The Conditional Validity of Risk-Adjusted Discounting

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Abstract: The constellation of the initiatives of ERM, Solvency II, and International Accounting traces back through capital management to modern finance and portfolio theory. These supposedly dynamic and market-oriented initiatives will eventually disappoint the (re)insurance industry, if they uncritically endorse risk-adjusted discounting. One's job is rendered more difficult, if not impossible, without the right tools. Building on earlier papers, the author will here show how a seminal academic paper from the 1960s contains the seeds of the downfall of risk-adjusted discounting. It is too much to expect a retraction, but hopefully, the emerging standards for these initiatives at least will not force risk-adjusted discounting upon the practitioners.

Keywords: present value, risk-adjusted discounting, stochastic cash flow.

1. INTRODUCTION

One standard textbook begins with this clear pronouncement about risk-adjusted discounting:

> To calculate present value, we discount expected payoffs by the rate of return offered by the equivalent investment alternative in the capital market. The rate of return is often referred to as the discount rate, hurdle rate, or opportunity cost of capital. [Brealey and Myers, 2002, p 15]

Having published several critiques of this principle,[1] we have challenged others to show where in the academic literature it has been rigorously derived. At length, someone directed our attention to Robichek and Myers [1966], whose co-author, Stewart C. Myers, is the same as the co-author of the textbook just cited.[2] To our surprise, this brief paper, far from deriving the principle, actually points out its "conceptual problems," as its title reveals. We are wholly in accord with its second paragraph:

> Since time and risk are logically separate variables, summing up their effects in the one number k requires a particular assumption about the actual relationship between the effects of time and risk on present value. The main purpose of this communication is to uncover this assumption and to point out that valuation errors may result if the risk-adjusted discount rate is used when this assumption does not hold.

After treating a simple example, its authors conclude:

> … the general conclusion [is] that the rate at which income is expected to be realized over time depends on the rate at which uncertainty is expected to be resolved over time. If uncertainly is

[1] See the author's publications in the References, especially Halliwell [2003], Appendix A.

[2] He is also the co-inventor of the Myers-Cohn insurance-pricing model, which has been used in Massachusetts rate hearings. Introductions to this model may be found in D'Arcy and Doherty [1988], Mahler [1998], and D'Arcy [1999].

> expected to be resolved at a constant rate over time, then the required rate of return k predicts accurately the rate at which income is expected to be realized. But this need not always be the case.

To this conclusion Philbrick [1994] agrees. But if risk-adjusted discounted depends on a certain resolution of uncertainty, how should one value a project (i.e., a stochastic cash flow) whose uncertainty resolves otherwise?[3] But putting this aside for now, we will test our theory, deriving risk-adjusted discounting from it in the case of a dividend-paying stock on the assumption of continuous risk resolution. First, we value the stock according to the prevailing theory.

2. THE DIVIDEND-GROWTH MODEL

The dividend-growth model (also called the Gordon, or Gordon-Shapiro, model) is the commonly accepted method of valuing a dividend-paying stock (Bowlin [1990, pp 96f], Brealey and Myers [2002, Chapter 4]). Because we will deal with continuous risk resolution, we will formulate a continuous version of this model. At time t the stock is *expected* to pay out a dividend at the rate:

$$dC(t) = \mu_0 e^{\gamma t} dt$$

We use '*C*' for cash instead of '*D*' for dividend, to avoid confusion with the differential '*d*' and the force of interest 'δ'. μ_0 is the instantaneous dividend flow (in units of currency per time) at time zero, from which it is expected continuously to grow at rate γ (in units of time^{-1}). We will discount this expected dividend stream at the cost of capital κ, so the discount function is $e^{-\kappa t}$. Hence, according to the principle of risk-adjusted discounting, the value of the stock at time t is:

$$V(t) = \int_{u=t}^{\infty} e^{-\kappa(u-t)} dC(u) = \int_{u=t}^{\infty} e^{-\kappa(u-t)} \mu_0 e^{\gamma u} du$$

Then we work out the integral:

[3] Halliwell [2001, Appendix D] shows that an asset should appreciate at a risk-free rate while uncertainly is not resolving, or more accurately "the price of an asset whose uncertainty is not changing remains proportional to the price of an asset whose future payment is certain."

$$V(t)=\int_{u=t}^{\infty}e^{-\kappa(u-t)}\mu_0 e^{\gamma u}du=\mu_0\int_{u=t}^{\infty}e^{-\kappa(u-t)}e^{\gamma(u-t)}du\cdot e^{\gamma t}$$
$$=\mu_0\int_{u=t}^{\infty}e^{-(\kappa-\gamma)(u-t)}d(u-t)\cdot e^{\gamma t}=\mu_0\int_{v=0}^{\infty}e^{-(\kappa-\gamma)v}dv\cdot e^{\gamma t}$$
$$=\frac{\mu_0}{\kappa-\gamma}e^{\gamma t}$$

Of course, for the integral to converge, the cost of capital must exceed the dividend growth rate, or $\kappa>\gamma$.

What is the instantaneous (expected) total return on the stock at time *t*, which we will call $\rho(t)$? It must consider both the dividend and the price appreciation. Therefore:

$$\rho(t)=\frac{1}{V(t)}\lim_{\Delta t\to 0}\frac{\Delta C(t)+\Delta V(t)}{\Delta t}=\frac{1}{V(t)}\frac{dC(t)}{dt}+\frac{1}{V(t)}\frac{dV(t)}{dt}$$

The first term is the instantaneous dividend yield, which we will call *y*, and the second is the instantaneous price appreciation. Simplifying to the utmost, we have:

$$\rho(t)=\frac{1}{V(t)}\frac{dC(t)}{dt}+\frac{1}{V(t)}\frac{dV(t)}{dt}$$
$$=\frac{\mu_0 e^{\gamma t}}{\left(\frac{\mu_0}{\kappa-\gamma}e^{\gamma t}\right)}+\frac{\frac{\mu_0}{\kappa-\gamma}e^{\gamma t}\gamma}{\left(\frac{\mu_0}{\kappa-\gamma}e^{\gamma t}\right)}$$
$$=(\kappa-\gamma)+\gamma$$
$$=\kappa$$

Accordingly, the total return is the dividend yield ($y=\kappa-\gamma$) plus the rate of appreciation γ.[4]

[4] Halliwell [1999, p 412]: '[A cash flow] is always earning its cost of capital ϱ, or working at "ϱ-power."'

3. THE ALTERNATIVE MODEL: STOCHASTIC CASH FLOWS

Next, we value the stock with our stochastic-cash-flow theory. The dividend stream we now regard as the Wiener process $dC(t) = \mu_0 e^{\gamma t} dt + \sigma_0 e^{\gamma t} dX(t)$, whose *mean* equals the $dC(t)$ of the dividend-growth model.[5] Although the drift and volatility functions are exponential in t, this equation represents arithmetic Brownian motion, rather than geometric. The arithmetic form allows for us to discount the increments and for their sum to be normally distributed.

Now $dC(t)$, the *actual* dividend received during interval $[t, t+dt]$, is normally distributed with mean $\mu_0 e^{\gamma t} dt$ and variance $\sigma_0^2 e^{2\gamma t} dt$, where the dimension of σ_0^2 is currency squared per time. If we let δ represent a flat and persistent risk-free force of interest, the discount function is $v(t) = e^{-\delta t}$. Hence, the present value of the stock at time t, which, we maintain [Halliwell, 2003, Section 3], should be considered as a random variable, is:

$$PV[C(t)] = \int_{u=t}^{\infty} v(u-t)dC(u) = \int_{u=t}^{\infty} e^{-\delta(u-t)} dC(u)$$

The discounted dividend received during interval $[u, u+du]$ from the standpoint of time t is normally distributed with mean $e^{-\delta(u-t)}\mu_0 e^{\gamma u} du$ and variance $e^{-2\delta(u-t)}\sigma_0^2 e^{2\gamma u} du$. Therefore, PV is normally distributed with mean:

$$\begin{aligned}
E[PV[C(t)]] &= \int_{u=t}^{\infty} e^{-\delta(u-t)}\mu_0 e^{\gamma u} du = \mu_0 \int_{u=t}^{\infty} e^{-\delta(u-t)} e^{\gamma(u-t)} du \cdot e^{\gamma t} \\
&= \mu_0 \int_{u=t}^{\infty} e^{-(\delta-\gamma)(u-t)} d(u-t) \cdot e^{\gamma t} = \mu_0 \int_{v=0}^{\infty} e^{-(\delta-\gamma)v} dv \cdot e^{\gamma t} \\
&= \frac{\mu_0}{\delta-\gamma} e^{\gamma t}
\end{aligned}$$

This agrees with the dividend-growth formula, except that δ takes the place of κ. Again, for

[5] We will not need the stochastic calculus, but introductions to it can be found in Wilmott [1995, pp 20-29] and Panjer [1998, Section 10.13].

convergence, $\delta > \gamma$.[6] Similarly, and due to the independence of the $dC(t)$, the variance of *PV* is:

$$Var\left[PV\left[C(t)\right]\right] = \int_{u=t}^{\infty} e^{-2\delta(u-t)}\sigma_0^2 e^{2\gamma u}du = \sigma_0^2 \int_{u=t}^{\infty} e^{-2\delta(u-t)}e^{2\gamma(u-t)}du \cdot e^{2\gamma t}$$

$$= \sigma_0^2 \int_{u=t}^{\infty} e^{-2(\delta-\gamma)(u-t)}d(u-t)\cdot e^{2\gamma t} = \sigma_0^2 \int_{v=0}^{\infty} e^{-2(\delta-\gamma)v}dv \cdot e^{2\gamma t}$$

$$= \frac{\sigma_0^2}{2(\delta-\gamma)}e^{2\gamma t}$$

So finally, the present value of the stochastic dividend flow at time t is a normal random variable with mean $\frac{\mu_0}{\delta-\gamma}e^{\gamma t}$ and variance $\frac{\sigma_0^2}{2(\delta-\gamma)}e^{2\gamma t}$.

We showed (Halliwell [2003, p. 66]) that the price of a quantum, or stand-alone, $N(\mu, \sigma^2)$ present-valued stochastic cash flow X is $q_X = \mu_X - a\sigma_X^2$ for an economic agent whose risk-aversion parameter is a. Accordingly, at time t such an agent will value the stock as:

$$V(t) = \frac{\mu_0}{\delta-\gamma}e^{\gamma t} - a(t)\frac{\sigma_0^2}{2(\delta-\gamma)}e^{2\gamma t}$$

6 It augurs well for our treatment of "time and risk [as] logically separate variables," as quoted above from Robichek and Myers, that it places a realistic constraint on growth, viz., that perpetual growth must be less than risk-free growth, a constraint that risk-adjusted discounting does not impose.

Furthermore, we argued elsewhere [Halliwell, 2001, Section 5 and Appendix D] that the product of one's risk aversion and expected wealth should remain constant. In this stand-alone realm, in which expected wealth is increasing by a factor of $e^{\gamma t}$, $a(t) = a_0 e^{-\gamma t}$. Therefore:

$$V(t) = \frac{\mu_0}{\delta - \gamma} e^{\gamma t} - a_0 e^{-\gamma t} \frac{\sigma_0^2}{2(\delta - \gamma)} e^{2\gamma t} = \left(\frac{\mu_0}{\delta - \gamma} - a_0 \frac{\sigma_0^2}{2(\delta - \gamma)} \right) e^{\gamma t} = V_0 e^{\gamma t}$$

The agent, in addition to receiving the dividend, will receive price appreciation at rate γ, as happens also according to the dividend-growth model.[7]

For the dividend-growth model and our theory to agree, the valuations must be equal, i.e., $\frac{\mu_0}{\kappa - \gamma} e^{\gamma t} = \left(\frac{\mu_0}{\delta - \gamma} - a_0 \frac{\sigma_0^2}{2(\delta - \gamma)} \right) e^{\gamma t}$, or $\frac{\mu_0}{\kappa - \gamma} = \frac{\mu_0}{\delta - \gamma} - a_0 \frac{\sigma_0^2}{2(\delta - \gamma)}$. Hence:

$$\begin{aligned} \kappa &= (\kappa - \gamma) + \gamma \\ &= \frac{\mu_0}{\left(\frac{\mu_0}{\kappa - \gamma} \right)} + \gamma \\ &= \frac{\mu_0}{\left(\frac{\mu_0}{\delta - \gamma} - a_0 \frac{\sigma_0^2}{2(\delta - \gamma)} \right)} + \gamma \end{aligned}$$

Therefore, there is a number κ defined in terms of the parameters of the stochastic-cash-flow model (i.e., in terms of μ_0, a_0, σ_0, δ, and γ) at which one can discount the expected dividend stream and arrive at the "correct" value. Furthermore, we can give a simple and meaningful interpretation to the right side of the last equation. The expected instantaneous dividend yield of the stock at time t, according to the stochastic theory, is:

[7] As a check, $V(t)$ decreases with increasing a. Since dividends can be negative in arithmetic Brownian motion, sufficient risk aversion will make $V(t)$ negative. In the case of risk-neutrality, when $a = 0$, $V(t) = E[PV[C(t)]]$.

$$
\begin{aligned}
E[y(t)] &= \frac{1}{V(t)} E\left[\frac{dC(t)}{dt}\right] \\
&= \frac{1}{V(t)} \frac{dE[C(t)]}{dt} \\
&= \frac{E\left[\mu_0 e^{\gamma t} dt + \sigma_0 e^{\gamma t} dX(t)\right] / dt}{\left(\dfrac{\mu_0}{\delta - \gamma} - a_0 \dfrac{\sigma_0^2}{2(\delta - \gamma)}\right) e^{\gamma t}} \\
&= \frac{\mu_0 e^{\gamma t} dt / dt}{\left(\dfrac{\mu_0}{\delta - \gamma} - a_0 \dfrac{\sigma_0^2}{2(\delta - \gamma)}\right) e^{\gamma t}} \\
&= \frac{\mu_0}{\left(\dfrac{\mu_0}{\delta - \gamma} - a_0 \dfrac{\sigma_0^2}{2(\delta - \gamma)}\right)}
\end{aligned}
$$

Obviously, since both the expected dividend and the price are growing at rate γ, the expected dividend rate $E[y\{t\}]$ is constant, or just $E[y]$. And this allows us to see that the expected total return, which Brealey and Myers call *inter alia* the cost of capital, is:

$$
\begin{aligned}
\kappa &= \frac{\mu_0}{\left(\dfrac{\mu_0}{\delta - \gamma} - a_0 \dfrac{\sigma_0^2}{2(\delta - \gamma)}\right)} + \gamma \\
&= E[y] + \gamma
\end{aligned}
$$

Therefore, risk-adjusted discounting is a special case of our theory. It is approximately correct, even as Newtonian mechanics is approximately correct vis-à-vis Special Relativity (and would be exactly correct, if the speed of light were infinite). The trouble is that the approximation is taken for the truth.

4. CONCLUSION

Therefore, risk-adjusted discounting is conditionally valid, sc., valid on the condition that the stochastic cash flow is continuously replicating itself on an exponentially-increasing scale.[8] To the reader we will leave to decide how often this condition applies to actual financial decisions, particularly to underwriting decisions. For some to counter that in the grand scheme every risk is but a drop in the ocean is as specious as for actuaries to argue from the central limit theorem that every distribution may be deemed normal. Robichek and Myers correctly state that "time and risk are logically separate variables."[9] We've all heard of a distinction without a difference. However, their claim that "valuation errors may result if … this assumption does not hold"[10] implies that this is one distinction that does make a difference.

[8] One might also add as ancillary conditions the independence of the flow from the (rest of) the agent's stochastic wealth, and the flatness of the yield curve.

[9] Even here, the adverb 'logically' is timid; time and risk are truly separate variables. So far as we know, the separation, or distinction, of time and risk is a basic principle only in Van Slyke [1995 and 1999] and Schnapp [2001]. Mango [2003] is ambiguous. However, agreement on this principle does not ensure agreement *in toto*. Van Slyke, in particular, urges that capital markets can and do synthesize the views of their participants into higher truths, a belief to which we do not subscribe. Nevertheless, Van Slyke [1995, p 587] is correct in rating the effect on finance of this principle as nothing short of revolutionary. Terms such as 'radical' and 'revolutionary' are bandied and overused; however, we regard this as much a revolution as the Copernican, which took a century finally to be settled. As the evidence mounted for heliocentrism, the old guard must have resorted to ploys like "Geo or helio, what's the difference? The day looks the same, anyway." So it is today in financial theory. But when the camel's nose gets under the tent, soon it will be overturned.

[10] Here again (see previous footnote), the auxiliary verb 'may' tones down. More accurately, valuation errors *will* result if the assumption does not hold, and it's just a matter of how serious these errors *may* be."

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