A Method for Efficient Simulation of the Collective Risk Model

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Abstract

The Collective Risk Model (CRM) constructs aggregate losses from a claim count distribution and a claim size distribution. The aggregate losses are $Z = X_1 + ... + X_N$, where the X_i are independent and identically distributed as well as independent from the claim counts N.

Simulating individual claims can be a lengthy process when the expected number of claims is large. Often it is sufficient to collect only individual claims greater than some threshold τ together with the aggregate smaller claims. This is the case when modeling the effects of excess of loss reinsurance.

The simulation run time can be significantly reduced, therefore, by simulating large losses individually and small losses in aggregate. The challenge in doing this is to preserve the risk characteristics of the original CRM, because the small losses and the large losses are not generally independent.

This paper shows how to do this by first simulating the total claim counts and then conditionally simulating both the individual large losses and an approximation to the aggregate small losses. In the case where the claim count distribution is a mixed Poisson, it is shown that the distribution of losses simulated from this method converges to the CRM distribution. This result is a generalization of the principle that the limiting behavior of a mixed Poisson CRM is controlled by the mixing distribution.

1 Introduction

The Collective Risk Model (CRM) constructs aggregate losses from a claim count distribution and a claim size distribution. The aggregate losses are $Z = X_1 + ... + X_N$, where the X_i are independent and identically distributed as well as independent from the claim counts N.

Simulating individual claims can be a lengthy process when the expected number of claims is large. Often it is sufficient to collect only individual claims greater than some threshold τ together with the aggregate smaller claims. This is the case when modeling the effects of excess of loss reinsurance, for example.

The simulation run time can be significantly reduced, therefore, by simulating large losses individually and small losses in aggregate. The challenge in doing this is to preserve the risk characteristics of the original CRM, because the small losses and the large losses are not generally independent. This paper shows how to do this by first simulating the total claim counts and then conditionally simulating both the individual large losses and an approximation to the aggregate small losses. The small losses are drawn from a *Conditional Aggregate Distribution (CAD)* so this method is referred to as the CAD method.

Section 2 provides a brief review of other methods of reflecting the dependence between large and small losses.

After providing some notation, definitions, and basic facts, Section 3 describes the CAD method for generating large and small losses in the CRM. An illustrative example shows that the method can be highly accurate.

Section 4 discusses mixed Poisson claim count distributions and proves a theorem that shows the distribution simulated from the CAD method converges to the CRM distribution when the claim counts arise from a mixed Poisson distribution. This provides theoretical support for the practical observation that the CAD method seems to work. Additionally, the theorem supports two other practical observations: (1) the particular choice of the conditional aggregate distribution used to approximate the small losses is to some extent immaterial and (2) the mixing distribution seems to control the overall aggregate distribution. These are related to ideas presented by Mildenhall [12] and their connections are discussed.

Section 5 provides a reinsurance application that uses only the total aggregate loss mean and variance together with large the claim size and count distributions.

Section 6 illustrates a multi-line example.

2 Brief Review of Methods for Reflecting Large-Small Dependence

Dependence between large and small losses as well as more general methods of reflecting dependencies have been discussed by several authors. The methods include: recursion, Fourier Transform, numerical integration, and simulation with copulas, as well as the Iman-Connover method [5].

Using two-dimensional Panjer recursions, Walhin [17] illustrates how different results are obtained when small and large losses are modelled independently as opposed to the dependence structure implicit in the CRM. Homer and Clark [3] perform similar calculations using two-dimensional Fourier Transforms. These methods are powerful and convenient when the expected claim counts are relatively small.

Other techniques discuss more generally the modeling of dependencies between random variates, but not specifically between the large and small losses of the CRM. Homer [4] shows how to extend Heckman and Meyers' [2] numerical integration to two dimensions. Numerical integration works effectively when the claim counts are high but requires extensive programming and lacks the flexibility of simulation.

Dependencies can be imposed in simulation exercises with tools like copulas or the Iman-Connover method. Wang [22] and Venter [16] discuss the use of copulas and Mildenhall [12] generalizes the Iman-Connover method to provide additional dependence structures.

3 The Conditional Aggregate Distribution (CAD) Method

The basic idea is to simulate the total claim count N and then conditionally simulate the large claim count N_L. The small claim count N_S follows as $N - N_L$. Large claims are simulated individually. Small claims are conditionally simulated in the aggregate from an approximating distribution, the *conditional aggregate distribution*.

It will be helpful to establish some notation and recall some basic facts of the CRM in order to describe the CAD method and show how the losses from the CAD method reproduce various moments of the CRM losses as well as the correlation between large and small losses.

3.1 Notation

The CRM losses are $Z = X_1 + ... + X_N$ where the X_i are independent, identically distributed (iid) severities with common distribution $F_X(x)$, N is the random claim count with distribution $Q_N(n)$, and independent of the X_i.

The losses X_i are partitioned into losses smaller than some threshold τ and losses greater than or equal to τ. The small claim count is N_S and the large count N_L with $N = N_S + N_L$. The aggregate large losses are the sum of the individual large losses $Z_L = X_{L,1} + ... + X_{L,N_L}$ and similarly for small losses Z_S, with $Z = Z_S + Z_L$.

The distributions of the individual small and large claim sizes respectively are

$$F_{X_S}(x) = \frac{F_X(x)}{F_X(\tau)}, \quad x \in (0, \tau), \tag{1}$$

and

$$F_{X_L}(x) = \frac{F_X(x) - F_X(\tau)}{1 - F_X(\tau)}, \quad x \in [\tau, \infty). \tag{2}$$

The large claim count distribution conditional on N total claims is a Binomial distribution because the claim sizes are iid and independent from the claim counts:

$$\Pr(N_L = m|N = n) = B(n, m, q) = \binom{n}{m} q^m (1-q)^{(n-m)}, \tag{3}$$

where $q = 1 - F_X(\tau)$ is the probability of a large loss.

Correlation of large and small losses: Large and small losses are correlated through the claim count random variable (r.v.). The value of the correlation coefficient [15] is given by

$$\rho(Z_S, Z_L) = \frac{q(1-q)\mathrm{E}[X_S]\mathrm{E}[X_L](\sigma^2(N) - \mathrm{E}[N])}{\sigma(Z_S)\sigma(Z_L)}, \tag{4}$$

where $\sigma(Y)$ denotes the standard deviation of the r.v. Y.

3.2 The CAD$_k$ Algorithm

The pseudo-code for a single trial is as follows:

1. Draw N the number of total claims from the total claim count distribution Q_N.
2. Draw N_L the number of large claims from the large claim count distribution conditional on N total claims using equation (3).
3. Set the small claims $N_S = N - N_L$.
4. Draw the individual large claims $\{X_1, ..., X_{N_L}\}$ from the claim size distribution conditional on $X_i > \tau$, given by equation (2).
5. Draw the aggregate small claims from a distribution parameterized by matching the first k moments of $Z_S|N_S$.

3.2.1 Preservation of Means, Variances and Correlations

To see how means, variances and large-small correlations are preserved consider how the large and small losses are constructed. The simulated losses in steps 1-4 are completely consistent with the CRM. In the last step an approximation is used: the small aggregate claims Z_S are simulated from an aggregate distribution with the matching k conditional moments. Denote this method with k matching moments by CAD$_k$. Further, let $\mathcal{F}$ represent the distributional family used in step 5, and set

$$\widehat{Z} := CAD_k(N, X, \mathcal{F})$$

to mean the total loss r.v. generated by CAD$_k$. Similarly, $\widehat{Z}_S$ is the small loss r.v. generated by CAD$_k$. The notation $\widehat{Z}_L$ is not needed since, by construction, $\widehat{Z}_L = Z_L$.

For $k \geq 2$, CAD$_k$ preserves the mean, variance, and correlation of large and small losses:

Claim 3.1 *For $j \leq k$, $E[\widehat{Z}_S^j] = E[Z_S^j]$, and for $k \geq 2$,*

$$\rho(\widehat{Z}_S, Z_L) = \rho(Z_S, Z_L). \tag{5}$$

Proof

$$\mathrm{E}[\widehat{Z}_S^j] = \underset{N,N_L}{\mathrm{E}}[\mathrm{E}[\widehat{Z}_S^j|N, N_L]] = \underset{N,N_L}{\mathrm{E}}[\mathrm{E}[Z_S^j|N, N_L]] = \mathrm{E}[Z_S^j], \tag{6}$$

by construction. To see that correlation is preserved, it suffices to show that $\mathrm{E}[\widehat{Z}_S Z_L] = \mathrm{E}[Z_S Z_L]$. This follows as above since $\widehat{Z}_S,\ Z_L$ are independent given $N,\ N_L$. $\square$

3.2.2 Selecting a Conditional Aggregate Distribution

The central limit theorem promises that the conditional small losses are asymptotically normal, but in fairly typical insurance situations, the r.v. $Z_S|N_S$ will carry significant skewness. It seems natural, then, to consider non-normal two-parameter families as well as three-parameter families to match the conditional moments of the aggregate small claims; i.e., consider CAD_2 and CAD_3 models.

The statistics used for fitting are generally the mean, variance, and skewness. The mean, variance, and skewness of conditional small claims are given by:

$$\mathrm{E}[Z_S|N_S] = N_S \mathrm{E}[X_S], \tag{7}$$

$$\sigma^2(Z_S|N_S) = N_S \sigma^2(X_S), \tag{8}$$

$$\gamma(Z_S|N_S) = \gamma(X_S)/\sqrt{N_S}. \tag{9}$$

Table 10 of Appendix A shows the parameterizations and method of moment fits for various distributions. In several instances, a shift is used to provide an extra parameter. Section 4 develops some theory showing that the form of the conditional aggregate distribution is in some sense immaterial.

3.3 Basic Example

The following example provides a comparison between direct simulation of the CRM and simulation using the CAD.

The severity distribution is a 1ognormal ($\mu = 9$ and $\sigma = 2$) censored at \$1,000,000. The frequency distribution is a negative binomial (mean=526.99 and variance=17884). These are the same parameters used by Mildenhall in [12], section 4.1.

The conditional aggregate distribution is a lognormal. (See formulae in Appendix A.)

Tables 1 and 2 summarize the claim size and claim count distributions.

Table 1: Claim Size Distribution

Claim Size	Incremental Probability	Cumulative Probability
0	0.0%	0.0%
10,000	54.2%	54.2%
20,000	13.2%	67.4%
30,000	6.9%	74.4%
40,000	4.4%	78.8%
50,000	3.1%	81.9%
60,000	2.3%	84.2%
70,000	1.8%	86.0%
80,000	1.4%	87.4%
90,000	1.2%	88.6%
100,000	1.0%	89.6%
200,000	5.0%	94.6%
300,000	1.9%	96.5%
400,000	1.0%	97.4%
500,000	0.6%	98.0%
600,000	0.4%	98.4%
700,000	0.3%	98.7%
800,000	0.2%	98.9%
900,000	0.2%	99.1%
1,000,000	0.9%	100.0%

Table 2: Negative Binomial Parameters

Mean	526.99
Variance	17,885

Table 3 provides a comparison of percentiles and statistics for the aggregate small and large losses, while Table 4 compares the total losses. CRM large and CAD large losses are drawn from the same distribution so they only differ due to different simulations. CRM small and CAD small losses look equally close; the CAD approximation seems to work well. The correspondence in Table 4 suggests that the dependence structure is preserved and this is further supported by Table 5 which shows the simulated and theoretical correlation for large and small losses. Table 6 shows the improved run-time using methods programmed in R [14].

Table 3: CRM and CAD Simulated Losses

Cumulative Probability	CRM Small Losses	CAD Small Losses	CRM Large Losses	CAD Large Losses
1.0%	8.0	8.0	1.9	2.0
2.0%	8.7	8.8	2.4	2.5
3.0%	9.3	9.3	2.8	2.8
4.0%	9.7	9.7	3.1	3.1
5.0%	9.9	10.0	3.4	3.4
10.0%	11.1	11.2	4.3	4.3
20.0%	12.7	12.7	5.5	5.4
30.0%	13.9	13.9	6.4	6.4
40.0%	15.0	15.0	7.3	7.3
50.0%	16.1	16.1	8.2	8.1
60.0%	17.3	17.3	9.0	9.0
70.0%	18.5	18.5	10.0	9.9
80.0%	20.2	20.0	11.2	11.1
90.0%	22.5	22.4	13.1	13.0
95.0%	24.5	24.6	14.6	14.5
99.0%	28.7	28.8	18.0	17.8
99.9%	33.5	33.4	22.3	21.5
Mean	16.5	16.5	8.5	8.4
Std	4.5	4.5	3.5	3.4

Table 4: CRM and CAD Simulated Losses

Cumulative Probability	CRM Total Losses	CAD Total Losses
1.0%	11.3	11.2
2.0%	12.5	12.4
3.0%	13.3	13.3
4.0%	14.0	13.9
5.0%	14.6	14.6
10.0%	16.5	16.5
20.0%	19.0	18.9
30.0%	20.9	20.8
40.0%	22.7	22.6
50.0%	24.3	24.3
60.0%	26.2	26.1
70.0%	28.3	28.1
80.0%	30.7	30.6
90.0%	34.3	34.3
95.0%	37.5	37.3
99.0%	44.0	43.9
99.9%	53.1	51.5
Mean	25.0	24.9
Std	7.1	7.0

Table 5: Theoretical, CRM, and CAD Small-Large Linear Correlation

	Correlation
Theoretical	57.3%
CRM	58.4%
CAD	57.0%

Table 6: CRM and CAD Simulation Run-Times

Trial Count	CRM	CAD	x Faster
5,000	1.08	0.13	8.31
10,000	2.15	0.22	9.77
20,000	4.33	0.44	9.84

Before moving on to some underlying theory, we note several properties of the CAD method for loss simulation modeling:

1. It captures individual large losses.

2. It is easy to program (wtih Excel\@Risk, or in R, for example) with fast run times.

3. It works well no matter the size of $\lambda = \mathrm{E}[N]$ (as long as $\lambda_L = \mathrm{E}[N_L]$ is manageable.)

4. It reflects the joint distribution of large and small losses.

5. It can be adapted to situations with incomplete knowledge (specifically when the severity distribution is not known or assumed; see the example in Section 5).

6. It is easy to incorporate into complex models (For example, CAD can be used for multiple lines of business correlated via the claim count r.v.; see the example in Section 6).

4 CAD with the Mixed Poisson Claim Count

The losses simulated from the CAD method can be shown to converge to the losses in the CRM when the claim count is a mixed Poisson. The particular conditional aggregate distribution used is somewhat immaterial while the mixing distribution of the Poisson controls the unconditional aggregate shape.

This section discusses mixed Poisson distributions and then proves a convergence theorem for the losses simulated with the CAD method.

4.1 Mixed Poisson Claim Counts

A Mixed Poisson distribution is just a Poisson distribution with a random parameter. Formally,

Definition: N is a mixed Poisson r.v. (Q_N is a mixed Poisson distribution) if $N \sim$ Poisson(λG) for $\lambda = \mathrm{E}[N]$ and non-negative G such that $\mathrm{E}[G] = 1$ and $\sigma^2(G) = c$. In this case we write $N = MP(\lambda, G)$.

The r.v. G is referred to as the *mixing distribution*, and c the *contagion parameter*. Note that for $N = MP(\lambda, G)$,

$$\sigma^2(N) = \lambda(1 + c\lambda) \tag{10}$$

and

$$\gamma(N) = \frac{1 + c\lambda(3 + \lambda\sqrt{c}\gamma(G))}{\sqrt{\lambda}(1 + c\lambda)^{3/2}} . \tag{11}$$

Thus mixed Poisson claim counts carry positive contagion in the sense that $c \geq 0$ and the *variance-to-mean ratio* $d = (1 + c\lambda) \geq 1$.

A convenient aspect of the mixed Poisson for ground-up claims is that large and small claim counts are also mixed Poisson with the same mixing distribution. Using $CRM(N, X) = Z = X_1 + ... + X_N$ as notation for the CRM losses and abbreviating the *coefficicent of variation* (c.v.) as $\nu(Y) = \sigma(Y)/\mathrm{E}[Y]$,

Claim 4.1 *If* $Z = CRM(MP(\lambda, G), X)$*, then*

$$Z_S = CRM(MP((1 - q)\lambda, G), X_S), \text{ and}$$

$$Z_L = CRM(MP(q\lambda, G), X_L),$$

where q *is the probability of a large loss. Furthermore,*

$$\rho(Z_S, Z_L) = c/[\nu(Z_S)\nu(Z_L)]. \tag{12}$$

Proof See Mildenhall [12]. Equation (12) follows from equation (4). $\square$

Recall that for $Z = CRM(N, X)$,

$$\begin{aligned} \mathrm{E}[Z] &= \lambda\mu(X) && (13)\\ \sigma^2(Z) &= \lambda\sigma^2(X) + \mu^2(X)^2\sigma^2(N) && (14)\\ \gamma(Z) &= \big[\mu^3(X)\gamma(N)\sigma^3(N) + 3\mu(X)\sigma^2(X)\sigma^2(N) \\ &\quad +\lambda\gamma(X)\sigma^3(X)\big]/\sigma^3(Z) && (15)\end{aligned}$$

Here and later it is convenient, in particular, to have $\lambda = \mathrm{E}[N]$ and, in general, to have $\mu(Y)$ denote $\mathrm{E}[Y]$ and $\mu'_j(Y)$ denote $\mathrm{E}[Y^j]$ for a r.v. Y.

We may now use equations (10) and (11) and (13)–(15) to derive expressions for the c.v. and skewness of $Z = CRM(MP(\lambda, G), X)$:

$$\nu(Z) = \sqrt{c + \frac{1+\nu^2(X)}{\lambda}} \tag{16}$$

$$\gamma(Z) = \frac{\mu'_3(X)/(\mu^3(X)\sqrt{\lambda}) + 3c\sqrt{\lambda}(1+\nu^2(X)) + (c\lambda)^{3/2}\gamma(G)}{(1+\nu^2(X)+c\lambda)^{3/2}}\ . \tag{17}$$

It follows that as long as G and X do not depend on λ, $\nu(Z) \to \nu(G) = \sqrt{c}$, and $\gamma(Z) \to \gamma(G)$ as $\lambda \to \infty$. We may thus infer that the choice of G wields critical influence on the properties of a mixed Poisson CRM. This intuition is confirmed by the convergence theorem and examples in section 4.4 (as well as by Proposition 1 of [12]).

4.2 Negative Binonial

The most common example of a mixed Poisson is the negaive binomial, arising from $G \sim$ gamma. The gamma mixing distribution has parameters $\alpha = 1/c$ and $\beta = c$. We specify the negative binomial in terms of the mean and variance-to-mean ratio, and write $N \sim NB[\lambda, d]$. Its pdf is given by

$$\Pr(N = n) = \frac{\Gamma(n + \lambda/(d-1))}{n!\Gamma(\lambda/(d-1))} d^{-\lambda/(d-1)} \left(\frac{d-1}{d}\right)^n .$$

In the mixed Poisson formulation ($d = 1 + c\lambda$) the Negative Binomial pdf becomes

$$\Pr(N = n) = \frac{\Gamma(n + 1/c)}{n!\Gamma(1/c)} (1 + c\lambda)^{-1/c} \left(\frac{c\lambda}{1+c\lambda}\right)^n .$$

This is the parameterization given in [10]. In [12], Mildenhall notes two types of negative binomial models, distinguished by their behavior as λ varies. In the *over-dispersed Poisson* (ODP) model, the variance-to-mean ratio is independent of λ. This forces the c parameter to depend on λ as $c = c_\lambda = (d-1)/\lambda$. In this case the c.v. $\nu(N) = \sqrt{c_\lambda + 1/\lambda} \to 0$ as $\lambda \to \infty$ (and $G = G_\lambda \xrightarrow{D} 1$). The *contagion* model, on the other hand, holds c fixed so that $d = d_\lambda \to \infty$ and $\nu(N) \to \sqrt{c}$ as $\lambda \to \infty$.

4.3 Other Mixing Distributions

Tables 11–13 in Appendix B show various choices for the mixing distribution G. A twist is that Tables 11–12 add shift and slope parameters s and m. So, the general form for G is $G = s + mH$, where H is the named distribution. Refer to the appendices of [6] for the standard parameterizations of the H-distributions. The parameters of H are then expressed in terms of the contagion c, and the (optional) parameters s and m. The parameters m and s are constrained by $0 <= s < 1$ and $m > 0$. They may be redundant or determined by the conditions $\mu(G) = 1$ and $\sigma^2(G) = c$.

Table 13 shows various ways to construct G from components G_i. In this case, c is expressed in terms of the contagions c_i of the components.

The second columns of Tables 11–13 show the skewness of G. Note the relationship $\mu_3'(G) = 1 + 3c + c^{3/2}\gamma(G)$ so that the symmetric distributions have third moment equal to $1 + 3c$. The skewness $\gamma(G)$ for a component distribution is expressed in terms of the $\gamma_i = \gamma(G_i)$

See the notes after Table 13 for a more detailed discussion.

Returning to our main context, the practitioner may have trustworthy estimates for the mean and c.v. of Z_S. This will rarely, if ever, be the case for the skewness $\gamma(Z_S)$. By equation (17), and Claim 4.1, the choice of G affords the opportunity to "take a view" of $\gamma(Z_S)$ in the limit $\lambda \to \infty$. For example, if one believes that the skewness will diversify away, then the continuous or discrete uniform might be the proper choice for G. Otherwise, consideration could be given to the ratio $\kappa(G) = \gamma(G)/\nu(G) = \gamma(G)/\sqrt{c}$ (the "skew-nu" ratio). For the unshifted Poisson, gamma, and inverse Gaussian, κ is constant ($\kappa = 1, 2, 3$, respectively). For the lognormal, $\kappa = 3 + c$. Choosing the shifted exponential or Pareto will result in much higher skewness for ordinarily encountered values of c. Adding the shift parameter allows for higher skewness with the more traditional choices. For example, the shifted gamma allows any skew-nu ratio ≥ 2. Another reason to add a shift is to reflect an assumption on the effective mimimum value of Z_S. That is, adding a shift to G will tend to increase the effective minimum of N_S and, therefore, of Z_S (Compare the simulated minimum values in Appendix C, Exhibit 5 to those in Exhibit 2).

4.4 Convergence Theorem

For the convergence theorem, we need the notions of characteristic function and weak convergence of distributions:

Definition:

1. The *characteristic function* of the r.v. Y is the complex-valued $\phi_Y(t) = \mathrm{E}[e^{itY}],\ t >$

$0, i = \sqrt{-1}$.

2. A sequence of distribution functions is said to *converge weakly* to a limit F (written $F_n \xrightarrow{D} F$) if $F_n(y) \to F(y)$ for all y that are continuity points of F. A sequence of random variables Y_n is said to converge weakly or *converge in distribution* to a limit Y ($Y_n \xrightarrow{D} Y$) if their distribution functions $F_{Y_n}(y)$ converge weakly.

Theorem 4.2 *Suppose we are given $N_\lambda = MP(\lambda, G)$, and r.v.'s Y_n such that $\mu(Y_n) = nm$, $\sigma^2(Y_n) <= n^j s^2$ for some j, $0 <= j < 2$, and fixed s. Define Y_{N_λ} by $Y_{N_\lambda}|(N_\lambda = n) = Y_n$. Then*

$$Y_{N_\lambda}/(\lambda m) \xrightarrow{D} G \text{ as } \lambda \to \infty.$$

Proof Without loss of generality we may assume $m = 1$, so that $\mu(Y_n) = n$. Set

$$\bar{Y}_\lambda = Y_{N_\lambda}/\lambda.$$

Applying the Continuity theorem (see Durrett, Theorem 3.4 [1], for example), which states that convergence of characteristic functions implies convergence in distribution, we need to show

$$L := \lim_{\lambda\to\infty} \phi_{\bar{Y}_\lambda}(t) = \phi_G(t).$$

Note that $\phi_{\bar{Y}_\lambda(t)} = \phi_{Y_\lambda}(\bar{t})$, where $\bar{t} = t/\lambda$. Define N_λ^G and L_λ^G by

$$N_\lambda^G = N_\lambda | G \ (\sim \text{Poisson}(\lambda G)),$$

$$L_\lambda^G = \mathop{\mathrm{E}}_{N_\lambda^G}[\phi_{Y_n}(\bar{t})|G, N_\lambda^G = n].$$

Then $L = \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_G[L_\lambda^G]$, and $|L_\lambda^G| \leq 1$ so by the Bounded Convergence Theorem it suffices to show that

$$\lim_{\lambda\to\infty} L_\lambda^G = e^{iGt}.$$

Now, if $Z_n = Y_n - n$ then $\mu(Z_n) = 0$ and $\mu_2'(Z_n) = \sigma^2(Y_n) = n^j s^2$. So, by Durrett, Theorem 3.8 [1],

$$\begin{aligned}
\lim_{\lambda\to\infty} L_\lambda^G &= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n}\phi_{Z_n}(\bar{t})|G, N_\lambda^G = n] \\
&= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n}(1 + n^j O(\bar{t}^2))|G, N_\lambda^G = n] \\
&= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n}|G, N_\lambda^G = n] \\
&\quad + \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n} n^j O(\bar{t}^2)|G, N_\lambda^G = n]. \qquad (18)
\end{aligned}$$

Note that $N_\lambda^G \sim \text{Poisson}(\lambda G)$ implies that $\mathrm{E}[(N_\lambda^G)^r] = O((\lambda G)^r)$, for all $r \geq 0$. With a second application of Durrett, Theorem 3.8 [1] to $e^{i\bar{t}n}$, we can evaluate the second term in 18 as

$$\begin{aligned}
L^* &= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n} n^j O(\bar{t}^2)|G, N_\lambda^G = n] \\
&= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[(1 + i\bar{t}n + n^2 O(\bar{t}^2)) n^j O(\bar{t}^2)|G, N_\lambda^G = n] \\
&= \lim_{\lambda\to\infty} [O((\lambda G)^j) O(\bar{t}^2) + iO((\lambda G)^{1+j}) O(\bar{t}^3) + O((\lambda G)^{2+j}) O(\bar{t}^4)] \\
&= 0, \text{ as } 0 \leq j < 2.
\end{aligned}$$

Finally, the Poisson characteristic function $\phi(t) = e^{\lambda(e^{it}-1)}$ and one more application of Durrett, Theorem 3.8 [1] show that

$$\begin{aligned}
\lim_{\lambda\to\infty} L_\lambda^G &= \lim_{\lambda\to\infty} \mathop{\mathrm{E}}_{N_\lambda^G}[e^{i\bar{t}n}|G, N_\lambda^G = n] \\
&= \lim_{\lambda\to\infty} e^{\lambda G(e^{i\bar{t}}-1)} \\
&= \lim_{\lambda\to\infty} e^{\lambda G(i\bar{t}+O(\bar{t}^2))} \\
&= e^{iGt}. \qquad \square
\end{aligned}$$

4.4.1 Convergence of CAD and CRM

If we set $Y_n = \sum_{i=1}^n X_i$, X_i iid, then $\sigma^2(Y_n) = n\sigma^2(X)$ and we have Proposition 1 of [12], i.e., for $Z = CRM(MP(\lambda, G), X)$,

$$Z/\mu(Z) \to G,$$

no matter the choice of X ("severity is irrelevent" [1]). In our context, setting $Y_n = \widehat{Z_S}|N_S = n$ shows that for $k >= 2$ and $\widehat{Z_S} = CAD_k(MP(\lambda(N_S), G), X_S, \mathcal{F})$,

$$\widehat{Z_S}/\mu(\widehat{Z_S}) \to G$$

no matter the choice of X or $\mathcal{F}$ (severity and conditional aggregate distribution are irrelevant). Putting the two cases together supports $\widehat{Z_S}$ as a good approximation for Z_S as each of these r.v.'s converge to G when normalized by the mean. The theorem equally applies to the CAD total losses $\widehat{Z}$ by setting $Y_n = \widehat{Z_S} + Z_L|(N_S = n - B, N_L = B)$, where $B \sim Bin(n, q)$. Thus, the CAD small, large (by construction), and total losses converge to those of the CRM.

[1] Mildenhall [12] explains in the context of a CRM that, "in some cases the actual form of the severity distribution is essentially irrelevant to the shape of the aggregate distribution."

4.4.2 Convergence to G - Examples

Of course, the theorem also applies to Z_L, but this is irrelevant to most insurance situations, due to the relatively small expected claim count. In this case, severity may be quite relevant. On the other hand, Z_S will take on the characteristics of G for moderately sized insurance portfolios. The top chart of Appendix C, Exhibit 1 shows the pdf of Z_S for a portfolio similar to the one in the Basic Example of Section 3.3 - with $\mu(Z) = \$25,000,000$ and large loss threshold of \$200,000 (solid area). The mixing distribution G is the three-point Hermite (Appendix B, notes). Overlaid is the pdf of $\widehat{Z_S}$ where $\widehat{Z_S}|N_S \sim$ shifted exponential (as in Appendix A, Table 10). It's interesting that the highly skewed, monotonic exponential distribution diversifies away to the symmetric, tri-modal Hermite. In fact the Table 10 shifted exponential, as a CAD$_2$ model, satisfies the convergence theorem with $j = 1$. If we match only the mean (i.e., use a CAD$_1$ model) we may reparameterize the shifted exponential as

$$N_s\mu(X_S) - \sqrt{N_S^j}\sigma(X_S) + \text{Exp}[\sqrt{N_S^j}\sigma(X_S)],$$

and this also satisfies the convergence theorem as long as $j < 2$. The bottom chart of Appendix C, Exhibit 1 shows the case $j = 1.5$ converging to G, but more slowly. Of course, a (CAD$_1$) model with $j = 0$ would converge to G too quickly to be useful in approximating the actual CRM. For example, such a model would have $\nu^2(\widehat{Z_S}) = \nu^2(X_S)/\lambda^2 + c + 1/\lambda$, so that the severity component $\to 0$ as $1/\lambda^2$ rather than $1/\lambda$ as in equation (16).

Exhibits 2-5 in Appendix C expand on the Basic Example in Section 3 in light of the convergence theorem. The claim count distribution in this example was a negative binomial wtih mean $\lambda = 527$ and and variance-to-mean ratio $d = 33.94$. Equivalently, this is a mixed Poisson with gamma mixing distribution and contagion $c = 0.0625$. This is the subject of Appendix C, Exhibit 2. We ran the CAD algorithm using the @Risk software with 30,000 iterations. We also simulated the small losses directly from the assumed claim count and lognormal severity distributions as a basis for comparison.

The top chart of Exhibit 2.1 shows the simulated pdf of the "true" losses (solid region) versus six different choices for the CAD distributional family $\mathcal{F}$. These include both CAD$_2$ and CAD$_3$ models. Visually the fits are excellent, even for exotic choices such as the shifted exponential and the (CAD$_3$) distribution on two points. The table at the bottom of Exhibit 2.1 is adapted from the standard @Risk "Detailed Statistcs" output. It shows moment and percentile statistics for each distribution. Convergence to the mixing distribution is evidenced by considering the ratio of skewness to the c.v. (the skew-nu ratio). For a gamma distribution, this ratio is equal to 2.

Exhibit 2.2 shows scatterplots of simulated large versus small losses. The top chart shows the true small losses (Z_L vs. Z_S), while the bottom chart generates small losses via tha CAD algorithm (Z_L vs. $\widehat{Z_S}$). The close similarity of the two plots indicates that CAD does a good job of reflecting the overall dependence of large and small losses, as well as matching the numerical correlation per Claim 3.1.

Exhibits 3-5 repeat Exhibit 2 for different choices of the mixing distribution. A lognormal mixing distribution is used in Exhibit 3 with similar results. Here, convergence to G is evidenced by a skew-nu ratio in the 3-ish range. Exhibits 4 and 5 reflect more unusual choices for the mixing distribution - a uniform and a three-point shifted binomial, respectively. The shifted binomial is parameterized to match the skewness of the gamma mixing distribution, i.e., $\gamma(G) = 0.5$. In these cases, due to the distinctive shapes of the pdf graphs, visual inspection serves as evidence of convergence to G. Once again, the large vs. small loss scatterplots match up extremely well. The scatterplot for the shifted binomial has three distinct regions, corresponding to the three possible values of G. Each region appears very nearly symmetric, reflecting the fact that $\rho(Z_S|G, Z_L|G) = 0$ by equation (4).

In [12] Mildenhall uses the Iman Conover (IC) method to model the dependence of large and small losses. This is a rank-order correlation method that has the advantage of being easy to use in spreadsheets and simulations. To apply IC, Mildenhall uses simulated output from method of moments fitted curves for both small and large losses. The curve used is a shifted gamma, i.e., a fit to the first three moments of the unconditional losses. In Appendix C, Exhibit 6-7, the IC method is applied with the shifted gamma fitted curve for small losses, but the actual CRM simulated output for large losses. For the gamma mixing distribution, IC appears to do a good job matching the pdf graphs and scatterplots from Exhibit 2. Note, however, that as long as the first three moments are kept constant, the small loss curve fit will not vary with a change in the mixing distribution. The result is a poor fit to the small loss pdf for the shifted binomial mixing distribution (Exhibit 7.1). The IC method also will not reproduce the three distinct regions of the large vs. small loss scatterplots in Exhibit 5.2. If we "cheat" by applying IC to CRM simulated output for *both* large and small losses, the resulting scatterplot will show three distinct regions (Exhibit 7.2). However, the rank-order construction will not replicate $\rho(Z_S|G, Z_L|G) = 0$, as can be seen by noting the positive slope within each region. That is, the CAD method reflects the conditional small/large indepence correctly, but the IC method does not.

5 CAD with Limited Information - A Reinsurance Example

The example considered in this section is typical of a reinsurance pricing exercise requiring simultaneous modeling of large and small losses. It is a reinsurance coverage with two sections - (1) a stop-loss on the cedant's "net" losses and (2) excess-of-loss (XoL) coverage. Here, net losses are losses limited to the large loss threshold τ. Excess losses include all amounts exceeding τ and limited to the policy limit. Aggregate net and excess loss are thus given by:

$$Z_{Net} = Z_S + N_L \tau$$

and

$$Z_{XoL} = Z_L - N_L \tau.$$

The stop-loss covers net losses excess of an annual aggregate deductible (AAD) and limited to the annual aggregate limit (AAL), that is

$$Z_{SL} = \min(AAL, \max(0, Z_{Net} - AAD)).$$

Finally, the reinsurance coverage will reimburse the total of the two coverage sections:

$$Z_{Re} = Z_{SL} + Z_{XoL}.$$

To evaluate and price such a reinsurance contract, it is clearly important to accurately reflect the dependence of large and small losses. For example, the large-small dependence may significantly impact downside risk measures such as Tail Value-at-Risk (TVaR). The CAD methodology is thus an excellent candidate for the loss modeling. We will continue to assume the underlying losses follow a mixed Poisson CRM, with contagion parameter c=0.0625. Various choices for the mixing distribution G will be considered.

To this point, the CAD method as presented requires the full (ground-up) severity and claim count distributions. In reinsurance applications, however, the available data may be insufficient to reasonably parameterize these distributions. We will demonstrate how to apply CAD with more limited input information.

For this example, the input data is limited to the mean and c.v. of total aggregate losses ($\mu(Z)$, $\nu(Z)$), the mean $\lambda(N_L)$ of the large loss claim count, and the large loss severity distribution F_{X_L}. This information set-up is fairly typical in reinsurance pricing. The parameters $\mu(Z)$, $\nu(Z)$ may have been estimated using aggregated data such as loss development triangles and historical loss ratios. The distribution F_{X_L} may have been derived by fitting a curve to the supplied large loss listing, with $\lambda(N_L)$ based on historical excess claim counts.

Alternatively, F_{X_L} may be an empirical distribution developed to replicate selected loss costs for several XoL layers. In this example, we do assume an empirical distribution for F_{X_L}, with the large loss threshold $\tau = \$200,000$. The large loss distribution and other parameter values are shown in Table 7.

Table 7: Initial Parameters for Reinsurance Example

Parameter		Value
τ		\$200,000
AAD		\$25,000,000
AAL		\$20,000,000
$\mu(Z)$		\$25,000,000
$\nu(Z)$		0.28
$\lambda(N_L)$		21.5
Contagion c		0.0625
Large Loss Severity($F(X_L)$)		
Claim Size	Incremental Probability	Cumulative Probability
200,000	19.6%	19.6%
300,000	25.2%	44.8%
400,000	14.1%	58.9%
500,000	8.9%	67.8%
600,000	6.1%	73.9%
700,000	4.4%	78.3%
800,000	3.3%	81.6%
900,000	2.6%	84.2%
1,000,000	15.8%	100.0%
Implied Large Loss Statistics		
$\mu(X_L)$		\$490,900
$\nu(X_L)$		0.5691
$\mu(Z_L)$		\$10,554,350
$\nu(Z_L)$		0.3522

The large loss values in the bottom portion of Table 7 are easily computed from $F(X_L)$, $\lambda(N_L)$, and c. The c.v. $\nu(Z_L)$ is derived with equation (16) for Z_L and X_L, noting that Z_L is also a mixed Poisson CRM with contagion c.

The CAD algorithm also requires a value for the mean total claim count λ. It may be that sufficient historical data is available for a reliable estimate of λ. If this is not the case, we *posit* a value for λ. For this example, we set $\lambda = 500$.

Given a choice for the mixing distribution G, CAD steps 1-4 may now be executed. This will generate simulated values for Z_L, N_L, and N_S. To simulate values for Z_S in step 5, we need to derive expressions for the mean and c.v. of $Z_S|N_S$. Note that $\mu(Z_S) = \mu(Z) - \mu(Z_L)$, $\lambda(N_S) = \lambda(N) - \lambda(N_L)$, and

$$\mu(Z_S|N_S) = N_S\mu(X_S) = N_S\mu(Z_S)/\lambda(N_S). \tag{19}$$

By equations (8) and (9), $\nu(Z_S|N_S) = \nu(X_S)/\sqrt{N_S}$. Equation (16) applied to Z_S can be used with equations (12) and (19) and the fact that $\sigma^2(Z_S) = \sigma^2(Z) - \sigma^2(Z_L) - 2\rho(Z_S, Z_L)\sigma(Z)\sigma(Z_L)$ to eliminate $\nu(X_S)$ from the expression for $\nu(Z_S|N_S)$. After some algebra, the formula for $\nu(Z_S|N_S)$ becomes:

$$\nu(Z_S|N_S) = \sqrt{\frac{\lambda(N_S)\left[\mu^2(Z)(\nu^2(Z)-c) - \mu^2(Z_L)(\nu^2(Z_L)-c)\right] - \mu^2(Z_S)}{N_S\mu^2(Z_S)}}. \tag{20}$$

Equations (19) and (20) now allow for the method of moments fit in step 5 without referring to the small loss severity r.v. X_S. This limited information version of the algorithm is strictly a CAD_2 excercise. To derive an expression for $\gamma(Z_S|N_S)$, say, would involve an *a priori* estimate of the skewness $\gamma(Z)$ - rarely, if ever, available. Table 8 substitutes the known values from Table 7 into equations (19) and (20).

Table 8: Small Loss Model

$\mu(Z_S)$	\$14,455,650
$\lambda(N_S)$	478.50
$\mu(Z_S\|N_s)$	$= 30,189.45N_S$
$\nu(Z_S\|N_S)$	$= 2.46/\sqrt{N_S}$

We may now run the CAD algorithm to determine an appropriate premium for the coverage of Z_{Re}. The premium P is set as $P = \mu(Z_{Re}) + u\Phi$, where Φ is a downside risk measure and u is the load factor. For this exercise, $\Phi = TVaR(Z_{Re}, 0.99)$, the Tail Value-at-Risk of Z_{Re} at the 99th percentile. The load factor is set equal to 10%. Table 9 shows the results of running the CAD algorithm with 30,000 iterations and various choices of the mixing distribution G. There is some variation in $\mu(Z_{SL})$ and significant variation in the TVaR values as G varies. This results in a smaller, but still significant variation in indicated premium.

Care should be taken in applying the limited information CAD method. The choice of the parameters λ, and c, along with the input information will impute values for some of the other loss statistics. The preceding example imputes values for the small loss statistics $\mu(X_S)$, $\sigma^2(X_S)$, $\mu(Z_S)$, $\sigma^2(Z_S)$, and also for $\sigma^2(Z_L)$ (through the choice of c). However, there is no a priori guarantee that, say, $\sigma^2(X_S) > 0$. There may also be a more subtle inconsistency, such as $\mu(X_S) > \mu(X_L)$. The practitioner should include these types of consistency checks when applying the limited information CAD.

It is possible that input information such as found in Table 7 is internally consistent but inconsistent with the mixed Poisson CRM. Informally, we say that the input information *admits a mixed Poisson CRM* if there is a choice of λ and c resulting in no inconsistencies.

Table 9: Simulation Results from Different Mixing Distributions

Mixing Distribution	Uniform	Gamma	Log-normal	Shifted Gamma	S. Log-normal	Expo-nential	Pareto	Beta	S. Bi-nomial
$\mu(Z_L)$	10.5	10.6	10.6	10.6	10.5	10.6	10.5	10.6	10.5
$\mu(Z_S)$	14.4	14.4	14.4	14.4	14.4	14.4	14.4	14.4	14.4
$\mu(Z_{Net})$	18.7	18.7	18.7	18.7	18.7	18.7	18.7	18.7	18.7
$\mu(Z_{XoL})$	6.2	6.3	6.3	6.3	6.2	6.3	6.2	6.3	6.2
$\mu(Z_{SL})$	1.6	1.5	1.5	1.4	1.2	1.4	1.1	1.5	1.7
$\mu(Z_{Re})$	7.8	7.7	7.7	7.7	7.5	7.7	7.4	7.7	7.9
TVaR(ZRe, 99)	22.3	27.3	28.5	33.0	34.6	33.1	34.3	31.8	26.5
Premium	10.1	10.5	10.6	11.0	10.9	11.0	10.8	10.9	10.6

6 CAD with Multiple Lines of Business

This section adapts the CAD method to model multiple lines of business and impose correlation between lines. In this context, let Z_i, $i = 1 \dots m$, be the aggregatee loss r.v. for the ith line, and τ_i the large loss threshhold. All other notations ($Z_{i,S}$, $Z_{i,L}$, etc.) carry through. As in the previous section we allow for limited information, but say that Z_i admits a mixed Poisson CRM with parameters λ_i and c_i. Note that by equation (16), $c_i < \min(\nu^2(Z_i) - 1/\lambda(N_i), \nu^2(Z_{i,S}) - 1/\lambda(N_{i,S}), \nu^2(Z_{i,L}) - 1/\lambda(N_{i,L}))$.

6.1 Common Shock CAD

Of course, one can extend the CAD method to m lines of business simply by iterating m times. For the multi-line mixed Poisson CRM, it's natural to impose correlation via a common shock component on the mixing distributions G_i [11]. As noted in Appendix B, the *twisted product* construction is well-suited to this purpose.

With notation as above set $c_{\min} = \min\{c_i, i = 1 \dots m\}$, and take w such that $0 \le w \le 1$. The parameter w is the weight given to the common shock component. We now assume that the mixing distribution G_i has the form

$$G_i[c_i] = G_1 \bullet G_{2,i} = G_1[wc_{\min}]G_{2,i}[(c_i - wc_{\min})/G_1].$$

Here, G_1 is the common (or industry) component and $G_{2,i}$ is the line-specific component, with contagion parameter "distorted" by G_1. By the discussion in Appendix B, $\sigma^2[G_i] = wc_{\min} + c_i - wc_{\min} = c_i$, as required.

Programatically, step 1 of the CAD algorithm becomes

Step 1^{CS}: Draw G_1 from $G_1[wc_{\min}]$. Then, for each i, draw N_i from $MP(\lambda_i G_1, G_{2,i}[(c_i - wc_{\min})/G_1])$.

Steps 2-5 then proceed unchanged for each line. By analogy with equation (12), the common shock CAD results in the following correlations for $i \neq j$:

$$\rho(\widehat{Z_{i,S}}, \widehat{Z_{j,S}}) = wc_{\min}/(\nu(Z_{i,S})\nu(Z_{j,S}))$$

$$\rho(\widehat{Z_{i,S}}, Z_{j,L}) = wc_{\min}/(\nu(Z_{i,S})\nu(Z_{j,L}))$$

$$\rho(Z_{i,L}, Z_{j,L}) = wc_{\min}/(\nu(Z_{i,L})\nu(Z_{j,L})).$$

6.2 Common Shock CAD with Conditional Correlation

In [11], Meyers employs a common shock model acting on the severity distributions, in addition to a claim count model similar to that described above. The CAD method suppresses reference to the small loss severity, especially in the case of limited information. To generate a second source of between-line correlation, we specify a fixed correlation matrix to be applied to the $Z_{i,S}|N_{i,S}$ in step 5 of the CAD algorithm. Step 5 is then replaced by

Step 5^{Corr}: Draw aggregate small losses for each line from a joint distribution $[\widehat{Z_{1,S}}|N_{1,s} \ldots \widehat{Z_{m,S}}|N_{m,s}]$ with correlation matrix $\Gamma = [r_{ij}]$ and such that the marginals $\widehat{Z_{i,S}}|N_{i,s}$ are parameterized by matching the first k moments of $Z_{i,S}|N_{i,S}$.

For $i \neq j$, Step 5^{Corr} implies that

$$\text{Cov}(\widehat{Z_{i,S}}|N_{i,S}, \widehat{Z_{j,S}}|N_{j,S}) = r_{ij}\sqrt{N_{i,S}N_{j,S}}\sigma(X_{i,S})\sigma(X_{j,S}).$$

It follows that for common shock CAD with conditional correlation:

$$\mathop{\text{E}}_{N_{i,S},N_{j,S}}[\text{Cov}(\widehat{Z_{i,S}}|N_{i,S}, \widehat{Z_{j,S}}|N_{j,S})] \approx h_{ij}r_{ij}\sqrt{\lambda(N_{i,S})\lambda(N_{j,S})}\sigma(X_{i,S})\sigma(X_{j,S}),$$

where $h_{ij} = \text{E}[\sqrt{G_iG_j}] = \mathop{\text{E}}_{G_1}[\sqrt{G_{2,i}G_{2,j}}|G_1]$, using $\text{E}[\sqrt{N}] \approx \sqrt{\lambda}$ for N Poisson. Furthermore,

$$\begin{aligned}
\text{Cov}[\text{E}[\widehat{Z_{i,S}}|N_{i,S}], \text{E}[\widehat{Z_{j,S}}|N_{j,S}]] &= \mu(X_{i,S})\mu(X_{j,S})\text{Cov}[N_{i,s}, N_{j,s}] \\
&= \mu(X_{i,S})\mu(X_{j,S})wc_{\min}\lambda(N_{i,S})\lambda(N_{j,S}) \\
&= wc_{\min}\mu(Z_{i,S})\mu(Z_{j,S}).
\end{aligned}$$

Using equation (16) to eliminate the small loss severity we find;

$$\begin{aligned}
\rho(\widehat{Z_{i,S}}, \widehat{Z_{j,S}}) &= \frac{\mathop{\text{E}}_{N_{i,S},N_{j,S}}[\text{Cov}(\widehat{Z_{i,S}}|N_{i,S}, \widehat{Z_{j,S}}|N_{j,S})] + \text{Cov}[\text{E}[\widehat{Z_{i,S}}|N_{i,S}], \text{E}[\widehat{Z_{j,S}}|N_{j,S}]]}{\sigma(Z_{i,s})\sigma(Z_{i,s})} \\
&\approx \frac{wc_{\min} + h_{ij}r_{ij}\prod_{\iota=i,j}\sqrt{\nu^2(Z_{\iota,S}) - c_\iota - 1/\lambda(N_{\iota,s})}}{\nu(Z_{i,S})\nu(Z_{j,S})}. \qquad (21)
\end{aligned}$$

Note that $h_{ij} = 1$ if $w = 1$ and $c_i = c_j = c_{\min}$. In particular, if Z_i and Z_j are identical, then write $c_i = c_j = t(\nu^2(Z_{i,S}) - 1/\lambda(N_{i,S}))$, and (21) reduces to

$$\begin{aligned} \rho(\widehat{Z_{i,S}}, \widehat{Z_{j,S}}) &\approx (t + r_{ij}(1-t))[1 - 1/(\nu^2(Z_{i,S})\lambda(N_{i,S}))] \\ &\approx (t + r_{ij}(1-t)), \end{aligned}$$

if $\lambda(N_{i,S}) >> 1/\nu^2(Z_{i,S})$.

7 Conclusion

The CAD method provides a way to efficiently simulate the CRM while preserving the inherent dependencies between large and small losses. These dependencies are fundamentally driven by the claim counts and the theorem presented herein shows how the mixed Poisson CRM and CAD method model will converge as the expected claim count grows. This provides theoretical support for the practical oservation that the CAD method does a good job approximating the CRM.

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A Conditional Aggregate Distributions

Table 10: CAD_2 and CAD_3 Fits to $Z_S|N_S$

Distribution	Statistics	Fit
Normal $(\widehat{\mu},\widehat{\sigma})$	$\mu=\widehat{\mu}$ $\sigma^2=\widehat{\sigma}^2$	$\widehat{\mu}=N_S\mu(X_S)$ $\widehat{\sigma}=\sqrt{N_S}\sigma(X_S)$
Uniform on $(\widehat{\mu}-\widehat{r},\widehat{\mu}+\widehat{r})$	$\mu=\widehat{\mu}$ $\sigma^2=\widehat{r}^2/3$	$\widehat{\mu}=N_S\mu(X_S)$ $\widehat{r}=\sqrt{3N_S}\sigma(X_S)$
Lognormal $(\widehat{\mu},\widehat{\sigma})$	$\mu=e^{\widehat{\mu}+\widehat{\sigma}^2/2}$ $\sigma^2=\mu^2(e^{\widehat{\sigma}^2}-1)$	$\widehat{\mu}=\ln[N_S\mu(X_S)]-\widehat{\sigma}^2/2$ $\widehat{\sigma}=\sqrt{\ln[1+\sigma^2(X_S)/(N_S\mu^2(X_S))]}$
Gamma $(\widehat{\alpha},\widehat{\beta})$	$\mu=\widehat{\alpha}\widehat{\beta}$ $\sigma^2=\widehat{\alpha}\widehat{\beta}^2$	$\widehat{\alpha}=N_S/\nu^2(X_S)$ $\widehat{\beta}=\mu(X_S)\nu^2(X_S)$
Shifted Exponential $(\widehat{s},\widehat{\theta})$	$\mu=\widehat{\theta}+\widehat{s}$ $\sigma^2=\widehat{\theta}^2$	$\widehat{\theta}=\sqrt{N_S}\sigma(X_S)$ $\widehat{s}=N_S\mu(X_S)-\widehat{\theta}$
2-Point (CAD_3) (P$(\widehat{\mu}-\widehat{a})=p$ P$(\widehat{\mu}+\widehat{b})=1-p$)	$\mu=\widehat{\mu}$ $\sigma^2=p\widehat{a}^2+(1-p)\widehat{b}^2$ $\gamma=\dfrac{(1-p)\widehat{b}^3-p\widehat{a}^3}{\sigma^{3/2}}$	$\widehat{\mu}=N_S\mu(X_S)$ $\widehat{s}=\sqrt{N_S}\sigma(X_S)$ $p=\left(1+\gamma(X_S)\sqrt{\dfrac{1}{4N_S+\gamma^2(X_S)}}\right)\Big/2$ $\widehat{a}=\widehat{s}\sqrt{(1-p)/p}$ $\widehat{b}=\widehat{a}p/(1-p)$
Shifted Lognormal $(\widehat{\mu},\widehat{\sigma},\widehat{s})$	$\mu=\widehat{s}+e^{\widehat{\mu}+\widehat{\sigma}^2/2}$ $\sigma^2=(e^{\widehat{\sigma}^2}-1)(\mu-\widehat{s})^2$ $\gamma=\eta(\eta^2+3)$, where $\eta=\sqrt{e^{\widehat{\sigma}^2}-1}$	$\widehat{\mu}=\ln(N_S\mu(X_S)-\widehat{s})-\widehat{\sigma}^2/2$ $\widehat{\sigma}=\sqrt{\ln\left[1+\dfrac{N_S\sigma^2(X_S)}{(N_S\mu(X_S)-\widehat{s})^2}\right]}$ $\widehat{s}=N_S\mu(X_S)-\sqrt{N_S}\sigma(N_S)/(\zeta-1/\zeta)$, where $\zeta=[\sqrt{4+\gamma^2(X_S)/N_S}+\gamma(X_S)/(2\sqrt{N_S})]^{1/3}$
Shifted Gamma $(\widehat{\alpha},\widehat{\beta},\widehat{s})$	$\mu=\widehat{s}+\widehat{\alpha}\widehat{\beta}$ $\sigma^2=\widehat{\alpha}\widehat{\beta}^2$ $\gamma=2/\sqrt{\widehat{\alpha}}$	$\widehat{\alpha}=4N_S/\gamma^2(X_S)$ $\widehat{\beta}=\gamma(X_S)\sigma(X_S)/2$ $\widehat{s}=N_S\mu(X_S)-\widehat{\alpha}\widehat{\beta}$
Generalized Beta $(\widehat{\alpha},\widehat{\beta},\widehat{m}(=\max))$ (min=0)	$\mu=\widehat{\alpha}\widehat{m}/(\widehat{\alpha}+\widehat{\beta})$ $\sigma^2=\mu^3\widehat{\beta}/[\widehat{\alpha}(\mu+\widehat{\alpha}\widehat{\beta})]$ $\gamma=2\mu\sigma(\widehat{\alpha}-\widehat{\beta})/\eta$, $\eta=\sigma^2\widehat{\alpha}+\mu^2\widehat{\beta}$	$\widehat{\alpha}=(1-1/\zeta)N_S/\nu^2(X_S)-1/\zeta$ $\widehat{\beta}=\widehat{\alpha}(\zeta-1)$ $\widehat{m}=\zeta N_S\mu(X_S)$, where $\zeta=1+\nu(X_S)\dfrac{\gamma(X_S)\nu(X_S)+2N_S}{2\nu(X_S)-\gamma(X_S)}$

B Poisson Mixing Distributions

B.1 Tables of distributions

Table 11: Continuous Mixing Distributions

Family and Equation	Skewness
Gamma: $G = s + \text{Gamma}\left[\frac{(1-s)^2}{c}, \frac{c}{(1-s)}\right]$	$\frac{2\sqrt{c}}{1-s}$
Lognormal (Logn): $G =$ $s + \text{Logn}\left[\ln\left(\frac{(1-s)^2}{\sqrt{(1-s)^2+c}}\right), \sqrt{\ln\left(1+\frac{c}{(1-s)^2}\right)}\right]$	$\frac{\sqrt{c}}{1-s}\left(3+\frac{c}{(1-s)^2}\right)$
Exponential (Exp): $G = 1-\sqrt{c}+\text{Exp}[\sqrt{c}],\ c<1$	2
Inverse Gaussian (IG): $G = s + \text{IG}\left[(1-s), \frac{(1-s)^3}{c}\right]$	$\frac{3\sqrt{c}}{1-s}$
Pareto (Par): $G = 1-\sqrt{\frac{c}{k}}+\text{Par}\left[\sqrt{\frac{c}{k}}\left(\frac{k+1}{k-1}\right), \frac{2k}{k-1}\right]$ where $max(1,c) < k < 3$	$\frac{2}{\sqrt{k}}\left(\frac{3k-1}{3-k}\right)$
Uniform (U): $G = \text{U}\left[1-\sqrt{3c}, 1+\sqrt{3c}\right],\ c<1/3$	0
Generalized Beta on (s,M+s) (GB): $G = \text{GB}\left[\alpha, (M-1+s)\alpha/(1-s), s, M+s\right]$, where $\alpha = (1-s)[(1-s)(M-1+s)/c-1]/M$	$\frac{2\sqrt{c}(M-2(1-s))}{((1-s)(M-1+s)+c)}$

Table 12: Discrete Mixing Distributions

Family and Equation	Skewness
Discrete Uniform on 2m+1 points: $G = D[\Delta, p, m]$, defined by $\mathrm{P}(1) = p,\ \mathrm{P}(1 \pm j\Delta) = \dfrac{1-p}{2m},\ j \le m$ $\Delta = \sqrt{\dfrac{6c/(1-p)}{(m+1)(2m+1)}},\ 1 - m\Delta > 0$	0
Poisson (Psn): $G = s + \dfrac{c}{(1-s)}\mathrm{Psn}[(1-s)^2/c]$	$\dfrac{\sqrt{c}}{1-s}$
Negative Binomial (NB[λ,d]): $G = s + \dfrac{c}{d(1-s)}\mathrm{NB}[d(1-s)^2/c, d]$ M an integer ≥ 1	$\dfrac{(2-1/d)\sqrt{c}}{1-s}$
Binomial (Bin): $G = s + \dfrac{(1-s)^2 + cM}{M(1-s)}\mathrm{Bin}\left[M, \dfrac{(1-s)^2}{(1-s)^2 + cM}\right]$ M an integer ≥ 1	$\dfrac{\sqrt{c}}{1-s} - \dfrac{1-s}{M\sqrt{c}}$

Table 13: Component Mixing Distributions

Family and Equation	Skewness
Weighted Sum: $G[c] = pG_1[c_1] + (1-p)G_2[c_2]$ $c = p^2 c_1 + (1-p)^2 c_2$	$\dfrac{pc_1^{3/2}\gamma_1 + (1-p)c_2^{3/2}\gamma_2}{c^{3/2}}$
Straight Product: $G[c] = G_1[c_1]G_2[c_2]$, G_1, G_2 independent. $c = c_1 + c_2 + c_1 c_2$	$\dfrac{c_1 c_2[6 + 3(\sqrt{c_1}\gamma_1 + \sqrt{c_2}\gamma_2) + \sqrt{c_1 c_2}\gamma_1\gamma_2]}{c^{3/2}}$
Twisted Product: $G[c] = G_1[c_1]G_2[c_2/G_1]$ $c = c_1 + c_2$	$\dfrac{\mu_3'(G_1)f(G_1, G_2) - 1 - 3c}{c^{3/2}}$, where $f(G_1, G_2) = \underset{G_1}{\mathrm{E}}(\mu_3'(G_2[c_2/G_1]\vert G_1))$

B.2 Additional Notes

1. **Products of Mixing Distributions**. In several papers ([8],[10], for example), Meyers presents count r.v.'s of the form $N = N^*[G_1[c_1]\lambda, d(G_1))]$, where G_1 is a mixing distribution, and N^* is a family depending on λ, and d (i.e., $N^* \sim NB[\lambda, d]$). We consider the case $N^* \sim MP(\lambda, G_2[c_2])$, with $d = d_2 = 1 + c_2\lambda$. Then N is also mixed Poisson, with $N \sim MP(\lambda, G_1G_2)$. If G_1 and G_2 are independent then we call $G = G_1G_2$ a *straight product.* In this case the contagion parameter for G is $c = c_1 + c_2 + c_1c_2$. The conditional r.v. $N|G_1$ has variance-to-mean ratio $d(G_1) = 1 + c_2G_1\lambda$. Should we wish to hold $d(G_1)$ constant, we may drop the independence of G_1, G_2, and assume that G_2 depends on G_1 as $G_2 = G_2^*[c_2/G_1]$ where G_2^* is a family of mixing distributions. With a slight abuse of notation, we drop the * and define the *twisted product* as $G_1 \bullet G_2 = G_1G_2[c_2/G_1]$. For a twisted product, $c = c_1 + c_2$, and $d|G_1 = d_2 = 1 + c_2\lambda$.

The claim count presented in [8] is concisely described as $N = NB[G_1\lambda, d]$. As d is fixed with respect to G_1, this is equivalent to $N = MP(\lambda, G_1 \bullet G_2)$, with $G_2 \sim$ Gamma and $c_2 = (d-1)/\lambda$. Now, its also the case that d is fixed with respect to λ, and thus the underlying negative binomial model (i.e., $N|G_1 = 1$) is of the ODP type. On the other hand, if $G_1 \sim$ gamma, then $N_1 = MP(\lambda, G_1)$ is a negative binomial model of the contagion type. If we set $c_1 = wc$, for some $0 <= w <= 1$, then $c = c_1 + c_2$ implies that $c_2 = (1-w)c$. Thus N can be considered a sort of credibility weighting between the ODP and contagion models.

The straight product formulation is seen in the "common shock" method for modeling correlation over several lines of business. This method assigns to the ith line of business the claim count $N_i = MP(\lambda_i, G_1G_{2,i})$. Here, G_1 is the common ("industy-based" in [10]) component and the $G_{2,i}$ are the line-specific components. As in equation (12), this generates a correlation of $\rho_{ij} = c_1/(\nu_i\nu_j)$ between lines i and j, $i \neq j$. A twisted product is also well-suited to this purpose, and produces the same correlations. As above, $c = c_1 + c_2$ allows us to consider the model as a credibility weighting, now between the common and line-specific components.

We do not have a closed-form formula for the skewness of $G = G_1 \bullet G_2$. However, suppose $\mu_3'(G_2) = \sum_{i=0}^{3} a_i c_2^i$. This is the case for $G_2 \sim$ gamma, and several others, but not for $G_2 \sim$ exponential. (The exponential is not a special case of gamma–unless $c = 1$–as the shift $s = 1 - \sqrt{c}$ is forced.) Then $\mu_3'(G) = a_0\mu_3'(G_1) + a_1(1+c_1)c_2 + a_2c_2^2 + a_3c_2^3$, from which $\gamma(G)$ can be computed.

2. **Discrete Mixing Distributions** - The three-point "Hermite" distribution given by $Pr(1 + k\sqrt{3c}) = 2/3 - |k|/6,\ k = -1, 0, 1$ is used in [10]. This is an instance of the general discrete uniform with a mass at $G = 1$. A Poisson *mixing* distribution is an important limiting case of the framework presented in [19] and [18]. This is one example of *infinitely*

divisible mixing distributions, in which case the claim count can also be represented as a compound Poisson in the sense of [6].

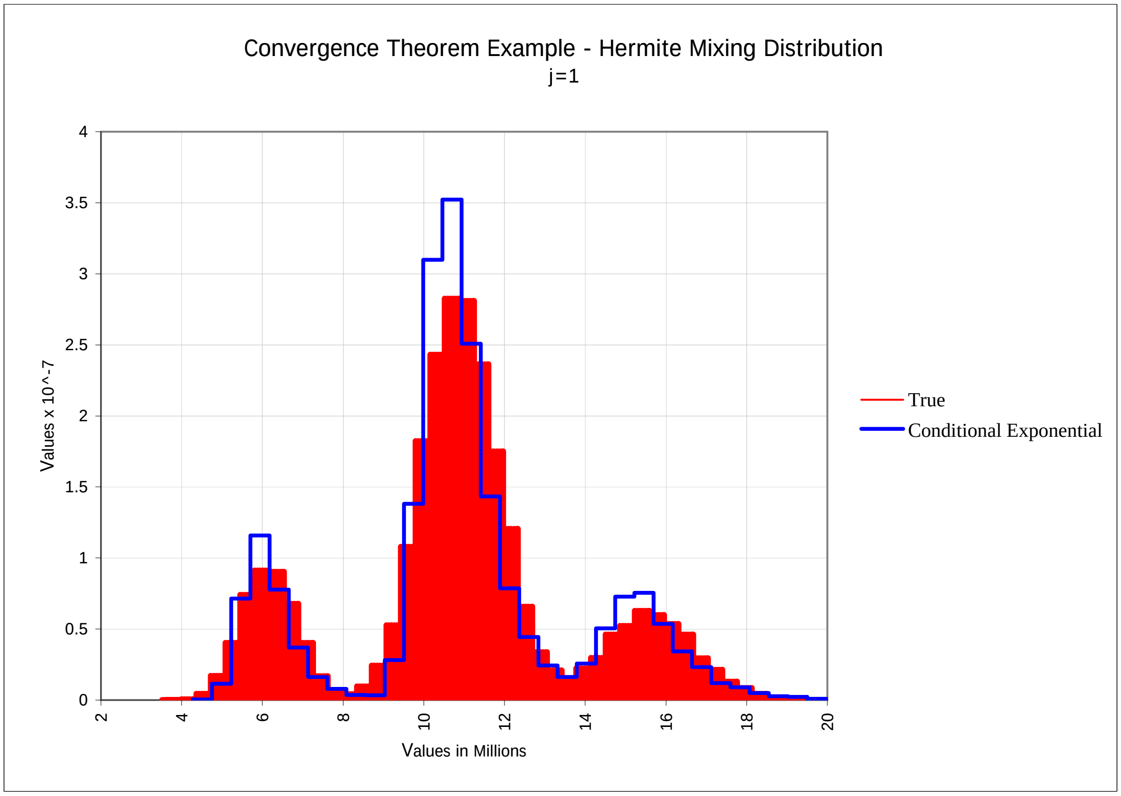
The shifted binomial is a very flexible choice for G. It converges to a Poisson as the the integer parameter $M \to \infty$, s fixed. For a given value of M, with $M \leq 1/c$, setting $s = 1 - \sqrt{Mc}$ results in a symmetric distribution different from the discrete uniform. In fact, $G \to$ normal as $c \to 0$ with $M = [1/c]$, and $s = 1 - \sqrt{Mc}$. In general, for any skewness value $\gamma > 0$, there is s such that $\gamma(G) = \gamma$, as long as $M\sqrt{c}(\sqrt{c} - \gamma) < 1$. (Note that this condition is satisfied trivially for $\gamma \geq \sqrt{c}$.)

In [21], Simar gives an algorithm for constructing a non-parametric maximum likelihood estimator (NPMLE) based on claim count observations. The NPMLE is then a finite mixing distribution whose size depends on the number of observations.

3. **Other Continuous Mixing Distributions** - The inverse Gaussian as a mixing distribution is the subject of [20] and is mentioned in [22], [18], and [12]. The resulting claim count is the Poisson-inverse Gaussian, or PIG. Given its populartiy as a model for aggregate distributions the lognormal is also a natural candidate as a mixing distribution.

C CAD Examples

Exhibit 1


Convergence Theorem Example - Hermite Mixing Distribution
j=1
Values x 10^-7
Values in Millions
True
Conditional Exponential
0
0.5
1
1.5
2
2.5
3
3.5
4
2
4
6
8
10
12
14
16
18
20

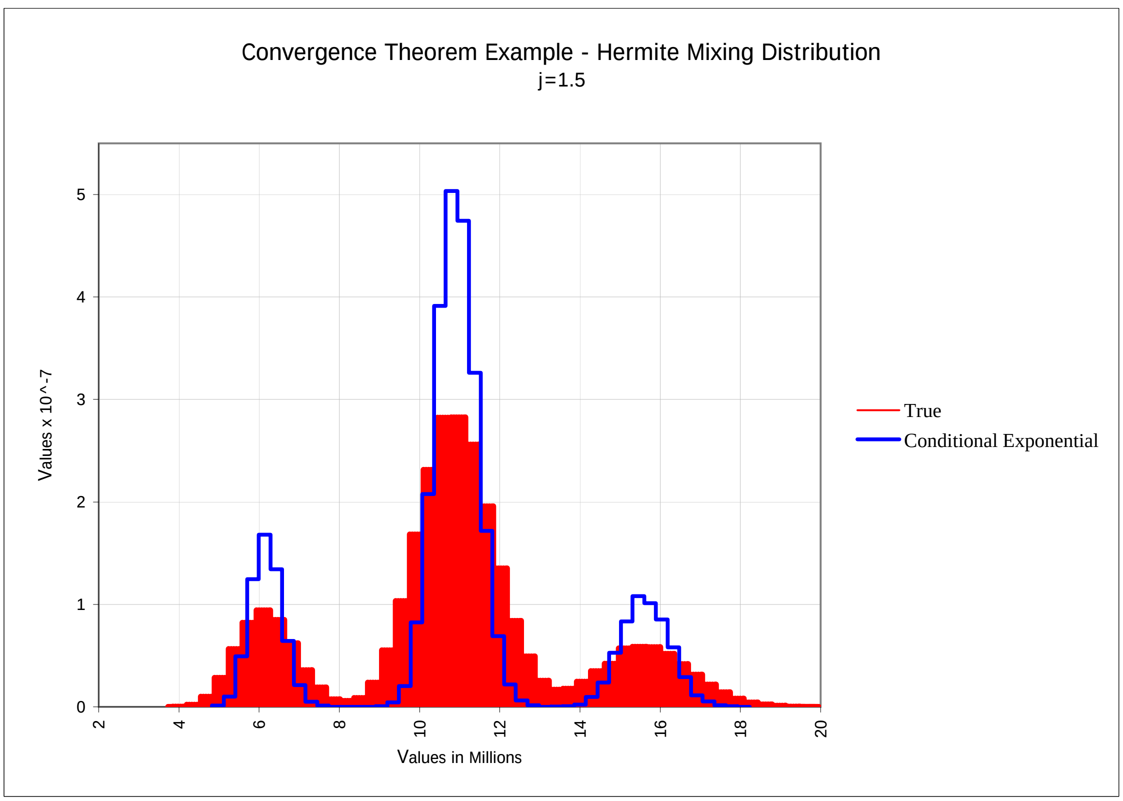
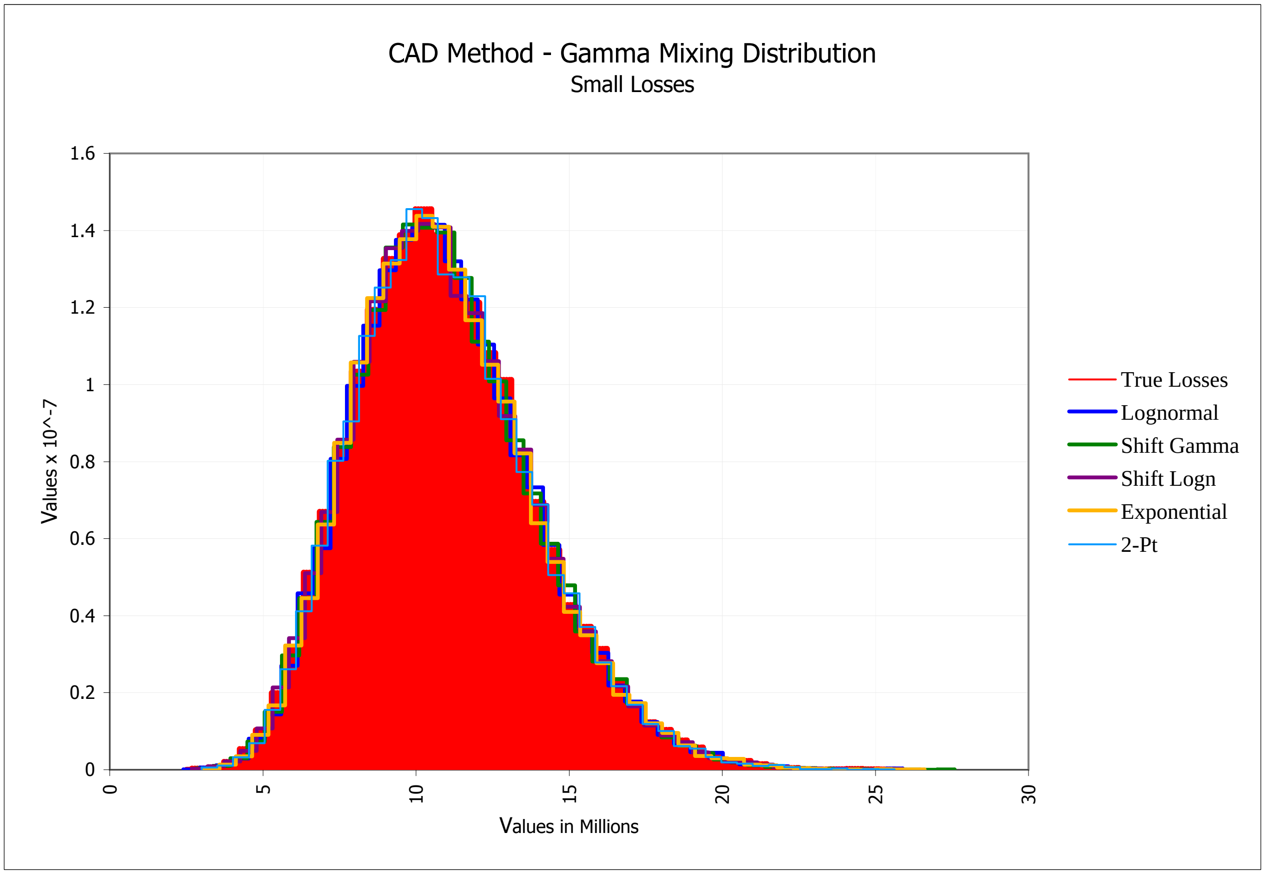
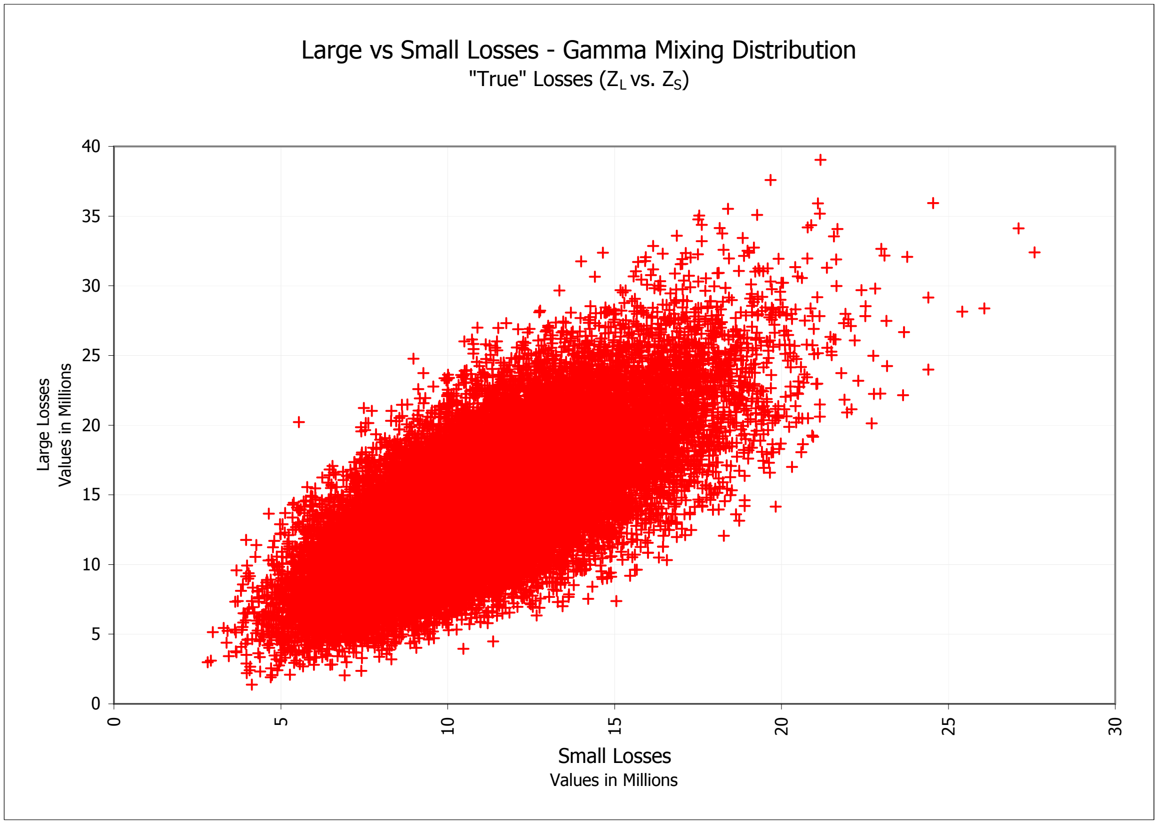

Convergence Theorem Example - Hermite Mixing Distribution
j=1.5
Values x 10^-7
Values in Millions
True
Conditional Exponential
0
1
2
3
4
5
2
4
6
8
10
12
14
16
18
20

Exhibit 2.1

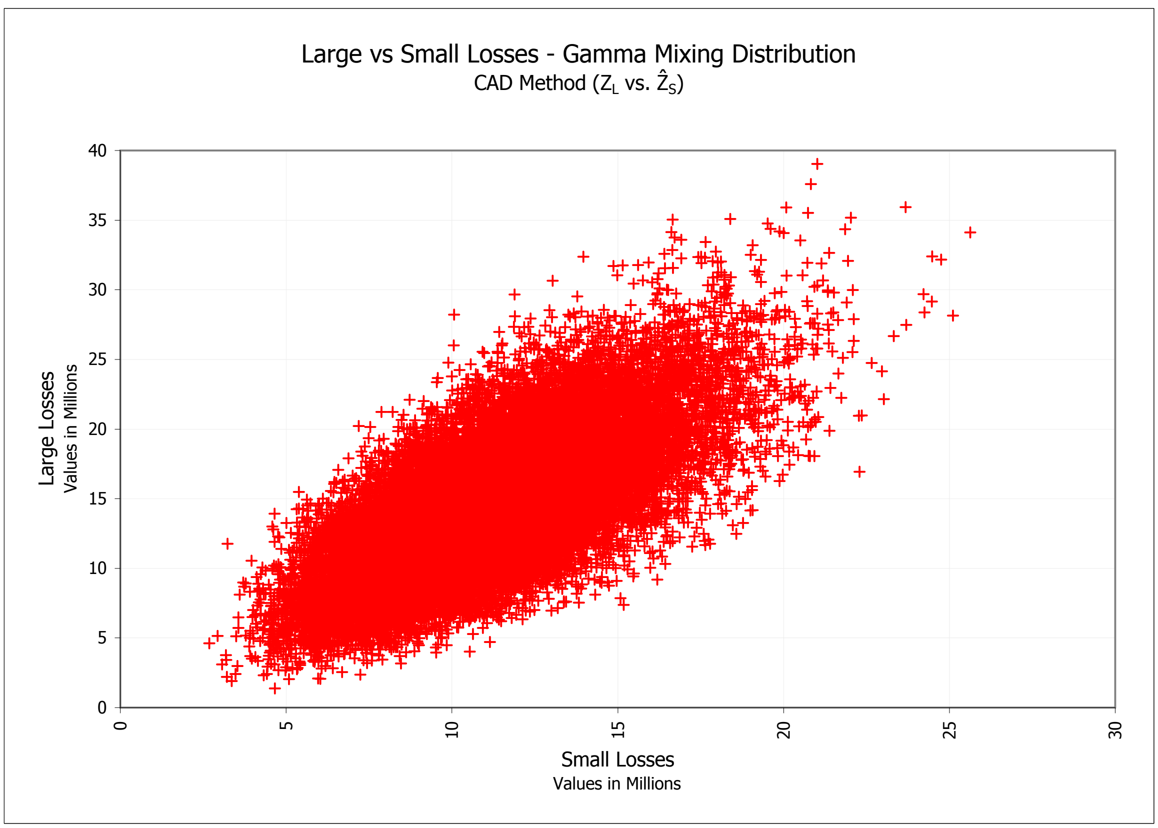


Detail Stats - Gamma Mixing							
Loss Type	Large	Small	Small	Small	Small	Small	Small
Method	"True" (CRM Sim)	CAD Logn.	CAD S. Gamma	CAD S. Logn	CAD Exponential	CAD 2-pt.	"True" (CRM Sim)
Minimum	1,385,975	2,398,609	2,808,190	2,685,245	2,998,957	3,045,139	2,682,782
Maximum	39,042,540	25,898,300	27,588,630	25,894,870	25,626,020	26,605,610	25,627,990
Mean	14,140,170	10,938,800	10,937,300	10,943,990	10,939,010	10,937,040	10,931,920
Std Deviation	4,652,660	2,898,627	2,893,873	2,895,765	2,894,886	2,891,107	2,889,024
Variance	2.16473E+13	8.40204E+12	8.3745E+12	8.38546E+12	8.38037E+12	8.3585E+12	8.34646E+12
Skewness	**0.532**	**0.5071**	**0.5040**	**0.4819**	**0.4904**	**0.5250**	**0.4949**
CV	**0.329**	**0.2650**	**0.2646**	**0.2646**	**0.2646**	**0.2643**	**0.2643**
Skew-Nu	**1.618**	**1.9139**	**1.9050**	**1.8213**	**1.8531**	**1.9860**	**1.8725**
Mode	11,839,840	10,443,110	10,444,170	9,741,620	9,137,559	10,569,540	10,023,330
5% Perc	7,280,964	6,617,527	6,616,314	6,616,352	6,625,957	6,646,187	6,628,636
10% Perc	8,478,519	7,405,594	7,398,611	7,405,873	7,405,128	7,443,271	7,402,793
15% Perc	9,427,957	7,977,141	7,977,279	7,996,510	7,976,839	7,995,743	7,983,588
20% Perc	10,153,280	8,455,025	8,456,465	8,461,667	8,455,437	8,454,707	8,465,613
25% Perc	10,815,190	8,869,101	8,878,454	8,888,060	8,855,242	8,867,610	8,873,186
30% Perc	11,444,160	9,251,367	9,269,360	9,269,123	9,255,394	9,250,547	9,259,721
35% Perc	12,006,930	9,625,078	9,627,173	9,644,261	9,634,785	9,618,922	9,630,174
40% Perc	12,584,930	9,981,641	9,984,575	9,983,185	9,993,351	9,983,671	9,988,756
45% Perc	13,161,940	10,341,280	10,341,520	10,339,860	10,323,750	10,333,640	10,332,560
50% Perc	13,720,500	10,687,030	10,689,310	10,694,390	10,687,300	10,673,790	10,677,960
55% Perc	14,293,300	11,059,730	11,045,600	11,041,930	11,070,400	11,032,790	11,041,920
60% Perc	14,890,360	11,433,110	11,409,440	11,445,670	11,436,400	11,406,070	11,427,120
65% Perc	15,557,820	11,838,890	11,807,440	11,862,800	11,844,620	11,814,010	11,830,620
70% Perc	16,231,410	12,259,280	12,251,480	12,282,940	12,255,120	12,252,290	12,260,860
75% Perc	17,038,840	12,749,390	12,735,450	12,767,710	12,742,260	12,740,690	12,741,150
80% Perc	17,900,140	13,276,090	13,296,010	13,312,140	13,294,120	13,272,500	13,256,540
85% Perc	18,937,040	13,942,170	13,935,960	13,938,120	13,950,790	13,907,760	13,903,870
90% Perc	20,365,650	14,781,560	14,794,910	14,778,790	14,808,720	14,764,230	14,745,370
95% Perc	22,465,010	16,101,710	16,124,400	16,093,960	16,095,310	16,130,030	16,089,070

Exhibit 2.2

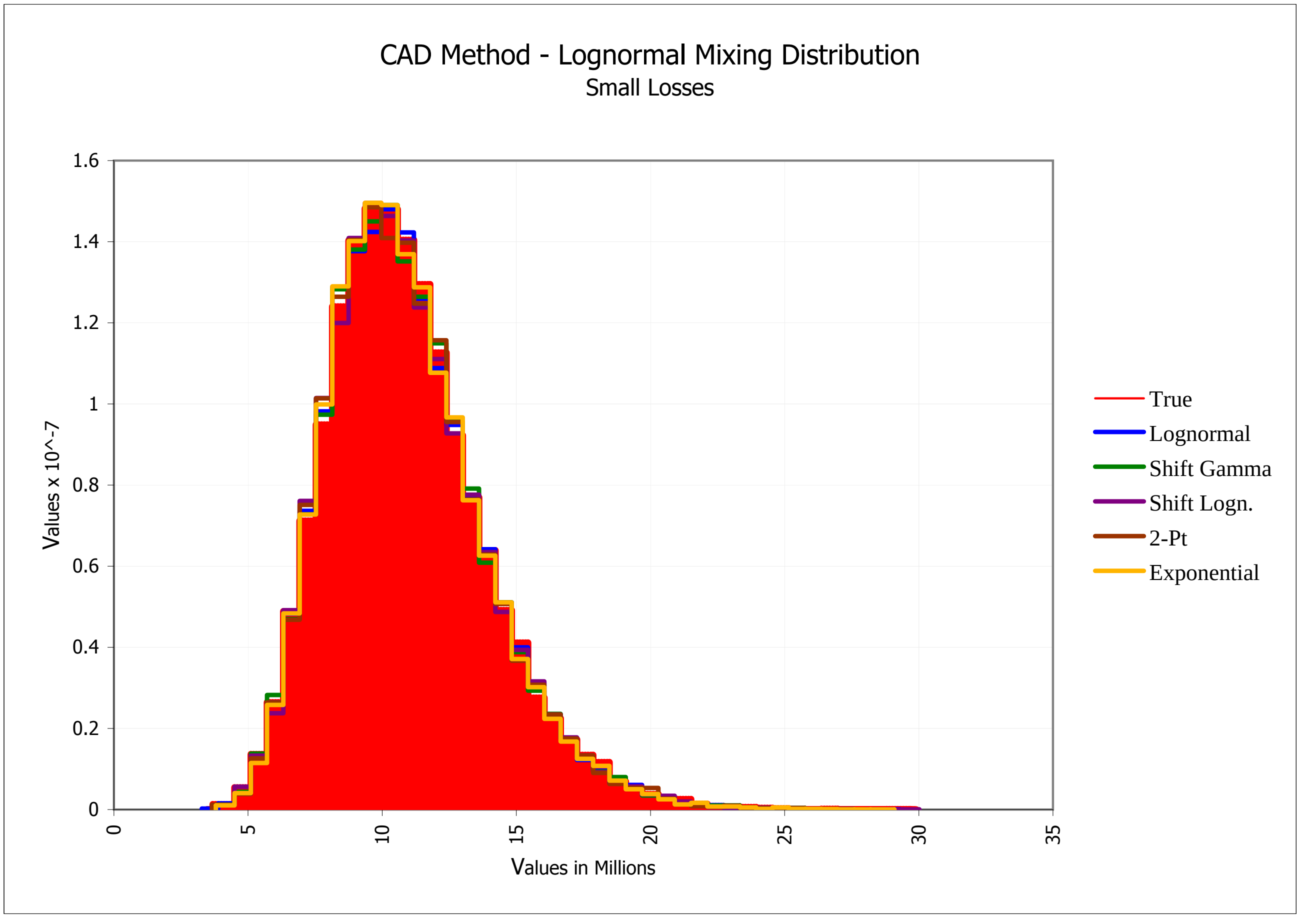


Large vs Small Losses - Gamma Mixing Distribution
"True" Losses (Z_L vs. Z_S)
Large Losses
Values in Millions
0
5
10
15
20
25
30
35
40
0
5
10
15
20
25
30
Small Losses
Values in Millions



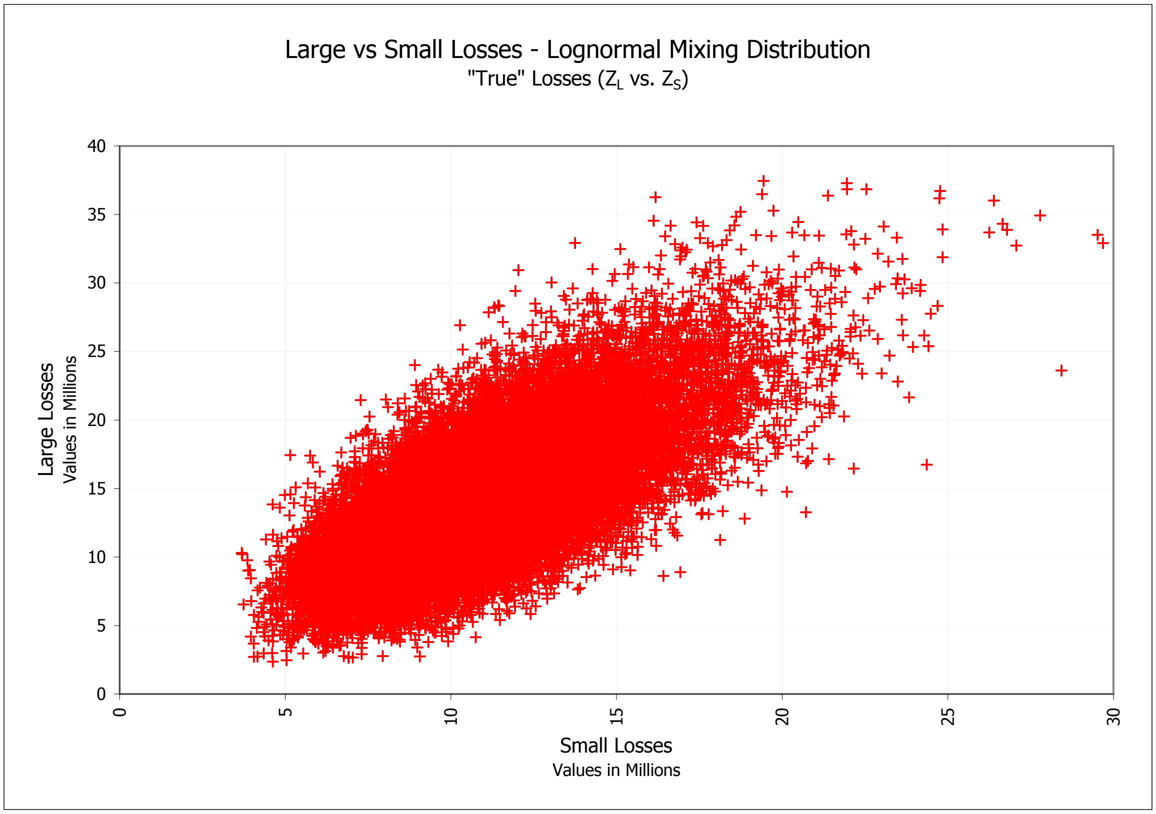
Large vs Small Losses - Gamma Mixing Distribution
CAD Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
0
5
10
15
20
25
30
35
40
0
5
10
15
20
25
30
Small Losses
Values in Millions

Exhibit 3.1



Detail Stats - Lognormal Mixing							
Loss Type	Large	Small	Small	Small	Small	Small	Small
Method	"True" (CRM Sim)	CAD Logn.	CAD S. Gamma	CAD S. Logn	CAD Exponential	CAD 2-pt.	"True" (CRM Sim)
Minimum	2,354,636	3,279,025	3,784,197	3,671,143	3,656,671	3,801,383	3,686,909
Maximum	37,452,880	29,297,280	28,652,320	30,017,500	27,943,100	29,091,590	29,686,280
Mean	14,086,670	10,905,680	10,904,890	10,901,090	10,899,620	10,897,280	10,906,430
Std Deviation	4,663,644	2,896,549	2,898,968	2,905,355	2,888,524	2,884,288	2,899,240
Variance	2.17496E+13	8.39E+12	8.40402E+12	8.44109E+12	8.34357E+12	8.31912E+12	8.40559E+12
Skewness	**0.657**	**0.7345**	**0.7532**	**0.7511**	**0.7555**	**0.7820**	**0.7548**
CV	**0.331**	**0.2656**	**0.2658**	**0.2665**	**0.2650**	**0.2647**	**0.2658**
Skew-Nu	**1.985**	**2.7654**	**2.8332**	**2.8182**	**2.8509**	**2.9546**	**2.8393**
Mode	13,312,870	9,229,467	10,256,040	10,247,740	9,900,903	9,298,220	9,593,592
5% Perc	7,379,982	6,807,512	6,786,931	6,814,347	6,839,257	6,830,828	6,783,801
10% Perc	8,522,062	7,512,230	7,506,557	7,503,218	7,531,037	7,537,819	7,511,076
15% Perc	9,390,949	8,032,948	8,046,047	8,007,588	8,034,989	8,046,408	8,052,007
20% Perc	10,102,340	8,449,887	8,447,854	8,445,220	8,455,437	8,468,788	8,484,241
25% Perc	10,756,510	8,834,509	8,828,217	8,834,182	8,813,141	8,837,245	8,866,154
30% Perc	11,354,550	9,198,359	9,198,009	9,201,309	9,186,013	9,198,724	9,204,506
35% Perc	11,937,640	9,551,106	9,542,255	9,528,743	9,529,370	9,527,572	9,562,121
40% Perc	12,484,700	9,891,079	9,887,307	9,866,483	9,868,970	9,867,238	9,890,388
45% Perc	13,034,730	10,224,090	10,223,390	10,216,780	10,210,100	10,201,090	10,234,600
50% Perc	13,589,010	10,570,880	10,557,930	10,549,800	10,563,310	10,539,160	10,571,090
55% Perc	14,173,760	10,913,890	10,924,320	10,898,780	10,914,400	10,887,340	10,918,910
60% Perc	14,742,170	11,276,170	11,304,320	11,270,320	11,281,810	11,266,530	11,280,100
65% Perc	15,378,140	11,680,180	11,694,860	11,669,240	11,685,900	11,647,070	11,666,960
70% Perc	16,102,160	12,113,640	12,118,480	12,111,900	12,106,920	12,094,540	12,089,420
75% Perc	16,888,410	12,602,800	12,576,840	12,575,800	12,572,790	12,572,670	12,563,000
80% Perc	17,763,110	13,162,890	13,152,030	13,162,970	13,135,720	13,120,340	13,138,160
85% Perc	18,831,010	13,835,790	13,811,970	13,845,860	13,791,750	13,804,000	13,818,980
90% Perc	20,265,510	14,747,880	14,734,090	14,753,650	14,695,780	14,693,850	14,735,270
95% Perc	22,529,300	16,170,290	16,180,220	16,216,460	16,162,980	16,163,700	16,199,130

Exhibit 3.2


Large vs Small Losses - Lognormal Mixing Distribution
"True" Losses (Z_L vs. Z_S)
Large Losses
Values in Millions
Small Losses
Values in Millions
0
5
10
15
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40

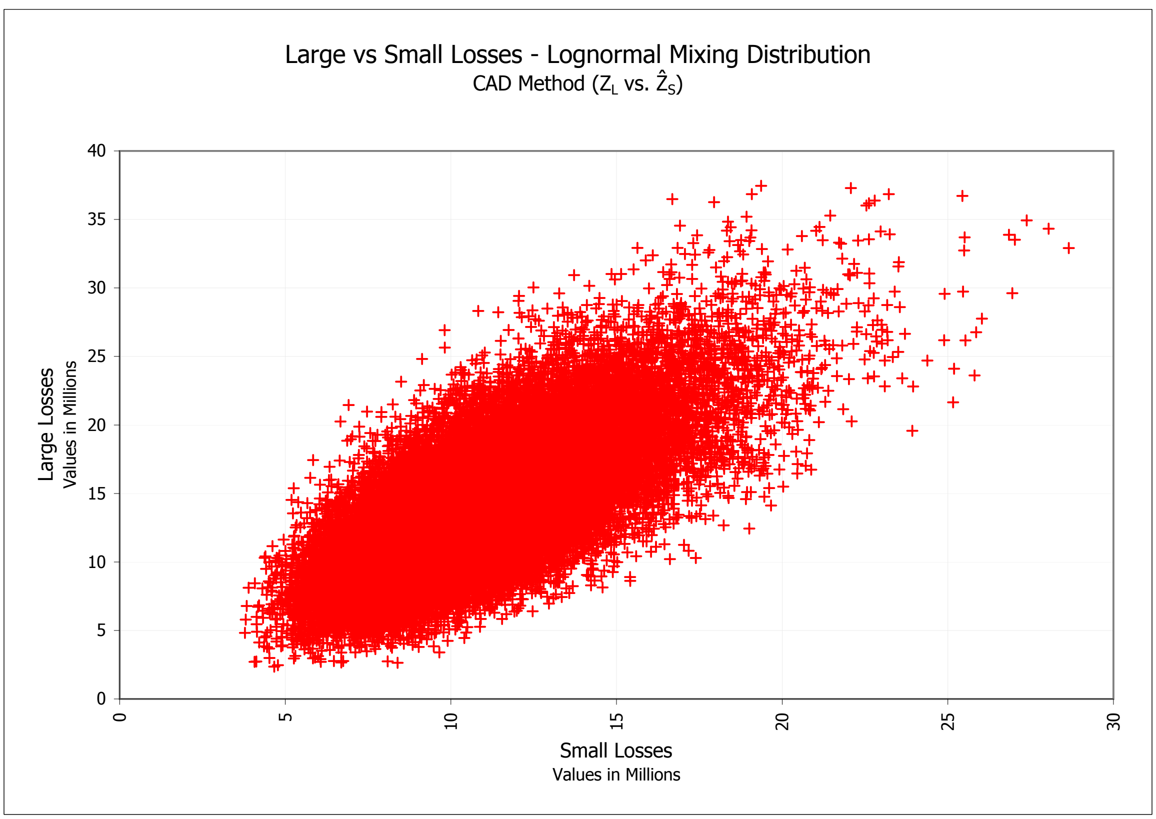
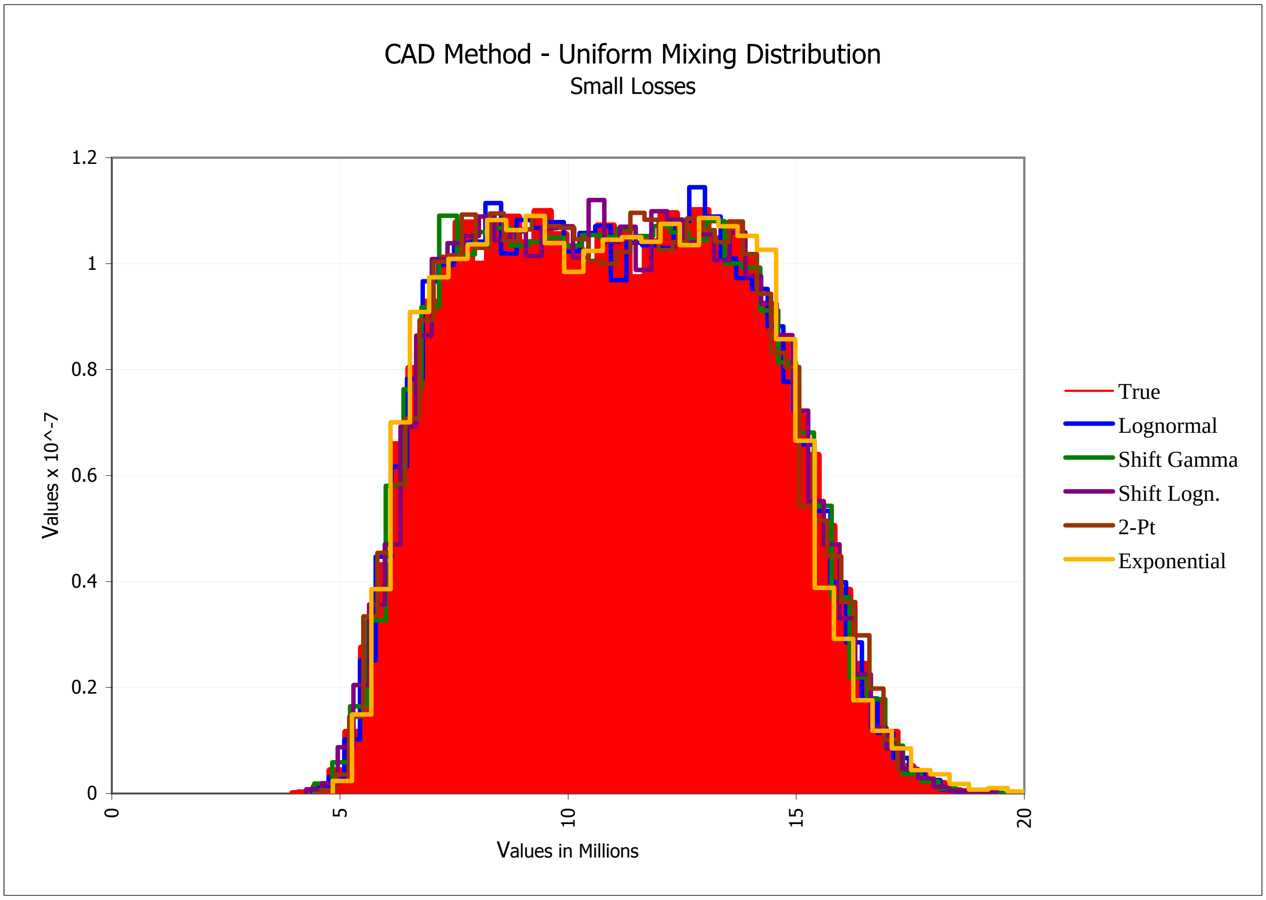
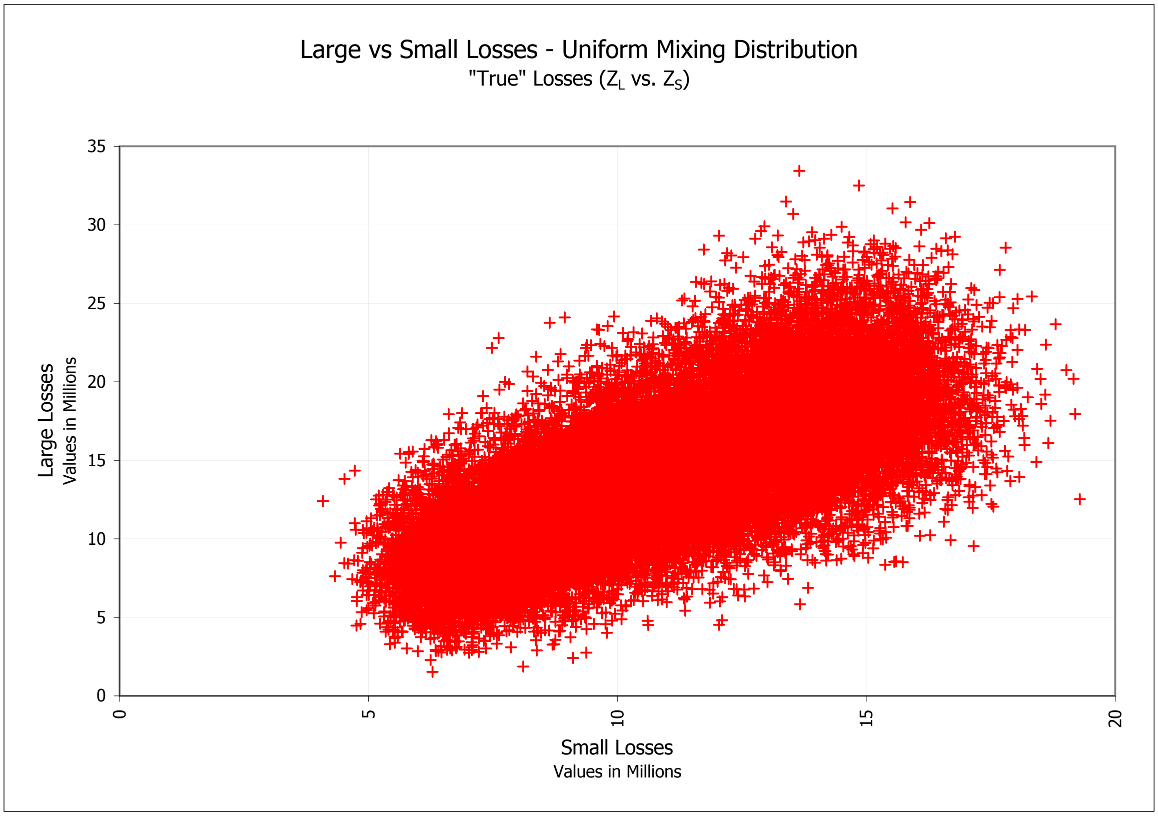

Large vs Small Losses - Lognormal Mixing Distribution
CAD Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
Small Losses
Values in Millions
0
5
10
15
20
25
30
35
40

Exhibit 4.1



Detail Stats - Uniform Mixing							
Loss Type	Large	Small	Small	Small	Small	Small	Small
Method	"True" (CRM Sim)	CAD Logn.	CAD S. Gamma	CAD S. Logn	CAD Exponential	CAD 2-pt.	"True" (CRM Sim)
Minimum	2,622,778	4,238,870	4,287,148	4,468,165	4,670,314	4,702,374	4,487,489
Maximum	32,122,340	18,258,020	18,495,580	19,186,070	17,875,910	22,198,040	18,737,170
Mean	14,053,220	10,889,670	10,907,560	10,898,670	10,890,100	10,895,280	10,893,480
Std Deviation	4,635,183	2,851,964	2,868,312	2,871,454	2,853,996	2,866,448	2,870,310
Variance	2.14849E+13	8.1337E+12	8.22722E+12	8.24525E+12	8.14529E+12	8.21653E+12	8.23868E+12
Skewness	**0.317**	**0.0601**	**0.0552**	**0.0624**	**0.0553**	**0.1008**	**0.0575**
CV	**0.330**	**0.2619**	**0.2630**	**0.2635**	**0.2621**	**0.2631**	**0.2635**
Skew-Nu	**0.960**	**0.2295**	**0.2099**	**0.2367**	**0.2108**	**0.3832**	**0.2182**
Mode	13,955,570	11,017,940	6,980,460	10,483,140	8,609,236	14,108,960	12,211,590
5% Perc	7,048,361	6,452,205	6,455,139	6,397,590	6,378,900	6,456,264	6,419,141
10% Perc	8,138,355	7,055,768	7,013,859	7,003,973	7,048,737	7,014,243	7,021,224
15% Perc	9,061,992	7,556,159	7,537,291	7,552,863	7,552,022	7,506,401	7,519,121
20% Perc	9,832,303	8,057,059	8,037,449	8,029,881	8,068,227	7,994,043	8,006,109
25% Perc	10,508,630	8,513,653	8,513,914	8,529,380	8,570,468	8,496,666	8,504,713
30% Perc	11,221,460	9,005,443	9,001,551	8,979,073	9,023,686	8,992,163	9,005,018
35% Perc	11,867,710	9,487,384	9,519,624	9,481,191	9,508,290	9,477,224	9,504,768
40% Perc	12,520,510	9,952,474	9,959,408	9,980,229	9,972,253	9,973,597	9,957,272
45% Perc	13,151,170	10,395,700	10,469,080	10,447,890	10,437,390	10,457,010	10,423,910
50% Perc	13,770,240	10,893,210	10,916,910	10,885,080	10,868,990	10,902,820	10,891,530
55% Perc	14,387,300	11,351,630	11,342,420	11,337,470	11,302,950	11,362,770	11,371,680
60% Perc	15,052,900	11,818,450	11,799,030	11,827,650	11,747,140	11,827,170	11,831,270
65% Perc	15,751,770	12,266,250	12,275,180	12,301,380	12,233,950	12,246,880	12,247,990
70% Perc	16,481,540	12,716,240	12,744,370	12,728,420	12,705,610	12,714,500	12,729,500
75% Perc	17,263,490	13,187,060	13,228,660	13,177,340	13,203,610	13,171,750	13,199,210
80% Perc	18,107,500	13,637,550	13,672,250	13,665,560	13,675,010	13,672,670	13,665,310
85% Perc	19,074,990	14,168,400	14,169,310	14,175,580	14,130,870	14,144,830	14,148,740
90% Perc	20,253,470	14,693,240	14,774,350	14,737,180	14,695,780	14,665,920	14,733,240
95% Perc	22,030,010	15,450,310	15,511,660	15,494,220	15,508,650	15,398,370	15,511,520

Exhibit 4.2


Large vs Small Losses - Uniform Mixing Distribution
"True" Losses (Z_L vs. Z_S)
Large Losses
Values in Millions
35
30
25
20
15
10
5
0
0
5
10
15
20
Small Losses
Values in Millions

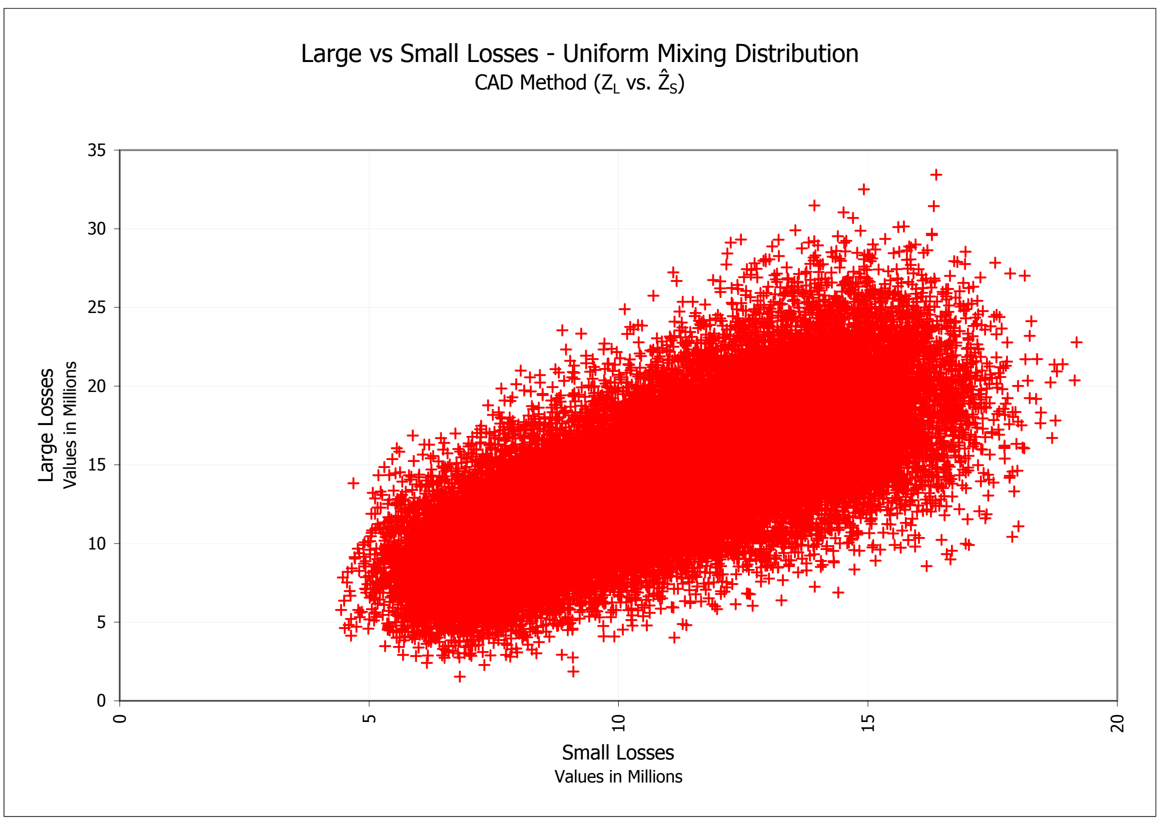
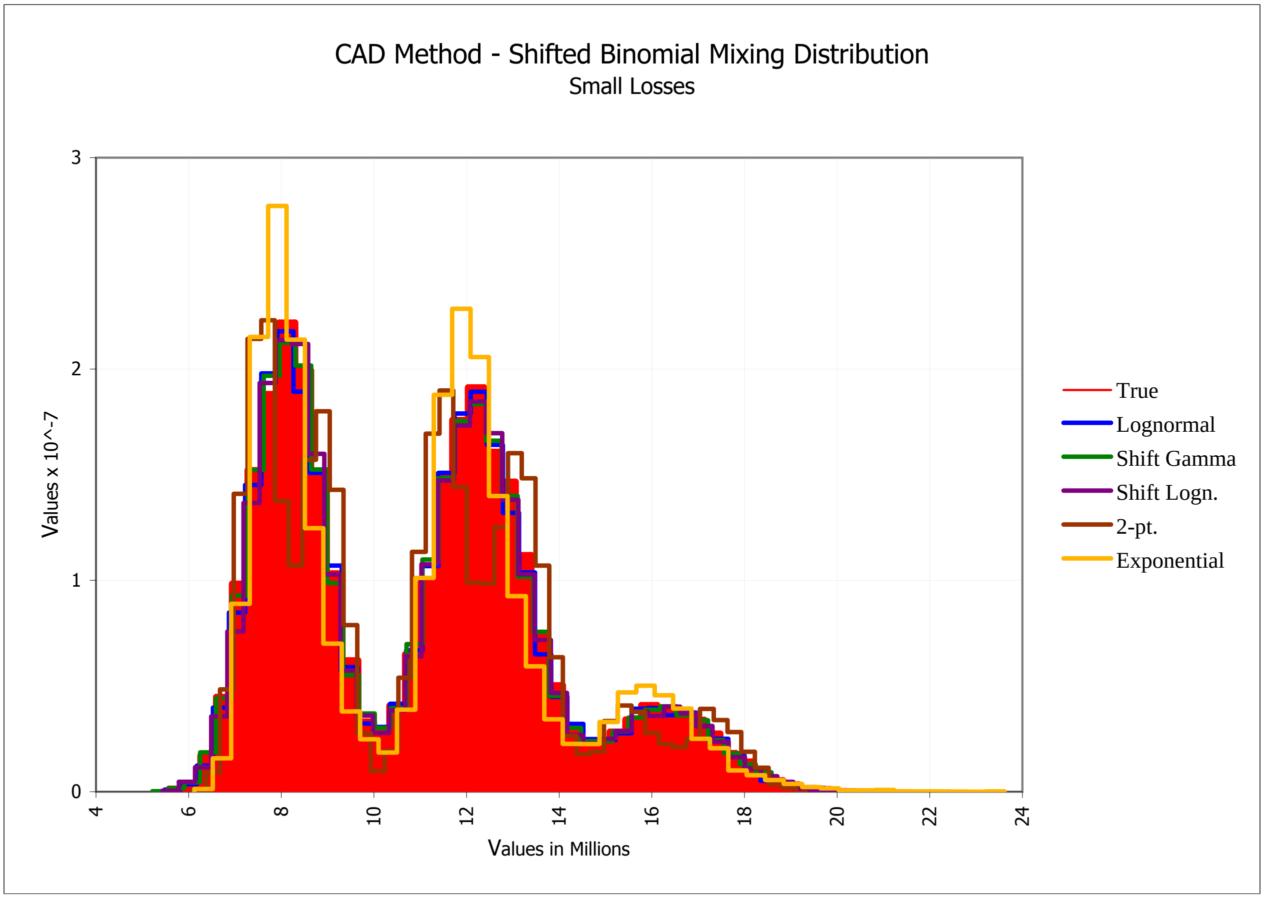

Large vs Small Losses - Uniform Mixing Distribution
CAD Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
35
30
25
20
15
10
5
0
0
5
10
15
20
Small Losses
Values in Millions

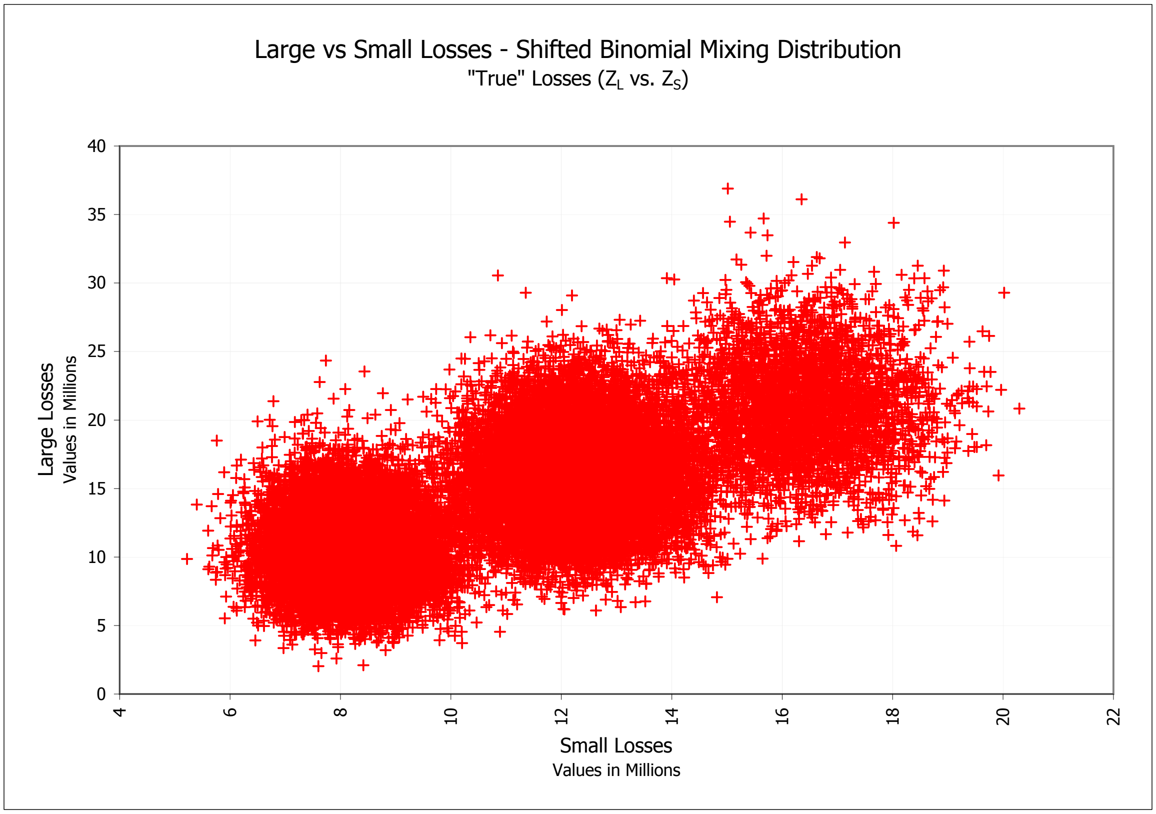
Exhibit 5.1



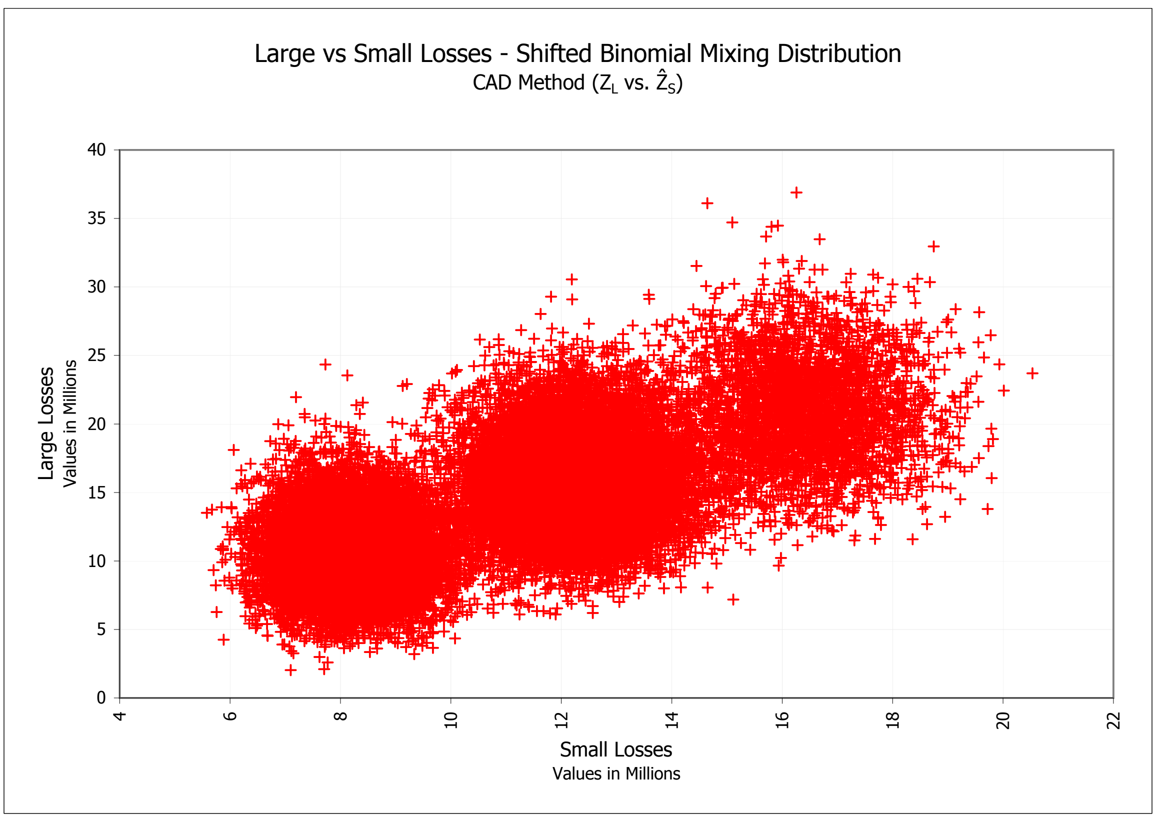
Detail Stats - Shifted Biniomial

Loss Type Method	Large "True" (CRM Sim)	Small CAD Logn.	Small CAD S. Gamma	Small CAD S. Logn	Small CAD Exponential	Small CAD 2-pt.	Small "True" (CRM Sim)
Minimum	2,029,372	5,480,892	5,220,685	5,438,516	6,086,327	6,125,506	5,578,668
Maximum	36,899,030	20,782,580	20,297,310	20,802,180	19,114,040	23,614,350	20,532,800
Mean	14,126,680	10,956,630	10,947,030	10,951,330	10,953,510	10,946,910	10,949,270
Std Deviation	4,631,549	2,889,085	2,882,983	2,889,601	2,896,795	2,889,096	2,880,868
Variance	2.14513E+13	8.34681E+12	8.31159E+12	8.3498E+12	8.39142E+12	8.34688E+12	8.2994E+12
Skewness	**0.515**	**0.4845**	**0.4885**	**0.4872**	**0.4863**	**0.5301**	**0.4920**
CV	**0.328**	**0.2637**	**0.2634**	**0.2639**	**0.2645**	**0.2639**	**0.2631**
Skew-Nu	**1.571**	**1.8374**	**1.8550**	**1.8465**	**1.8386**	**2.0086**	**1.8700**
Mode	12,964,350	8,087,866	8,148,018	8,147,161	7,299,558	8,102,896	8,147,660
5% Perc	7,483,617	7,227,355	7,225,197	7,235,228	7,216,417	7,361,047	7,229,344
10% Perc	8,535,075	7,570,712	7,577,371	7,579,443	7,468,091	7,599,695	7,574,011
15% Perc	9,306,652	7,828,013	7,841,232	7,836,373	7,677,954	7,797,241	7,843,340
20% Perc	9,965,355	8,066,383	8,081,884	8,077,192	7,929,945	7,978,476	8,080,636
25% Perc	10,604,400	8,292,712	8,314,742	8,302,414	8,350,285	8,161,311	8,308,913
30% Perc	11,226,030	8,556,097	8,553,878	8,543,078	8,707,333	8,385,700	8,549,825
35% Perc	11,833,320	8,867,372	8,849,983	8,827,609	8,980,929	8,692,694	8,862,549
40% Perc	12,426,840	9,308,444	9,295,946	9,276,051	9,299,908	9,299,985	9,303,648
45% Perc	13,021,220	10,554,790	10,493,120	10,543,550	10,690,050	10,919,380	10,509,370
50% Perc	13,661,470	11,250,190	11,224,580	11,240,710	11,176,100	11,368,280	11,244,800
55% Perc	14,291,840	11,625,320	11,602,130	11,615,890	11,450,980	11,632,780	11,611,090
60% Perc	14,945,040	11,923,280	11,906,950	11,918,820	11,725,980	11,850,840	11,900,920
65% Perc	15,622,390	12,183,350	12,177,440	12,192,390	12,085,760	12,071,050	12,163,850
70% Perc	16,370,580	12,450,640	12,452,890	12,462,020	12,593,970	12,294,140	12,441,100
75% Perc	17,150,500	12,752,340	12,755,160	12,758,750	12,977,290	12,571,500	12,758,940
80% Perc	17,999,210	13,127,510	13,112,140	13,122,760	13,294,120	12,960,450	13,111,080
85% Perc	19,036,000	13,675,010	13,649,720	13,648,090	13,678,680	13,653,300	13,627,890
90% Perc	20,423,720	15,070,390	15,084,120	15,098,890	15,033,470	15,360,840	15,105,940
95% Perc	22,467,470	16,539,630	16,510,020	16,539,500	16,659,090	16,393,870	16,502,230

Exhibit 5.2

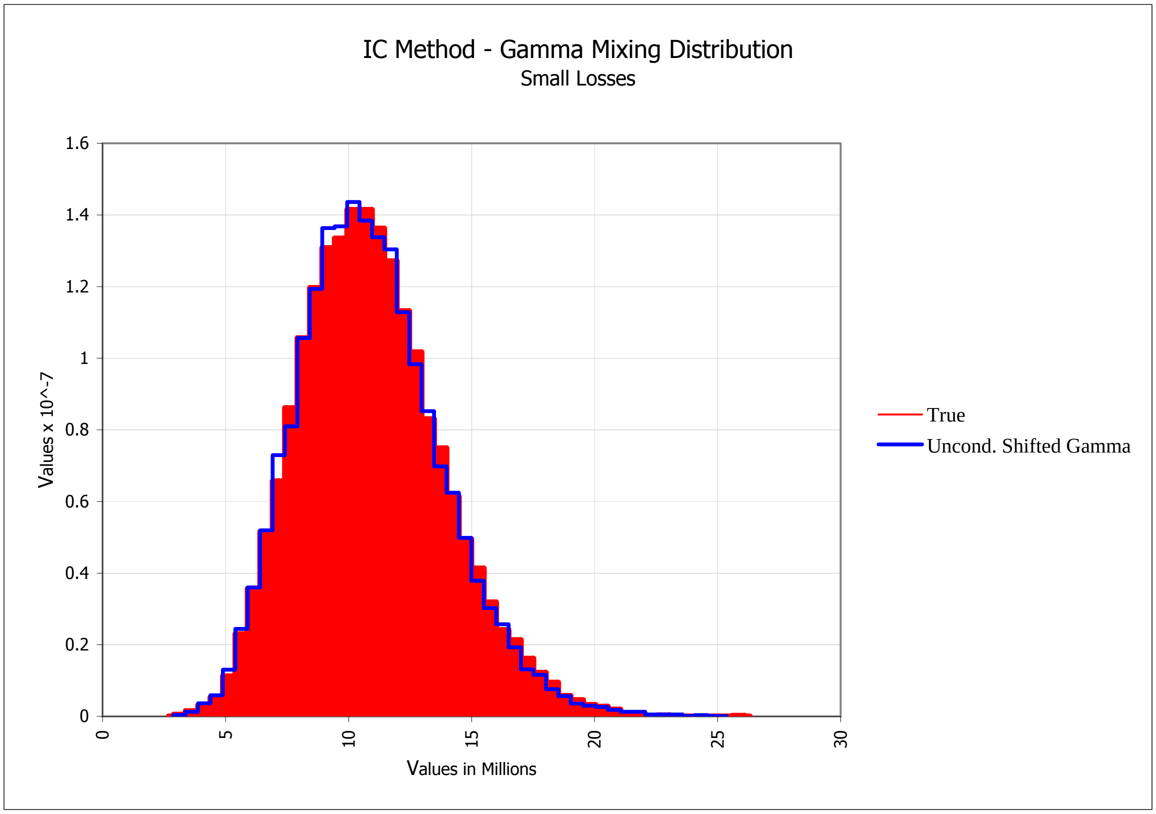


Large vs Small Losses - Shifted Binomial Mixing Distribution
"True" Losses (Z_L vs. Z_S)
Large Losses
Values in Millions
Small Losses
Values in Millions
0
5
10
15
20
25
30
35
40
4
6
8
10
12
14
16
18
20
22

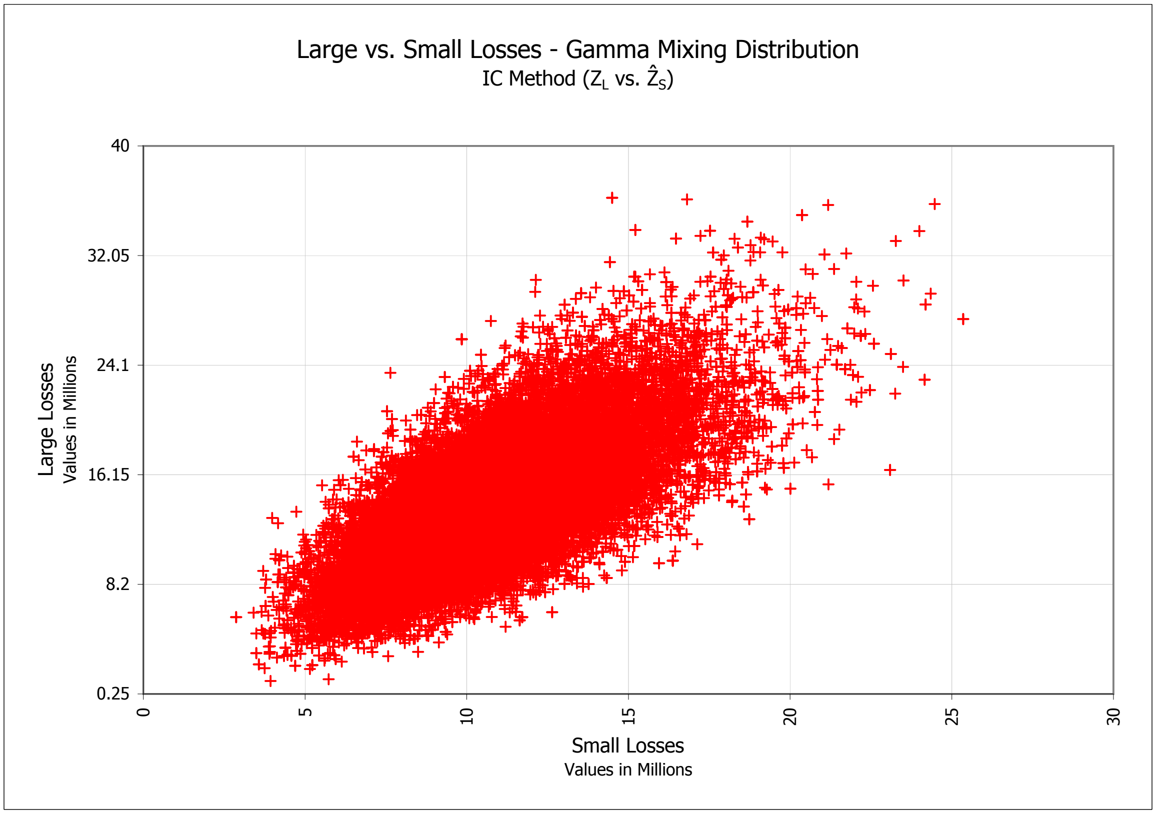


Large vs Small Losses - Shifted Binomial Mixing Distribution
CAD Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
Small Losses
Values in Millions
0
5
10
15
20
25
30
35
40
4
6
8
10
12
14
16
18
20
22

Exhibit 6

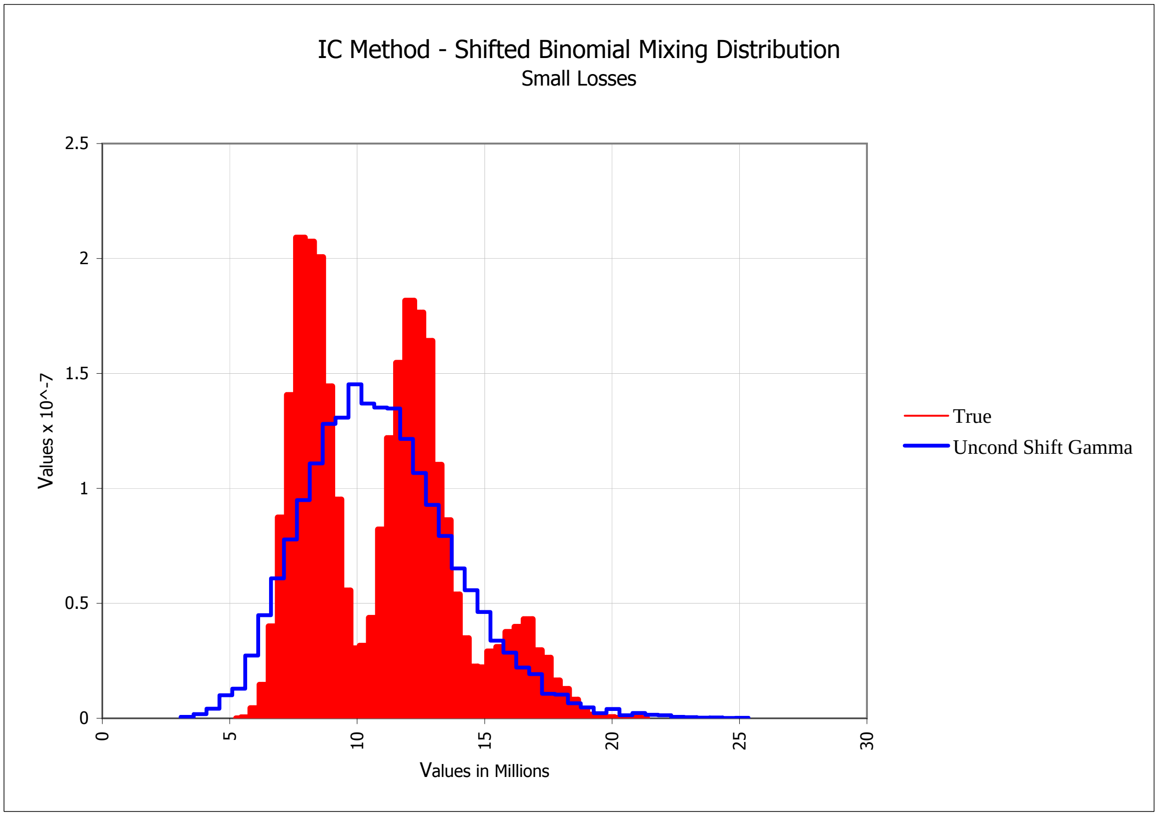


IC Method - Gamma Mixing Distribution
Small Losses
Values x 10^-7
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
Values in Millions
True
Uncond. Shifted Gamma

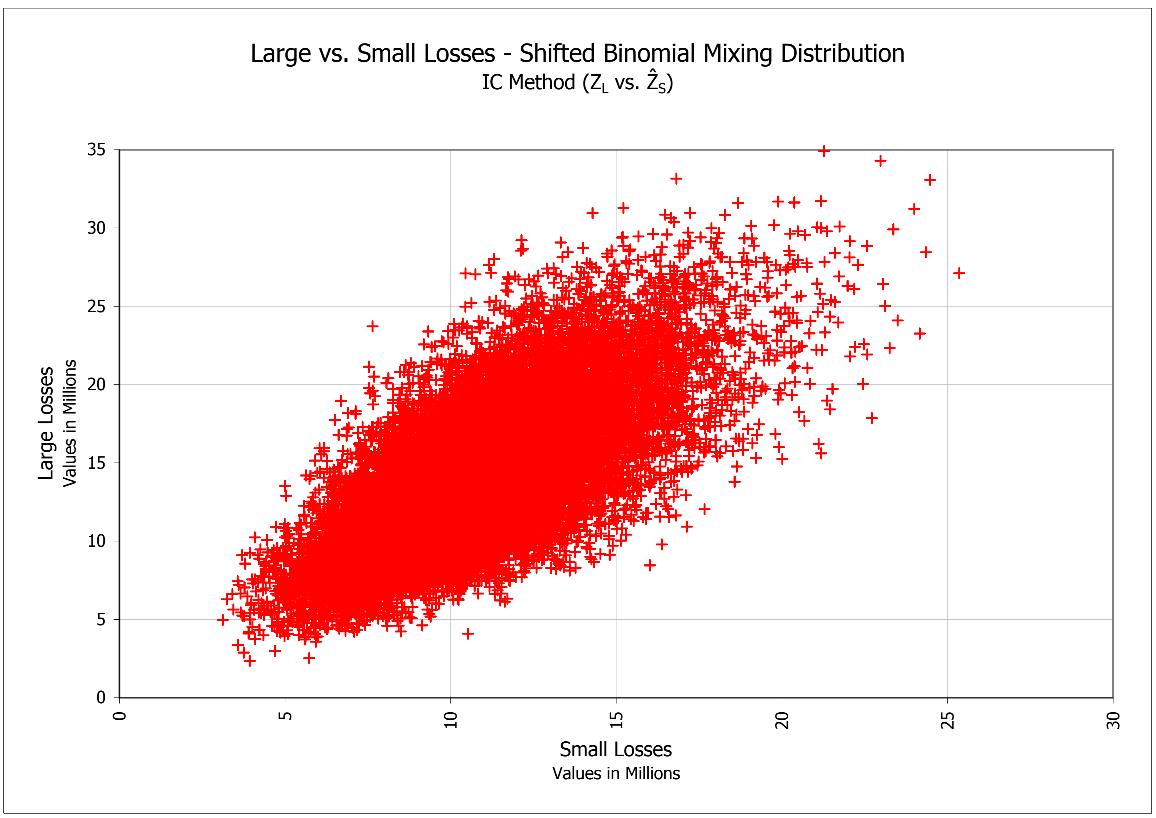


Large vs. Small Losses - Gamma Mixing Distribution
IC Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
40
32.05
24.1
16.15
8.2
0.25
0
5
10
15
20
25
30
Small Losses
Values in Millions

Exhibit 7.1

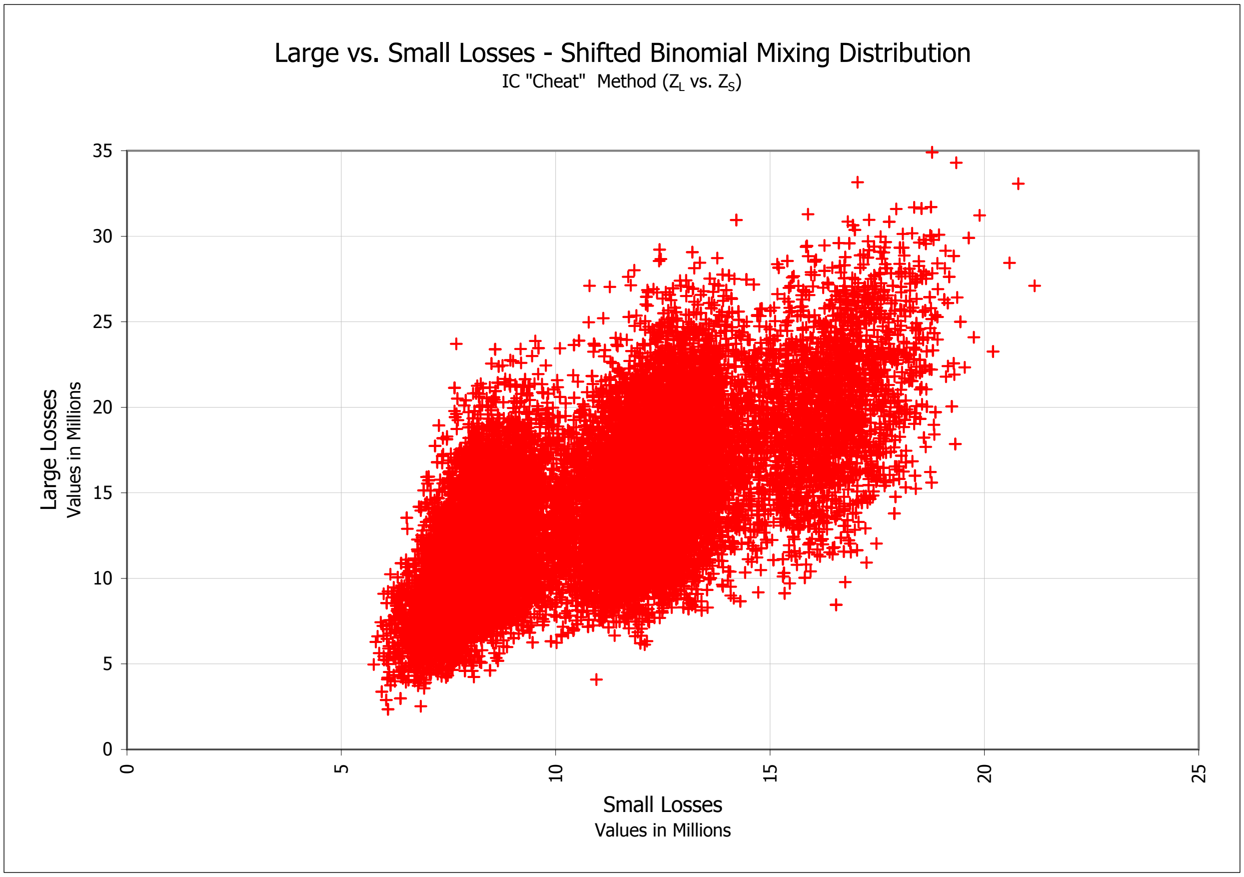


IC Method - Shifted Binomial Mixing Distribution
Small Losses
Values x 10^-7
2.5
2
1.5
1
0.5
0
0
5
10
15
20
25
30
Values in Millions
True
Uncond Shift Gamma



Large vs. Small Losses - Shifted Binomial Mixing Distribution
IC Method (Z_L vs. $\hat{Z}_S$)
Large Losses
Values in Millions
35
30
25
20
15
10
5
0
0
5
10
15
20
25
30
Small Losses
Values in Millions

Exhibit 7.2



Large vs. Small Losses - Shifted Binomial Mixing Distribution
IC "Cheat" Method (Z_L vs. Z_S)
Large Losses
Values in Millions
0
5
10
15
20
25
30
35
0
5
10
15
20
25
Small Losses
Values in Millions