Index Clause for Aggregate Deductibles and Limits in Non-Proportional Reinsurance

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Abstract

Index clauses currently in place in the market do not specify how Annual Aggregate Deductibles (AAD) and Annual Aggregate Limits (AAL) should be indexed, which result in inconsistency when indexed deductibles and limits are in place.

In this paper, concepts of indexed deductible and limit will be revisited for developing indexing methods for AAD and AAL. Formal mathematical proofs and numerical examples will be presented. The introduced AAL indexing methods enable determination of paid reinstatement premium when index clause for per-claim deductible and limit is in place.

The choice of appropriate method for indexing AAD and AAL shall take into account both theoretical and practical soundness.

Keywords. Reinsurance; Excess (Non-Proportional); Deductibles, Retentions and Limits.

1. INTRODUCTION

Index clause (or "Stability clause") have become a standard clause in non-proportional (NP) reinsurance contracts for long-tail classes in many international markets. Index clause handles the leveraged effect of inflation on excess layer loss cost by adjusting the per-claim deductible and limit so that effect of inflation is shared between the primary insurer and NP reinsurer equally.

However, most index clauses do not specify how Annual Aggregate Deductibles (AAD) and Annual Aggregate Limits (AAL) should be adjusted for inflation. In practice many NP reinsurance contracts try to mitigate this inherent problem by

(a) Specifying unlimited reinstatements or an AAL that is much greater than the per-claim limit, or

(b) Simply endorsing that the AAL is "un-indexed", although per-claim deductible and limit are still subject to index clause adjustment

While AAD's are becoming more common for long-tail NP reinsurance (for various purposes, e.g., reinsurance premium saving or fulfilling sufficient risk transfer), the above mitigating measures do not provide good solutions for AAD's.

1.1 Research Context

The method for indexing a per-claim deductible and limit has been explained in Ferguson [1]. It has become a standard calculation method specified in the index clause of long-tail excess of loss reinsurance contracts in many markets. Implementation of index clause, wording, and pricing has been discussed in that paper as well.

Feldblum [6] and Feldblum [8] suggested a different method for indexing per-claim deductibles and limits, by making use of internal rate of return concept. However this method has not been widely adopted. Further, calculation procedures with this method can be complicated in a varying inflation environment.

1.2 Objective

This paper will propose two methods for indexing AAD's and AAL's, both based on achieving the goals of "equitable share of inflation effect" and "equitable share of deflated payments and actual payments" between primary insurer and reinsurer. The current per-claim deductibles and limit indexation method will be briefly revisited. The proposed methods for indexing AAD's and AAL's will be developed in a manner consistent with the per-claim indexing approach.

1.3 Outline

The remainder of the paper proceeds as follows. In section 2 the concept of index clause currently in place in the market will be revisited. In section 3 two methods for indexing AAD's and AAL's will be introduced, first through intuitive arguments from a retrocessionaire's point of view, then numerical examples and formal mathematical proofs will be presented. In section 4 practical issues will be discussed, including implementing an AAD and AAL index clause, pricing approaches, and calculating paid-reinstatement premium.

2. INDEX CLAUSE – REVISITING THE CONCEPTS

The following is an example of index clause wording that explains how per-claim deductibles and per-claim limits are indexed:

Each loss payment shall be brought back separately to its respective value at the base date according to the indices prevailing on the date the loss payments are made, by means of the following formula:

$$\frac{\textit{actual amount of payment} \times \textit{index at base date}}{\textit{index at date of payment}} = \textit{adjusted payment value}$$

All actual payments and adjusted payment values shall then be separately totaled and deductible and limit shall be multiplied by the following fraction:

$$\frac{\textit{Total of actual payments}}{\textit{Total of adjusted payment values}}$$

in order to determine the overall indexed deductible and, where applicable, limit of indemnity, and thus the amount recoverable in accordance with the provisions of this clause.

2.1 Index Clause in Practice – An Example

An excess of loss reinsurance program has a per-claim deductible of $3 million and per-claim limit of $5 million, both subject to an index clause. Let time 0 denote the base date for index clause calculation. Two incremental payments have been made for a claim at time 1 and 2 (in years), as shown in the following table.

Incremental Actual Payment ($000s)

payment time	0	1	2	row sum
claim 1	$0.0	$3,180.0	$1,308.0	$4,488.0

Next, adjusted payments are calculated:

Incremental Adjusted Payment ($000s)

payment time	0	1	2	row sum
claim 1	$0.0	$3,000.0	$1,200.0	$4,200.0
Index	100	106	109	

For example, adjusted payments for time 1 = $3.18 million × 100/106 = $3 million, which can be interpreted as the deflated value at time 0 of an actual payment $3.18 million paid at time 1 according to the specified index. Hereafter in this paper, the author will use the term "deflated value" or "deflated payment" which take the same technical calculation steps as "adjusted payment" presented above, but the author believes the term "deflated" better represents inflation measurement and sharing concepts underlying index clause calculations.

In the third step, the indexed deductible and indexed limit are calculated.

Indexed deductible = \$3 million × 4,488/4,200 = \$3.206 million.

Indexed limit = \$5 million × 4,488/4,200 = \$5.343 million.

In the final step, the amount to be paid by the NP reinsurer to the primary insurer for claim 1 = (\$4.488 – \$3.206) million = \$1.282 million.

In practice, at time 2 when the primary insurer notifies claim 1 to the NP reinsurer, the indexed deductible and limit will be calculated immediately, whether the claim is fully settled at time 2 or not. The NP reinsurer then needs to make a payment to the primary insurer if the total of all actual payments exceeds the indexed deductible calculated at time 2.

At time 3 another payment is made:

Incremental Actual Payment (\$000s)

payment time	0	1	2	3	row sum
claim 1	\$0.0	\$3,180.0	\$1,308.0	\$2,808.0	\$7,296.0

Similarly, adjusted payments are calculated:

Incremental Adjusted Payment (\$000s)

payment time	0	1	2	3	row sum
claim 1	\$0.0	\$3,000.0	\$1,200.0	\$2,400.0	\$6,600.0
Index	100	106	109	117	

Indexed deductible = \$3 million × 7,296/6,600 = \$3.316 million

Indexed limit = \$5 million × 7,296/6,600 = \$5.527 million

The cumulative amount to be paid by NP reinsurer to primary insurer for claim 1 = (\$7.296 – \$3.316) million = \$3.980 million. Therefore NP reinsurer pays (\$3.980 – \$1.282) million = \$2.698 million at time 3 to primary insurer.

To check whether effect of inflation is shared between the primary insurer and NP reinsurer equally, first consider inflation for the gross claim:

$$\frac{\text{Total of all actual payments}}{\text{Total of all deflated payments}} \tag{2.1}$$

Inflation of gross claim = 7.296M / 6.600M – 1 = 10.5%

Similarly, consider inflation of NP reinsurer's excess of loss payments:

$$\frac{\text{Total of all actual payments} - \text{indexed deductible}}{\text{Total of all deflated payments} - \text{unindexed deductible}} \tag{2.2}$$

Finally, inflation of primary insurer's retained claim:

$$\frac{\text{indexed deductible}}{\text{unindexed deductible}} \tag{2.3}$$

Inflation of the NP reinsurer's payment = 3.980M / 3.600M – 1 = 10.5%

Inflation of the primary insurer's retained claim = 3.316M / 3.000M – 1 = 10.5%

2.2 Generalizing the Principles of Index Clause

In the numerical example in section 2.1, inflation for the gross claim, inflation for the NP reinsurer's payment, and inflation for the primary insurer's retained claim are the same. It can be verified that the three inflation measurements in equations (2.1), (2.2), and (2.3) are equal in general.

Notations:

i = claim identifier, $i = 1, 2, 3, \ldots$

t = time of payment, $t = 0, 1, 2, 3, \ldots$ base date is denoted by $t = 0$

v_t = deflating factor for payment made at time t

= value of index clause index at time 0 ÷ value of index clause index at time t

$X_{i,t}$ = Actual dollar payment of i^{th} claim at time t

$X_i^T = \sum_{t=1}^{T} X_{i,t} \cdot v_t$ = Total of all deflated payments made between time 0 and time T

d = un-indexed deductible per claim

l = un-indexed limit per claim

2.2.1 Indexed Deductible and Indexed Limit

According to equation (2.1), given incremental payment information up to time T for the i^{th} claim, indexed deductible is calculated as:

$$d'_{i,T} = d \times \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \tag{2.4}$$

Similarly, indexed limit for the i^{th} claim is calculated as:

$$l'_{i,T} = l \times \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \tag{2.5}$$

2.2.2 Cumulative Payments Paid by NP Reinsurer and Incremental Payments Paid by Primary Insurer

Cumulative payments paid by the NP reinsurer to the primary insurer at time T for the i^{th} claim is:

$$Y_{i,T} = \min\{\max\{(\sum_{t=1}^{T} X_{i,t}) - d'_{i,T}, 0\}, l'_{i,T}\} \tag{2.6}$$

It can be proved that, under the conditions $X_{i,T+1} \geq 0$ and $v_{t+1} \leq v_t \quad \forall t \leq T$, then $d'_{i,T+1} \geq d'_{i,T}$, $l'_{i,T+1} \geq l'_{i,T}$ and $Y_{i,T+1} \geq Y_{i,T}$. Proof of the third inequality is shown in Appendix A. The third inequality means that if the following two conditions are fulfilled:

(1) The primary insurer's incremental payment for the next period is a net outflow for any claim, and

(2) There is no deflation along the claim payment time horizon

then in the next period, incremental payments made by the NP reinsurer to the primary insurer is net outflow as well, meaning that the NP reinsurer would not request a payback from primary insurer. It is desirable to observe the third inequality because the primary insurer may be concerned that an increase in the indexed deductible over time could offset or exceed the increase in cumulative gross payment. That would not happen as indicated by the inequality $Y_{i,T+1} \geq Y_{i,T}$.

Next, consider incremental payments paid by primary insurer net of recoveries from the NP reinsurer at time T+1 for the i^{th} claim:

$$X_{i,T+1} - Y_{i,T+1} + Y_{i,T} \tag{2.7}$$

Under the conditions $X_{i,T+1} \geq 0$ and $v_{t+1} \leq v_t \quad \forall t \leq T$, then $(X_{i,T+1} - Y_{i,T+1} + Y_{i,T}) \geq 0$. That means at time T+1 primary insurer's gross incremental payment is always greater than the incremental recovery from the NP reinsurer. The proof is shown in Appendix A.

Sections 3.2 and 3.3 will discuss whether indexed AAD and AAL demonstrate similar desirable properties as well.

2.2.3 Inflation of Gross Claims, NP Reinsurer's Payments and Primary Insurer's Retained Claims

In Ferguson [1], "equitable share of inflation effect" means that applying the indexed deductible and indexed limit on gross claims will result in equal inflations for the primary insurer's retained claim and the NP reinsurer's claim payments.

According to equation (2.1), at time T, inflation for the i^{th} gross claim can be rewritten as:

$$\frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} = \frac{1}{w_{i,T}} \tag{2.8}$$

The notation $w_{i,T}$ represents the reciprocal of inflation at time T for the i^{th} gross claim.

Inflation for NP reinsurer's excess of loss payments is as follows:

$$\frac{\min\{\max\{(\sum_{t=1}^{T} X_{i,t}) - d'_{i,T}, 0\}, l'_{i,T}\}}{\min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}} \tag{2.9}$$

Inflation for the primary insurer's retained claim is as follows:

$$\frac{\min\{(\sum_{t=1}^{T} X_{i,t}), d'_{i,T}\}}{\min\{\sum_{t=1}^{T} X_{i,t} \cdot v_t), d\}} \tag{2.10}$$

It can be proved that, at time T for the i^{th} claim, the gross claim's inflation equals inflation for the NP reinsurer's excess of loss payments and also equals inflation for the primary insurer's retained claim. That means:

$$\frac{1}{w_{i,T}} = \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} = \frac{\min\{\max\{(\sum_{t=1}^{T} X_{i,t}) - d'_{i,T}, 0\}, l'_{i,T}\}}{\min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}} = \frac{\min\{(\sum_{t=1}^{T} X_{i,t}), d'_{i,T}\}}{\min\{\sum_{t=1}^{T} X_{i,t} \cdot v_t), d\}} \tag{2.11}$$

The proof is shown in Appendix B.

2.2.4 An Alternative View: Indexing Deductibles and Limits by Principle of Equitable Sharing of Deflated Payments and Actual Payments

In Ferguson [1], "equitable share of deflated payments and actual payments" means that the ratio of the NP reinsurer's actual claim payment to actual gross claim equals the ratio of the NP reinsurer's deflated claim payment to deflated gross claim. This concept can be applied to explain equations (2.4) and (2.5).

If the index clause's selected index correctly reflects claims inflation at each payment time, then the following expression represents the value of the i^{th} claim as if all its future partial payments were paid at time 0.

$$\sum_{t=1}^{T} X_{i,t} \cdot v_t \tag{2.12}$$

Similarly, the following expression represents the NP reinsurer's payment to primary insurer as if all future partial payments of the i^{th} claim were paid at time 0:

$$\min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} \tag{2.13}$$

That means that the un-indexed deductible and limit are directly applied to the total of all deflated payments for calculating excess layer loss.

What should be the NP reinsurer's share in the total actual payment of the i^{th} claim ($\sum_{t=1}^{T} X_{i,t}$)? The NP reinsurer should pay the proportion of $\sum_{t=1}^{T} X_{i,t}$, which is the same as the ratio of expression in (2.13) to expression in (2.12). That means NP reinsurer's share in the total actual payment is as follows:

$$\sum_{t=1}^{T} X_{i,t} \times \frac{\min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \tag{2.14}$$

It can be verified that the above expression is identical to the right-hand side of equation (2.6), and therefore results in the same formulas for indexed deductible and indexed limit in equations (2.4) and (2.5).

Also, equation (2.14) shows the following relationships between the NP reinsurer's actual cumulative payment ($Y_{i,T}$) and the total deflated gross payments ($\sum_{t=1}^{T} X_{i,t} \cdot v_t$) for the i^{th} claim:

(1) $Y_{i,T} = 0$ when $\sum_{t=1}^{T} X_{i,t} \cdot v_t \le d$

That is, the NP reinsurer makes no payment if the total of deflated gross payments is below the un-indexed deductible.

(2) $Y_{i,T} > 0$ when $\sum_{t=1}^{T} X_{i,t} \cdot v_t > d$

That is, the NP reinsurer makes payment if the total of deflated gross payments is greater than the un-indexed deductible.

(3) $\sum_{t=1}^{T} X_{i,t} = d'_{i,T}$ if and only if $\sum_{t=1}^{T} X_{i,t} \cdot v_t = d$

That is, total deflated gross payments equal the un-indexed deductible if and only if the total actual gross payments equal the indexed deductible. Once reaching this condition for a particular claim, the NP reinsurer will start paying immediately after the primary insurer makes another payment for that claim in the future.

The third relationship is particularly useful for understanding reasonableness of the indexed deductible and limit formula. In section 3, in order to verify the formulas for indexing AAD's and AAL's, it will be checked whether total actual payments and total deflated payments on aggregate basis hold similar relationships.

3. INDEX CLAUSE FOR AGGREGATE DEDUCTIBLES AND LIMITS

AAD's are becoming more common for long-tail NP reinsurance. Without a proper index clause for AAD's, many NP reinsurance contracts simply endorse an "un-indexed" AAD, however, the per-claim deductible and limit are still subject to index clause adjustment. Such un-indexed AAD's in practice are simple to implement, but there are two problems. First, an un-indexed AAD may provide a misleading picture of how the NP reinsurer's expected loss will be reduced relative to "no-AAD". Second, while an index clause for per-claim deductible and limit is used to share effect of inflation equitably between primary insurer and reinsurer, the goal cannot be achieved without an indexed AAD.

Consider this example: a NP reinsurance contract has per-claim limit of $5 million and per-claim deductible of $3 million both subject to an index clause adjustment, but with an un-indexed AAD of $5 million in place. Inflation is 4% per annum. If a claim is settled at $10 million by single payment in year 5, this claim is a total loss to the excess of loss layer. The per-claim limit and per-claim deductible are indexed to become $6.083 million and $3.650 million, respectively. If there was no AAD in place, the NP reinsurer would have paid $6.083 million to the primary insurer for this claim. With the $5 million un-indexed AAD, the NP reinsurer now pays $1.083 million. In this example, the primary insurer's additional retention under the un-indexed AAD provision is less than the occurrence of first total loss to the excess layer. This is not the expected outcome if one simply and carelessly interprets the structure to be $5mil xs $3mil xs $5mil, ignoring the gap between an indexed per-claim limit and un-indexed AAD.

3.1 Intuitive Arguments: Retrocessionaire's Point of View

Consider this example: a primary insurer purchases NP reinsurance $5 million xs $3 million with unlimited free reinstatements. An index clause will be applied to both per-claim deductible and limit.

The reinsurer wants to limit its potential frequency risk arising from this NP reinsurance contract, and decides to purchase a retrocession that caps the aggregate loss amount to the NP contract at an AAL equivalent to four times the per-claim-limit, which is $20 million. From the retrocessionaire's point of view, there should be an index clause for AAL as well, in order to share the effect of inflation between the reinsurer and retrocessionaire equitably.

The retrocessionaire now considers what factors shall and shall not enter into the AAL indexing formula. To start with, consider how the original index clause affects the transactions between the NP reinsurer, the primary insurer, and the original policyholder(s). If the original policy has a $5,000 policyholder retention, and the policyholder incurs one loss of $5,000,000, then the primary insurer will only pay $4,995,000. From the NP reinsurer's point of view, the actual amount paid by the primary insurer, $4,995,000, should be used for calculating the indexed per-claim deductible instead of the original policyholder's incurred loss of $5,000,000. Similarly, the retrocessionaire will only use the actual amount paid by the NP reinsurer (i.e., the difference between $4,995,000 and the indexed per-claim deductible) for calculating the indexed AAL, not the ground-up claim size of $4,995,000 paid by the primary insurer. Therefore, all claims below the indexed per-claim deductible should not be used for calculating the indexed AAL.

Following similar logics, the retrocessionaire makes a comparison between the original excess of loss program and the retrocession program:

	Original Excess of Loss program	Retrocession program
Deductible and Limit	Deductible and limit applied per claim.	"Aggregate" means that AAL is applied to sum of all excess layer losses. The sum of all excess layer losses is determined with index-clause-adjusted deductibles and limits

		applied to each claim separately
Basis of Inflation Measurement	Measured separately for each "claim", depending on the coverage basis. For example, if it is a per accident excess of loss, a "claim" may involve multiple claimants from the same accident and the sum of all claimant's claim amounts is used to determine loss to layer for each accident.	Measured for all losses to the excess of loss program combined together.
Measuring Inflation	Ratio of sum of all actual payments paid by the primary insurer to the policyholder(s) that belong to a "claim", to sum of all deflated payments that belong to the "claim"	Ratio of sum of all actual payments paid by the reinsurer to primary insurer according to the excess of loss program, to sum of all deflated payments to the excess of loss program

Conclusively, the AAL will be indexed by the formula:

$$\text{unindexed AAL} \times \frac{\text{Total of all actual payments to excess of loss layer}}{\text{Total of all deflated payments to excess of loss layer}} \tag{3.1}$$

Note that potential claim payouts by the above retrocessionaire are identical to the situation where a reinsurer sells a NP reinsurance \$5 million xs \$3 million with AAD \$20 million all subject to index clause. The method for indexing AAL can be applied for indexing AAD as well.

3.2 Indexing AAD and AAL Method 1: Matching Deflated Excess Loss with Deflated Gross Loss Per Claim

Additional notations are introduced, along with the notations in section 2.2:

D = un-indexed AAD

L = un-indexed AAL

D_T' = indexed AAD, given payment information up to time T for all claims

$L_T^{'}$ = indexed AAL, given payment information up to time T for all claims

The numerator of the fraction in equation (3.1), total of all actual payments to excess of loss layer, equals:

$$\sum_i Y_{i,T} \tag{3.2}$$

Next, consider the denominator of the fraction in equation (3.1), total of all deflated payments to excess layer. If gross, excess layer and retained claims are matched together, then deflated excess layer loss equals the difference between deflated gross loss and primary insurer's retention (un-indexed). For the i th claim, deflated excess layer loss equals $\min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}$, which is identical to $Y_{i,T} \cdot w_{i,T}$. As a result the denominator is:

$$\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} = \sum_i Y_{i,T} \cdot w_{i,T} \tag{3.3}$$

The formula for indexed AAL is as follows:

$$L_T^{'} = L \times \frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} \tag{3.4}$$

The formula for indexed AAD is as follows:

$$D_T^{'} = D \times \frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} \tag{3.5}$$

3.2.1 Inflation of Claims Before and After Application of Indexed AAD and AAL

Taking the retrocessionaire's point of view as described in section 3.1, the objective is to show that the following three programs have equal average inflation:

(1) Average inflation of total payments made by the NP reinsurer underlying the original excess of loss contract, assuming unlimited reinstatements, equals:

$$\frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} \tag{3.6}$$

(2) Average inflation of total payments of the retrocession program that indemnifies the NP reinsurer portion of aggregate loss exceeding the indexed AAL equals:

$$\frac{\max\{(\sum_i Y_{i,T}) - L'_T, 0\}}{\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\}} \tag{3.7}$$

(3) Average inflation of total payments made by the NP reinsurer underlying the original excess of loss contract, with the aggregate payments capped by the indexed AAL, equals:

$$\frac{\min\{(\sum_i Y_{i,T}), L'_T\}}{\min\{(\sum_i Y_{i,T} \cdot w_{i,T}), L\}} \tag{3.8}$$

As illustrated in section 3.1, inflation is measured for all claims to the excess of loss program combined, not measured for each claim separately. The three expressions in equations (3.6), (3.7), and (3.8) have very similar forms compared to the expressions in equations (2.8), (2.9), and (2.10) respectively.

Proof of the following equality:

$$\frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} = \frac{\max\{(\sum_i Y_{i,T}) - L'_T, 0\}}{\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\}} = \frac{\min\{(\sum_i Y_{i,T}), L'_T\}}{\min\{(\sum_i Y_{i,T} \cdot w_{i,T}), L\}} \tag{3.9}$$

is outlined below. A detailed proof is shown in Appendix C.

First consider equation (3.8). It can be shown that $(\sum_i Y_{i,T}) \leq L'_T$ if and only if $(\sum_i Y_{i,T} \cdot w_{i,T}) \leq L$. Therefore, when the expression in (3.8) equals $L'_T \div L$, by using definition of L'_T in equation (3.4), it can be shown that $L'_T \div L$ equals $(\sum_i Y_{i,T}) \div (\sum_i Y_{i,T} \cdot w_{i,T})$, which is equal to the expression in (3.6). Otherwise, the expression in (3.8) equals $(\sum_i Y_{i,T}) \div (\sum_i Y_{i,T} \cdot w_{i,T})$. Again, this equals the expression in (3.6).

After proving equality of expressions in (3.6) and (3.8), it can be noted that the numerator in (3.6) equals the sum of the numerators in (3.7) and (3.8). Similarly, the denominator in (3.6) equals the sum of the denominators in (3.7) and (3.8) as well. Based on these facts, the expressions in (3.7) must equal the expressions in (3.6).

Conclusively, the equality in (3.9) holds. Inflation of the NP reinsurer's claims before and after application of indexed AAL (and AAD) are the same.

3.2.2 An Alternative View: Indexing AAD and AAL by Principle of Equitable Sharing of Deflated Payments and Actual Payments

First, note that the NP reinsurer's deflated aggregate excess layer payments without any AAL equal the total of the deflated gross partial payments with un-indexed deductibles and un-indexed limits applied to each claim separately. This can be represented by equation (3.3), $\sum_i Y_{i,T} \cdot w_{i,T} = \sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}$.

Similarly, the following expression represents the retrocessionaire's payment to the NP reinsurer if future partial payments of all claims were paid at time 0:

$$\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\} = \max\{(\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}) - L, 0\} \tag{3.10}$$

What should be the retrocessionaire's share in the total actual excess layer payment ($= \sum_i Y_{i,T}$)? The retrocessionaire should pay the proportion of $\sum_i Y_{i,T}$ that is same as the ratio of expression in (3.10) to expression in (3.3). That means that the retrocessionaire's share in the total actual payment is:

$$\sum_i Y_{i,T} \times \frac{\max\{(\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}) - L, 0\}}{\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}} \tag{3.11}$$

It can be verified that the above expression is identical to the numerator of the expression in (3.7) and therefore results in the same formulas for indexed AAL in equation (3.4).

Also, equation (3.11) shows the following relationships between the retrocessionaire's actual cumulative payment $(\max\{(\sum_i Y_{i,T}) - L'_T, 0\})$ and the NP reinsurer's deflated aggregate excess layer loss before applying AAL $(\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\})$:

(1) $\max\{(\sum_i Y_{i,T}) - L'_T, 0\} = 0$ when $\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} \le L$
$\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} \le L$

That is, the retrocessionaire makes no payment if the NP reinsurer's aggregate deflated payments (before applying AAL) is below the un-indexed AAL.

(2) $\max\{(\sum_i Y_{i,T}) - L'_T, 0\} > 0$ when $\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} > L$

That is, the retrocessionaire makes payment if the NP reinsurer's aggregate deflated payments (before applying AAL) is greater than the un-indexed AAL.

(3) $\sum_i Y_{i,T} = L'_T$ if and only if $\sum_i \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} = L$

That is, the NP reinsurer's aggregate deflated payments (before applying AAL) equals the un-indexed AAL if and only if the NP reinsurer's aggregate actual payments (before applying AAL) equals the indexed AAL. Once reaching this condition, the retrocessionaire will start paying immediately after the NP reinsurer makes another payment in the future.

3.2.3 Monotonicity Property of Retrocessionaire's Cumulative Payments

Incremental payments that the retrocessionaire makes to the NP reinsurer in the next period are considered net outflow and the retrocessionaire will not request a payback from the NP reinsurer, as long as the following two conditions are fulfilled:

(1) No deflation occurs along the claim payment time horizon.

(2) The total of all actual claims payments exceed the indexed deductible during current payment period.

The notation S'_T represents the retrocessionaire's cumulative actual payments to NP reinsurer at time T:

$$S'_T = \max\{(\sum_i Y_{i,T}) - L'_T, 0\} \tag{3.12}$$

Therefore, the proposition means that, $S'_{T+1} \geq S'_T$ under the conditions $v_{t+1} \leq v_t \quad \forall t \leq T$ and $X_{i,T+1} \geq 0 \quad \forall i$ (implying that $(\sum_i Y_{i,T+1}) - (\sum_i Y_{i,T}) \geq 0$). The retrocessionaire's cumulative payment with indexed AAD and AAL is monotonically increasing over time. The proof is shown in Appendix D.

3.2.4 Monotonicity Property of Indexed AAD and AAL

In section 2.2.2, it was indicated that $d'_{i,T+1} \geq d'_{i,T}$ and $l'_{i,T+1} \geq l'_{i,T}$ under the conditions $v_{t+1} \leq v_t \quad \forall t \leq T$ and $X_{i,T+1} \geq 0 \quad \forall i$. The proof is shown in Appendix E.

Indexed AAD and AAL calculated using equations (3.4) and (3.5), however, are not monotonically increasing over time, even given the conditions $v_{t+1} \leq v_t$ $\forall t \leq T$ and $X_{i,T+1} \geq 0$ $\forall i$. Indexed AAD and AAL are neither monotonically increasing nor decreasing over time. This is an undesirable property under practical considerations, which will be illustrated with a numerical example in section 3.4.4.

3.3 Indexing AAD and AAL Method 2: Deflating Incremental Excess Loss According to Payment Time

Another method to calculate deflated excess of loss payment is to multiply the deflating factor v_t with incremental actual payments of the NP reinsurer $(Y_{i,t} - Y_{i,t-1})$ and use as denominator of equation (3.1). Therefore:

$$\text{Total of all deflated payments to excess of loss layer} = \sum_i \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t . \tag{3.13}$$

The formula for indexed AAL and AAD becomes:

$$L_T'' = L \times \frac{\sum_i Y_{i,T}}{\sum_i \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t} \tag{3.14}$$

$$D_T'' = D \times \frac{\sum_i Y_{i,T}}{\sum_i \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t} \tag{3.15}$$

By rewriting the numerator $\sum_i Y_{i,T}$ as $\sum_{t=1}^{T} \sum_i (Y_{i,t} - Y_{i,t-1})$, equation (3.14) can be compared with equation (2.5):

	Equation (2.5): Indexed per-claim limit	Equation (3.14): Indexed AAL
Indexed limit	$l_{i,T}'$	L_T''
Numerator	$\sum_{t=1}^{T} X_{i,t}$	$\sum_{t=1}^{T} [\sum_i (Y_{i,t} - Y_{i,t-1})]$
Denominator	$\sum_{t=1}^{T} X_{i,t} \cdot v_t$	$\sum_{t=1}^{T} [\sum_i (Y_{i,t} - Y_{i,t-1}) \cdot v_t]$

Therefore, the concepts in sections 2.2.2, 2.2.3, and 2.2.4 can be applied to verify properties of the retrocessionaire's payment underlying the formula for indexed AAL in equation (3.14).

3.3.1 Inflation of Gross Claims, NP Reinsurer's Payments and Primary Insurer's Retained Claims

The following three programs have equal average inflation:

(1) Average inflation of the total payments made by the NP reinsurer underlying the original excess of loss contract, assuming unlimited reinstatements:

$$\frac{\sum_i Y_{i,T}}{\sum_i \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t} \tag{3.16}$$

(2) Average inflation of total payments of the retrocession program that indemnifies the NP reinsurer portion of aggregate loss exceeding the indexed AAL:

$$\frac{\max\{(\sum_i Y_{i,T}) - L''_T, 0\}}{\max\{(\sum_{t=1}^{T}\sum_i (Y_{i,t} - Y_{i,t-1})\cdot v_t) - L, 0\}} \tag{3.17}$$

(3) Average inflation of total payments made by the NP reinsurer underlying the original excess of loss contract, with the aggregate payments capped by the indexed AAL:

$$\frac{\min\{(\sum_i Y_{i,T}), L''_T\}}{\min\{(\sum_{t=1}^{T}\sum_i (Y_{i,t} - Y_{i,t-1})\cdot v_t), L\}} \tag{3.18}$$

3.3.2 Indexing AAD and AAL by Principle of Equitable Sharing of Deflated Payments and Actual Payments

The following expression represents the retrocessionaire's payment to the NP reinsurer if future partial payments of all claims were paid at time 0:

$$\max\{(\sum_{t=1}^{T}\sum_i (Y_{i,t} - Y_{i,t-1})\cdot v_t) - L, 0\} \tag{3.19}$$

The retrocessionaire should pay the proportion of $\sum_i Y_{i,T}$, which is the same as the ratio of the expression in (3.19) to the expression in (3.13). That means that the retrocessionaire's share in the total actual payment is:

$$\sum_i Y_{i,T} \times \frac{\max\{(\sum_{t=1}^{T}\sum_i (Y_{i,t} - Y_{i,t-1})\cdot v_t) - L, 0\}}{\sum_i \sum_{t=1}^{T}(Y_{i,t} - Y_{i,t-1})\cdot v_t} \tag{3.20}$$

It can be verified that the above equation agrees with the formula for indexed AAL in equation (3.14), therefore showing the following relationships between the retrocessionaire's actual cumulative payment $\max\{(\sum_i Y_{i,T}) - L''_T, 0\}$ and NP reinsurer's aggregate deflated payments according to the original excess of loss program before applying AAL ($\sum_i \sum_{t=1}^{T}(Y_{i,t} - Y_{i,t-1})\cdot v_t$):

(1) $\max\{(\sum_i Y_{i,T}) - L''_T, 0\} = 0$ when $\sum_i \sum_{t=1}^{T}(Y_{i,t} - Y_{i,t-1})\cdot v_t \le L$

That is, the retrocessionaire does not make any payment if the NP reinsurer's aggregate deflated payments (before applying AAL) is below the un-indexed AAL.

(2) $\max\{(\sum_i Y_{i,T}) - L''_T, 0\} > 0$ when $\sum_i \sum_{t=1}^{T}(Y_{i,t} - Y_{i,t-1})\cdot v_t > L$

That is, the retrocessionaire makes payment if NP reinsurer's aggregate deflated payments (before applying AAL) is greater than the un-indexed AAL.

(3) $\sum_i Y_{i,T} = L_T''$ if and only if $\sum_i \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t = L$

That is, the NP reinsurer's aggregate deflated payments (before applying AAL) equals the un-indexed AAL if and only if the NP reinsurer's aggregate actual payments (before applying AAL) equals the indexed AAL. Once reaching this condition, the retrocessionaire will start paying immediately after the NP reinsurer makes another payment in the future

3.3.3 Monotonicity Properties of Retrocessionaire's Cumulative Payments, Indexed AAD and AAL

The notation S_T'' represents the retrocessionaire's cumulative actual payments to the NP reinsurer at time T:

$$S_T'' = \max\{(\sum_i Y_{i,T}) - L_T'', 0\} \tag{3.21}$$

It can be proved that $S_{T+1}'' \geq S_T''$ under the conditions $v_{t+1} \leq v_t \quad \forall t \leq T$ and $X_{i,T+1} \geq 0 \quad \forall i$. Retrocessionaire's cumulative payment with indexed AAD and AAL [using equations (3.14) and (3.15)] is monotonically increasing over time. The proof is similar to the proof of $Y_{i,T+1} \geq Y_{i,T}$, shown in Appendix A.

Also under the conditions $v_{t+1} \leq v_t \quad \forall t \leq T$ and $X_{i,t} \geq 0 \quad \forall i$ and $\forall t$, the two sequences $\{L_t''\}$ and $\{D_t''\}$ are both monotonically increasing on t.

3.3.4 Incremental Payments Paid by the NP Reinsurer Net of Recoveries from the Retrocessionaire

Consider incremental aggregate payments paid by the NP reinsurer net of recoveries from the retrocessionaire at time T+1:

$$(\sum_i Y_{i,T+1} - \sum_i Y_{i,T}) - S_{T+1}'' + S_T'' \tag{3.22}$$

Under the conditions $X_{i,T+1} \geq 0 \quad \forall i$ and $v_{t+1} \leq v_t \quad \forall t \leq T$, then $(\sum_i Y_{i,T+1} - \sum_i Y_{i,T} - S_{T+1}'' + S_T'') \geq 0$. That is, at time T+1 the NP reinsurer's incremental payment to the primary insurer is always greater than the incremental recovery from the retrocessionaire. Practically, if the NP reinsurer is not making a recovery from the primary insurer, then any claims emerging from the original excess of loss program will not result in a net cash-inflow for the NP reinsurer with the retrocession program in place.

3.4 A Numerical Example

The original excess of loss reinsurance program has a per-claim deductible of $3 million and per-claim limit of $1 million, both subject to the index clause. The original program has unlimited free reinstatements.

At time $T = 4$, there are three large claims, as shown in the following table.

Incremental Actual Gross Payment ($000s)

payment time	0	1	2	3	4	row sum
claim 1	$0.0	$2,120.0	$1,090.0	$0.0	$1,230.0	$4,440.0
claim 2	$0.0	$2,120.0	$2,180.0	$0.0	$0.0	$4,300.0
claim 3	$0.0	$0.0	$0.0	$4,680.0	$0.0	$4,680.0

Adjusted payments (or deflated payments) are calculated as follows:

Incremental Adjusted Gross Payment ($000s)

payment time	0	1	2	3	4	row sum
claim 1	$0.0	$2,000.0	$1,000.0	$0.0	$1,000.0	$4,000.0
claim 2	$0.0	$2,000.0	$2,000.0	$0.0	$0.0	$4,000.0
claim 3	$0.0	$0.0	$0.0	$4,000.0	$0.0	$4,000.0
Index	100	106	109	117	123	

All three claims have deflated values equal to the un-indexed ceiling (sum of un-indexed deductible and limit). According to the three relationships among the NP reinsurer's actual cumulative payment per claim and total deflated gross payments per claim illustrated in section 2.2.4, all three claims are total losses to the excess of loss program after the deductible and limit are indexed. However, the NP reinsurer's cumulative actual payments at time $T = 4$ are different for these three claims, as shown in the following table:

NP Reinsurer's Cumulative Actual Payments ($000s) at time $T = 4$

	Cumulative Actual Payments	Indexed Deductible	Indexed Limit	NP Reinsurer's Cum. Payment
claim 1	$4,440.0	$3,330.0	$1,110.0	$1,110.0
claim 2	$4,300.0	$3,225.0	$1,075.0	$1,075.0
claim 3	$4,680.0	$3,510.0	$1,170.0	$1,170.0
Total	$13,420.0	-	-	$3,355.0

3.4.1 Indexed AAL with Method 1: Matching Deflated Excess Loss with Deflated Gross Loss Per Claim

Assume that the NP reinsurer purchases a retrocession capping its potential aggregate payments at $3 million AAL subject to indexed clause. From the above table, it seems straight forward that indexed AAL should be $3.355 million at time 4, that is, the total of the losses to excess layer of the three claims.

For time periods before $T = 4$, what are the values of indexed AAL at each stage? And what if other un-indexed AAL ($2 million, $1 million) were chosen instead?

Table 3-1: Indexed AAL with Method 1 at Each Payment Time ($000s)

payment time	1	2	3	4
un-indexed AAL = $3mil	$3,000.0	$3,225.0	$3,367.5	$3,355.0
un-indexed AAL = $2mil	$2,000.0	$2,150.0	$2,245.0	$2,236.7
un-indexed AAL = $1mil	$1,000.0	$1,075.0	$1,122.5	$1,118.3

The above table is compared with the NP reinsurer's cumulative actual payments and cumulative deflated payments at each time. Deflated payment to excess of loss layer for the i^{th} claim is calculated using equation (3.3).

Table 3-2: NP Reinsurer's Cumulative Actual Payment ($000s)

payment time	0	1	2	3	4
claim 1	$0.0	$0.0	$0.0	$0.0	$1,110.0
claim 2	$0.0	$0.0	$1,075.0	$1,075.0	$1,075.0
claim 3	$0.0	$0.0	$0.0	$1,170.0	$1,170.0
Total	$0.0	$0.0	$1,075.0	$2,245.0	$3,355.0

Table 3-3: NP Reinsurer's Cumulative Deflated Payment with Method 1 ($000s)

payment time	0	1	2	3	4
claim 1	$0.0	$0.0	$0.0	$0.0	$1,000.0
claim 2	$0.0	$0.0	$1,000.0	$1,000.0	$1,000.0
claim 3	$0.0	$0.0	$0.0	$1,000.0	$1,000.0
Total	$0.0	$0.0	$1,000.0	$2,000.0	$3,000.0

3.4.2 Observations: Why Indexed AAL Changes over Time upon New Claims

From Table 3-3, the NP reinsurer's aggregate cumulative deflated payment equals $1 million at time 2. If an un-indexed AAL = $1 million was chosen, then indexed AAL at time 2 equals $1.075 million according to Table 3-1.

At time 3, the indexed AAL increases to $1.1225 million from $1.075 million at time 2. The increase in index AAL may sound intuitively incorrect, because if the purpose of AAL is to "limit" the NP reinsurer's aggregate payment, then it seems contradictory to observe an increase in the indexed AAL, even after one total loss has been observed at time 2. The phenomenon can be explained by considering the concepts of indexed per-claim deductible and limit as follows:

- Recall that $d'_{i,T+1} \geq d'_{i,T}$ and $l'_{i,T+1} \geq l'_{i,T}$ under the conditions $v_{t+1} \leq v_t \quad \forall t \leq T$ and $X_{i,T+1} \geq 0 \quad \forall i$. Therefore, indexed per-claim deductible and limit can increase over time, even if a claim is already a "total loss to excess layer" at a certain stage.

- To measure the effect of inflation, the formulas for indexed per-claim deductible and limit consider all gross payments known for a claim, irrespective of whether the claim has already become a total loss to excess layer in the past.

Similar arguments can be used to explain why indexed AAL can increase upon new payments made by the NP reinsurer at time 3.

3.4.3 Observations: Conditions when Indexed AAL by Method 1 Decreases upon New Claims

Now if un-indexed AAL = \$2 million was chosen, then indexed AAL at time 3 equals \$2.245 million according to Table 3-1. This is consistent with Table 3-3, indicating that two total losses are observed and that the NP reinsurer's aggregate cumulative actual payment equals \$2.245 million at time 3. However, at time 4, indexed AAL drops down to \$2.2367 million.

In general, if one can accept that per-claim indexed deductible and limit can increase over time when there is no deflation and that there is no gross claims recovery, it is reasonable to expect that similar monotonicity property shall be observed for indexed AAL. However, the above numerical example disproves any monotonicity property. It can be explained from two angles:

(1) In order to analyze why the indexed AAL at time 4 decreases, consider the change in average inflation. At time 3, claim 1 is still not observed as a loss to excess layer, and average inflation for claim 2 and claim 3 combined is 12.25% (= \$2.245mil ÷ \$2.000mil – 1). At time 4, claim 1 is observed as a total loss to the excess layer with average inflation 11.0%, which is lower than 12.25%. Average inflation for claims 1, 2, and 3 combined decreases to 11.83%, and therefore indexed AAL decreases accordingly.

(2) From Table 3-3, two total losses to excess layer are observed at time 3, therefore the NP reinsurer retains all cumulative actual payments to the excess of loss program (= \$2.245mil for claim 2 and claim 3 combined). At time 4, the NP reinsurer shall retain two-thirds of the aggregate cumulative actual payments to the excess of loss program when three total losses to excess layer are observed. Since claim 1's actual payment (\$1.110 million) is less than claims 2 and 3 combined average (\$1.1225 million), therefore two-thirds of the NP reinsurer's cumulative aggregate actual payment is only \$2.2367 million (= \$3.355 million × 2/3) and indexed AAL is adjusted

downward accordingly.

Conclusively, from time T to T+1, indexed AAL decreases when the average inflation of aggregate claims to the original excess of loss program paid at time T+1 (e.g., consider claim 3, claim 2 and claim 1 combined) is lower than average inflation of aggregate claims to the original excess of loss program up to time T (e.g. consider claim 3 and claim 2 combined).

Although Method 1 for indexing AAL has no monotonicity property, the indexed AAL are at each stage correctly reflecting the split between the NP reinsurer's and the retrocessionaire's payments, according to principles of equitable sharing of inflation and equitable sharing of deflated payments, assuming that deflated value of gross, excess layer, and retained claims should be matched together.

3.4.4 Indexed AAL with Method 2: Deflating Incremental Excess Loss According to Payment Time

A deflating factor v_t is multiplied with incremental excess loss according to payment time. Resulting in tables of indexed AAL's and NP reinsurer's cumulative deflated payments that are different from the corresponding tables in section 3.4.1.

Table 3-4: Indexed AAL with Method 2 at Each Payment Time ($000s)

payment time	1	2	3	4
un-indexed AAL = $3mil	$3,000.0	$3,270.0	$3,390.8	$3,484.3
un-indexed AAL = $2mil	$2,000.0	$2,180.0	$2,260.6	$2,322.9
un-indexed AAL = $1mil	$1,000.0	$1,090.0	$1,130.3	$1,161.4

Table 3-5: NP Reinsurer's Cumulative Deflated Payment with Method 2 ($000s)

payment time	0	1	2	3	4
claim 1	$0.0	$0.0	$0.0	$0.0	$902.4
claim 2	$0.0	$0.0	$986.2	$986.2	$986.2
claim 3	$0.0	$0.0	$0.0	$1,000.0	$1,000.0
Total	$0.0	$0.0	$986.2	$1,986.2	$2,888.7

To compare Method 2 with Method 1, consider if the un-indexed AAL = $2 million was chosen.

Under Method 2, the NP reinsurer's cumulative aggregate deflated payment at time 3 equals $1.9862 million, which is less than the un-indexed AAL of $2 million. Therefore, at time 3, indexed AAL ($2.2606 million) should be greater than the NP reinsurer's cumulative aggregate actual payment ($2.245 million).

However, Table 3-2 indicates that the NP reinsurer's actual payments for claim 2 and claim 3 are both total loss to excess layer, since both payments equal their corresponding indexed per-claim limit. Ideally, indexed AAL should equal $2.245 million at this stage, such that the retrocessionaire will start to pay immediately after another excess layer claim is observed. Comparing the two methods for indexing AAL, Method 1 can always satisfy such a requirement, because when deflating excess layer loss, Method 1 takes into account matching of gross, retained, and excess layer payments. Method 2, however, generally cannot satisfy such a requirement as it ignores the link between gross and excess layer payments.

Despite the above advantage, Method 1 has a major shortcoming. At time 4, the NP reinsurer pays $1.110 million to the primary insurer but receives $1.1183 million (= $3.355mil – $2.2367mil) from the retrocessionaire, therefore resulting in net cash-inflow for the NP reinsurer despite claim emergence. Scenarios similar to this are problematic because often the primary insurer practically takes up the role of the retrocessionaire: that means the NP reinsurer is only liable up to the indexed AAL and then the primary insurer will be responsible for the portion of aggregate claims above. In this numerical example, the primary insurer makes a payment of $1.230 million to its policyholder for claim 1, and also makes a net payment of $8,300 (= $1.1183mil – $1.110mil) to the NP reinsurer. Practically, the primary insurers may not be convinced to make the payment to NP reinsurer under an indexed AAL, especially since they will not need to do so if the AAL is simply un-indexed. Under Method 2, however, the situation becomes different: at time 4, the NP reinsurer makes a net payment of $77,900 to the primary insurer (= $1.110mil – [$3.355mil – $2.3229mil]). As illustrated in section 3.3.4, the NP reinsurer's incremental payment to the primary insurer is always greater than the incremental recovery from the retrocessionaire.

4. PRACTICAL ISSUES

4.1 Which Method to Use for Indexing AAD and AAL: Method 1 or Method 2?

It is not straightforward to decide whether Method 1 or Method 2 is the correct method simply by relying on principles of equitable sharing of inflation and equitable sharing of deflated payments. Each method uses its own way to determine deflated excess loss, therefore, equitable sharing can be "achieved by definition". A comparison from both a theoretical and practical point of view is shown below:

	Method 1: Matching Deflated Excess Loss with Deflated Gross Loss Per Claim	Method 2: Deflating Incremental Excess Loss According to Payment Time
Advantages	(1) Theoretical: Indexed AAL match with Indexed per-claim limit (2) Practical: If un-indexed AAL is chosen to be *k* times un-indexed per-claim limit, then occurrence of *k* total losses, but not more, will be exactly covered under indexed AAL	(1) Practical: As long as no deflation, indexed AAD and AAL increase over time when claims emerge (2) Practical: NP reinsurer always has net cash-outflow when claims emerge which sounds reasonable
Disadvantages	(1) Practical: Indexed AAD and AAL may decrease over time, which can be difficult to explain to primary insurers (2) Practical: Decreasing AAL may require the primary insurer (who takes the retrocessionaire's role) to make extra payment to the NP reinsurer besides paying the gross claim	(1) Theoretical: Indexed AAL mismatch with Indexed per-claim limit (2) Practical: If un-indexed AAL is chosen to be *k* times un-indexed per-claim limit, occurrence of *k* total losses generally result in indexed AAL greater than total of the indexed per-claim limit of the *k* total losses

Although Method 1 appears to be more appropriate from a theoretical point of view by matching both actual and deflated excess loss with gross loss and retained loss, in practice the importance of such theoretical advantage is not easily observable. Generally, when AAL is exhausted it is more likely to observe a mix of partial losses and total losses to the excess layer rather than purely total losses. The theoretical advantage only has more meaning in terms of coverage interpretation: un-indexed AAL equals k times un-indexed per-claim limit implies that exactly k total losses will be covered.

Practically Method 2 will likely receive higher level of acceptance by the market. It is because under Method 2 indexed AAL retains most of the desirable properties that are observed in indexed per-claim deductible and limit, including:

- Equitable sharing of inflation and equitable sharing of deflated payments (although "equitable sharing" depends on excess loss deflating method assumption).
- Indexed AAL "increases with claims inflation" (indexed AAL increases over time when claims emerge and inflation is positive).
- All parties (primary insurer, NP reinsurer, retrocessionaire) have net cash-outflow when claims emerge (and inflation is positive).

Indexed AAD and AAL under Method 2 are generally greater than that under Method 1. Therefore primary insurer may prefer to use Method 2 for indexed AAL, and the NP reinsurer may prefer to use Method 2 for indexed AAD.

4.2 Pricing Excess of Loss Reinsurance with Indexed AAD and AAL

The objective of pricing is to estimate expected loss cost for the prospective quotation year, and express the estimated value as a percentage of Gross Net Premium Income (GNPI) for the quotation year. This percentage is often called risk rate of the reinsurance program.

In this section, the view of the retrocessionaire as illustrated in section 3 will be taken. In taking this view, the objective is to estimate the expected value of aggregate loss cost to the original excess layer program that exceed the "indexed AAL". Using the notations in section 3, the expected value of the random variable S_T'' (or S_T' if Method 1 for indexing AAL is chosen) will be calculated. In the following discussion of various pricing approaches, it is assumed that Method 2 for indexing the AAL is chosen. Nevertheless most procedures and observations are appropriate for both Method 1 and Method 2.

Additional assumptions and notations are as follows:

T = time when all claims to the original excess of loss program are settled, assuming that T is not a random variable (e.g., one can choose T to be 50 years or even 100 years if the line of business has an extremely long tail, but practically 20 years or 25 years shall be reasonable choices)

$X_{i,t}$ = random variables for incremental loss payment, from losses that occur during the prospective quotation year (but revalued as if the occurrence date is the average accident date). As a result, $Y_{i,T}$ (loss to excess layer) and S_T'' (aggregate loss excess of indexed AAL) are random variables too. In addition, $d_{i,t}'$, $l_{i,t}'$ (indexed deductible and limit) and L_T'' are random variables as well.

In practice the distribution of $\sum_{t=1}^{T} X_{i,t}$ (ultimate ground-up loss random variable) is often modeled first, then the payment pattern at time t ($X_{i,t} \div \sum_{t=1}^{T} X_{i,t}$) is estimated.

N = number of loss random variable for the prospective quotation year. The definition of "loss occurrence" needs to match with the distribution of $\sum_{t=1}^{T} X_{i,t}$. For modeling convenience, loss occurrence can be defined as the event when $\sum_{t=1}^{T} X_{i,t}$ exceeds the indexed deductible, therefore it is then only necessary to model the severity of large losses that hit the excess of loss program.

p = GNPI for the prospective quotation year. Assume that p can be forecasted accurately at inception.

S_T'' = $\max\{(\sum_{i=1}^{N} Y_{i,T}) - L_T'', 0\}$ = the random variable of aggregate loss cost to the original excess layer program that exceeds the indexed AAL (i.e., aggregate loss cost to the retrocession program)

$\dfrac{\mathrm{E}[S_T'']}{p}$ = risk rate of the retrocession program = ratio of expected value of S_T'' to GNPI of the prospective quotation year

4.2.1 Empirical Approach

The empirical approach (also called burning cost approach) uses claims and GNPI in historical observation year(s):

- Step 1: historical ground-up claim sizes are revalued for claims inflation. For long-tail classes, claim payments for future development years and Pure IBNR need to be forecasted.

- Step 2: by using the deductible and limit for the prospective quotation year, indexed deductibles and limits are determined for calculating excess layer loss for each claim.
- Step 3: aggregate (as-if) actual excess loss is determined for each payment time, and therefore aggregate (as-if) deflated excess loss can be determined for each payment time as well, in order to determined indexed AAL (L_T'') at final settlement time T.
- Step 4: risk rate of the retrocession program is estimated as the ratio of aggregate (as-if) actual excess loss exceeding L_T'' to on-level GNPI of a historical year. If more than one historical year is available, weighted average of the ratios is taken as the risk rate of the retrocession program.

Notations

N^h = number of loss random variable for a historical observation year

$Y_{i,T}^h$ = random variable for as-if loss to the excess layer, by revaluing historical ground-up loss random variable in an observation year for claims inflation

p^h = on-level GNPI for a historical observation year

Underlying the empirical approach, it is assumed that if on-level GNPI and claim sizes are revalued appropriately, then expected historical loss frequency (= number of claims per on-level GNPI) equals prospective quotation year's expected loss frequency:

$$\frac{\mathrm{E}[N^h]}{p^h} = \frac{\mathrm{E}[N]}{p} \tag{4.1}$$

Generally $p^h \neq p$. For example, when portfolio growth is not due to rate increases, then $p^h < p$ and $\mathrm{E}[N^h] < \mathrm{E}[N]$. For this reason, if one uses the following expression to estimate risk rate of the retrocession program:

$$\frac{1}{p^h} \times \mathrm{E}[\max\{(\sum_{i=1}^{N^h} Y_{i,T}^h) - L \times \frac{\sum_{i=1}^{N^h} Y_{i,T}^h}{\sum_{i=1}^{N^h}\sum_{t=1}^{T}(Y_{i,t}^h - Y_{i,t-1}^h)\cdot v_t}, 0\}] \tag{4.2}$$

Then risk rate will likely be underestimated since $\mathrm{E}[\sum_{i=1}^{N^h} Y_{i,T}^h]$ is less than $\mathrm{E}[\sum_{i=1}^{N} Y_{i,T}]$ but the same L (= un-indexed AAL) is used.

Often a conventional solution is to modify L by multiplying with p^h/p. As a result risk rate of the retrocession program is estimated by the expression:

$$\frac{1}{p^b} \times \mathrm{E}[\max\{(\sum_{i=1}^{N^b} Y_{i,T}^b) - \frac{L \cdot p^b}{p} \times \frac{\sum_{i=1}^{N^b} Y_{i,T}^b}{\sum_{i=1}^{N^b} \sum_{t=1}^{T} (Y_{i,t}^b - Y_{i,t-1}^b) \cdot v_t}, 0\}] \tag{4.3}$$

However, the expression in (4.3) is still a biased estimator of the risk rate. The proof is straightforward by considering the short-tail case, which means per-claim deductible, limit, and AAL will not be indexed.

There are other shortcomings with the empirical approach. For example, when pricing an excess of loss layer without any AAD or AAL, often more than one claim is observed in each accident year on average. Observing 10-years experience can generally provide a reasonably large sample size. However, under the empirical approach, each observation year is only considered to be one sample. Overall, the empirical approach is not highly accurate for estimating the risk rate for the retrocession program.

4.2.2 Simulation Approach

One option is to apply a "historical simulation" approach:

- Step 1: realized values of pairs of $Y_{i,T}^b$ (revalued ultimate actual excess loss) and $\sum_{t=1}^{T} (Y_{i,t}^b - Y_{i,t-1}^b) \cdot v_t$ (revalued ultimate deflated excess loss) from all observation years are collected to form a pool of sample losses. Forecast of future claim payment development may be needed. Equal weights can be assigned to each realized pair.

- Step 2: on-level GNPI (p^b) and realized values of N^b are used to estimate E[N] and/or other parameters for distribution of N. An allowance for Pure IBNR may be needed.

- Step 3: in each simulated scenario, the number of losses are simulated from distribution of N. Then loss sizes are sampled randomly from the pool of actual and deflated loss pairs, which would then allow calculation of simulated values of $\sum_{i=1}^{N} Y_{i,T}$ (aggregate actual loss cost to the original excess layer program), $\sum_{i=1}^{N^b} \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t$ (aggregate deflated loss cost to the original excess layer program), and L_T'' (indexed AAL) and finally S_T'' (aggregate loss cost to the retrocession program).

- Step 4: repeat scenario generations in Step 3 until sufficiently large number of scenarios are generated. Then take the average of the simulated S_T'' divided by p as the risk rate for the retrocession program.

Historical simulation approach can be viewed as a refinement of empirical approach, by making use of empirical distribution of historical loss sizes (actual and deflated) while matching with prospective quotation year's loss frequency through the simulation procedure. Historical simulation approach is an appropriate choice when a large reliable sample of historical losses is available.

Another simulation approach alternative is to model the severity distribution of $\sum_{t=1}^{T} X_{i,t}$ (ultimate ground-up loss random variable) as well as the payment pattern at time t ($X_{i,t} \div \sum_{t=1}^{T} X_{i,t}$). After ground-up severity and payment pattern are simulated, $\sum_{i=1}^{N} Y_{i,T}$ and $\sum_{i=1}^{N^b} \sum_{t=1}^{T} (Y_{i,t} - Y_{i,t-1}) \cdot v_t$ can be calculated as well.

Options for modeling payment pattern include:

(1) Deterministic payment pattern: every simulated ground-up loss has the same payment pattern between time t = 1 and T.

(2) Stochastic payment pattern that is independent of $\sum_{t=1}^{T} X_{i,t}$

(3) Stochastic payment pattern that varies with $\sum_{t=1}^{T} X_{i,t}$. For example, large claims generally take a longer time to reach full settlement than small claims. However, the modeler should judge the strength of dependency between claim size and payment pattern for claims that penetrate the excess layer, and thus whether it is necessary to insert such extra complexity in the simulation procedure.

If it is decided to model the payment pattern stochastically, one simplification is to model "average settlement time". It is assumed that each claim is settled fully with a single payment at some time between 1 and T. However, remember that with this simplification, the indexed AAL calculated under Method 2 will always be the same as the indexed AAL calculated under Method 1. Therefore, it is not recommended to use such simplification if Method 2 for indexing AAL is chosen.

Even if the same deterministic payment pattern is applied for all ground-up claims, different excess layer payment patterns will still be observed for claims of different sizes: larger claims will have shorter average excess layer payment patterns. The implications are very different for indexing AAL with Method 1 or Method 2. If Method 1 for indexing AAL is chosen, then for each claim the ratio of actual excess loss to deflated excess loss equals $Y_{i,T} \div (Y_{i,T} \cdot w_{i,T})$ and is the same for all ground-up claim sizes. However, if Method 2 for indexing AAL's is chosen, then for each claim the ratio of actual excess loss to deflated excess loss equals $Y_{i,T} \div (\sum_{t=1}^{T}(Y_{i,t} - Y_{i,t-1}) \cdot v_t)$, and the ratio is higher for smaller claims to the excess layer. The above observations do not add extra complexity to the simulation approach if Method 2 for indexing AAL is chosen, but it is necessary to consider whether the implications reasonably reflect the reality.

4.2.3 Collective Risk Model

For short-tail classes, in order to estimate expected aggregate loss cost to the original excess layer program that exceed an un-indexed AAL, it is often convenient to adopt a collective risk model approach as follows:

- Step 1: the distribution of actual loss to excess layer random variable $Y_{i,T}$ is approximated by a discrete distribution.
- Step 2: some choice of distribution for number of loss random variable N (e.g., any (a,b,0) class distribution) allows a recursive formula to be used for determining distribution of aggregate loss cost to the original excess layer program $\sum_{i=1}^{N} Y_{i,T}$.
- Step 3: since un-indexed AAL is a constant, it is straightforward to calculate expected value of $\max\{(\sum_{i=1}^{N} Y_{i,T}) - L, 0\}$.

For long-tail classes, however, it is not that straightforward to calculate the expected value of $S_T'' = \max\{(\sum_{i=1}^{N} Y_{i,T}) - L_T'', 0\}$ because L_T'' is a random variable dependent on $Y_{i,t}$'s. Similar to section 4.2.2, there are several options to model payment patterns such that the distribution of L_T'' can be simplified as follows:

(1) Modeling L_T'' stochastically: assume L_T'' equals L multiplied by a random variable M. The expected value of M shall equal the average ratio of actual aggregate excess loss to deflated aggregate excess loss, and M is assumed to be independent of $Y_{i,t}$'s. It is reasonable to choose M to be lower bounded by 1 and to have an upper bound. Then the expected value of $\mathrm{E}[S_T'']$ can be calculated using conditional expectation:

$$\mathrm{E}[S_T''] = \mathrm{E}_M[\,\mathrm{E}[S_T'' \mid M]\,] = \mathrm{E}_M[\,\mathrm{E}[\max\{(\sum_{i=1}^{N} Y_{i,T}) - L \cdot M, 0\} \mid M]\,] \tag{4.4}$$

(2) Deterministic payment pattern (Excess): every excess layer loss has the same payment pattern between time t = 1 and T. This is appropriate when Method 2 is chosen for indexing AAL, because L_T'' is then no longer a random variable. L_T'' is calculated as L multiplied by the reciprocal of deflated value of $1 using the selected deterministic payment pattern. Expected value of S_T'' can then be calculated easily like in the short-

tail case.

(3) Deterministic payment pattern (Ground-up): every ground-up loss has the same payment pattern between time t = 1 and T. This is appropriate when Method 1 is chosen for indexing AAL, because $L_T^{'}$ is then no longer a random variable.

4.2.4 Allowance for Investment Income

Most loss payments of the retrocession program are paid long after the quotation year. Risk premium of the retrocession program calculated by any of the pricing approaches in section 4.2.1 to 4.2.3 shall be reduced by investment income that can be earned by the retrocessionaire between the time of premium installments and the time of loss payments.

One option is to determine an average payment pattern of the retrocession program's loss payments. Then a discount for investment income can be calculated deterministically, and multiplied with $\mathrm{E}[S_T^{''}]$ determined from the selected pricing approach.

A second option is to incorporate investment income allowance directly into stochastic modeling of loss to the retrocession program. Recall that $S_T^{''} = \max\{(\sum_{i=1}^{N} Y_{i,T}) - L_T^{''}, 0\}$ is the random variable of aggregate loss cost to the retrocession program before including allowance for investment income, then $\mathrm{E}[S_T^{''} - S_{T-1}^{''}]$ equals the expected loss payment of the retrocession program at time T. Therefore, assuming all retrocession premiums are received on the base date, risk premium for the retrocession program including investment income allowance equals:

$$\sum_{t=1}^{T} \frac{\mathrm{E}[S_t^{''}] - \mathrm{E}[S_{t-1}^{''}]}{(1+r_t)^t} \tag{4.5}$$

Where r_t denotes the annualized investment return from time 0 to t.

The second option is more practical if simulation pricing approach is used, for which not much extra modeling complexity will be added to the simulation procedures.

If a collective risk model pricing approach is used with $L_T^{''}$ modeled stochastically, much effort is needed in determining $\mathrm{E}[S_t^{''}] - \mathrm{E}[S_{t-1}^{''}]$ for all t between 1 and T, since it is necessary to define distributions of $L_t^{''} = L \cdot M_t$ for all t between 1 and T.

4.3 Limited Reinstatement and Calculating Paid Reinstatement Premium with Indexed AAL

4.3.1 Revision: Calculating Paid Reinstatement Premium for Short-Tail Classes

In short-tail classes, paid reinstatement premium are most often paid "at 100% additional premium as to time but pro rata as to amount reinstated only" (also called "100% pro-rata capita"). It means that upon occurrence of any claim to the excess layer with ground-up size X, irrespective of the time of loss occurrence or loss payment, the primary insurer pays an additional reinstatement premium to the NP reinsurer of the following amount:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \frac{\min\{X-d, l\}}{l} \tag{4.6}$$

Paid reinstatement provision is often associated with limited number of reinstatements. For example, if "two full reinstatements" are offered, it is identical to state that the excess layer has an AAL that equals three times the per-claim limit. In general, relationship between annual aggregate limit L, per-claim limit l and number of reinstatements k can be represented by the equation:

$$\text{number of reinstatements} = k = \frac{L}{l} - 1 \tag{4.7}$$

The maximum possible amount of total reinstatement premium paid by the primary insurer equals:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \frac{L-l}{l} \tag{4.8}$$

Therefore, the primary insurer is not required to pay reinstatement premium for the portion of aggregate excess layer loss that exceeds $(L - l)$. Here the author introduces the term "Annual Aggregate Reinstatement Limit", or AARL, to describe the value $(L - l)$.

To generalize, if N claims are observed each with ground-up size X_i, then total reinstatement premium paid by primary insurer equals:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \min\{(\sum_{i=1}^{N} \frac{\min\{\max\{X_i - d, 0\}, l\}}{l}), k\} \tag{4.9}$$

4.3.2 Calculating Paid Reinstatement Premium for Long-Tail Classes with Indexed Per-Claim Deductible, Limit and Method 1 for Indexing AAL

Recall that in equation (2.6), $Y_{i,T} = \min\{\max\{(\sum_{t=1}^{T} X_{i,t}) - d'_{i,T}, 0\}, l'_{i,T}\}$ represents the cumulative excess layer loss at time T for the i^{th} claim, and that $l'_{i,T}$ represents the indexed limit for the i^{th} claim. Modifying equation (4.9) so as to fit into long-tail environment implies that, at time *T,* the cumulative total reinstatement premium paid by the primary insurer equals:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \min\{(\sum_{i=1}^{N} \frac{Y_{i,T}}{l'_{i,T}}), k\} \tag{4.10}$$

It can be easily verified that equation (4.10) is identical to the following:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \min\{(\sum_{i=1}^{N} \frac{Y_{i,T}}{L'_T \times \frac{l}{L}}), \frac{L-l}{l}\} \tag{4.11}$$

Equation (4.11) indicates that the primary insurer is not required to pay reinstatement premium for the portion of aggregate excess layer loss that exceeds $(L-l) \times \frac{L'_T}{L}$ (= indexed value of AARL).

In practice, equation (4.10) is the easier method to represent how paid reinstatement should be calculated, but it brings up several issues:

(1) It is possible and reasonable that under some circumstances the NP reinsurer is required to pay the primary insurer for excess claim, but the primary insurer is not required to pay any reinstatement premium at the same time.

To demonstrate this, use the numerical example in section 3.4. For example, if four full reinstatements each at 100% pro-rata capita is offered. At time 4, three total excess layer losses are observed, therefore $\sum_{i=1}^{N}(Y_{i,T} \div l'_{i,T})$ equals 300%. Assume at time 5, no other claims are reported, but the primary makes another payment for claim 1. As a result, indexed limit for claim 1 at time 5 (= $l'_{1,5}$) is greater than that at time 4 (= $l'_{1,4}$), and the NP reinsurer is required to pay primary insurer the difference between $l'_{1,5}$ and $l'_{1,4}$. However, $(Y_{1,5} \div l'_{1,5}) = (Y_{1,4} \div l'_{1,4}) = 100\%$ and therefore $\sum_{i=1}^{N}(Y_{i,T} \div l'_{i,T})$ at time 5 is unchanged at 300%, which means that primary insurer is not required to pay additional reinstatement premium at time 5.

The above observation sounds contradictory to the reinstatement premium calculation performed in the short-tail case, where reinstatement premium is received by the NP reinsurer every time an excess claim is paid until the AARL is used up.

The author suggests, however, that a broader view should be taken to interpret the reinstatement premium calculation if it is to compare with the short-tail case. The reinstatement premium is received by the NP reinsurer every time a per-claim limit needs to be reinstated. In the numerical example, the difference between $l'_{1,5}$ and $l'_{1,4}$ simply reflects an adjustment of claim 1's indexed limit due to the updated average inflation information for this claim, but does not involve any portion of the limit being used from time 4 to 5. Therefore no limit needs to be reinstated. The portion of per-claim limit is used and needs to be reinstated if and only if $(Y_{i,t} \div l'_{i,t}) > (Y_{i,t-1} \div l'_{i,t-1})$ but not just under the condition $Y_{i,t} > Y_{i,t-1}$.

(2) As a result, the NP reinsurer's loss payment should not be constrained by whether the condition $\sum_{i=1}^{N}(Y_{i,T} \div l'_{i,T}) \geq k$ has been met at a particular point of time, but should only be capped by the AAL.

(3) Method 1 for indexing AAL is the method that is consistent with equation (4.10). This means that when $\sum_{i=1}^{N}(Y_{i,T} \div l'_{i,T})$ (= total of ratios of actual excess claim to indexed per-claim limit) exactly equals the number of full reinstatements offered, then the remaining "unused AAL" will be sufficient to pay exactly one more total loss to the excess layer (or equivalent) that will emerge in the future.

(4) When an index clause applies to per-claim deductible and limit only but not AAL, it can be problematic if paid reinstatement provision is in place. For example, it is possible that aggregate excess layer loss exceeds the un-indexed AAL, but cumulative total reinstatement premium has not yet reached the maximum according to equation (4.10). Further, if aggregate excess layer loss is less than the un-indexed AAL at time *T*–1 but exceeds the un-indexed AAL at time *T*, then what should be the amount of reinstatement premium to be paid at time *T*?

4.3.3 Calculating Paid Reinstatement Premium for Long-Tail Classes with Indexed Per-Claim Deductible, Limit and Method 2 for Indexing AAL

When Method 2 for indexing AAL (that means L''_T calculated using equation (3.14)) is chosen, then $\sum_{i=1}^{N}(Y_{i,T} \div l'_{i,T})$ is not a measure of "used limit" that is consistent with L''_T. Consider an example, when the number of total excess layer losses occurred equals *k* (= number of full reinstatements offered), then the remaining "unused AAL" is sufficient to pay future occurrences of one more total excess layer loss plus another partial loss to the excess layer.

Attempting to correct the inconsistency, modifying equation (4.11) can result in the following formula for cumulative total reinstatement premium paid by the primary insurer at time *T*:

$$\text{GNPI} \times \text{reinsurance premium rate} \times \min\{(\sum_{i=1}^{N} \frac{Y_{i,T}}{L_T'' \times \frac{l}{L}}), k\} \tag{4.12}$$

However, the above formula only corrects the inconsistency partially. Further comparing with equation (4.10), it is much more difficult to explain the concept and reasonableness of equation (4.12) when the calculation of the reinstatement premium in short-tail case had already been widely accepted in the market.

Conclusively, it is still reasonable in practice to use equation (4.10) to calculate reinstatement premium even if Method 2 for indexing AAL is chosen.

5. CONCLUSIONS

Two methods for indexing AAD and AAL are presented in this paper: Method 1 matches deflated excess loss with deflated gross loss per claim, and Method 2 deflates incremental excess loss according to payment time. The two methods are developed with concepts that are closely linked to the concepts underlying indexation of per-claim deductible and limit.

In comparing the advantages and disadvantages of the two methods from a practical point of view, indexed AAL's with Method 2 retain most of the desirable properties that are observed in the indexed per-claim deductible and limit. Method 2 will likely receive a higher level of acceptance by the market.

For the various proposed pricing approaches, the empirical approach (burning cost approach) is less preferable than the simulation or collective risk model approaches. In fact, the accuracy of the empirical approach is questionable even in the short-tail case with un-indexed AAD and AAL.

Finally, the method for calculating reinstatement premium is applicable whether Method 1 or Method 2 for indexing AAD and AAL is chosen.

Acknowledgment

The author acknowledges Paul Vendetti and Gary Blumsohn for their comments on the draft of this paper.

Appendix A : Monotonicity Properties of NP Reinsurer's Cumulative Payment and Primary Insurer's Net Cash-flow under Indexed Per-Claim Deductible and Limit

Proposition 1: NP reinsurer's cumulative payment made to primary insurer is monotonically increasing, that is, $Y_{i,T+1} \geq Y_{i,T}$, under the conditions $X_{i,T+1} \geq 0$ and $v_{t+1} \leq v_t \ \forall t \leq T$.

Proof:

By using equations (2.4), (2.5) , and (2.6), express $Y_{i,T}$ in terms of $X_{i,t}$, v_t, d, and l:

$$Y_{i,T} = (\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}$$

And similarly $Y_{i,T+1} = (\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times \min\{\max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\}, l\}$

Next, consider three cases of $\sum_{t=1}^{T} X_{i,t}$. (I) $\sum_{t=1}^{T} X_{i,t} > l'_{i,T} + d'_{i,T}$; (II) $\sum_{t=1}^{T} X_{i,t} \leq d'_{i,T}$; and (III) $d'_{i,T} < \sum_{t=1}^{T} X_{i,t} \leq l'_{i,T} + d'_{i,T}$;

Case (I): when $\sum_{t=1}^{T} X_{i,t} > l'_{i,T} + d'_{i,T}$

From equations (2.4) and (2.5) $\Rightarrow \sum_{t=1}^{T} X_{i,t} > (l+d) \times (\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t)$

$\Rightarrow \sum_{t=1}^{T} X_{i,t} \cdot v_t > l + d$

$$\begin{aligned}
\therefore Y_{i,T+1} - Y_{i,T} &= (\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times l - (\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times l \\
&= l \times \frac{(\sum_{t=1}^{T+1} X_{i,t}) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t}) \times (\sum_{t=1}^{T+1} X_{i,t} \cdot v_t)}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} \\
&= l \times X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_{T+1})}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} \\
&\geq l \times X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_t)}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} = 0
\end{aligned}$$

Case (II): when $\sum_{t=1}^{T} X_{i,t} \leq d'_{i,T}$

From equations (2.4) and (2.5) $\Rightarrow \sum_{t=1}^{T} X_{i,t} \cdot v_t \leq d \ \Rightarrow \ Y_{i,T} = 0$

$\therefore \ Y_{i,T+1} \geq Y_{i,T}$

Case (III): when $d'_{i,T} < \sum_{t=1}^{T} X_{i,t} \leq l'_{i,T} + d'_{i,T}$

From equations (2.4) and (2.5) $\Rightarrow$ $d < \sum_{t=1}^{T} X_{i,t} \cdot v_t \le d + l$

$$
\begin{aligned}
\therefore Y_{i,T+1} - Y_{i,T} \quad &= (\textstyle\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times \min\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, l\} \\
&\quad - (\textstyle\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\ge (\textstyle\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times \min\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, l\} \\
&\quad - (\textstyle\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\ge (\textstyle\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\quad - (\textstyle\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\ge 0
\end{aligned}
$$

Conclusion:

Combining cases (I), (II), and (III), under all situations $Y_{i,T+1} \ge Y_{i,T}$ holds, when the conditions $X_{i,T+1} \ge 0$ and $v_{t+1} \le v_t \ \forall t \le T$ can be fulfilled.

Proposition 2: $X_{i,T+1} - Y_{i,T+1} + Y_{i,T} \geq 0$ under the conditions $X_{i,T+1} \geq 0$ and $v_{t+1} \leq v_t \ \forall t \leq T$

Proof:

First, by using equations (2.4), (2.5), and (2.6):

$$X_{i,T+1} - Y_{i,T+1} + Y_{i,T} = X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times \min\{\max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\}, l\}$$
$$+ \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \times \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\}$$

Next, consider five cases:

Case (I): when $\sum_{t=1}^{T} X_{i,t} \cdot v_t - d < 0$ and $\sum_{t=1}^{T+1} X_{i,t} \cdot v_t - d < l$

$$\begin{aligned}
X_{i,T+1} - Y_{i,T+1} + Y_{i,T} &= X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times \max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\} + 0 \\
&\geq X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t - d) + X_{i,T+1} \cdot v_{T+1}] \\
&\geq X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times [X_{i,T+1} \cdot v_{T+1}] \quad (\because \sum_{t=1}^{T} X_{i,t} \cdot v_t - d < 0) \\
&\geq X_{i,T+1} - 1 \times [X_{i,T+1}] \quad (\because v_{t+1} < v_t \quad \forall t) \\
&= 0
\end{aligned}$$

Case (II): when $\sum_{t=1}^{T} X_{i,t} \cdot v_t - d < 0$ and $\sum_{t=1}^{T+1} X_{i,t} \cdot v_t - d \geq l$

$$\begin{aligned}
X_{i,T+1} - Y_{i,T+1} + Y_{i,T} &= X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times \min\{\max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\}, l\} + 0 \\
&\geq X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times \max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\} + 0 \\
&\geq 0
\end{aligned}$$

Case (III): when $\sum_{t=1}^{T} X_{i,t} \cdot v_t - d \geq 0$ and $\sum_{t=1}^{T+1} X_{i,t} \cdot v_t - d < l$

$$\begin{aligned}
X_{i,T+1} - Y_{i,T+1} + Y_{i,T} &= X_{i,T+1} - (\sum_{t=1}^{T+1} X_{i,t} \div \sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d] \\
&\quad + (\sum_{t=1}^{T} X_{i,t} \div \sum_{t=1}^{T} X_{i,t} \cdot v_t) \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&= X_{i,T+1} - \sum_{t=1}^{T+1} X_{i,t} + \sum_{t=1}^{T} X_{i,t} + d'_{i,T+1} - d'_{i,T} \\
&\geq 0 \quad (\because d'_{i,T+1} \geq d'_{i,T})
\end{aligned}$$

Case (IV): when $0 \leq \sum_{t=1}^{T} X_{i,t} \cdot v_t - d < l$ and $\sum_{t=1}^{T+1} X_{i,t} \cdot v_t - d \geq l$

$$\begin{aligned}
X_{i,T+1} - Y_{i,T+1} + Y_{i,T} &= X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times \min\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, l\} \\
&\quad + \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\geq X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times [(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d] \\
&\quad + \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \times [(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d] \\
&\geq 0
\end{aligned}$$

Case (V): when $\sum_{t=1}^{T} X_{i,t} \cdot v_t - d \geq l$ and $\sum_{t=1}^{T+1} X_{i,t} \cdot v_t - d \geq l$

$$\begin{aligned}
X_{i,T+1} - Y_{i,T+1} + Y_{i,T} &= X_{i,T+1} - \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} \times l + \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t} \times l \\
&= X_{i,T+1} - l \times X_{i,T+1} \frac{\sum_{t=1}^{T} X_{i,t} \cdot v_t - v_{T+1} \times \sum_{t=1}^{T} X_{i,t}}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} \\
&= X_{i,T+1} \times \frac{[(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - l] \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t) + l \times v_{T+1} \times \sum_{t=1}^{T} X_{i,t}}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} \\
&\geq 0 \quad (\because \sum_{t=1}^{T+1} X_{i,t} \cdot v_t \geq l + d)
\end{aligned}$$

Conclusion:

Combining the five cases: $X_{i,T+1} - Y_{i,T+1} + Y_{i,T} \geq 0$ under the conditions $X_{i,T+1} \geq 0$ and $v_{t+1} \leq v_t \ \forall t \leq T$

Appendix B : Equal Inflation of Gross Claims, NP Reinsurer's Payments under Indexed Per-Claim Deductible and Limit, and Primary Insurer's Retained Claim

Proposition: at time T for the i^{th} claim, gross claim's inflation equals inflation for NP reinsurer's excess of loss payments and also equals inflation for the primary insurer's retained claim, which is represented by equation (2.11):

$$\frac{1}{w_{i,T}}=\frac{\sum_{t=1}^{T}X_{i,t}}{\sum_{t=1}^{T}X_{i,t}\cdot v_t}=\frac{\min\{\max\{(\sum_{t=1}^{T}X_{i,t})-d'_{i,T},0\},l'_{i,T}\}}{\min\{\max\{(\sum_{t=1}^{T}X_{i,t}\cdot v_t)-d,0\},l\}}=\frac{\min\{(\sum_{t=1}^{T}X_{i,t}),d'_{i,T}\}}{\min\{\sum_{t=1}^{T}X_{i,t}\cdot v_t),d\}}$$

Proof:

Firstly, the case of $\sum_{t=1}^{T}X_{i,t}\le d'_{i,T}$ (which also implies $\sum_{t=1}^{T}X_{i,t}\cdot v_t\le d$) can be ignored.

Next, when $d'_{i,T}<\sum_{t=1}^{T}X_{i,t}\le l'_{i,T}+d'_{i,T}$ (which also implies $d<\sum_{t=1}^{T}X_{i,t}\cdot v_t\le l+d$.)

Inflation for primary insurer's retained claim $=\dfrac{\min\{(\sum_{t=1}^{T}X_{i,t}),d'_{i,T}\}}{\min\{\sum_{t=1}^{T}X_{i,t}\cdot v_t),d\}}$

$$=\frac{d'_{i,T}}{d}=d\times\frac{\sum_{t=1}^{T}X_{i,t}}{\sum_{t=1}^{T}X_{i,t}\cdot v_t}\times\frac{1}{d}=\frac{1}{w_{i,T}}$$

Inflation for NP reinsurer's excess of loss payments $=\dfrac{(\sum_{t=1}^{T}X_{i,t})-d'_{i,T}}{(\sum_{t=1}^{T}X_{i,t}\cdot v_t)-d}$

$$=\frac{(\sum_{t=1}^{T}X_{i,t})-d\times(\sum_{t=1}^{T}X_{i,t})\div(\sum_{t=1}^{T}X_{i,t}\cdot v_t)}{(\sum_{t=1}^{T}X_{i,t}\cdot v_t)-d}=\frac{\sum_{t=1}^{T}X_{i,t}}{\sum_{t=1}^{T}X_{i,t}\cdot v_t}=\frac{1}{w_{i,T}}$$

Finally, when $\sum_{t=1}^{T}X_{i,t}>l'_{i,T}+d'_{i,T}$ (which also implies $\sum_{t=1}^{T}X_{i,t}\cdot v_t>d+l$,)

Inflation for NP reinsurer's excess of loss payments $=\dfrac{l'_{i,T}}{l}=\dfrac{d'_{i,T}}{d}=\dfrac{1}{w_{i,T}}$

Conclusion:

$$\frac{1}{w_{i,T}}=\frac{\sum_{t=1}^{T}X_{i,t}}{\sum_{t=1}^{T}X_{i,t}\cdot v_t}=\frac{\min\{\max\{(\sum_{t=1}^{T}X_{i,t})-d'_{i,T},0\},l'_{i,T}\}}{\min\{\max\{(\sum_{t=1}^{T}X_{i,t}\cdot v_t)-d,0\},l\}}=\frac{\min\{(\sum_{t=1}^{T}X_{i,t}),d'_{i,T}\}}{\min\{\sum_{t=1}^{T}X_{i,t}\cdot v_t),d\}}$$

Appendix C: Equal Inflation of Claims Before and After Application of Indexed AAD and AAL with Method 1

Proposition: equal inflation of claims before and after application of indexed AAD and AAL can be represented by the equation in (3.9):

$$\frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} = \frac{\max\{(\sum_i Y_{i,T}) - L'_T, 0\}}{\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\}} = \frac{\min\{(\sum_i Y_{i,T}), L'_T\}}{\min\{(\sum_i Y_{i,T} \cdot w_{i,T}), L\}}$$

Proof:

Note that $(\sum_i Y_{i,T}) \le L'_T \quad \Leftrightarrow \quad (\sum_i Y_{i,T}) \le L \times \dfrac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} \quad \Leftrightarrow \quad (\sum_i Y_{i,T} \cdot w_{i,T}) \le L$

Case (I): when $(\sum_i Y_{i,T}) \le L'_T$

This case can be ignored because the retrocessionaire makes no payment.

Case (II): when $(\sum_i Y_{i,T}) > L'_T$

From equation (3.8),

$$\frac{\min\{(\sum_i Y_{i,T}), L'_T\}}{\min\{(\sum_i Y_{i,T} \cdot w_{i,T}), L\}} = \frac{L'_T}{L} = L \times \frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} \times \frac{1}{L} = \frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}}$$

From equation (3.7),

$$\frac{\max\{(\sum_i Y_{i,T}) - L'_T, 0\}}{\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\}} = \frac{(\sum_i Y_{i,T}) - L \times \dfrac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}}}{(\sum_i Y_{i,T} \cdot w_{i,T}) - L} = \frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}}$$

Conclusion:

The equality in (3.9) holds:

$$\frac{\sum_i Y_{i,T}}{\sum_i Y_{i,T} \cdot w_{i,T}} = \frac{\max\{(\sum_i Y_{i,T}) - L'_T, 0\}}{\max\{(\sum_i Y_{i,T} \cdot w_{i,T}) - L, 0\}} = \frac{\min\{(\sum_i Y_{i,T}), L'_T\}}{\min\{(\sum_i Y_{i,T} \cdot w_{i,T}), L\}}$$

Appendix D: Monotonicity Property of Retrocessionaire's Cumulative Payment with Indexed AAD and AAL with Method 1

Proposition: According to section 3.2.2, under two conditions:

(1) $X_{i,T+1} \geq 0 \ \forall i$, and

(2) $v_{t+1} \leq v_t \ \forall t \leq T$

then $S'_{T+1} \geq S'_T$, where $S'_T = \max\{(\sum_i Y_{i,T}) - L'_T, 0\}$ as defined in equation (3.12). S'_T represents retrocessionaire's cumulative payment at time T.

Proof:

First it is to prove three inequalities:

$$\sum_i Y_{i,T+1} \geq \sum_i Y_{i,T} \tag{D.1}$$

$$\sum_i Y_{i,T+1} \cdot w_{i,T+1} \geq \sum_i Y_{i,T} \cdot w_{i,T} \tag{D.2}$$

$$w_{i,T} \geq w_{i,T+1} \quad \forall i \tag{D.3}$$

For (D.1), it has been proved in Appendix A that $Y_{i,T+1} \geq Y_{i,T}$.

Therefore $\sum_i Y_{i,T+1} \geq \sum_i Y_{i,T}$ is trivial.

For (D.2), consider $Y_{i,T+1} \cdot w_{i,T+1} - Y_{i,T} \cdot w_{i,T}$ and refer to equation (3.3):

$$\begin{aligned}
& Y_{i,T+1} \cdot w_{i,T+1} - Y_{i,T} \cdot w_{i,T} \\
&= \min\{\max\{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) - d, 0\}, l\} - \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} \\
&\geq \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} - \min\{\max\{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - d, 0\}, l\} = 0.
\end{aligned}$$

For (D.3), consider $w_{i,T} - w_{i,T+1}$ and refer to equation (2.8):

$$\begin{aligned}
w_{i,T} - w_{i,T+1} &= \frac{\sum_{t=1}^{T} X_{i,t} \cdot v_t}{\sum_{t=1}^{T} X_{i,t}} - \frac{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t}{\sum_{t=1}^{T+1} X_{i,t}} \\
&= \frac{(\sum_{t=1}^{T+1} X_{i,t})(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t})(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t)}{(\sum_{t=1}^{T} X_{i,t})(\sum_{t=1}^{T+1} X_{i,t})}
\end{aligned}$$

$$= X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_{T+1})}{(\sum_{t=1}^{T} X_{i,t})(\sum_{t=1}^{T+1} X_{i,t})}$$

$$\geq X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_t)}{(\sum_{t=1}^{T} X_{i,t})(\sum_{t=1}^{T+1} X_{i,t})} = 0$$

Next for proving $S'_{T+1} \geq S'_T$, consider both cases of $L'_{T+1} \leq L'_T$ and $L'_{T+1} > L'_T$:

Case (I): when $L'_{T+1} \leq L'_T$

$S'_{T+1} - S'_T = \max\{(\sum_i Y_{i,T+1}) - L'_{T+1}, 0\} - \max\{(\sum_i Y_{i,T}) - L'_T, 0\}$ from equation (3.12)

$\geq \max\{(\sum_i Y_{i,T+1}) - L'_T, 0\} - \max\{(\sum_i Y_{i,T}) - L'_T, 0\}$

$\geq \max\{(\sum_i Y_{i,T}) - L'_T, 0\} - \max\{(\sum_i Y_{i,T}) - L'_T, 0\}$ from equation (D.1)

$= 0$

Case (II): when $L'_{T+1} > L'_T$ and $S'_T = 0$

$S'_{T+1} - S'_T \geq 0$ is trivial to prove

Case (III): when $L'_{T+1} > L'_T$ and $S'_T > 0$

$S'_{T+1} - S'_T = \max\{(\sum_i Y_{i,T+1}) - L'_{T+1}, 0\} - [(\sum_i Y_{i,T}) - L'_T]$ from equation (3.12)

$\geq (\sum_i Y_{i,T+1}) - L'_{T+1} - (\sum_i Y_{i,T}) + L'_T$

$= (\frac{L'_{T+1}}{L}) \times (\sum_i Y_{i,T+1} \cdot w_{i,T+1} - L) - (\frac{L'_T}{L}) \times (\sum_i Y_{i,T} \cdot w_{i,T} - L)$ from equation (3.4)

$\geq (\frac{L'_{T+1}}{L}) \times (\sum_i Y_{i,T} \cdot w_{i,T} - L) - (\frac{L'_T}{L}) \times (\sum_i Y_{i,T} \cdot w_{i,T} - L)$ from equation (D.2)

≥ 0 since $L'_{T+1} > L'_T$

Conclusion:

Combining cases (I), (II), and (III), under all situations $S'_{T+1} \geq S'_T$ holds, when the conditions $v_{t+1} \leq v_t \; \forall t \leq T$ and $X_{i,T+1} \geq 0 \; \forall i$ can be fulfilled.

Appendix E: Monotonicity Properties of Indexed Per-Claim Deductible and Limit

Proposition: According to section 3.2.4, under two conditions:

(1) $X_{i,T+1} \geq 0 \ \forall i$, and

(2) $v_{t+1} \leq v_t \ \forall t \leq T$

then $d'_{i,T+1} \geq d'_{i,T}$ and $l'_{i,T+1} \geq l'_{i,T}$ with $d'_{i,T}$ and $l'_{i,T}$ as defined in equations (2.4), (2.5)

Proof of: $d'_{i,T+1} \geq d'_{i,T}$ by considering $(d'_{i,T+1} - d'_{i,T}) \div d$

$$\frac{d'_{i,T+1} - d'_{i,T}}{d} = \frac{\sum_{t=1}^{T+1} X_{i,t}}{\sum_{t=1}^{T+1} X_{i,t} \cdot v_t} - \frac{\sum_{t=1}^{T} X_{i,t}}{\sum_{t=1}^{T} X_{i,t} \cdot v_t}$$ from equation (2.4)

$$= X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_{T+1})}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)}$$

$$\geq X_{i,T+1} \times \frac{(\sum_{t=1}^{T} X_{i,t} \cdot v_t) - (\sum_{t=1}^{T} X_{i,t} \cdot v_t)}{(\sum_{t=1}^{T+1} X_{i,t} \cdot v_t) \times (\sum_{t=1}^{T} X_{i,t} \cdot v_t)} = 0$$

Similar logic can be applied for proving $l'_{i,T+1} \geq l'_{i,T}$.

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Abbreviations and notations

AAD, annual aggregate deductible
AAL, annual aggregate limit
AARL, annual aggregate reinstatement limit
NP, non proportional
XS, excess of
GNPI, gross net premium income

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Ka Chun Yeung is senior pricing actuary at Gen Re in Sydney. He has worked in Gen Re's Hong Kong and Cologne offices before taking up his role in the Sydney P&C treaty team. He received his BS in Actuarial Science from the University of Hong Kong. He is a Fellow of the Casualty Actuarial Society, Fellow of the Institute of Actuaries of Australia, and RMS Certified Catastrophe Risk Analyst. He is a member of the CAS Committee on Reinsurance Research. The opinions expressed here are those of the author and not necessarily Gen Re.

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