it increases the reinsurer's, the total capital cost is reduced, which benefits primary policyholders. A treaty, or a set of treaties, is optimal, if it minimizes the total capital cost. Such optimal reinsurance arrangements are the subject of this paper.

Numerous authors have written about optimal reinsurance and have proposed various optimality criteria. My approach is noticeably different. Usually an optimal reinsurance is defined from the ceding company's point of view. The ceding insurer seeks a treaty to maximize its risk-adjusted return (Lampaert and Walhin 2005, Fu and Khury 2010), to minimize the variance of its net loss (Kaluszka 2001, Lampaert and Walhin 2005), or to minimize the tail risk of the net loss (Gajek and Zagrodny 2004, Cai and Tan 2007), under the constraint of a given premium principle that links the ceded premium to the ceded loss. This line of research is valuable. However, it does not pay enough attention to the profit target of the reinsurer. Although the proposed premium principles usually include risk margins reflecting the volatility of the ceded loss, they generally ignore the fact that the reinsurer needs to put up more capital thus incurring greater capital costs. My approach places the ceding insurer and the reinsurer on an equal footing and addresses the capital costs of both directly. A reinsurance arrangement that minimizes the total capital is the best deal for the combined welfare of primary insurers, reinsurers and policyholders.

Under reasonable assumptions, minimization of the total capital cost is equivalent to minimization of the total amount of capital carried by all companies. This latter problem may be directly solved by simulating insurers' and reinsurers' losses. A remarkable fact, however, is that this type of optimal reinsurance need not be solved by any one party. (In fact, neither the ceding insurer nor the reinsurer can obtain the full knowledge of the joint probability distribution of losses of both parties.) Market forces automatically push the insurer and the reinsurer to select treaties with less total capital costs. In other words, an optimal reinsurance arrangement is an attracting equilibrium.

The capital requirement will be set by a risk measure. In this paper, I assume that the risk measure is coherent, as defined in Artzner et al. (1999). For such a risk measure, there is an absolute lower bound for the total capitals. Regardless of reinsurance arrangements, the total capital must be greater than this lower bound. It can be shown that if the losses of the insurers have a certain correlation called comonotonicity (defined in Section 5), then the total capital attains the lower bound. This observation leads to a discussion on the relationship between optimal reinsurance and correlated losses. Evidence suggests that an optimal treaty is one that makes the losses of insurers and reinsurers as correlated as possible. (Such correlation needs only occur at the tail.)

The main part of the paper is organized as follows. In Section 2, I prove that minimization of the total primary insurance premium leads to minimization of the total capital. I then show in Section 3 that price competition tends to produce this type of optimal reinsurance. Coherent risk measures are discussed in Section 4. In Section 5, I point out that a lower bound exists for the total required capital, and in some cases an inverse relationship exists between the sum of capitals and the correlation between losses. Section 6 contains a general formulation of the optimal reinsurance problem. Examples are given in Section 7 to further examine the link between the sum of capitals and correlation. Section 8 concludes the paper.

2 Why Minimize the Total Required Capital?

In this section I will rigorously prove that, if a reinsurance arrangement minimizes the total capital cost, then it minimizes the aggregate premium of primary policyholders. I will also point out the exact conditions under which minimization of the total capital cost is equivalent to minimization of the total amount of capital.

Policyholders purchase insurance to protect themselves against unexpected losses. At the same time, they also provide funds to cover all operating costs of the insurance company, including underwriting and claim expenses, income taxes, agency costs and reinsurance costs. The reinsurance costs, in turn, cover the reinsurer's expenses, taxes and agency costs, and *its* reinsurance costs (costs of retrocession). Ultimately, it is the primary insurance policyholders that bear the operating costs of primary insurers and reinsurers. For the insurance/reinsurance market as a whole, reinsurance treaties rearrange these costs among all insurers and reinsurers. Some reinsurance arrangements result in lower total costs than others. A reinsurance arrangement is optimal if the total cost is minimized, in which case the primary policyholders pay the lowest aggregate premium.

This paper focuses on minimizing the total capital cost, consisting of income taxes and agency costs.[1] To cleanly study the capital cost, I assume that the aggregate underwriting and claim expenses remain constant under various reinsurance arrangements. Therefore, these expenses can be excluded from consideration. The gross insurance premium of a policy can be decomposed into the following components

$$p = \mathrm{PV(Loss)} + \mathrm{PV(Tax)} + \mathrm{PV(Agency\ Cost)} + \mathrm{Reinsurance\ Premium}. \quad (2.1)$$

The p in (2.1) represents the fair premium, which is the exact amount to fund all insurer's costs related to the policy. Equation (2.1) is a version of the net present value principle. Slightly different formulas for the fair premium have appeared in the literature (Myers and Cohn 1987, Taylor 1994, Vaughn 1998). Each term on the right-hand side of (2.1) provides the exact amount to cover that specific type of cost. The PV's represent risk-adjusted present values. The loss in the first term is the net loss. It is assumed here that the present value of insured loss satisfies the following two basic requirements of the fair value accounting: (1) The value PV(Loss) is independent of the carrier of the insurance policy.[2] (2) The function PV( ) is additive. The two conditions together eliminate the possibility

[1] Agency costs include any cost associated with the issue of "separation of ownership and control", as discussed in Jensen and Meckling (1976), Perold (2005), like monitoring and bonding expenditures and other losses in profits due to misalignment of managers' decisions and shareholders' welfare.

[2] The risk-adjusted PV can be viewed as the risk-free discounted expected cash flow plus a risk margin, where the risk margin reflects the market, or the systematic risk of the cash flow. It is sometimes argued that the fair value of losses should be affected by its carrier's default risk. In this paper, I only consider insurance firms that hold the required level of capital and whose risk of default is negligible.

of arbitrage. In particular, they imply that PV(Gross Loss) = PV(Net Loss) + PV(Ceded Loss).

I now examine the relationship between the gross fair premium and the total amount of capital held by insurers and reinsurers. Consider a one-year model containing only one loss to be shared between a primary insurer and a reinsurer. Let p be the gross premium charged by the primary insurer at the beginning of the year and L the random gross loss paid at the end of the year. The primary insurer collects the premium p then cedes an amount p_c to the reinsurer, retaining $p_n = p - p_c$. Similarly for losses, $L_n = L - L_c$, where L_c is the ceded loss and L_n the net loss.

The total income tax is the sum of two charges, one on the income generated by premiums, which equals the underwriting profit plus the investment income on premiums, and the other on the investment income generated by capital. To write premium formulas in a concise way, I use the following notations

e_{Pr}	:	capital carried by the primary insurer
e_{Re}	:	capital carried by the reinsurer
t_{Pr}	:	average tax rate for the primary insurer
t_{Re}	:	average tax rate for the reinsurer

The present value of tax for the primary insurer is of the form $t_{Pr}(p_n - \mathrm{PV}(L_n)) + u_{Pr}e_{Pr}$, and that for the reinsurer is $t_{Re}(p_c - \mathrm{PV}(L_c)) + u_{Re}e_{Re}$, where the u's are constants: if r_f represents the risk-free rate, then $u_{Pr} = t_{Pr} \cdot r_f/(1+r_f)$ and $u_{Re} = t_{Re} \cdot r_f/(1 + r_f)$. (A derivation of the multiplier $r_f/(1 + r_f)$ can be found in Cummins 1990). Agency costs generally increase with the amount of capital.[3] For simplicity, I assume there is a linear relationship: for some constants s_{Pr} and s_{Re}, the present value of agency cost is $s_{Pr}e_{Pr}$ for the primary company and $s_{Re}e_{Re}$ for the reinsurer.

Following (2.1), for the primary insurer, we have

$$p = \mathrm{PV}(L_n) + t_{Pr}(p_n - \mathrm{PV}(L_n)) + u_{Pr}e_{Pr} + s_{Pr}e_{Pr} + p_c \tag{2.2}$$

and, for the reinsurer (if there is no retrocession),

$$p_c = \mathrm{PV}(L_c) + t_{Re}(p_c - \mathrm{PV}(L_c)) + u_{Re}e_{Re} + s_{Re}e_{Re}. \tag{2.3}$$

[3] An important type of capital cost is the cost of financial distress, which increases as capital becomes more insufficient. But firms considered in this paper satisfy a given capital requirement. So the cost of financial distress is ignored.

An equation for the fair gross premium p can be obtained by substituting (2.3) into (2.2). p is the sum of the following four terms.

1. The present value of loss: $\mathrm{PV}(L_n) + \mathrm{PV}(L_c) = \mathrm{PV}(L)$, which does not vary with reinsurance.

2. The tax on the incomes generated by premium: $t_{Pr}(p_n - \mathrm{PV}(L_n)) + t_{Re}(p_c - \mathrm{PV}(L_c))$. On the condition that the tax rates are equal, $t_{Pr} = t_{Re} = t$, this term is $t(p - \mathrm{PV}(L))$, which decreases as p decreases.

3. The tax on the incomes generated by capital: $u_{Pr}e_{Pr} + u_{Re}e_{Re}$. If the applicable tax rates are the same, then $u_{Pr} = u_{Re} = u$, and the term equals $u(e_{Pr} + e_{Re})$, which decreases if a reinsurance contract lowers the sum of capitals, $e_{Pr} + e_{Re}$.

4. The agency cost: $s_{Pr}e_{Pr} + s_{Re}e_{Re}$. If the cost factors are equal, $s_{Pr} = s_{Re} = s$, then the term equals $s(e_{Pr} + e_{Re})$, again a direct function of the total capital $e_{Pr} + e_{Re}$.

To sum up, as reinsurance varies, the loss component $\mathrm{PV}(L)$ remains constant, while the fair premium p varies because taxes and agency costs vary. p is lower if the present values of taxes and agency costs are lower. Under the above assumptions, this is equivalent to a less amount of total capital, $e_{Pr} + e_{Re}$. The optimal reinsurance is then defined as the one that minimizes $e_{Pr} + e_{Re}$. An optimal treaty creates the least gross premium, so is best for the policyholder.

This definition can be generalized to an insurance market with many primary insurers and reinsurers, and many primary policyholders. Assume each primary insurer covers a given set of policyholders. There are a great number of ways in which each insurer buys reinsurance and each reinsurer enters retrocession agreements. A set of reinsurance/retrocession arrangements is called optimal if it minimizes the total capital cost of the insurers and reinsurers. With the condition that all companies have identical tax rates and agency cost factors, this criterion is equivalent to minimizing the total amount of capital.[4]

[4] It has been pointed out to me that reinsurers usually have a different tax rate than primary companies. If tax rates or agency cost factors are not all equal, or the costs are not all linear to the capital, then the optimal treaty is one that minimizes some increasing function of the capitals.

3 Market Competition Produces Lower Total Capital

Minimization of the total capital cost is a new optimality criterion. Criteria in the existing literature are very different; see Kaluszka (2001), Gajek and Zagrodny (2004), Lampaert and Walhin (2005), Cai and Tan (2007) and Fu and Khury (2010) for a sample of recent papers. In these papers, reinsurance is considered optimal if it minimizes the risk of the net loss under a given constraint on the reinsurance cost (or a constraint on the ceded premium). This line of research is valuable for reinsurance purchase decisions but is incomplete. A major concern of reinsurance has been missing. The reinsurer needs additional capital to accommodate the increased risk from assumed losses, which increases its capital cost. This extra cost is transferred to the ceding company through reinsurance pricing. To the ceding company, if this extra cost is not offset by the reduction of its own capital cost, the deal is not acceptable. My method treats the ceding insurer and the reinsurer equally. The optimal treaty is fair to both firms and is the most beneficial to the primary policyholder. Obviously, an optimal reinsurance treaty so defined cannot be calculated by either company since one company cannot model the other company's aggregate loss distribution. Fortunately, it is not necessary to explicitly calculate the optimal treaty terms. As long as each company correctly prices its own policies, the optimal treaty is automatically attained through price competition. I will use a few examples to illustrate the working of this market force.

Let us begin with a simple scenario. Assume a primary insurer has written a line of business and would like to cede a part of it. Denote by f_{Pr} the amount of capital cost saved by reinsurance. The reinsurer incurs extra capital costs associated with the assumed loss. It charges the primary insurer an additional premium, denoted by f_{Re}, to cover these costs.[5] So the primary insurer pays an amount of premium f_{Re} to save an amount of cost f_{Pr}. The reinsurance only makes sense if $f_{Re} \le f_{Pr}$, which means the sum of the capital costs of both companies must decrease.

Assume further that there are two competing reinsurers; a treaty placed with reinsurer 1 costs the primary insurer a premium $f_{Re,1}$ to save a capital cost $f_{Pr,1}$, and one placed with reinsurer 2 costs $f_{Re,2}$ to save a capital cost $f_{Pr,2}$. The immediate (present value) benefits from the treaties are $f_{Pr,1} - f_{Re,1}$ and $f_{Pr,2} - f_{Re,2}$, respectively. The insurer would choose the reinsurer with the greater benefit,

[5] Rigorously, f_{Pr} and f_{Re} represent risk-adjusted present values of the corresponding capital cost cash flows.

which is the one producing the lower total capital cost.

Now look at an example where primary insurers choose reinsurance to compete with each other for business. Suppose that a line of business is on the market and two insurers are bidding. Suppose each insurer has a set of available reinsurance options. As proved in Section 2, the fair gross premium includes a capital cost component that equals the present value of the total capital cost of the insurer and the reinsurer. To win the bid, an insurer looks for a reinsurance treaty that can produce the lowest possible total capital cost. Eventually, the business will go to the insurer able to secure a reinsurance with so low a total capital cost that the other cannot match. Obviously, an insurer's ability to get a more competitive reinsurance deal depends on its existing business and capital structure.

The above analysis shows that market competition always favors a reinsurance structure that produces less total capital cost. Consequently, a reinsurance structure with the least total capital cost is an attracting equilibrium.

4 Capital Requirement Defined by a Coherent Risk Measure

Suppose a uniform capital requirement is imposed on all insurers by regulation. I will only deal with the loss risk, that is, the risk that L becomes very large. The required capital can be defined by a risk measure on the loss distribution. A class of risk measures considered desirable are the coherent risk measures. According to Artzner et al. (1999), risk measure ρ is called coherent if it satisfies the following conditions:

- Monotonicity: For any two losses, L_1 and L_2, if $L_1 \leq L_2$, then $\rho(L_1) \leq \rho(L_2)$
- Positive homogeneity: For any loss L and a constant $a > 0$, $\rho(aL) = a\rho(L)$
- Translation invariance: For any loss L and a constant b, $\rho(L+b) = \rho(L)+b$
- Subadditivity: For any two losses, L_1 and L_2, $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$

All these properties have simple intuitive meanings. Most important to this study is subadditivity. Subadditivity implies diversification: When two risks are pooled together, the required capital of the pool is less than the sum of the required capitals of each risk.

A typical property/casualty loss is a continuous random variable, that is, its cumulative distribution function $F_L(x)$ is continuous. The p-quantile of L is defined by

$$Q_p(L) = \min\{x | F_L(x) \geq p\}, \qquad p \in (0, 1), \tag{4.1}$$

and the tail value at risk (TVaR) at level p is

$$\text{TVaR}_p(L) = E[L | L \geq Q_p(L)], \qquad p \in (0, 1). \tag{4.2}$$

The TVaR is the most well-known coherent risk measure for continuous risks. (The quantile, also called the value at risk, does not always respect subadditivity.) The TVaR will be used in my illustrative examples.

Suppose a coherent risk measure ρ is selected by the regulator. Then $\rho(L)$ is the amount of assets a company is required to hold. In a one-year model, the premium provides part of the assets at the beginning of the year; the required capital thus equals the required assets minus the premium. Following Section 2, I examine reinsurance structures that minimize the sum of the required capitals of the insurer and the reinsurer. This is equivalent to the problem of minimizing the sum of their required assets,[6] i.e., minimizing the sum of their risk measures. Note that the required assets should be calculated from the loss distribution at the end of the year and discounted back to the beginning of the year. I ignore the discounting here for simplicity.

5 Lower Bound of Total Capitals and Comonotonicity

Reconsider the simplified model with a single loss L, one primary insurer and one reinsurer. The primary insurer issues a policy to cover the entire loss L and cedes part of it to the reinsurer. Thus, L is split between the two insurers, $L = L_{Pr} + L_{Re}$. For a given coherent risk measure ρ, by the rule of subadditivity, $\rho(L) \leq \rho(L_{Pr}) + \rho(L_{Re})$. This inequality provides an absolute lower bound for the sum of capitals: however L is split between the two insurers, the sum of their required assets is no less than $\rho(L)$. To minimize the total required capital is to get the sum $\rho(L_{Pr}) + \rho(L_{Re})$ as close to $\rho(L)$ as possible.

The lower bound can be attained by many reinsurance arrangements. One trivial case is that $L_{Pr} = L$ and $L_{Re} = 0$, or $L_{Pr} = 0$ and $L_{Re} = L$, that is, only one insurer holds all of L. This fact is no surprise, for if there is only one insurer

[6] This can be explained using equations (2.2) and (2.3). The assets for the insurer are $p_n + e_{Pr}$, and that for the reinsurer are $p_c + e_{Re}$. It is proved in Section 2 that the total (gross) fair premium $p_n + p_c$ decreases as the total capital $e_{Pr} + e_{Re}$ decreases. If a reinsurance treaty minimizes the total required assets, it must simultaneously minimize the total required capital and the total fair premium.

and all losses are insured with it, the effect of diversification is maximized, and the least amount of capital is required. An extension of this fact is that an insurance market with few insurers requires less total amount of capital than a market with many insurers. But few insurers means less competition, and insurers have less incentive to price policies fairly.

The lower bound is also reached by the quota share reinsurance. If a is the quota share ceding fraction ($0 < a < 1$), then $L_{Pr} = (1-a)L$ and $L_{Re} = aL$. The equality $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$ follows from the rule of positive homogeneity of ρ. More generally, if two losses L_1 and L_2 are perfectly linearly correlated, that is, their linear (Pearson) correlation coefficient equals 1, then $\rho(L_1 + L_2) = \rho(L_1) + \rho(L_2)$. Therefore, if a reinsurance treaty splits L into two linearly correlated parts, then the sum of their required capitals is minimized. The condition of perfect linear correlation can rarely be fulfilled. Fortunately, it can be much relaxed in the following two steps. First, although some kind of perfect correlation has to exist between two losses, L_1 and L_2, for their risk measures to add up, the correlation does not have to be linear—any monotonic and increasing relationship suffices. Second, a perfect correlation only needs to exist at the tail, for large values of L_1 and L_2. Mathematically, both these issues have been well treated in the literature, as explained below.

Let me first give the definition of comonotonicity. Two random variables, X and Y, are perfectly linearly correlated (the linear correlation coefficient of X and Y equals 1) if and only if their support lies in a straight line with a positive slope. (Recall that the support is the set of all possible values of X and Y in the (x, y)-plane. It can be visualized by drawing a scatter plot. A scatter plot of a pair of random variables is merely a small, random subset of its support.) Comonotonicity is an extention of perfect linear correlation. Two random variables, X and Y, are called comonotonic, if their support lies in a one-dimensional curve that is never decreasing. More precisely, the support of a pair of comonotonic random variables satisfies the following condition: if, for any two points in the support, (x_1, y_1) and (x_2, y_2), $x_1 < x_2$ implies $y_1 \leq y_2$ and $y_1 < y_2$ implies $x_1 \leq x_2$. A good overview of comonotonicity and its application in risk theory is Dhaene et al. (2006), where comonotonicity is defined for any number of random variables. Comonotonicity can be considered a perfect nonlinear correlation. For example, if X is a positive random variable, then X and X^2 are comonotonic but not linearly correlated. The support of (X, X^2) is contained in the graph of parabola $y = x^2$. The Spearman rank correlation coefficient is a more meaningful measure than the linear correlation coefficient for characterizing such a nonlinear relationship. The rank correlation coefficient of two comonotonic random variables equals 1 (see Wang 1998), while their linear correlation coefficient is typically less than 1.

The TVaR is a coherent risk measure and is also additive for comonotonic risks: If two losses L_1 and L_2 are comonotonic, then $\mathrm{TVaR}_p(L_1 + L_2) = \mathrm{TVaR}_p(L_1) + \mathrm{TVaR}_p(L_2)$ for any p (Dhaene et al. 2006).[7] In the one-insurer-one-reinsurer model, assume the required asset is determined by a risk measure ρ that is coherent and additive for comonotonic risks. If L is split in such a way that L_{Pr} and L_{Re} are comonotonic, then $\rho(L_{Pr}) + \rho(L_{Re})$ reaches its lower bound $\rho(L)$. We have seen that the quota share reinsurance splits the loss this way. Another example is the stop-loss reinsurance, which is defined by

$$L_{Pr} = \min(L, k), \qquad L_{Re} = \max(L - k, 0), \tag{5.1}$$

where $k > 0$ is the attachment point. It is easy to check that the three variables L, L_{Pr} and L_{Re} are comonotonic, and $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$.

Risk measures like $Q_p(L)$ and $\mathrm{TVaR}_p(L)$ are determined by large values of L. When looking for a way to split L into L_{Pr} and L_{Re} to minimize the total capital, one should focus on large losses. The condition of comonotonicity requires the entire support of the random vector to be in a one-dimensional non-decreasing curve. This condition is too strong. Cheung (2009) introduces the concept of upper comonotonicity, only requiring the condition to be satisfied in the upper tail. If L_{Pr} and L_{Re} are upper comonotonic, then $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$, where ρ is either Q_p or TVaR_p and p is sufficiently close to 1. In general, the amount of total capital corresponding to a reinsurance structure is determined by large losses only.

6 Optimal Reinsurance in a General Setting

I now apply the concepts developed so far to formulate a general problem about optimal reinsurance. I have discussed the problem of splitting a single loss L between an insurer and a reinsurer. In the real world, a primary insurer does not have the option or the intension to cover its entire book with a reinsurance treaty. It only attempts to cede some unwanted lines or accounts. On the other hand, a reinsurer assumes losses from many insurers and reinsurers. A new treaty adds losses to its existing book. When determining the optimal reinsurance, one needs

[7] There are other risk measures that are coherent and additive for comonotonic risks, e.g., the concave distortion risk measures. The VaR is additive for comonotonic risks but is not coherent (Dhaene et al. 2006).

to consider these "other" loss portfolios of the ceding insurer and the reinsurer, in addition to the loss portfolio to be ceded. The following model includes all these sets of losses.

Assume a primary insurer initially carries losses $X + Z$, where X will be entirely retained and Z may be partially ceded. The reinsurer holds a loss Y before assuming any part of Z. A reinsurance treaty splits Z into a net and a ceded part, $Z = Z_n + Z_c$. Before reinsurance, the total required asset of the insurer and the reinsurer is $\rho(X + Z) + \rho(Y)$. After reinsurance, the total required asset is $\rho(X + Z_n) + \rho(Y + Z_c)$. A treaty is optimal if the latter sum is minimized.

If ρ is a coherent risk measure, an absolute lower bound for $\rho(X + Z_n) + \rho(Y + Z_c)$ is $\rho(X + Y + Z)$. In general, the distributions of losses X, Y and Z and correlations between them are complex. There is no ceding arrangement that can bring down the sum $\rho(X + Z_n) + \rho(Y + Z_c)$ to anywhere near this lower bound. Moreover, in the reinsurance market, only a few types of treaties are commonly placed, like the quota share, excess of loss, catastrophe and stop loss treaties. This further limits how low $\rho(X + Z_n) + \rho(Y + Z_c)$ can become. Minimizing the sum $\rho(X + Z_n) + \rho(Y + Z_c)$ for a given set of available treaties is mathematically a constrained optimization problem.

From the preceding section, we learned that if a ceding arrangement makes $X + Z_n$ and $Y + Z_c$ comonotonic (upper comonotonicity suffices), then the sum of required capitals attains its minimum value $\rho(X + Y + Z)$. In other words, the minimum sum of capitals corresponds to the maximum correlation between the losses (their rank correlation equals 1). This suggests that the value of $\rho(X + Z_n) + \rho(Y + Z_c)$ may be inversely related to the correlation between $X + Z_n$ and $Y + Z_c$. A reinsurance contract that makes the total capital small must make the correlation large. This observation, if it can be proved in certain circumstances, should be very interesting. I will examine some examples where a linkage between the total capital and the correlation does exist. In the appendix, I will provide a graphic reasoning to further support this relationship.

7 Examples

In the rest of the paper, examples are provided to examine how closely the total capital is related to the correlation between the ceding insurer's and the reinsurer's losses. The first example uses normally distributed losses, where the optimal ceding terms can be obtained in closed form. The second example is more general and has to be solved numerically. The optimal cedings are calculated based on simulation results.

7.1 A multivariate normal example

Let X, Y and Z be three jointly normally distributed variables. X and Z are losses written by the primary insurer, X will be retained and Z partially ceded; Y is the existing loss of the reinsurer. Suppose only quota share treaties may be placed on Z. Although this is not a realistic situation (actual losses do not take negative values as the normal distribution does), discussion of this tractable example can provide us valuable insights.

Let X, Y and Z have the following parameters: means μ_x, μ_y and μ_z, standard deviations σ_x, σ_y and σ_z, and pairwise correlation coefficients γ_{xz}, γ_{yz} and γ_{xy}. If a quota share treaty is placed and a is the ceding fraction, then the primary company's net loss is $L_{Pr} = X + (1-a)Z$, and the reinsurer's total loss is $L_{Re} = Y + aZ$. These two losses are also normal random variables. Their means and standard deviations are as follows.

$$\begin{aligned}
\mu_{Pr} &= E(L_{Pr}) = \mu_x + (1-a)\mu_z \\
\sigma^2_{Pr} &= \mathrm{Var}(L_{Pr}) = \sigma_x^2 + (1-a)^2\sigma_z^2 + 2(1-a)\gamma_{xz}\sigma_x\sigma_z \\
\mu_{Re} &= E(L_{Re}) = \mu_y + a\mu_z \\
\sigma^2_{Re} &= \mathrm{Var}(L_{Re}) = \sigma_y^2 + a^2\sigma_z^2 + 2a\gamma_{yz}\sigma_y\sigma_z
\end{aligned}$$

For a given confidence level p, the risk measures Q_p and TVaR_p of a normal random variable can be easily obtained. In fact, they can be written as $Q_p = \mu + h_p\sigma$ and $\mathrm{TVaR}_p = \mu + k_p\sigma$, where h_p and k_p are constants independent of μ and σ. For example, $Q_{0.99} = \mu + 2.33\sigma$ and $\mathrm{TVaR}_{0.99} = \mu + 2.67\sigma$. Therefore, if the risk measure ρ is of the quantile or the TVaR type, minimizing the sum $\rho(L_{Pr})+\rho(L_{Re})$ is equivalent to minimizing the sum $\sigma_{Pr} + \sigma_{Re}$. The latter problem will be solved below.

The variances of the insurer and the reinsurer can be written in a simpler form

$$\begin{aligned}
\sigma^2_{Pr} &= \sigma_z^2((a - A_{Pr})^2 + B^2_{Pr}) \\
\sigma^2_{Re} &= \sigma_z^2((a + A_{Re})^2 + B^2_{Re}),
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
A_{Pr} = 1 + \gamma_{xz}\sigma_x/\sigma_z, &\qquad B^2_{Pr} = (1-\gamma^2_{xz})\sigma_x^2/\sigma_z^2 \\
A_{Re} = \gamma_{yz}\sigma_y/\sigma_z, &\qquad B^2_{Re} = (1-\gamma^2_{yz})\sigma_y^2/\sigma_z^2.
\end{aligned} \tag{7.2}$$

The sum of standard deviations is thus

$$\sigma_{Pr} + \sigma_{Re} = \sigma_z\left(((a - A_{Pr})^2 + B^2_{Pr})^{1/2} + ((a + A_{Re})^2 + B^2_{Re})^{1/2}\right).$$

To minimize this sum is to minimize the following function $f(a)$

$$f(a) = ((a - A_{Pr})^2 + B_{Pr}^2)^{1/2} + ((a + A_{Re})^2 + B_{Re}^2)^{1/2},$$

where the ceding fraction a is between 0 and 1. The derivative of $f(a)$ is

$$f'(a) = \frac{a - A_{Pr}}{((a - A_{Pr})^2 + B_{Pr}^2)^{1/2}} + \frac{a + A_{Re}}{((a + A_{Re})^2 + B_{Re}^2)^{1/2}}.$$

Setting the right-hand side of the equation equal to zero, moving one of the terms to the other side and squaring the terms, we have

$$\frac{(A_{Pr} - a)^2}{(a - A_{Pr})^2 + B_{Pr}^2} = \frac{(a + A_{Re})^2}{(a + A_{Re})^2 + B_{Re}^2}.$$

Simplifying this gives

$$(A_{Pr} - a)^2 B_{Re}^2 = (a + A_{Re})^2 B_{Pr}^2.$$

Let us assume that $\gamma_{xz} \geq 0$ and $\gamma_{yz} \geq 0$, meaning that the losses X, Y and Z are not negatively correlated, a condition likely to be true in the real world. Mathematically, this implies $A_{Pr} \geq 1$ and $A_{Re} \geq 0$. If we assume $-A_{Re} \leq a \leq A_{Pr}$, then $A_{Pr} - a \geq 0$ and $a + A_{Re} \geq 0$. Taking the square root in the above equation, we get the solution

$$a^* = \frac{A_{Pr} B_{Re} - A_{Re} B_{Pr}}{B_{Pr} + B_{Re}}. \tag{7.3}$$

This is the unique zero of $f'(a)$ between $-A_{Re}$ and A_{Pr} and the unique minimum point of $f(a)$. The function $f(a)$ strictly decreases from $-A_{Re}$ to a^* and strictly increases from a^* to A_{Pr}. Note that the optimal ceding fraction does not depend on how X and Y are correlated.

Now let us examine a few special cases. First, suppose Z is uncorrelated with both X and Y, that is, $\gamma_{xz} = \gamma_{yz} = 0$. From the equations (7.2), $A_{Pr} = 1$, $B_{Pr} = \sigma_x/\sigma_z$, $A_{Re} = 0$ and $B_{Re} = \sigma_y/\sigma_z$. Using (7.3), we obtain the optimal ceding fraction $a^* = \sigma_y/(\sigma_x + \sigma_y)$. So, in this case, to minimize $\sigma_{Pr} + \sigma_{Re}$, Z should be shared between the primary insurer and the reinsurer in proportion to the standard deviations of their "fixed" losses, σ_x and σ_y.

A more interesting case is when Z is highly correlated to X but almost uncorrelated to Y. Then $\gamma_{xz} \approx 1$ and $\gamma_{yz} \approx 0$. These imply that $A_{Pr} \approx 1 + \sigma_y/\sigma_z$, $B_{Pr} \approx 0$, $A_{Re} \approx 0$ and $B_{Re} \approx \sigma_y/\sigma_z$. By (7.3), $a^* \approx 1 + \sigma_x/\sigma_z$. This a^* is greater than 1. Thus, to minimize $\sigma_{Pr} + \sigma_{Re}$, Z should be 100 percent ceded. On the other hand, since Z and X are highly correlated, the more Z is ceded to the reinsurer, the greater is the (linear) correlation between $X+(1-a)Z$ and $Y+aZ$. This correlation is maximized at $a = 100\%$. In this example, the reinsurance is optimized at the same ceded ratio where the correlation between the losses is maximized.

A parallel result is that, if Z is highly correlated to Y but almost uncorrelated to X, then the optimal ceded ratio is 0 percent. At this ceded ratio, the correlation between the losses is again maximized.

Now let us plug in some numerical values. Assume $\sigma_x = 300$, $\sigma_y = 500$ and $\sigma_z = 100$; $\gamma_{xz} = 0.4$, $\gamma_{yz} = 0.4$ and $\gamma_{xy} = 0.2$. Using (7.2), we compute $A_{Pr} = 2.20$, $B_{Pr} = 2.75$, $A_{Re} = 2.00$ and $B_{Re} = 4.58$. Substituting these into (7.3), we obtain the optimal ceding fraction $a^* = 62.5\%$. However, this a^* does not provide the maximum correlation between $X + (1-a)Z$ and $Y + aZ$. Using simulation, we get that the maximum linear correlation coefficient is 0.290 and is reached at the ceded ratio of 30.5 percent. Therefore, the minimum total capital does not always correspond to the maximum correlation. As mentioned before, this result is not really a surprise because the capital is determined by large losses, while the linear or rank correlation coefficient does not distinguish between large and small losses (or even negative losses, which is the case in this example).

7.2 A numerical example

If the joint distribution of losses X, Y and Z is known, and a set of available reinsurance treaties is given, the optimal treaty can be found by simulation. To have an easy control on correlations between the losses, I will assume the losses are jointly lognormal. I will look at two common types of treaties, the quota share and the stop loss.

Let the variables X, Y and Z be jointly lognormal, in the sense that $\ln(X)$, $\ln(Y)$ and $\ln(Z)$ are jointly normal. The mean μ^0 and the standard deviation σ^0 of these normal variables are as follows.

	$\ln(X)$	$\ln(Y)$	$\ln(Z)$
μ^0	19.5	20.0	17.0
σ^0	0.16	0.25	1.10

The mean, the standard deviation and quantiles of X, Y and Z can be computed from the above table with simple formulas. I will denote a parameter for a normal random variable with a superscript 0, and the same parameter for the corresponding lognormal variable without a superscript. For example, μ_x^0 is the mean of $\ln(X)$ and μ_x the mean of X. These formulas are well known:

$\mu_x = \exp(\mu_x^0 + (\sigma_x^0)^2/2)$, and $\sigma_x = \exp(\mu_x^0 + (\sigma_x^0)^2/2)(\exp((\sigma_x^0)^2) - 1)^{1/2}$. The p-quantile of X can be written as $Q_p(X) = \exp(\mu_x^0 + h_p \sigma_x^0)$, where h_p is the p-quantile of the standard normal distribution. More complex measures of the lognormals, like $\text{TVaR}_p(X)$ or the standard deviation of $X + Y + Z$, are more conveniently estimated using simulation. Some useful statistics for X, Y and Z are listed in the following table (loss amounts are in millions).

	X	Y	Z
μ	298	501	44
σ	48	127	68
CV	0.16	0.25	1.53
$Q_{0.99}$	427	868	312
$\text{TVaR}_{0.99}$	451	941	477

I will choose $\rho = \text{TVaR}_{0.99}$ as the risk measure. In addition to the known μ and σ, if the linear correlation coefficients γ_{xz}, γ_{yz} and γ_{xy} are also given, then the distribution of the triplet (X, Y, Z) is completely determined. Following our naming convention, γ_{xz}^0 is the linear correlation coefficient between $\ln(X)$ and $\ln(Z)$. γ_{xz}^0 determines γ_{xz}, and vise versa. A greater γ_{xz}^0 corresponds to a greater γ_{xz}. The strongest correlation between X and Z is attained when $\ln(X)$ is a linear function of $\ln(Z)$ with a positive slope. In this case, $\gamma_{xz}^0 = 1$ but γ_{xz} is generally less than 1.[8]

A straightforward sampling method is used to find the optimal ceding term. For μ and σ in the above table and known γ_{xz}, γ_{yz} and γ_{xy}, a large random sample of (X, Y, Z) is drawn (using Excel with the @RISK add-in or with a macro performing the Cholesky decomposition). Applying a given reinsurance treaty on the sample data, we get samples of losses of the primary insurer and the reinsurer, from which the TVaR of the losses can be estimated. Table 1 displays results for quota share treaties. Five scenarios of different γ_{xz}^0, γ_{yz}^0 and γ_{xy}^0 are analyzed. For each scenario, a set of 20,000 sample points of the triplet (X, Y, Z) is drawn; 101 quota share fractions, a, ranging from 0 to 100 percent with 1 percent increments, are applied; the measures $\rho(X + (1 - a)Z)$ and $\rho(Y + aZ)$ are estimated; and the least sum of them is found by comparison, which gives the optimal quota share term. (Loss amounts in Table 1 are in millions.)

[8] The exact formula is $\gamma_{xz} = [\exp(\sigma_x^0 \sigma_z^0 \gamma_{xz}^0) - 1]/[(\exp((\sigma_x^0)^2) - 1)(\exp((\sigma_z^0)^2) - 1)]^{1/2}$. When $\gamma_{xz}^0 = 1$, γ_{xz} is generally less than 1, but the Spearman rank correlation coefficient between X and Z equals 1.

Table 1: Optimal Quota Share Fractions

	(1)	(2)	(3)	(4)	(5)
γ^0_{xz}	0.9	0.9	0	0.1	0
γ^0_{yz}	0	0.1	0.99	0.9	0
γ^0_{xy}	0	0	0	0	0
$\rho(X+Y+Z)$	1,540	1,570	1,764	1,721	1,422
a^* (optimal ceding)	100%	100%	0%	36%	75%
$\rho(X+(1-a^*)Z)+\rho(Y+a^*z)$	1,596	1,624	1,771	1,756	1,529

In the table, $\rho(X+Y+Z)$ is the absolute lower bound of the total required asset, for any type of reinsurance. In scenario (3), the optimal total asset $\rho(X+(1-a^*)Z)+\rho(Y+a^*Z)$ is close to $\rho(X+Y+Z)$. But, in general, the difference between the two is sizable. In scenarios (1) and (2), Z is strongly correlated to X but weakly correlated Y. Ceding out the entire Z ($a=100\%$) would maximize the correlation between $X+(1-a)Z$ and $Y+aZ$.[9] This supports the claim that the optimal treaty is the one that creates the strongest correlation between the insurer's and the reinsurer's losses. A similar relationship holds in scenario (3), where Z is strongly correlated to Y but weakly correlated to X. The optimal term is to cede nothing, which again corresponds to the strongest correlation between the two losses. However, in scenario (5), the optimal ceding ratio is 75 percent, while, as can be shown, the maximum correlation is reached at $a=55\%$. The two ratios are different.

I now consider the same five correlation scenarios and perform a similar analysis for stop-loss treaties. In each scenario, let the primary insurer's retention, k, vary from 20 million to 250 million, with 5 million increments. The ceded loss is $Z_c=\max(Z-k,0)$, and the retained loss $Z_n=Z-Z_c=\min(Z,k)$. Comparing the total asset $\rho(X+Z_n)+\rho(Y+Z_c)$ for all these k, we get the optimal retention k^*. The results are summarized in Table 2 (loss amounts are in millions).

In the first two scenarios, Z is highly correlated to X; in the next two scenarios, it is highly correlated to Y. Thus, intuitively, in the first two scenarios, the correlation (at the right tail) between $X+Z_n$ and $Y+Z_c$ increases as more of Z is ceded. In fact, the sample linear correlation is indeed the highest at $k=20$.

[9] It can be proved mathematically that, if γ_{xz} is very close to 1, then the greater the ceded ratio a, the greater the linear correlation between $X+(1-a)Z$ and $Y+aZ$. The intuition behind this result is that, if Z behaves very similarly to X, then $Y+Z$, for an arbitrary variable Y, behaves more similarly to X than Y does.

Table 2: Optimal Stop-Loss Retentions

	(1)	(2)	(3)	(4)	(5)
γ^0_{xz}	0.9	0.9	0	0.1	0
γ^0_{yz}	0	0.1	0.99	0.9	0
γ^0_{xy}	0	0	0	0	0
$\rho(X+Y+Z)$	1,540	1,570	1,764	1,721	1,422
k^* (optimal retention)	20	20	250	250	85
$\rho(X+Z^*_n)+\rho(Y+Z^*_c)$	1,598	1,625	1,804	1,770	1,557

$Z^*_n = \min(Z, k^*)$, $Z^*_c = \max(Z - k^*, 0)$

This again supports the claim that the optimal treaty maximizes the correlation. This statement holds true in the next two scenarios, where the optimal treaty is to cede the least of Z. However, in scenario (5), the maximum linear correlation is attained at the retention $k = 115$, which is different from the optimal retention $k^* = 85$.

Finally, let us look at scenario (5) and compare the two types of treaties. The optimal total required asset for the stop-loss treaties is 1,557, and for the quota share treaties it is 1,529. So the quota share is more effective in cutting the total capital.[10] This appears to contradict the general belief that a stop-loss treaty reduces volatility more effectively than a quota share treaty. The fact is, however, although the stop-loss treaty cuts more capital from the primary insurer, it adds even more to the reinsurer, which results in an increase in the total required capital. In general, which type of treaty reduces the total capital more effectively depends on the joint distribution of all losses.

8 Conclusions

I have proposed to call a reinsurance arrangement optimal if it minimizes the total capital of the primary insurer and the reinsurer. This optimal reinsurance produces the lowest price for primary insurance policies, so is an attracting equilibrium under market competition. An interesting relationship is observed between the total capital and the tail correlation between the losses of the insurer and the

[10] The quota share structure is better in the other four scenarios as well, but those results are of no surprise. As the stop-loss retention is limited to between 20 and 250, ceding the whole of Z and ceding none of Z are excluded, yet the optimal quota share terms in these scenarios fall into these extremes.

reinsurer. A multivariate normal model and a numerical example are analyzed to get more insight into the nature of an optimal treaty.

This paper fills a gap in the existing literature on optimal reinsurance, in which the capital cost of the reinsurer has not been adequately addressed. My approach establishes a close link between reinsurance and pricing of insurance and reinsurance policies. In a competitive market, reinsurance not only provides the ceding insurer a tool of risk transfer, but also satisfies the reinsurer with a fair amount of profit and benefits primary policyholders by reducing their costs.

Tail correlation between losses has been widely discussed in relation to risk measurement and management. In this paper, it is linked to the size of the total capital. This seems to be an interesting area of research.

References

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath. "Coherent Measures of Risk." *Mathematical Finance* 9: (1999) 203–228.

Cai, Jun, and Ken Seng Tan. "Optimal Retention for a Stop-Loss Reinsurance Under the VaR and CTE Risk Measures." *ASTIN Bulletin* 37, 1: (2007) 93–112.

Chandra, Victor, and Michael Sherris. "Capital Management and Frictional Costs in Insurance." *Australian Actuarial Journal* 12, 4: (2006) 399–448.

Cheung, Ka Chun. "Upper Comonotonicity." *Insurance: Mathematics and Economics* 45, 1: (2009) 35–40.

Cummins, J. David. "Asset Pricing Models for Insurance Ratemaking." *ASTIN Bulletin* 20, 2: (1990) 125–166.

Dhaene, Jan, Steven Vanduffel, M. J. Goovaerts, Rob Kaas, Qihe Tang, and David Vyncke. "Risk Measures and Comonotonicity: A Review." *Stochastic Models* 22: (2006) 573–606.

Fu, Luyang, and C.K. "Stan" Khury. "Optimal Layers for Catastrophe Reinsurance." *Variance* 4, 2: (2010) 191–208.

Gajek, Leslaw, and Dariusz Zagrodny. "Reinsurance Arrangements Maximizing Insurer's Survival Probability." *Journal of Risk and Insurance* 71, 3: (2004) 421–435.

Jensen, Michael, and William Meckling. "Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure." *Journal of Financial Economics* 3, 4: (1976) 305–340.

Kaluszka, Marek. "Optimal Reinsurance Under Mean-Variance Premium Principles." *Insurance: Mathematics and Economics* 28: (2001) 61–67.

Lampaert, Ilse, and Jean-Francois Walhin. "On the Optimality of Proportional Reinsurance." *Casualty Actuarial Society Forum* 2005, spring: (2005) 93–114.

Myers, Stewart S., and Richard A. Cohn. "A Discounted Cash Flow Approach to Property-Liability Insurance Rate Regulation." In *Fair Rates of Return in Property-Liability Insurance*, edited by J. David Cummins, and Scott E. Harrington. Dordrecht: Kluwer Nijhoff Publishing, 1987, 55–77.

Perold, Andre. "Capital Allocation in Financial Firms." *Journal of Applied Corporate Finance* 17, 3: (2005) 110–118.

Taylor, Greg C. "Fair Premium Rating Methods and the Relations Between Them." *Journal of Risk and Insurance* 61: (1994) 592–615.

Vaughn, Trent. "The Impact of Investment Strategy on the Market Value and Pricing Decisions of a Property/Casualty Insurer." *Proceedings of the Casualty Actuarial Society* LXXXV: (1998) 211–244.

Wang, Shaun. "Aggregation of Correlated Risk Portfolios: Models and Algorithms." *Proceedings of the Casualty Actuarial Society* LXXXV: (1998) 848–939.

Zhang, Yingjie. "Economic Impact of Capital Level in an Insurance Company." *Variance* 2, 1: (2008) 39–51.

Appendix. More on the Linkage Between the Total Capital and Correlation

I have shown that the TVaR is a subadditive risk measure: If $\rho = \text{TVaR}_p$, then $\rho(X) + \rho(Y) \geq \rho(X+Y)$, and the equality holds if X and Y (representing the losses of a primary insurer and a reinsurer) are comonotonic. Following this fact, I propose that a linkage exists between the total asset $\rho(X) + \rho(Y)$ and the correlation between X and Y, that is, the greater the tail correlation, the closer is $\rho(X) + \rho(Y)$ to $\rho(X+Y)$. In this appendix, I will use the scatter plot to further explain why there should be such a link.

Figures 1 through 3 provide scatter plots of a pair of losses X and Y correponding to three different correlation scenarios. (The correlations are actually only different at the right tail.) Each loss is in the range $[0, 100)$. In Figure 1, X and Y are comonotonic at the tail. In Figure 2, they are not comonotonic but are still highly correlated at the tail: as X moves up from about 80, Y generally moves up as well, although it sometimes moves in the opposite direction (down) slightly. In Figure 3, X and Y have little correlation at the tail.

Let the risk measure be $\rho = \text{TVaR}_{0.9}$. There are 100 sample points in each figure. The point labeled A has the 11th largest x coordinate, and the one labeled B has the 11th largest y coordinate. The quantile $Q_{0.99}(X)$ is the x coordinate of A, and $Q_{0.99}(Y)$ the y coordinate of B. $\rho(X)$ is the average of the x coordinates of the points to the right of A, and $\rho(Y)$ the average of the y coordinates of the points higher than B. $\rho(X+Y)$ is the average of the largest 10 $x+y$ of all points.

In Figure 1, A and B are actually the same point $(78, 76)$ (coordinates are rounded), and the points to the right of A are the same as those higher than A, which are also the 10 points with the largest $x+y$. Thus, $Q_{0.99}(X)+Q_{0.99}(Y) = Q_{0.99}(X+Y) = 78+76 = 154$, and $\rho(X)+\rho(Y) = \rho(X+Y)$ $(= 178)$. This explains that if X and Y are perfectly correlated at the tail, then $\rho(X)+\rho(Y) = \rho(X+Y)$.

In Figure 2, the upper-right tail is a rather "thin" set. Thus the two points A and B are close to each other. Further, the following three sets of points are similar (contain mostly the same points): those to the right of A, those higher than B, and the ten points with the largest $x+y$. This implies that $\rho(X)+\rho(Y)$ is close to $\rho(X+Y)$. (Here $\rho(X) = 95.3$, $\rho(Y) = 88.8$ and $\rho(X+Y) = 183.9$.) This example shows that if X and Y are highly correlated at the tail, then $\rho(X)+\rho(Y)$ is (greater than but) close to $\rho(X+Y)$.

The upper-right tail in Figure 3 is not a thin set, and the two points A and B are generally far apart. Also, it is likely that the three sets—the one to the right of A, the one higher than B and the one with the largest $x+y$—contain very different points. So $\rho(X)+\rho(Y)$ can be much larger than $\rho(X+Y)$. (Here $\rho(X) = 95.3$, $\rho(Y) = 82.4$ and $\rho(X+Y) = 174.1$.) This is what normally happens when X and Y are not correlated at the tail.

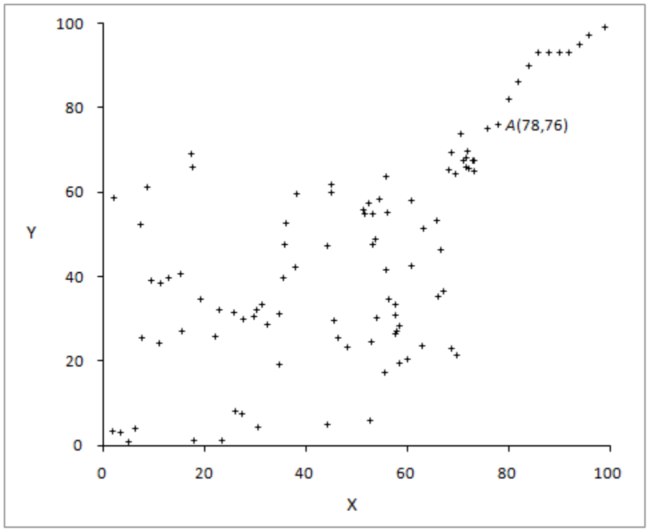

Figure 1: X and Y are comonotonic at the tail; $\rho(X) + \rho(Y) = \rho(X + Y)$

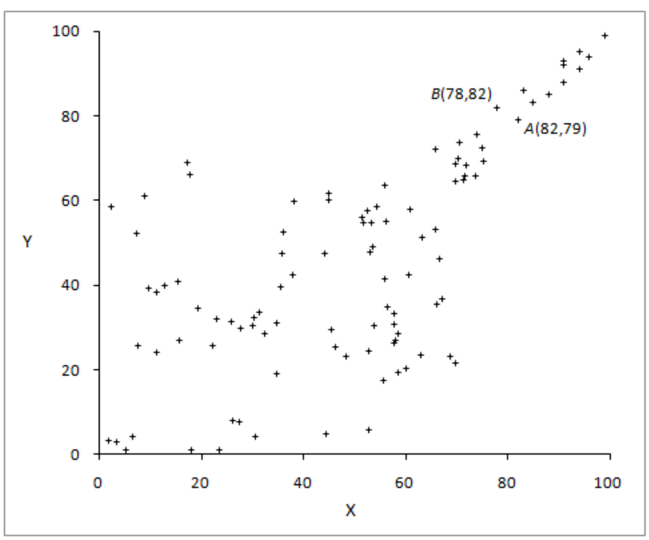

Figure 2: X and Y are highly correlated at the tail; $\rho(X) + \rho(Y)$ is close to (but greater than) $\rho(X + Y)$

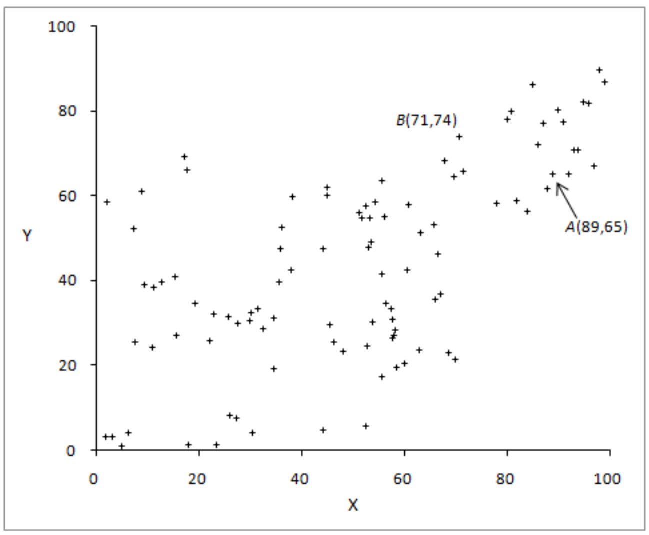

Figure 3: X and Y are not correlated at the tail; $\rho(X) + \rho(Y)$ is generally much greater than $\rho(X + Y)$