Comparison of Minimum Bias and Maximum Likelihood Methods for Claim Severity

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Abstract
The objective of this study is to compare the methods of minimum bias and maximum likelihood by using a weighted equation on claim severity data. The advantage of using the weighted equation is that the fitting procedure provides a faster convergence compared to the classical procedure introduced by Bailey and Simon [1] and Bailey [2]. Furthermore, the fitting procedure may be extended to other models in addition to the multiplicative and additive models, as long as the function of the fitted value is written in a specified linear form. In this study, the minimum bias and maximum likelihood methods will be compared and fitted on three types of claim severity data; the Malaysian data, the U.K. data from McCullagh and Nelder [3] and the Canadian data from Bailey and Simon [1].

Keywords: Minimum bias; maximum likelihood; claim severity; multiplicative; additive.

1. INTRODUCTION

The process of establishing premium rates for insuring uncertain events requires estimates which are made of two important elements; the probabilities or frequencies associated with the occurrence of insured event, and the magnitude or severities of such event. The process of grouping risks of similar risk characteristics for frequencies or severities is known as risk classification where its goal is to group homogeneous risks and charge each group a premium commensurate with the expected average loss. Failure to achieve this goal may lead to adverse selection to insureds and economic losses to insurers. The risks may be categorized according to risk or rating factors; in motor insurance for instance, driver's gender and claim experience, or vehicle's make and capacity, may be considered as rating factors.

In the last forty years, actuarial researchers suggested various statistical procedures for risk classification. For instance, Bailey and Simon [1] suggested the minimum chi-squares, Bailey [2] proposed the zero bias, Jung [4] produced a heuristic method for minimum modified chi-squares, Ajne [5] applied the method of moments also for minimum modified chi-squares, Chamberlain [6] used the weighted least squares, Coutts [7] produced the method of orthogonal weighted least squares with logit transformation, Harrington [8] suggested the maximum likelihood procedure for models with functional form, and Brown [9] proposed the bias and likelihood functions.

In the recent actuarial literature, research on risk classification methods is still continuing and developing. For example, Mildenhall [10] studied the relationship between the minimum bias and the Generalized Linear Models (GLMs), Feldblum and Brosius [11] provided the minimum bias procedures for practicing actuaries, Anderson et al. [12] provided practical insights for the GLMs analysis also for practicing actuaries, Fu and Wu [13] developed and generalized the minimum bias models, Ismail and Jemain [14] bridged the minimum bias and maximum likelihood methods for claim frequency data, and Ismail and Jemain [15] suggested the Negative Binomial and the Generalized Poisson regressions as alternatives to handle over-dispersion in claim frequency or count data.

The objective of this study is to compare the methods of minimum bias and maximum likelihood by using a weighted equation on claim severity data. Although the weighted equation was previously suggested by Ismail and Jemain [14], the application was implemented on claim frequency data. Therefore, this study differs such that the weighted equation will be applied to estimate claim severity or average claim cost which is equivalent to the total claim costs divided by the number of claims. Since the nature of claim frequency and severity is different, the approach taken is also slightly modified. In fact, with a few modifications, the same weighted equation may also be used for loss cost or pure premium which is equal to the total claim costs divided by the exposures, and for loss ratio which is equal to the total claim costs divided by the premiums. However, the weight generally used for fitting loss cost and loss ratio is the exposures.

Several studies have been carried out on claim severity data in the actuarial literature. Since it is well established that the claim cost distributions generally have positive support and are positively skewed, the distributions of Gamma and Lognormal have been used by practitioners for modeling claim severities. As a comparison, several actuarial studies also reported severity results from the Normal distribution. For example, Baxter et al. [16] fit the U.K. own damage costs for privately owned and comprehensively insured vehicles to the weighted linear regression (additive model) by assuming that the variance is constant within classes, McCullagh and Nelder [3] reanalyzed the same data by fitting the Gamma regression model and assuming that the coefficient of variation is constant within classes and the mean is linear on reciprocal scale (inverse model), Brockman and Wright [17] fit the U.K. own damage costs for comprehensive policies also to the Gamma model by using a log-linear regression (multiplicative model), Renshaw [18] fit the U.K. motor insurance claim severity also to the Gamma log-linear regression model, and Fu and Moncher [19] applied several Monte Carlo simulation techniques to examine the unbiasedness and stability of the Gamma, Lognormal and Normal distributions which were fitted on the severity data obtained from Mildenhall [10].

The advantage of using the weighted equation suggested in this study is that the fitting procedure provides a faster convergence compared to the classical procedure introduced by Bailey and Simon [1] and Bailey [2]. Furthermore, the fitting procedure may be extended to other models in addition to the multiplicative and additive models, as long as the function of the fitted value is written in a specified linear form.

In this study, the minimum bias and maximum likelihood methods will be compared and fitted on three types of claim severity data; the Malaysian data, the U.K. data from McCullagh and Nelder [3] and the Canadian data from Bailey and Simon [1].

2. REGRESSION MODEL

In the actuarial literature, various methods have been studied and implemented by actuarial researchers and practitioners for classifying risks. Most of these methods, which also include the minimum bias and maximum likelihood, may be written as a regression model where the explanatory variables are the risk or rating factors. In this study, the regression methods of minimum bias and maximum likelihood will be compared and fitted on claim severity data.

The related data sets for claim severity regression model are (c_i, y_i), where c_i and y_i denotes the average claim cost already adjusted for inflation and the claim count for the ith rating class, $i = 1,2,...,n$, so that the total claim cost is equal to the product of claim count and average claim cost, $y_i c_i$. The response variable and weight for the regression model is the average claim cost, c_i, and the claim count, y_i, respectively.

Consider a regression model with n observations of average claim cost and p explanatory variables inclusive of an intercept and dummy variables. Next, consider a data of average claim costs involving three rating factors, each respectively with three, two, and three rating classes. Thus, the data has a total of $n = 18$ observed average claim costs with $p = 6$ explanatory variables.

Let $\mathbf{c}$ denotes the vector of average claim cost vector, $\mathbf{y}$ the vector of claim count, $\mathbf{X}$ the matrix of explanatory variables where the *i*th row is equivalent to vector $\mathbf{x}_\mathbf{i}^\mathbf{T}$, and $\boldsymbol{\beta}$ the vector of regression parameters. If x_{ij}, $i = 1,2,...,18$, $j = 1,2,...,6$, is the *ij*th element of matrix $\mathbf{X}$, the value for x_{ij} is either one or zero. Table 1 summarizes the regression model for the claim severity data.

Table 1. Data summary

i	c_i	y_i	x_{i1}	x_{i2}	x_{i3}	x_{i4}	x_{i5}	x_{i6}
1	c_1	y_1	1	0	0	0	0	0
2	c_2	y_2	1	0	0	0	1	0
3	⋮	⋮	1	0	0	0	0	1
4			1	0	0	1	0	0
5			1	0	0	1	1	0
6			1	0	0	1	0	1

7			1	1	0	0	0	0
8			1	1	0	0	1	0
9			1	1	0	0	0	1
10			1	1	0	1	0	0
11			1	1	0	1	1	0
12			1	1	0	1	0	1
13			1	0	1	0	0	0
14			1	0	1	0	1	0
15			1	0	1	0	0	1
16			1	0	1	1	0	0
17			1	0	1	1	1	0
18	c_{18}	y_{18}	1	0	1	1	0	1

Moreover, let $\mathbf{f}$, a function of $\mathbf{X}$ and $\boldsymbol{\beta}$, denote the vector of fitted average claim costs. If the function of the fitted average claim cost is log-linear (multiplicative model), the fitted value in the i th rating class is equivalent to

$$f_i = \exp(\mathbf{x}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}), \qquad (1)$$

if the function is linear (additive model), the fitted average claim cost in the i th rating class is equal to

$$f_i = \mathbf{x}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}, \qquad (2)$$

and if the function is inverse (inverse model), the fitted average claim cost in the i th rating class is

$$f_i = \left(\mathbf{x}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}\right)^{-1}. \qquad (3)$$

In fact, a variety of regression models may be created and fitted, as long as the function of the fitted value is written as

$$f_i = \left(\sum_{j=1}^{p}\beta_j x_{ij}\right)^b, \qquad -1 \le b < 0,\ 0 < b \le 1. \qquad (4)$$

Thus, the objective of risk classification is to have the fitted average claim cost, f_i, be as close as possible to the observed average claim cost, c_i, for all i.

3. MINIMUM BIAS

Bailey and Simon [1] were among the pioneer researchers that consider bias in risk classification. They introduced the minimum bias method and proposed a famous list of four criteria for an acceptable set of classification rates:

- The rates should reproduce experience for each class and overall, i.e., they should be balanced for each class and overall.
- The rates should reflect the relative credibility of various classes.
- The rates should provide minimum amount of departure from the raw data.
- The rates should produce a rate for each class close enough to the experience so that the differences could reasonably be caused by chance.

3.1 Zero Bias

Bailey and Simon [1] proposed a suitable test for the first criteria by calculating,

$$\frac{\sum_i y_i f_i}{\sum_i y_i c_i}, \tag{5}$$

for each *j* and total. Thus, a set of rates is balanced, i.e., zero bias, if Equation (5) equals 1.00 and automatically, zero bias for each class implies zero bias for all classes.

From this test, Bailey [2] derived a minimum bias model by setting the average difference between the observed and the fitted rates to be equal to zero. In the case of claim severity regression model, the zero bias equation for each *j* can be written in the form of a weighted difference between the observed and the fitted average claim cost,

$$\sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,\ldots, p, \tag{6}$$

where w_i is equal to $y_i x_{ij}$.

3.2 Minimum Chi-squares

Bailey and Simon [1] also suggested the chi-squares statistics, χ^2, as an appropriate test for the fourth criteria,

$$\chi^2 = K \sum_i \frac{y_i}{f_i} (c_i - f_i)^2,$$

where K is a constant dependent on the data. The same test is also suitable for the second and third criteria as well.

By minimizing the chi-squares, another minimum bias model was derived. For each *j*, the minimum chi-squares equation could be written in the form of a weighted difference between the observed and the fitted average claim cost,

$$\frac{\partial \chi^2}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1, 2, \ldots, p, \tag{7}$$

where w_i is $\frac{y_i(c_i + f_i)}{f_i^2} \frac{\partial f_i}{\partial \beta_j}$.

If the function is log-linear (multiplicative model), the first derivative of the fitted value is equal to

$$\frac{\partial f_i}{\partial \beta_j} = f_i x_{ij}, \tag{8}$$

if the function is linear (additive model), the first derivative is

$$\frac{\partial f_i}{\partial \beta_j} = x_{ij}, \tag{9}$$

and if the function is inverse (inverse model), the first derivative is

$$\frac{\partial f_i}{\partial \beta_j} = -f_i^2 x_{ij}. \tag{10}$$

4. MAXIMUM LIKELIHOOD

Let $T_i = y_i C_i$ be the random variable for total claim costs and assume that the *i*th total claim cost, $y_i c_i$, comes from a distribution whose probability density function is $g(c_i; f_i)$. A maximum likelihood method maximizes the likelihood function,

$$L = \prod_i g(c_i; f_i),$$

or equivalently, the log likelihood function,

$$\ell = \log L = \sum_i \log\left(g(c_i; f_i)\right).$$

Thus, the regression parameters can be obtained by setting $\frac{\partial \ell}{\partial \beta_j} = 0$ for each j, $j = 1, 2, \ldots, p$.

4.1 Normal

If $T_i = y_i C_i$ is assumed to follow Normal distribution with mean $E(T_i) = y_i f_i$ and variance $Var(T_i) = \sigma^2$, the probability density function is (Brown [9])

$$g(c_i; f_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(y_i c_i - y_i f_i \right)^2 \right\}.$$

The regression parameters may be solved by using the likelihood equation

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,..., p, \tag{11}$$

where w_i is $y_i^2 \frac{\partial f_i}{\partial \beta_j}$. The first derivative of the fitted value is equal to equation (8) for a log-linear function (multiplicative), equation (9) for a linear function (additive), and equation (10) for an inverse function.

4.2 Poisson

If $T_i = y_i C_i$ is Poisson distributed with mean $E(T_i) = y_i f_i$, the probability density function is

$$g(c_i; f_i) = \frac{\exp(-y_i f_i)(y_i f_i)^{y_i c_i}}{(y_i c_i)!}.$$

As a result, the likelihood equation for each j is equal to

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,..., p, \tag{12}$$

but w_i is now equivalent to $\frac{y_i}{f_i} \frac{\partial f_i}{\partial \beta_j}$.

The same weighted equation could also be used to show that the Poisson is actually equivalent to the zero bias if the function of the fitted value is in a log-linear form (multiplicative model). By substituting Equation (8) into Equation (12), the likelihood equation for the Poisson is now equal to

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,..., p,$$

where w_i is $y_i x_{ij}$, and this likelihood equation is equivalent to the zero bias.

4.3 Exponential

Let $T_i = y_i C_i$ be exponential distributed with mean $E(T_i) = y_i f_i$. The probability density function is

$$g(c_i;\, f_i) = \frac{1}{y_i\, f_i} \exp\left(-\frac{c_i}{f_i}\right),$$

and the regression parameters may be solved by using the likelihood equation

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,\ldots,p, \tag{13}$$

where w_i is $\frac{1}{f_i^2}\frac{\partial f_i}{\partial \beta_j}$.

4.4 Gamma

If $T_i = y_i C_i$ is Gamma distributed with mean $E(T_i) = y_i\, f_i$ and variance $Var(T_i) = v^{-1} y_i^2 f_i^2$, the probability density function is

$$g(c_i;\, f_i) = \frac{1}{y_i c_i \Gamma(v)} \left(\frac{v c_i}{f_i}\right)^v \exp\left(-\frac{v c_i}{f_i}\right),$$

where v denotes the index parameter. Assuming that v is allowed to vary within classes and written as $v_i = \sigma^{-2} y_i$, the likelihood equation is

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,\ldots,p, \tag{14}$$

where w_i is $\frac{y_i}{f_i^2}\frac{\partial f_i}{\partial \beta_j}$.

4.5 Inverse Gaussian

The derivation of the weighted equation for an Inverse Gaussian distribution is slightly different. Instead of using the random variable for total claim cost, $T_i = y_i C_i$, the random variable for average claim cost, C_i, is used. Let the random variable for average claim cost, C_i, be distributed as Inverse Gaussian with mean $E(C_i) = f_i$ and variance $Var(C_i) = f_i^3 \tau$. The probability distribution function is (see Mildenhall [10] and Renshaw [18])

$$g(c_i;\, f_i) = \frac{1}{\sqrt{2\pi c_i^3 \tau}} \exp\left\{-\frac{1}{2 c_i\, f_i^2 \tau}(c_i - f_i)^2\right\},$$

where τ denotes the scale parameter. If τ is allowed to vary within classes and written as $\tau_i = \sigma^2 y_i^{-1}$, the likelihood equation is

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,\ldots,p, \tag{15}$$

where w_i is $\frac{y_i}{f_i^3}\frac{\partial f_i}{\partial \beta_j}$.

4.6 Lognormal

The derivation of the weighted equation for a Lognormal distribution is also slightly different. Let the average claim cost, C_i, be distributed as Lognormal with parameters f_i and $\sigma^2 y_i^{-1}$. Thus, the logarithm of the average claim cost, $\log C_i$, is Normal distributed with mean f_i and variance $\sigma^2 y_i^{-1}$ and the probability density function is now equivalent to

$$g(\log c_i; f_i) = \frac{1}{c_i\sqrt{2\pi\sigma^2 y_i^{-1}}}\exp\left\{-\frac{y_i(\log c_i - f_i)^2}{2\sigma^2}\right\}.$$

The likelihood equation can be written as,

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i(\log c_i - f_i) = 0, \qquad j = 1,2,\ldots p. \tag{16}$$

where w_i is $y_i\frac{\partial f_i}{\partial \beta_j}$. Compared to the likelihood equation for other distributions shown by Equations (6), (7), (11), (12), (13), (14) and (15), the Lognormal likelihood equation is slightly different.

5. OTHER MODELS

5.1 Least Squares

The weighted equation may also be extended to other error functions as well. For example, if the sum squares error is defined as (Brown [9])

$$S = \sum_i y_i(c_i - f_i)^2,$$

the regression parameters may be solved by using the least squares equation

$$\frac{\partial S}{\partial \beta_j} = \sum_i w_i(c_i - f_i) = 0, \qquad j = 1,2,\ldots, p, \tag{17}$$

where w_i is $y_i\frac{\partial f_i}{\partial \beta_j}$.

The same weighted equation could also be used to show that the least squares is actually equivalent to the zero bias if the function of the fitted value is in a linear form (additive model). By substituting Equation (9) into Equation (17), the likelihood equation for the least squares is

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,..., p ,$$

where w_i is equal to $y_i x_{ij}$, and this likelihood equation is equivalent to the zero bias.

5.2 Modified Chi-squares

If the function of error is a modified chi-squares which is defined as

$$\chi^2_{\text{mod}} = \sum_i \frac{y_i}{c_i}(c_i - f_i)^2 ,$$

the weighted equation is equal to

$$\frac{\partial \chi^2_{\text{mod}}}{\partial \beta_j} = \sum_i w_i (c_i - f_i) = 0, \qquad j = 1,2,..., p , \tag{18}$$

where w_i is $\frac{y_i}{c_i}\frac{\partial f_i}{\partial \beta_j}$.

Table 2 summarizes the weighted equations for all of the models discussed above.

Table 2: Weighted equations

Models	w_i for weighted equation, $\sum_i w_i(c_i - f_i) = 0$
Zero bias	$y_i x_{ij}$
Minimum χ^2	$\frac{y_i(c_i + f_i)}{f_i^2} \frac{\partial f_i}{\partial \beta_j}$
Normal	$y_i^2 \frac{\partial f_i}{\partial \beta_j}$
Poisson	$\frac{y_i}{f_i} \frac{\partial f_i}{\partial \beta_j}$
Exponential	$\frac{1}{f_i^2} \frac{\partial f_i}{\partial \beta_j}$
Gamma	$\frac{y_i}{f_i^2} \frac{\partial f_i}{\partial \beta_j}$
Inverse Gaussian	$\frac{y_i}{f_i^3} \frac{\partial f_i}{\partial \beta_j}$
Least squares	$y_i \frac{\partial f_i}{\partial \beta_j}$
Minimum modified χ^2	$\frac{y_i}{c_i} \frac{\partial f_i}{\partial \beta_j}$

Models	w_i for weighted equation, $\sum_i w_i(\log c_i - f_i) = 0$
Lognormal	$y_i \frac{\partial f_i}{\partial \beta_j}$

From Table 2, the following conclusions can be drawn:

- If the function of fitted value is in a linear form (additive), the zero bias and the least squares are equivalent.

- If the function of fitted value is in a log-linear form (multiplicative), the zero bias and the Poisson are equivalent.
- The weighted equation, which is in the form of a weighted difference between the observed and the fitted average claim cost, shows that all models are similar and can be distinguished by its weight.
- Since the weighted equation for all models are similar, the regression parameters for all models are expected to be similar. However, the Lognormal regression parameters are expected to be different from the rest of other models because its weighted equation is in the form of a weighted difference between the logarithm of the observed value and the fitted value.

6. FITTING PROCEDURE

The regression fitting procedure suggested in this study provides a faster convergence compared to the classical procedure introduced by Bailey and Simon [1] and Bailey [2]. In the classical procedure, each regression parameter, β_j, $j = 1,2,..., p$, is calculated individually in each iteration whereas in the regression procedure, all of the regression parameters are calculated simultaneously in each iteration.

In the regression fitting procedure, the parameters, β_j, are solved by minimizing,

$$\sum_i w_i (c_i - f_i)^2 , \tag{19}$$

or by equating,

$$\sum_i w_i (c_i - f_i) \frac{\partial f_i}{\partial \beta_j} = 0 , \qquad j = 1,2,..., p . \tag{20}$$

It can be seen that Equation (20) is equivalent to the weighted equation for the minimum bias and maximum likelihood methods shown by Equations (6), (7), (11), (12), (13), (14), (15), (17) and (18). As for Equation (16), the equation is equivalent to the same weighted equation if the value of c_i is replaced by $\log c_i$.

By using Taylor series approximation, it can be shown that the value of vector $\boldsymbol{\beta}$ in the first iteration is

$$\boldsymbol{\beta}_{(1)} = \left(\mathbf{Z}_{(0)}^{\mathbf{T}} \mathbf{W}_{(0)} \mathbf{Z}_{(0)} \right)^{-1} \mathbf{Z}_{(0)}^{\mathbf{T}} \mathbf{W}_{(0)} \left(\mathbf{c} - \mathbf{s}_{(0)} \right), \tag{21}$$

where $\boldsymbol{\beta}_{(0)}$ is the initial value of vector $\boldsymbol{\beta}$, $\mathbf{Z}_{(0)}$ the $n \times p$ matrix whose ij th element is equal to the first derivative of the fitted value evaluated at $\boldsymbol{\beta}_{(0)}$,

$$z_{ij(0)} = \left.\frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{(0)}},$$

$\mathbf{W}_{(0)}$ the $n \times n$ diagonal weight matrix evaluated at $\boldsymbol{\beta}_{(0)}$, and $\mathbf{s}_{(0)}$ the $n \times 1$ vector whose i th row is equal to

$$s_i = f_i(\boldsymbol{\beta}_{(0)}) - \sum_{j=1}^{p} \beta_{j(0)} z_{ij(0)}.$$

In the first iteration, the vector of initial values, $\boldsymbol{\beta}_{(0)}$, are required to calculate $\boldsymbol{\beta}_{(1)}$. The process of iteration is then repeated until the solution converges. Since the regression parameters are represented by vector $\boldsymbol{\beta}$, the regression model solves them simultaneously and thus, providing a faster convergence compared to the classical approach.

As an example, the fitting procedure for the least squares additive whereby the weighted equation is equivalent to

$$\sum_i y_i (c_i - f_i) \frac{\partial f_i}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, p, \tag{22}$$

will be discussed here. By comparing the least squares weighted equation, i.e., Equation (22), with the regression fitting equation, i.e., Equation (20), the i th diagonal element of the weight matrix, $\mathbf{W}_{(0)}$, is equal to y_i and this value is free of $\boldsymbol{\beta}_{(0)}$.

For an additive model, the ij th element of matrix $\mathbf{Z}_{(0)}$ is

$$z_{ij(0)} = \left.\frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{(0)}} = x_{ij},$$

and this value is also free of $\boldsymbol{\beta}_{(0)}$.

Therefore,

$$\mathbf{Z}_{(0)} = \mathbf{X},$$

and

$$\mathbf{s}_{(0)} = \mathbf{f}(\boldsymbol{\beta}_{(0)}) - \mathbf{X}\boldsymbol{\beta}_{(0)} = 0,$$

and Equation (21) for the least squares additive may now be simplified into

$$\boldsymbol{\beta}_{(1)} = \boldsymbol{\beta} = (\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{c}. \tag{23}$$

It can be seen that Equation (23) is equivalent to the Normal equation in standard linear regression and the equation also indicates that the regression parameters for the least squares additive may be solved without any iteration.

However, for a multiplicative model, the ij th element of matrix $\mathbf{Z}_{(0)}$ is equivalent to

$$z_{ij(0)} = \left.\frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{(0)}} = f_i(\boldsymbol{\beta}_{(0)}) x_{ij} .$$

Therefore, matrix $\mathbf{Z}_{(0)}$ may be written as

$$\mathbf{Z}_{(0)} = \mathbf{F}_{(0)}\mathbf{X} , \qquad (24)$$

where $\mathbf{F}_{(0)}$ is the $n \times n$ diagonal matrix whose i th diagonal element is $f_i(\boldsymbol{\beta}_{(0)})$. Vector $\mathbf{s}_{(0)}$ may now be written as

$$\mathbf{s}_{(0)} = \mathbf{f}(\boldsymbol{\beta}_{(0)}) - \mathbf{F}_{(0)}\mathbf{X}\boldsymbol{\beta}_{(0)} .$$

Besides multiplicative and additive models, the fitting procedure suggested in this study can also be extended to other regression models and thus, allowing a variety of regression model to be created and applied as long as the function of the fitted value is written as

$$f_i = \left(\sum_{j=1}^{p} \beta_j x_{ij} \right)^b , \qquad -1 \le b < 0, \;\; 0 < b \le 1.$$

As an example, if the fitted average claim cost is assumed to follow an inverse function, i.e., $b = -1$, the ij th element of matrix $\mathbf{Z}_{(0)}$ is equal to

$$z_{ij(0)} = \left.\frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{(0)}} = -\left\{ f_i(\boldsymbol{\beta}_{(0)}) \right\}^2 x_{ij} .$$

Therefore, the equation for matrix $\mathbf{Z}_{(0)}$ may also be written as Equation (24), but the i th diagonal element of matrix $\mathbf{F}_{(0)}$ is equal to $-\{ f_i(\boldsymbol{\beta}_{(0)})\}^2$.

An example of S-PLUS programming for the least squares multiplicative is given in Appendix A. Similar programming can also be used for all of the multiplicative, additive and inverse models proposed in this study. Each programming should be differentiated only by the following three elements:

- The vector of fitted average claim cost is equal to $\mathbf{f} = \exp(\mathbf{X}\boldsymbol{\beta})$ for a multiplicative model, $\mathbf{f} = \mathbf{X}\boldsymbol{\beta}$ for an additive model, and $\mathbf{f} = (\mathbf{X}\boldsymbol{\beta})^{-1}$ for an inverse model.

- The equation for $\mathbf{Z}_{(0)}$ is $\mathbf{Z}_{(0)} = \mathbf{X}$ for an additive model, and $\mathbf{Z}_{(0)} = \mathbf{F}_{(0)}\mathbf{X}$ for both multiplicative and inverse models. However, the i th diagonal element of matrix $\mathbf{F}_{(0)}$ is equal to $f_i(\boldsymbol{\beta}_{(0)})$ for a multiplicative model, and $-\{f_i(\boldsymbol{\beta}_{(0)})\}^2$ for an inverse model.
- Each model has its own weight matrix.

7. EXAMPLES

7.1 Malaysian Data

In this study, the methods of minimum bias and maximum likelihood will be compared and fitted on three types of claim severity data; the Malaysian data, the U.K. data from McCullagh and Nelder [3] and the Canadian data from Bailey and Simon [1]. For the Malaysian data, the weighted equation will be applied on a set of private car Third Party Property Damage (TPPD) claim costs obtained from an insurer in Malaysia which covers the legal liability of third party property loss or damage caused by or arising out of the use of an insured motor vehicle. The Malaysian data was based on 170,000 private car policies (1998-2000). The claims, which include both paid and outstanding, were already adjusted for inflation and were provided in Ringgit Malaysia (RM) currency.

The risks for the Malaysian claims were associated with five rating factors namely scope of coverage, vehicle make, vehicle use and gender of driver, vehicle year, and location. Altogether, there were $2 \times 2 \times 3 \times 4 \times 5 = 240$ cross-classified rating classes of claim severities to be estimated. Appendix B shows the rating factors, claim counts and average claim costs for the Malaysian data.

The fitting procedure involves only 108 data points because 132 of the rating classes have zero claim count (or weight). In addition, the models were evaluated using two different tests; the chi-squares and the average absolute difference. The average absolute difference,

$$\frac{\sum_i y_i |c_i - f_i|}{\sum_i y_i c_i},$$

was suggested by Bailey and Simon [1] as a suitable test for the third criteria whereas the chi-squares,

$$\chi^2 = K \sum_i \frac{y_i}{f_i}(c_i - f_i)^2,$$

was proposed by Bailey and Simon [1] as a suitable test for the fourth criteria.

Table 3 and Table 4 give the results of the regression parameters, chi-square values and average absolute difference for the multiplicative and additive models of the Malaysian data. Based on the results, the following conclusions can be made:

- For multiplicative models, the regression parameters for the *Poisson* are equivalent to the *zero bias*.
- For additive models, the regression parameters for the *least squares* are equal to the *zero bias*.
- Except for Lognormal, the regression parameters for multiplicative and additive models are similar. The reason is that the observed average claim costs, c_i, in the Lognormal were replaced by the logarithm of the average claims costs, $\log c_i$.
- Except for Lognormal, the smallest chi-square value is given by the *minimum chi-squares* for both additive and multiplicative models.
- Except for Lognormal, the smallest absolute difference is given by the *least squares* for both additive and multiplicative models.

Table 3: Multiplicative models for Malaysian data

Regression parameters	Zero bias	Minimum χ^2	Normal	Exponential	Poisson	Gamma	Inverse Gaussian	Least squares	Minimum modified χ^2	Lognormal
$\exp(\beta_1)$ Intercept	9242.10	9233.24	9278.99	8938.08	9242.10	9257.87	9281.65	9233.38	9267.88	9.14
$\exp(\beta_2)$ Non-comp	1.16	1.18	1.14	1.17	1.16	1.16	1.16	1.16	1.11	1.01
$\exp(\beta_3)$ Foreign	1.08	1.08	1.07	1.21	1.08	1.09	1.09	1.08	1.08	1.01
$\exp(\beta_4)$ Female	0.90	0.90	0.93	0.80	0.90	0.89	0.88	0.90	0.88	0.99
$\exp(\beta_5)$ Business	0.19	0.19	0.20	0.21	0.19	0.19	0.19	0.19	0.19	0.81
$\exp(\beta_6)$ 2-3 years	0.78	0.78	0.78	0.73	0.78	0.77	0.77	0.78	0.77	0.97
$\exp(\beta_7)$ 4-5 years	0.69	0.69	0.68	0.65	0.69	0.68	0.68	0.69	0.68	0.96
$\exp(\beta_8)$ 6+ years	0.72	0.72	0.71	0.71	0.72	0.72	0.72	0.71	0.72	0.96
$\exp(\beta_9)$ North	0.94	0.94	0.93	0.92	0.94	0.94	0.94	0.94	0.93	0.99
$\exp(\beta_{10})$ East	0.86	0.88	0.84	0.88	0.86	0.87	0.88	0.85	0.83	0.98
$\exp(\beta_{11})$ South	0.94	0.94	0.94	1.04	0.94	0.94	0.93	0.94	0.93	0.99
$\exp(\beta_{12})$ East M'sia	0.94	0.97	0.94	1.06	0.94	0.94	0.93	0.95	0.89	0.99
χ^2	476,081	471,147	492,026	844,318	476,081	477,160	480,147	477,541	517,605	8.16
Absolute difference ($\times 10^3$)	65.62	66.12	66.60	115.76	65.62	66.15	66.63	65.19	66.58	7.83

Table 4: Additive models for Malaysian data

Parameters	Zero bias	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	Least Squares	Minimum modified χ^2	Lognormal
β_1 Intercept	9167	9165	9254	9165	9006	9166	9171	9167	9170	9.13
β_2 Non-comp	1038	1201	931	1034	712	1031	1026	1038	684	0.13
β_3 Foreign	557	597	523	582	1274	606	628	557	555	0.08
β_4 Female	-765	-775	-592	-805	-1537	-838	-863	-765	-884	-0.12
β_5 Business	-4988	-4981	-4908	-4992	-4858	-4997	-5002	-4988	-5024	-1.64
β_6 2-3 years	-1976	-1983	-1968	-1987	-2315	-2000	-2013	-1976	-2009	-0.26
β_7 4-5 years	-2793	-2793	-2896	-2810	-2937	-2832	-2855	-2793	-2850	-0.39
β_8 6+ years	-2505	-2532	-2615	-2508	-2458	-2511	-2515	-2505	-2458	-0.33
β_9 North	-467	-446	-525	-459	-484	-451	-447	-467	-481	-0.07
β_{10} East	-1046	-833	-1084	-946	-782	-869	-815	-1046	-1193	-0.16
β_{11} South	-479	-452	-471	-467	314	-457	-449	-479	-496	-0.07
β_{12} East M'sia	-431	-257	-439	-452	269	-477	-504	-431	817	-0.09
χ^2	468,589	462,541	482,320	467,208	780,662	467,624	469,012	468,589	507,891	8.14
Absolute difference ($\times 10^3$)	64.65	65.42	65.42	64.97	109.19	65.50	66.02	64.65	66.48	7.81

7.2 U.K. Data

The U.K. data provides information on the Own Damage claim counts and average claim costs for privately owned and comprehensively insured vehicles (McCullagh and Nelder [3]). The average claim costs (in Pound Sterling) were already adjusted for inflation and the risks were associated with three rating factors: policyholder's age, car group, and vehicle age. Altogether, there were $8 \times 4 \times 4 = 128$ cross-classified rating classes of claim severities to be estimated. However, the fitting procedure involved only 123 data points because five of the rating classes have zero claim count. In addition to multiplicative and additive models, the severities were also fitted to the inverse models. The results of inverse models were compared to those of McCullagh and Nelder [3], who have applied Gamma regression model on the same severity data by assuming that the regression effects were linear on reciprocal scale.

Table 5, Table 6 and Table 7 give the results of the regression parameters, chi-square values, and average absolute difference for the U.K. data. As expected, except for Lognormal, the regression parameters for each of the multiplicative, additive, and inverse models are similar. In addition, the regression parameters for the Gamma whose fitted value is in the form of an inverse function are equal to the regression parameters produced by the McCullagh and Nelder [3]. The smallest chi-square value for additive, multiplicative and inverse models is provided by the *minimum chi-square*, whereas the smallest absolute difference for additive, multiplicative and inverse models is given by the *Gamma*.

7.3 Canadian Data

The Canadian data was obtained from Bailey and Simon [1] and it provides information on the liability claim counts and average claim costs for private passenger automobile insurance. The data involves two rating factors, namely merit and class, and altogether there were $4 \times 5 = 20$ cross-classified rating classes of claim severities to be estimated. In this study, the claim severities were fitted to the multiplicative and additive models.

Table 8 and Table 9 give the results of the regression parameters, chi-square values, and average absolute difference for the Canadian data. As expected, each of the multiplicative and additive models gives similar estimates for the regression parameters. The smallest chi-square value is provided by the *minimum chi-squares* for both additive and multiplicative models, whereas the smallest absolute difference is given by the *Normal* for both additive and multiplicative models.

Table 5: Multiplicative models for UK data

Parameters	Zero bias	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	Least squares	Minimum modified χ^2	Lognormal
$\exp(\beta_1)$ Intercept	297.57	313.59	279.34	297.57	302.38	286.75	276.52	309.81	257.91	5.61
$\exp(\beta_2)$ 21-24 years	0.98	0.95	1.05	0.98	0.90	1.00	1.02	0.94	1.08	1.01
$\exp(\beta_3)$ 25-29 years	0.91	0.87	0.97	0.91	1.01	0.94	0.97	0.88	1.04	1.00
$\exp(\beta_4)$ 30-34 years	0.88	0.84	0.96	0.88	0.75	0.89	0.90	0.86	1.01	0.99
$\exp(\beta_5)$ 35-39 years	0.70	0.67	0.75	0.70	0.72	0.73	0.76	0.67	0.79	0.95
$\exp(\beta_6)$ 40-49 years	0.77	0.73	0.81	0.77	0.76	0.79	0.80	0.75	0.89	0.97
$\exp(\beta_7)$ 50-59 years	0.78	0.75	0.83	0.78	0.79	0.80	0.82	0.76	0.89	0.97
$\exp(\beta_8)$ 60+ years	0.78	0.74	0.82	0.78	0.75	0.80	0.81	0.77	0.90	0.97
$\exp(\beta_9)$ B	0.99	0.99	0.96	0.99	1.06	1.00	1.01	0.98	0.99	1.00
$\exp(\beta_{10})$ C	1.16	1.16	1.14	1.16	1.17	1.17	1.18	1.15	1.16	1.03
$\exp(\beta_{11})$ D	1.48	1.50	1.53	1.48	1.60	1.49	1.50	1.48	1.45	1.07
$\exp(\beta_{12})$ 4-7 years	0.91	0.91	0.95	0.91	0.89	0.92	0.92	0.90	0.91	0.98
$\exp(\beta_{13})$ 8-9 years	0.70	0.70	0.74	0.70	0.66	0.71	0.72	0.69	0.69	0.94
$\exp(\beta_{14})$ 10+ years	0.49	0.51	0.50	0.49	0.48	0.50	0.50	0.48	0.46	0.87
χ^2	31,410	30,722	32,685	31,410	45,003	31,250	31,948	31,344	34,046	24.03
Absolute difference ($\times 10^3$)	81.30	82.05	83.06	81.30	106.90	80.74	81.24	83.46	82.73	14.56

Table 6: Additive models for UK data

Parameters	Zero bias	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	Least squares	Minimum modified χ^2	Lognormal
β_1 Intercept	298.67	303.94	273.49	288.34	291.89	278.98	270.03	298.67	241.88	5.60
β_2 21-24 years	-5.60	-7.53	17.58	0.31	-10.84	4.96	9.08	-5.60	34.01	0.04
β_3 25-29 years	-24.64	-30.52	-2.01	-16.95	15.31	-9.91	-2.61	-24.64	26.37	-0.01
β_4 30-34 years	-33.22	-43.39	-7.76	-29.34	-47.35	-26.59	-24.37	-33.22	14.17	-0.06
β_5 35-39 years	-87.89	-89.26	-64.78	-75.74	-44.23	-64.82	-53.72	-87.89	-33.45	-0.27
β_6 40-49 years	-66.99	-75.55	-50.51	-60.27	-45.84	-54.15	-47.87	-66.99	-13.68	-0.18
β_7 50-59 years	-63.35	-70.12	-45.49	-55.64	-36.19	-48.60	-41.39	-63.35	-10.87	-0.17
β_8 60+ years	-63.15	-72.15	-47.39	-56.91	-44.32	-51.10	-44.79	-63.15	-10.32	-0.17
β_9 B	-2.46	-0.50	-7.03	-0.21	8.19	2.04	4.11	-2.46	-0.30	0.00
β_{10} C	34.18	35.05	33.89	35.45	25.86	36.41	36.83	34.18	35.84	0.16
β_{11} D	108.66	113.74	123.07	108.76	97.83	108.90	108.62	108.66	96.09	0.39
β_{12} 4-7 years	-24.21	-21.98	-10.57	-21.54	-30.60	-19.62	-18.39	-24.21	-20.39	-0.08
β_{13} 8-9 years	-76.75	-71.63	-59.08	-72.26	-96.51	-69.12	-67.12	-76.75	-74.38	-0.35
β_{14} 10+ years	-126.63	-118.78	-111.15	-121.21	-147.85	-117.94	-116.35	-126.63	-128.54	-0.72
χ^2	34,060	33,200	35,487	33,547	48,796	33,954	35,059	34,060	37,670	24.34
Absolute difference ($\times 10^3$)	87.22	85.47	86.61	85.33	114.54	85.07	86.26	87.22	88.42	14.66

Table 7: Inverse models for UK data

Parameters **(10^4)**	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	Least squares	Minimum modified χ^2	Lognormal
β_1 Intercept	31.23	35.10	32.79	33.24	34.11	35.37	31.30	37.44	1782.06
β_2 21-24 years	3.12	-0.28	2.41	6.04	1.01	-0.11	4.16	-0.74	-7.66
β_3 25-29 years	6.11	2.25	4.75	3.53	3.50	2.30	6.26	0.46	8.40
β_4 30-34 years	6.81	2.50	5.30	11.78	4.62	4.23	6.39	0.83	20.99
β_5 35-39 years	16.11	12.41	14.97	16.36	13.70	12.64	16.61	11.36	93.71
β_6 40-49 years	11.73	8.41	10.28	12.04	9.69	9.25	11.12	5.97	63.00
β_7 50-59 years	11.30	7.78	9.96	11.00	9.16	8.47	10.98	5.89	58.45
β_8 60+ years	11.26	7.88	9.75	12.39	9.20	8.81	10.58	5.32	58.59
β_9 B	0.68	2.06	0.70	-2.82	0.38	-0.08	0.93	0.65	0.34
β_{10} C	-5.60	-5.11	-5.68	-6.51	-6.14	-6.70	-5.29	-5.95	-52.43
β_{11} D	-13.90	-14.27	-13.77	-18.24	-14.21	-14.83	-13.55	-13.60	-123.83
β_{12} 4-7 years	3.99	2.65	3.95	3.39	3.66	3.38	4.21	3.88	28.61
β_{13} 8-9 years	16.33	15.45	16.83	17.42	16.51	16.21	17.14	17.95	123.20
β_{14} 10+ years	38.52	43.50	41.74	33.78	41.54	41.38	41.97	47.09	275.82
χ^2	30,699	32,744	31,032	42,866	31,166	31,731	31,304	33,693	23.80
Absolute difference ($\times 10^3$)	81.29	81.93	80.18	99.61	79.23	79.33	81.46	80.30	14.45

Table 8: Multiplicative models for Canadian data

Parameters	Zero bias	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	Least squares	Minimum modified χ^2	Lognormal
$\exp(\beta_1)$ Intercept	292.00	291.97	291.08	292.00	294.57	291.92	291.84	292.10	292.07	5.68
$\exp(\beta_2)$ Merit X	0.99	0.99	1.00	0.99	0.97	0.99	0.99	0.99	0.98	1.00
$\exp(\beta_3)$ Merit Y	0.99	0.99	0.99	0.99	1.00	0.99	0.99	0.99	0.99	1.00
$\exp(\beta_4)$ Merit B	1.06	1.06	1.07	1.06	1.05	1.06	1.06	1.05	1.06	1.01
$\exp(\beta_5)$ Class 2	1.09	1.09	1.09	1.09	1.12	1.09	1.09	1.08	1.08	1.01
$\exp(\beta_6)$ Class 3	1.02	1.02	1.03	1.02	0.98	1.02	1.02	1.02	1.02	1.00
$\exp(\beta_7)$ Class 4	1.17	1.17	1.18	1.17	1.16	1.17	1.17	1.17	1.17	1.03
$\exp(\beta_8)$ Class 5	0.92	0.92	0.92	0.92	0.92	0.92	0.92	0.92	0.92	0.99
χ^2	49,520	49,470	54,461	49,520	80,313	49,542	49,657	49,609	49,895	27.51
Absolute difference $(\times 10^3)$	10.66	10.59	7.84	10.66	20.38	10.42	10.20	10.94	10.94	1.81

Table 9: Additive models for Canadian data

Parameters	Zero bias	Minimum χ^2	Normal	Poisson	Exponential	Gamma	Inverse Gaussian	least squares	Minimum modified χ^2	Lognormal
β_1 Intercept	291.95	291.83	291.06	291.87	294.77	291.80	291.74	291.95	291.94	5.68
β_2 Merit X	-4.24	-3.38	0.59	-4.05	-10.11	-3.92	-3.82	-4.24	-5.37	-0.02
β_3 Merit Y	-3.45	-3.51	-3.95	-3.58	1.00	-3.68	-3.74	-3.45	-3.71	-0.01
β_4 Merit B	17.11	17.58	20.28	17.53	15.49	17.92	18.28	17.11	17.44	0.06
β_5 Class 2	25.16	25.75	25.13	25.35	35.64	25.54	25.73	25.16	24.63	0.08
β_6 Class 3	4.71	4.80	8.26	4.68	-6.92	4.65	4.62	4.71	4.43	0.02
β_7 Class 4	51.08	51.28	53.30	51.18	47.12	51.30	51.42	51.08	51.01	0.16
β_8 Class 5	-22.92	-22.79	-23.60	-22.99	-25.33	-23.05	-23.11	-22.92	-23.38	-0.08
χ^2	46,776	46,665	51,049	46,713	82,024	46,722	46,790	46,776	47,074	27.22
Absolute difference ($\times 10^3$)	10.08	9.79	7.17	9.86	20.51	9.66	9.49	10.08	10.10	1.79

8. CONCLUSION

This study compares several minimum bias and maximum likelihood methods by using a weighted equation which is written as a weighted difference between the observed and the fitted values. The weighted equation was applied to estimate claim severity or average claim cost which is equivalent to the total claim costs divided by the number of claims.

The weighted equations are summarized in Table 2. Based on the weighted equations, it can be concluded that the equations for *zero bias* and *least squares* are equal if the function of fitted value is linear (additive model) and the equations for *zero bias* and *Poisson* are equal if the function of fitted value is log-linear (multiplicative model). It can also be shown from the weighted equations that all models are similar and can be distinguished by its own weight, except for Lognormal where the observed average claim costs, c_i, were replaced by the logarithm of the average claim costs, $\log c_i$.

The fitting procedure was suggested to be carried out using a regression approach. The advantage of using the regression fitting procedure is that it provides a faster convergence compared to the classical procedure introduced by Bailey and Simon [1] and Bailey [2]. Furthermore, the fitting procedure may also be extended to other models in addition to the multiplicative and additive models, as long as the function of fitted value is written in a specified linear form. A similar programming for the fitting procedure may also be used for all of the multiplicative, additive and inverse models proposed in this study. Each model should be differentiated only by three elements: the vector of fitted average claim cost, $\mathbf{f}$; the equation for matrix $\mathbf{Z}$; and the equation for weight matrix, $\mathbf{W}$.

In this study, the minimum bias and maximum likelihood methods were applied to fit three types of severity data: the Malaysian data, the U.K. data from McCullagh and Nelder [3], and the Canadian data from Bailey and Simon [1]. The models were tested based on the average absolute difference and the chi-square value. Based on the results, except for Lognormal, the smallest chi-square value is given by the *minimum chi-squares*. As for the absolute difference, the smallest value for the Malaysian, U.K., and Canadian data is provided by the *least squares*, *Gamma* and *Normal*, respectively. The U.K. data also showed that the regression parameters for Gamma with an inverse fitted function are equivalent to those produced by the McCullagh and Nelder [3].

When this study was carried out, two main targets were outlined: to provide strong basic statistical justification for the available models, and to search for a match point that is able to merge the available parametric and nonparametric models into a more generalized form. It is hoped that a more friendly and efficient computation approach can be created through both of these targets. As a result, this study managed to not only offer more models which include both parametric and nonparametric approaches, but also a friendlier computation method.

Even though the approach taken in this study was based on statistical parametric theory, the theory can be matched with nonparametric theory as well. For the proposed models, the actuary does not really have to determine the statistical distribution appropriate for the available data; all he needs to do is just determine the weight. Therefore, the main principle which is applied in this approach is the selection of an appropriate weight suitable for the available data. The proposed models may be more flexible and at the same time able to attend both streams of thought in statistics; nonparametric and parametric.

Besides modeling aspects, the suggested regression approach may build a base for efficient computation as well as analysis. The reason is that the regression approach allows the data to be analyzed, interpreted, and predicted with a similar manner to the data analysis, interpretation, and prediction of the regression analysis.

Finally, rewriting the equations of minimum bias and maximum likelihood as a weighted equation has several advantages:

- The mathematical concept of the weighted equation is simpler and hence, providing an easier understanding particularly for insurance practitioners.
- The weighted equation allows the usage of a regression model as an alternative programming algorithm to calculate the regression parameters.
- The weighted equation provides a basic step to further understand the more complex distributions such as Gamma, Inverse Gaussian, and Lognormal.
- The weights of each of the multiplicative, additive and inverse models show that the models have similar regression parameters.

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Appendix A: S-PLUS programming for least squares multiplicative

```
leastsquares.multi <- function(data)
{
#  To identify matrix X, vector cost and vector count from the data
   X <- as.matrix(data[,-(1:2)])
   cost <- as.vector(data[,1])
   count <- as.vector(data[,2])
#  To set initial values for vector beta
   new.beta <- c(10, rep(c(0.01), dim(X)[2]))
#  To start iteration
   for (i in 1:20)
   {
      beta <- new.beta
      fitted <- as.vector(exp(X%*%beta))
      Z <- diag(fitted)%*%X
      W <- diag(count)
      r.s <- cost-fitted+as.vector(Z%*%beta)
      new.beta <- as.vector(solve(t(Z)%*%W%*%Z)%*%t(Z)%*%W%*%r.s)
   }
#  To calculate fitted values, chi-square and absolute difference
   fitted <- as.vector(exp(X%*%new.beta))
   chi.square <- sum((count*(cost-fitted)^2)/fitted)
```

```
   abs.difference <- sum(count*abs(cost-fitted))/sum(count*cost)
#  To list programming output
   list  (expbeta= exp(new.beta), chi.square= chi.square,
         abs.difference= abs.difference)
}
```

Appendix B: Malaysian data

Rating factors					Claim count	Average claim cost (RM)
Scope of coverage	Vehicle make	Vehicle use & gender of driver	Vehicle year	Location		
Comprehensive	Local	Private-male	0-1 year	Central	381	9290
				North	146	8775
				East	44	6447
				South	161	8484
				East Malaysia	8	7785
			2-3 year	Central	422	7220
				North	203	6713
				East	41	6461
				South	164	7298
				East Malaysia	19	4037
			4-5 year	Central	276	6558
				North	145	5220
				East	29	6415
				South	115	5554
				East Malaysia	17	6937
			6+ year	Central	223	6678
				North	150	6230
				East	39	5372
				South	89	5915
				East Malaysia	33	5005
		Private-female	0-1 year	Central	165	9136
				North	55	7876
				East	12	7536
				South	23	6789
				East Malaysia	6	10306
			2-3 year	Central	147	6642
				North	72	5731
				East	12	5038
				South	39	6023
				East Malaysia	8	3977
			4-5 year	Central	56	5545
				North	36	4642
				East	7	4565
				South	23	5038
				East Malaysia	2	3818
			6+ year	Central	51	5709

			North	38	6272
			East	5	2869
			South	23	6243
			East Malaysia	9	3765
	Business	0-1 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		2-3 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		4-5 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		6+ year	Central	0	0
			North	1	1206
			East	0	0
			South	0	0
			East Malaysia	0	0
Foreign	Private-male	0-1 year	Central	94	8986
			North	47	9402
			East	21	7321
			South	38	9170
			East Malaysia	6	11507
		2-3 year	Central	202	8251
			North	85	6772
			East	21	5332
			South	65	5821
			East Malaysia	23	9503
		4-5 year	Central	157	6498
			North	85	8235
			East	15	8758
			South	73	6391
			East Malaysia	24	7047
		6+ year	Central	245	6923
			North	151	6777
			East	44	7563
			South	113	7266
			East Malaysia	64	7047
	Private-female	0-1 year	Central	29	10442
			North	11	7599
			East	2	9492
			South	17	9003

				East Malaysia	6	5867
			2-3 year	Central	46	6460
				North	41	5966
				East	5	3463
				South	13	7329
				East Malaysia	10	5222
			4-5 year	Central	39	4798
				North	15	4921
				East	0	0
				South	16	4384
				East Malaysia	11	6792
			6+ year	Central	47	5197
				North	35	7131
				East	6	6480
				South	9	5152
				East Malaysia	10	7718
		Business	0-1 year	Central	0	0
				North	0	0
				East	0	0
				South	0	0
				East Malaysia	0	0
			2-3 year	Central	0	0
				North	0	0
				East	0	0
				South	0	0
				East Malaysia	0	0
			4-5 year	Central	0	0
				North	0	0
				East	0	0
				South	0	0
				East Malaysia	0	0
			6+ year	Central	0	0
				North	0	0
				East	0	0
				South	0	0
				East Malaysia	0	0
Non-comprehensive	Local	Private-male	0-1 year	Central	0	0
				North	0	0
				East	0	0
				South	0	0
				East Malaysia	0	0
			2-3 year	Central	3	10225
				North	0	0
				East	0	0
				South	1	14265
				East Malaysia	0	0
			4-5 year	Central	1	3619

		North	5	5003
		East	0	0
		South	1	3375
		East Malaysia	0	0
	6+ year	Central	9	8736
		North	5	5142
		East	2	3598
		South	4	8673
		East Malaysia	2	17210
Private-female	0-1 year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	2-3 year	Central	0	0
		North	1	1563
		East	0	0
		South	0	0
		East Malaysia	0	0
	4-5 year	Central	0	0
		North	1	3619
		East	0	0
		South	0	0
		East Malaysia	0	0
	6+ year	Central	1	2003
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	1	4455
Business	0-1 year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	2-3 year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	4-5 year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	6+ year	Central	0	0
		North	0	0
		East	0	0
		South	0	0

			East Malaysia	0	0
Foreign	Private-male	0-1 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		2-3 year	Central	0	0
			North	3	6739
			East	0	0
			South	2	12657
			East Malaysia	0	0
		4-5 year	Central	0	0
			North	3	9796
			East	0	0
			South	0	0
			East Malaysia	3	13812
		6+ year	Central	49	7234
			North	71	7740
			East	6	9383
			South	56	8108
			East Malaysia	22	6207
	Private-female	0-1 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		2-3 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		4-5 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		6+ year	Central	14	6942
			North	15	7462
			East	2	6148
			South	6	10584
			East Malaysia	3	10168
	Business	0-1 year	Central	0	0
			North	0	0
			East	0	0
			South	0	0
			East Malaysia	0	0
		2-3 year	Central	0	0

		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	4-5 year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
	6+ year	Central	0	0
		North	0	0
		East	0	0
		South	0	0
		East Malaysia	0	0
Total			5,728	-