Uncertainty-Based Credibility and its Application to Excess-of-Loss Reinsurance

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Abstract

This paper proposes a methodology to calculate the credibility risk premium based on the uncertainty on the risk premium, as estimated by the standard deviation of the risk premium estimator. An optimal estimator based on the uncertainties involved in the pricing process is constructed.

The methodology is then applied to pricing layers of excess-of-loss reinsurance, and the behaviour of the credibility factor as a function of layer excess is analysed. Results are obtained for both the general case and the significant special case where the severity distribution is the same for all clients, for which it is proved that credibility is broadly constant across the reinsurance layers. A real-world application to pricing motor reinsurance is also discussed.

Although the methodology is especially useful when applied to reinsurance, the underlying ideas are completely general and can be applied to all contexts where the uncertainties in the pricing process can be calculated.

Keywords. uncertainty-based credibility, pricing horizon, excess-of-loss reinsurance pricing, market heterogeneity, error propagation analysis

1. INTRODUCTION

The experience-based calculation of the risk premium for an insurance or reinsurance account is affected by several sources of uncertainty, the most obvious – and perhaps the best understood – of which is the limited size of the historical database of losses of the client.

To make up for such uncertainty the analyst may use average, or typical, information from the market (the market risk premium) to replace or complement the client risk premium. The problem with this is that the market experience is not fully relevant to a particular client. This is usually captured by the spread, or heterogeneity, of the client risk premiums around the standard market rate. As an added complication, although the market rate is typically computed from a larger data set than that of a client, it, too, is based on a loss database of limited size and is therefore affected by the same type of uncertainty.

The considerations above apply both to direct insurance and to reinsurance, but the problems are felt more acutely with excess-of-loss (XL) reinsurance, as the data on large losses are scarcer. The existence of a layer structure adds one obvious difficulty to the pricing process: the accuracy with which we price each layer will typically decrease rapidly as a function of the layer excess. This is a consequence of relying on data from bottom layers to build a model that will be used to price the higher layers, beyond the limit for which our experience is relevant.

Given the scant supply of data that is typical of reinsurance, resorting to the market for an indication of rates is even more important. However, even at the market level the experience on large losses is limited and insufficient to price the higher layers of reinsurance accurately. Above a given layer excess the effect of uncertainty on the market reference rate may be comparable to the effects due to market heterogeneity.

The standard way to combine client and market information is credibility. The credibility risk premium is the convex combination of the client risk premium and the market risk premium:

Credibility risk premium = Z x Client risk premium+ (1-Z) x Market risk premium

where Z is a real number between 0 and 1, reflecting the relative weight that we give to the client's experience.

The idea of this paper is to use the standard deviation of the client risk premium estimator (σ_c) as a measure of (lack of) credibility, weighting this against the market heterogeneity (σ_h) and the standard deviation of the market risk premium estimator (σ_m). Furthermore, since the risk premium of the market is calculated based on data from the whole market, including in general the client itself, the two estimators for the market and the client are correlated ($\rho_{m,c}$). The resulting formula for the credibility factor (Proposition 1):

$$Z = \frac{\sigma_h^2 + \sigma_m^2 - \rho_{m,c}\sigma_m\sigma_c}{\sigma_h^2 + \sigma_m^2 + \sigma_c^2 - 2\rho_{m,c}\sigma_m\sigma_c} \tag{1.1}$$

can be easily generalised to be used for XL reinsurance pricing, by considering the value of the parameters for a specific layer (formula (4.1)). As a consequence, the credibility factor will depend on the layer. However, in the important special case where the severity distributions of the different clients can be assumed to be the same, the credibility factor defined as above is broadly constant across the layers (Propositions 2 and 3).

This methodology was applied to pricing UK motor XL reinsurance, which can be performed by modelling the frequency of large losses as Poisson and the loss amounts above a certain threshold as a Generalised Pareto distribution (GPD). For this application, a hybrid approach to credibility was found to be adequate, using the general uncertainty-based credibility for the lower layers and a single-severity distribution model for the higher layers.

1.1 Research Context and Objective

This paper presents a credibility methodology that we think is particularly appropriate for excess-of-loss reinsurance pricing, as it takes into account the uncertainty of the client and the market for different layers of reinsurance.

The modern approach to credibility – which stems from the works of Bühlmann and Straub (see Bühlmann [4]; Bühlmann & Straub [6] and the comprehensive book by Bühlmann & Gisler [5]) does not explicitly take the uncertainty on the market price into account in the formula for the credibility factor (see, e.g., theorem 3.7 in Bühlmann & Gisler [5], which gives results for both inhomogeneous and homogeneous credibility).

On the other hand Boor [3], who uses (as we do) uncertainty as a base for credibility, displays a credibility factor that contains an extra term for market uncertainty. This paper, however, focuses on a two-samples model (client v rest of the market) and attempts no analysis of the overall market heterogeneity/spread.

Credibility for excess-of-loss reinsurance was first examined by Straub [17]. This was extended by Patrik & Mashitz [13]. An implementation of this approach has been carried out by one of us for the UK motor reinsurance market [2].

All these works restrict their attention to the credibility of claim counts rather than considering aggregate losses, which are the real item of interest when pricing a reinsurance excess cover. Furthermore, these efforts have focused on the Poisson/Gamma credibility model applied to claim frequency.

An attempt to extend the ideas in [17] and [13] to provide a credibility formula for claim aggregate loss rather than claim frequency was made by Cockroft [7]. Cockroft provides a complex analytical solution involving infinite summations for the special case where the number of claims is Poisson with a Gamma prior distribution for the Poisson rate and the claim amounts are distributed according to a Pareto with a Gamma prior distribution for the power-law exponent.

Thus far, a simple general solution for calculating credibility for excess-of-loss reinsurance has not been provided in the literature. This paper argues that by using uncertainty as the main driver for credibility one is able to produce an intuitive and general method to calculate the credibility premium, which can be used both in insurance and in reinsurance.

1.2 Outline

Section 2 introduces a measure of uncertainty and outlines the various ways in which it can be calculated. Section 3 illustrates the methodology of uncertainty-based credibility in a general context, proving the basic result (Proposition 1) that gives the optimal value for the credibility factor. Section 4 illustrates the application of this result to reinsurance. Section 5 describes a real-world application of uncertainty-based credibility to pricing motor reinsurance. A detailed comparison with other methods is presented in Section 6. The limitations of the methodology are given in Section 7. Section 8 draws the conclusions.

2. THE RISK PREMIUM AND ITS UNCERTAINTY

2.1 Risk premium – definition and calculation

The risk premium φ is given by $\varphi = \frac{E(S)}{w}$ where $E(S)$ is the expected aggregate loss in a given period and w is the expected exposure in that same period.

Using the collective risk model assumption, the losses to an insurer in a given period can be modelled as a stochastic process $S = \sum_{i=1}^{N} X_i$ where N represents the number of losses in the period and $X_1,\ldots X_N$ represent their amounts. Both the number of losses and their amounts are random variables. The claims amounts $X_1,\ldots X_N$ are i.i.d. and independent of N.

Using the collective risk model, $E(S)$ can be written as $E(S) = E(N)E(X)$ where $E(N)$ is the expected number of claims and $E(X)$ is the expected claim amount. To derive $E(N)$ and $E(X)$, we need to know the underlying frequency and severity distributions with their exact parameter values (e.g., $N \sim Poi(\lambda w)$, $X \sim Exp(\mu)$ → $E(S) = \lambda\mu w$, $\varphi = \lambda\mu$).

However, the model is usually not so straightforward, since it is not always possible to express $E(S)$ in a simple analytical form. This may be due to policy modifications (excesses, limits, reinstatements…) and to the effect of settlement delay and discounting. Therefore, $E(S)$ will usually be appraised by a stochastic simulation or by an approximate formula.

2.2 Risk premium – sources and measures of uncertainty

In practice, we will only have an estimate of $E(S)$ and therefore of the risk premium. This estimate will be affected by several sources of uncertainty: the models for frequency and severity will not replicate reality perfectly (model uncertainty); the values of the model parameters will only be known approximately (parameter uncertainty); the data themselves are often reserve estimates rather than known quantities (data uncertainty).

Parameter uncertainty is the most important contribution to uncertainty and the one we will focus on in this paper. It depends on the fact that we only have a limited sample from which to estimate the parameters of the model. Data uncertainty has the effect of increasing parameter uncertainty; its effects, which can be studied by inspecting the IBNER distribution, will be analysed elsewhere [12]. Model uncertainty is difficult to quantify and will be usually dealt with in a low-profile fashion, by making sure that our models pass appropriate goodness-of-fit tests.

We will use the standard deviation as a measure of the uncertainty of an estimator. Although the standard deviation of an estimator is commonly denoted as "standard error", we will stick to the expression "standard deviation of the estimator" to avoid the ambiguity surrounding the term "standard error" in the literature[1].

We will refer to the standard deviation of the risk premium as shorthand for "the standard deviation of the estimator for the risk premium". In general, the standard deviation of the risk premium will therefore depend on the process by which the risk premium is estimated. Notice that the standard deviation of the risk premium estimator should not be confused with the standard deviation of S/w, the aggregate loss per unit of exposure!

Section 3.4.2 will give examples of how the standard deviation of the risk premium estimator can be calculated in practice.

3. UNCERTAINTY-BASED CREDIBILITY

Let φ_c be the "true" risk premium of the client. This is simply given by $\varphi_c = \dfrac{E(S_c)}{w_c}$ where $E(S_c)$ is the expected aggregate loss in a year and w_c is the exposure in the same year. According to the collective model, $E(S_c)$ can be written as $E(S_c) = E(N_c)E(X_c)$ where $E(N_c)$ is the expected number of claims and $E(X_c)$ is the expected claim amount. However, we will only have

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[1] As an example, "standard error" is used as "standard deviation of the estimator" or "estimated standard deviation of the estimator" depending on the author.

an estimate of $E(S_c)$. The goodness of this estimate will be affected by data uncertainty, parameter uncertainty and model uncertainty.

Let $\hat{\varphi}_c$ be the estimated risk premium of the client. This will typically be obtained by estimating the parameters of the frequency and severity distribution and by calculating the average frequency and severity based on those estimates. E.g., if frequency is a Poisson distribution: $N \sim Poi(\lambda \cdot w_c)$ and severity is an exponential distribution with (true) mean μ: $X \sim Exp(\mu)$, then the risk premium is given by $\hat{\varphi}_c = \hat{\lambda} \cdot \hat{\mu}$, where $\hat{\lambda}$ is the estimated rate per unit of exposure and $\hat{\mu}$ is the estimated mean of the exponential distribution.

We can also define φ_m (true risk premium) and $\hat{\varphi}_m$ (estimated risk premium) for the market. The estimated risk premium $\hat{\varphi}_m$ will be obtained in a similar fashion to $\hat{\varphi}_c$ but it will use data from all participating clients, including the data used to calculate $\hat{\varphi}_c$.

Credibility is a standard technique by which the estimated risk premium of the client, $\hat{\varphi}_c$, and the estimated risk premium for the market, $\hat{\varphi}_m$, are combined to provide another estimate $\hat{\varphi}$, called the credibility estimate, of the client's risk premium φ_c, via a convex combination:

$$\hat{\varphi} = Z \cdot \hat{\varphi}_c + (1 - Z) \cdot \hat{\varphi}_m \tag{3.1}$$

where $Z \in [0,1]$ is called the credibility factor.

In this section, we provide a means to calculate the credibility factor Z based on the uncertainty of the estimates $\hat{\varphi}_c$, $\hat{\varphi}_m$ and on the heterogeneity of the market. To do this we need an uncertainty model, i.e. a set of assumptions on how uncertainty affects the estimates.

3.1 The uncertainty model – Assumptions

1. The estimated risk premium of the market is described by a random variable $\hat{\varphi}_m$ with expected value φ_m (the true risk premium for the overall market) and variance σ_m^2. For readability, we write this as

$$\hat{\varphi}_m = \varphi_m + \sigma_m \varepsilon_m \tag{3.2}$$

where ε_m is a random variable with zero mean and unit variance: $E(\varepsilon_m) = 0$, $E(\varepsilon_m^2) = 1$. Notice that φ_m is *not* viewed as a random variable here. Despite the terminology above, which resembles that used for Gaussian random noise, no other assumption is needed on the shape of the distribution of ε_m.

2. The <u>true risk premium</u> φ_c of the client is described by a random variable with mean $E(\varphi_c) = \varphi_m$ (the true market risk premium) and variance $Var(\varphi_c) = \sigma_h^2$. In other terms,

$$\varphi_c = \varphi_m + \sigma_h \varepsilon_h \tag{3.3}$$

   where σ_h measures the spread (or heterogeneity) of the different clients around the mean market value, and $E(\varepsilon_h) = 0$, $E(\varepsilon_h^2) = 1$.

3. The <u>estimated risk premium</u> of the client, $\hat{\varphi}_c$, given the true risk premium, φ_c, is described by a random variable with mean $E(\hat{\varphi}_c \mid \varphi_c) = \varphi_c$, $Var(\hat{\varphi}_c \mid \varphi_c) = \sigma_c^2$. In other words,

$$\hat{\varphi}_c \mid \varphi_c = \varphi_c + \sigma_c \varepsilon_c \quad (\hat{\varphi}_c = \varphi_m + \sigma_h \varepsilon_h + \sigma_c \varepsilon_c) \tag{3.4}$$

   where ε_c is another random variable with zero mean and unit variance: $E(\varepsilon_c) = 0$, $E(\varepsilon_c^2) = 1$. Again, no other assumption is made on the distribution of ε_c. Notice that in this case both $\hat{\varphi}_c$ and φ_c are random variables.

4. Assume that ε_h is uncorrelated to both ε_m and ε_c: $E(\varepsilon_m \varepsilon_h) = 0$, $E(\varepsilon_c \varepsilon_h) = 0$.

We are now in a position to prove the following result.

Proposition 1. *Given assumptions 1-4 above, the value of* Z *that minimises the mean squared error* $E_{m,c,h}\left((\hat{\varphi} - \varphi_c)^2\right) = E_{m,c,h}\left((Z \cdot \hat{\varphi}_c + (1-Z) \cdot \hat{\varphi}_m - \varphi_c)^2\right)$, *where the expected value is taken on the joint distribution of* ε_m, ε_c, ε_h, *is given by*

$$Z = \frac{\sigma_h^2 + \sigma_m^2 - \rho_{m,c}\sigma_m\sigma_c}{\sigma_h^2 + \sigma_m^2 + \sigma_c^2 - 2\rho_{m,c}\sigma_m\sigma_c} \tag{3.5}$$

where $\rho_{m,c}$ *is the correlation between* ε_m *and* ε_c.

Proof. The result is straightforward once we express $\hat{\varphi} - \varphi_c$ in terms of ε_m, ε_c, ε_h only. The mean squared error is given by

$$E_{m,c,h}\left((Z \cdot \hat{\varphi}_c + (1-Z)\cdot \hat{\varphi}_m - \varphi_c)^2\right) = E_{m,c,h}\left(((Z-1)(\sigma_h \varepsilon_h - \sigma_m \varepsilon_m) + Z \cdot \sigma_c \varepsilon_c)^2\right)$$
$$= (Z-1)^2(\sigma_h^2 + \sigma_m^2) + Z^2 \sigma_c^2 - 2Z(Z-1)\rho_{m,c}\sigma_m\sigma_c \; .$$

where $\rho_{m,c} = E(\varepsilon_m \varepsilon_c)$. By minimising with respect to *Z* one obtains equation (3.5). □

The following sections will go into more detail as to the meaning of the assumptions and of this result.

3.1.1 Explaining the assumptions

Assumption 2 tries to capture market heterogeneity: different clients will have different risk premiums, reflecting the different riskiness of the accounts. This is similar to the risk factor in Bayesian and Buhlmann's approach to credibility. We do not need to know what the prior distribution of the risk premiums is, as long as we know its variance. In practice, this will be determined empirically.

Assumptions 1 and 3 try to capture the uncertainty inherent in the process of estimating the risk premium. The quantities σ_m and σ_c should not be confused with the standard deviation of the underlying aggregate loss distribution for the market and the client.

The random variable ε_h gives the prior distribution of the client price around a market value, whereas ε_m, ε_c are parameter uncertainties on the market and the client. Therefore, assumption 4 ($E(\varepsilon_m \varepsilon_h) = 0$, $E(\varepsilon_c \varepsilon_h) = 0$) is quite sound. The correlation between ε_m and ε_c, however, cannot be ignored. The reason for this is that the estimated risk premium of the market is based on data collected from different clients, including client *c*.

3.2 Is $\hat{\varphi}$ an unbiased estimator for φ_c?

It is important to notice that the expected value $E_{m,c,h}\left((\hat{\varphi} - \varphi_c)^2\right)$ is also taken over the distribution of ε_h. As a consequence, the mean squared error is not necessarily minimised for each individual client, but only over all possible clients.

For a given client *c*, $\hat{\varphi}$ is in general a biased estimator for φ_c. The bias is given by $bias(\hat{\varphi} \mid \varphi_c) = E_{m,c}(\hat{\varphi} \mid \varphi_c) - \varphi_c = (1-Z)(\varphi_m - \varphi_c) = -(1-Z)\sigma_h \varepsilon_h$. The expected value is in this case taken over the joint distribution of ε_m and ε_c. Averaging over ε_h, the bias disappears: $E_h(bias(\hat{\varphi} \mid \varphi_c)) = 0$.

Notice how the quest for an estimate $\hat{\varphi}$ of φ_c that is *collectively* unbiased is a common feature of credibility theory (see for example Bühlmann's approach as described in the book by Klugman et al. [10]).

The meaning of the formula for the bias, $bias(\hat{\varphi} \mid \varphi_c) = -(1-Z)\sigma_h \varepsilon_h$, is that when credibility is close to 1, the credibility estimate for the risk premium will be close to the client estimated price, $\hat{\varphi}_c$, and the bias will be close to zero. On the other hand, if the credibility is close to 0, the credibility estimate of the risk premium will be close to $\hat{\varphi}_m$, and the bias will be about $\sigma_h \varepsilon_h$ – i.e., the credibility estimate will be distributed randomly around the market risk premium with a standard deviation equal to σ_h – which is exactly what we expect to happen.

3.3 The effect of correlation

- In case the data for client *c* are not included in the market data set the correlation between ε_m and ε_c can be assumed to be zero. In this case the credibility factor simplifies to

$$Z = \frac{\sigma_h^2 + \sigma_m^2}{\sigma_h^2 + \sigma_m^2 + \sigma_c^2} \qquad (3.6)$$

  which is more intuitive than (3.5). This also suggests an alternative way to carry out the credibility calculations: for each client, first remove the client's data from the market database and then calculate $\hat{\varphi}$ as $\hat{\varphi} = Z \cdot \hat{\varphi}_c + (1-Z) \cdot \hat{\varphi}_{m-c}$, where $Z = \dfrac{\sigma'^2_h + \sigma^2_{m-c}}{\sigma'^2_h + \sigma^2_{m-c} + \sigma^2_c}$ (notice how the market heterogeneity itself, σ_h, has to be recalculated). However, this methodology is more lengthy and awkward than that implied by (3.5), as the rest-of-the-market parameters need to be recalculated for each client.

The effect of a positive correlation between ε_m and ε_c is to increase the credibility factor. This makes sense intuitively, as a larger correlation indicates a larger participation of the client in the market loss database. As a consequence, the market data will provide less useful information to that client.

- Note that the condition $Z \le 1$ can be translated into $\rho_{m,c} \dfrac{\sigma_m}{\sigma_c} \le 1$. As $\rho_{m,c} \le 1$, this is automatically satisfied if $\dfrac{\sigma_m}{\sigma_c} \le 1$, which will hold under non-pathological circumstances as it is normally the case that $\sigma_m << \sigma_c$, the market estimate being based on a far larger sample.

- Note also that if $\rho_{m,c} < \min\left\{\frac{\sigma_m^2+\sigma_h^2}{\sigma_m\sigma_c}, \frac{\sigma_m^2+\sigma_h^2+\sigma_c^2}{2\sigma_m\sigma_c}\right\}$ the credibility factor is guaranteed to be positive. Under non-pathological circumstances, $\sigma_m < \sigma_c$ (the market has a larger sample than the client) and $\sigma_c < \sigma_h$ (the uncertainty on the risk premium is smaller than the spread of prices across the market, otherwise it would make no sense to use the client risk premium at all). Therefore, both ratios inside the bracket are larger than 1 and the inequality above is automatically satisfied.

3.4 Practical considerations

In practice, the standard deviations σ_h, σ_m, σ_c and $\rho_{m,c}$ are not known and they must be estimated from the data. Therefore the credibility factor can also be written as:

$$Z \approx \frac{s_h^2 + s_m^2 - r_{m,c}s_m s_c}{s_h^2 + s_m^2 + s_c^2 - 2r_{m,c}s_m s_c} \tag{3.7}$$

where s_h is the *estimated* market heterogeneity, $r_{m,c}$ is the *estimated* correlation between the market and the client, s_m and s_c are the *estimated* standard deviations of the estimators for the market and client risk premiums.

3.4.1 Estimating market heterogeneity

Market heterogeneity can be estimated as the empirical variance of the risk premium for all available clients. This may be done in a weighted or in a non-weighted fashion. If the market premium is calculated by collecting all data from all clients, larger clients will inevitably weigh more, and the weighted version of the variance will have to be used for consistency:

$$s_h^2 = \frac{\sum_c W_c}{\left(\sum_c W_c\right)^2 - \sum_c W_c^2} \sum_c W_c \left(\hat{\varphi}_c - \hat{\varphi}_m\right)^2 \tag{3.8}$$

where $W_c = \sum_j w_c^j$ is the cumulative exposure of client *c* over all years *j* considered in the analysis.

3.4.2 Estimating the standard deviation of the risk premium estimator

As mentioned in Section 2.2, the standard deviation on the risk premium depends on the process by which the risk premium is calculated. This is best explained with the following simple example.

Suppose the frequency distribution is modelled as a Poisson process whose estimated rate is $\hat{\lambda} w_c$ and the severity distribution is modelled as an exponential distribution whose estimated mean is $\hat{\mu}$. The estimated risk premium will then be $\hat{\varphi} = \hat{\lambda}\hat{\mu}$. The standard error on $\hat{\varphi}$ will depend on the standard deviation of the estimators $\hat{\lambda}$ and $\hat{\mu}$: in this case we have the exact result $\frac{Var(\hat{\varphi})}{E(\hat{\varphi})^2} = \frac{Var(\hat{\lambda})}{E(\hat{\lambda})^2} + \frac{Var(\hat{\mu})}{E(\hat{\mu})^2} + \frac{Var(\hat{\lambda})Var(\hat{\mu})}{E(\hat{\lambda})^2 E(\hat{\mu})^2}$. The values of $Var(\hat{\lambda})$ and $Var(\hat{\mu})$ depend in turn on how the distribution parameters are calculated, and the expected values in the denominators will usually be approximated by their estimated value: $E(\hat{\varphi})^2 \approx \hat{\varphi}^2$, etc. E.g., if the mean of the exponential distribution is calculated by MLE based on the data sample $\{X_1, \ldots X_n\}$, then $\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n}$ and $Var^{est}(\hat{\mu}) = \frac{\hat{\mu}^2}{n}$ (there are two approximations here: one is the replacement of μ with $\hat{\mu}$ in the formula and the fact that this formula is only true asymptotically).

Usually, we cannot find an exact formula for $Var(\hat{\varphi})$. This may happen for two reasons.

- Except for very simple cases such as that illustrated above, the formula linking $\hat{\varphi}$ to the severity and frequency parameters will be too complex to propagate the uncertainties on the parameters exactly. In this case the standard deviation of $\hat{\varphi}$ can be estimated by drawing at random from the distribution of the parameters, which in the case of MLE is asymptotically known to be normal (or rather, multivariate normal). The correlations between the parameters must be taken into account. See Section 5.2 for a detailed example of how this is achieved.

- There might not even exist an analytical formula for $\hat{\varphi}$. This will often be the case when there are payment and settlement delays, complicated structures (excesses, limits, premium adjustments, premium reinstatements after a claim (reinsurance), etc). In this case $\hat{\varphi}$ may have to be estimated by a stochastic simulation. The stochastic simulation will then have to be run for different values of the parameters, according to the parameter distribution. As a consequence, the estimation of the standard deviation of $\hat{\varphi}$ will have a far larger computational complexity.

3.4.3 Estimating the correlation

How the correlation between the uncertainty on the client and on the market is calculated depends on the exact process to calculate the risk premium. The following is a simple example assuming that the risk premium is calculated in a burning cost fashion. This works by dividing the estimated total losses over a base period (typically, at least 10 years of data for long-tail business such as liability) by the total exposure over that period. The estimated total losses over the base period are corrected for inflation, IBNR, IBNER, etc.

Notice that we are interested in calculating the correlation between the *uncertainty* on φ_m and φ_c, therefore we can assume that φ_m and φ_c are fixed. Let $\hat{T}_m$ be the total losses for the market, and $\hat{T}_c$ the total losses for the client. Since the client is part of the market, we shall have $\hat{T}_m = \hat{T}_c + \hat{T}_{m-c}$ where $\hat{T}_{m-c}$ represents the losses of the rest of the market. The risk premium for the market and the client are defined respectively as $\hat{\varphi}_m = \frac{\hat{T}_m}{\sum_j w_m^j}$ and $\hat{\varphi}_c = \frac{\hat{T}_c}{\sum_j w_c^j}$, where w_c^j and w_m^j are respectively the client and market exposures in year *j*, the year index ranging over the base period. If we assume that[2] $Cov(\hat{T}_c, \hat{T}_{m-c}) = 0$, then $Cov(\hat{\varphi}_c, \hat{\varphi}_m) = \frac{Var(\hat{T}_c)}{\sum_j w_m^j \sum_j w_c^j}$ and $r_{m,c} = \sqrt{\frac{Var(\hat{T}_c)}{Var(\hat{T}_m)}}$. In the case of a compound Poisson distribution, this translates into:

$$r_{m,c} = \sqrt{\frac{\hat{\lambda}_c \sum_j w_c^j \left\langle X_c^2 \right\rangle}{\hat{\lambda}_m \sum_j w_m^j \left\langle X_m^2 \right\rangle}} \qquad (3.9)$$

Even when the risk premium is not obtained by the burning cost approach, this formula is still a good guidance as to the degree of correlation one may expect.

3.4.4 Updating the market statistics

Generally speaking, the client risk premium will have to be calculated at different times for different clients. Furthermore, once all contributors to the market have been priced there will still be a time lag between when the data for all clients are available and when the market-related statistics

[2] Notice that we are assuming φ_m and φ_c to be given (see Assumptions 1 and 3 of the credibility model in Section 3.1), therefore we can ignore the correlation of the aggregate losses of the client v the rest of the market, which of course exists (and motivates the credibility approach). We are focusing here on the correlation between the residual variations that exist because of parameter uncertainty.

(heterogeneity, uncertainty) are calculated. Typically, we will be comparing last year's market statistics with this year's clients. Therefore, the comparison will not be exactly like-for-like. At the very least, we will have to correct the market results for residual inflation.

4. APPLICATION TO REINSURANCE

In the case of excess-of-loss reinsurance, the quantity to be estimated will be $\varphi_c^{D,L}$, the "true" risk premium for layer $(D, D+L)$. Equation (3.1) can be rewritten as: $\hat{\varphi}^{D,L} = Z^{D,L} \cdot \hat{\varphi}_c^{D,L} + (1 - Z^{D,L}) \cdot \hat{\varphi}_m^{D,L}$, where:

- $\hat{\varphi}_c^{D,L}$, $\hat{\varphi}_m^{D,L}$ are the expected losses per unit of exposure for layer $(D, D+L)$ for the client and the market respectively;
- $Z^{D,L} \in [0,1]$ is the credibility factor for the client for layer $(D, D+L)$.

The problem here is to determine the value of $Z^{D,L}$ that minimises $E\left(\left(\hat{\varphi}^{D,L} - \varphi_c^{D,L}\right)^2\right) = E\left(\left(Z^{D,L} \cdot \hat{\varphi}_c^{D,L} + (1 - Z^{D,L}) \cdot \hat{\varphi}_m^{D,L} - \varphi_c^{D,L}\right)^2\right)$. By the same expansion of the mean squared error shown in the proof of Proposition 1, it is straightforward to show that the credibility factor for the layer (D, $D+L$) is

$$Z^{D,L} = \frac{\left(\sigma_h^{D,L}\right)^2 + \left(\sigma_m^{D,L}\right)^2 - \rho_{m,c}^{D,L}\sigma_m^{D,L}\sigma_c^{D,L}}{\left(\sigma_h^{D,L}\right)^2 + \left(\sigma_m^{D,L}\right)^2 + \left(\sigma_c^{D,L}\right)^2 - 2\rho_{m,c}^{D,L}\sigma_m^{D,L}\sigma_c^{D,L}} \tag{4.1}$$

where $Z^{D,L} \in [0,1]$.

The crucial question about credibility applied to reinsurance is the behaviour of the credibility risk premium – and therefore of $Z^{D,L}$ – as a function of the layer's characteristics. Since the most important dependency is that on the layer excess, D, the dependency on the layer limit can be removed by considering either infinitesimal layers ($Z^{D,0} = \lim_{L \to 0^+} Z^{D,L}$) or infinite layers ($Z^{D,\infty} = \lim_{L \to \infty} Z^{D,L}$).

The behaviour of $Z^{D,L}$ as a function of D and L depends on a combination of factors including:

a) The relative size of uncertainty for market and client;

b) The asymptotic behaviour of market severity v client severity;

c) How the market heterogeneity $\sigma_h^{D,L}$ depends on D, L.

The effect of this combination is quite complex and must in general be investigated empirically, as a general analytical expression will not always be available. Furthermore, estimating market heterogeneity for the highest layers is difficult, because market heterogeneity for a given layer is calculated from the expected aggregate losses of each client to that layer (as in (3.8)), and these are themselves affected by a large error. There is, however, a special case that is significant for the practitioner for which this behaviour simplifies. This is illustrated in the next section.

4.1 The "single severity" model

An important special case is obtained under the hypothesis that – although the frequency of large losses depends on the risk profile of the insurance company (e.g. age, sex), the severity distribution is unaffected by it, and the market severity curve can be used instead. As a consequence, market heterogeneity will be mostly due to heterogeneity in frequency. Empirical evidence supports this hypothesis for some kinds of portfolio, e.g. for the motor reinsurance portfolio, at least above a certain threshold (see Section 5). This also reflects general reinsurance practice where a market reference curve is used for most lines of business to price the higher layers.

The basic result (Proposition 2) is for the case where the market severity curve is known with infinite accuracy. Proposition 3 will then consider the amendments in the case where the market severity curve is known with limited accuracy.

Proposition 2 – Basic single severity model. . Let $\varphi_c^{D,L}$, $\hat{\varphi}_c^{D,L}$, $\hat{\varphi}_m^{D,L}$ and $\hat{\varphi}^{D,L}$ be as above. *Assume the validity of the collective risk model, and that:*

i. *The severity curve is the same for all clients (i.e., the market severity curve) above a threshold* μ*;*

ii. *the severity distribution of the market is known with infinite accuracy*

Then the credibility factor $Z^{D,L}$ *is independent of the layer* $(D, D+L)$ *and is equal to:*

$$Z = \frac{\left(\sigma_h^{\lambda}\right)^2 + \left(\sigma_m^{\lambda}\right)^2 - \rho_{m,c}^{\lambda}\sigma_c^{\lambda}\sigma_m^{\lambda}}{\left(\sigma_h^{\lambda}\right)^2 + \left(\sigma_m^{\lambda}\right)^2 + \left(\sigma_c^{\lambda}\right)^2 - 2\rho_{m,c}^{\lambda}\sigma_c^{\lambda}\sigma_m^{\lambda}} \tag{4.2}$$

where σ_h^{λ} *measures the heterogeneity of clients' frequencies;* σ_m^{λ} and σ_c^{λ} *are the standard deviations of the estimators of the market and the client frequency respectively;* $\rho_{m,c}^{\lambda}$ *is the correlation between the estimator of* λ_m *and that of* λ_c .

Proof. We need to calculate $\sigma_m^{D,L}$, $\sigma_c^{D,L}$, $\sigma_h^{D,L}$ and $\rho_{m,c}^{D,L}$ in general formula (4.1). Under the collective risk model applied to the losses for layer $(D, D+L)$, the mean aggregate loss is

$E(S^{D,L}) = E(N^{D,L})E(X^{D,L})$ where $E(N^{D,L})$ is the expected number of losses to the layer $(D, D+L)$ and $E(X^{D,L})$ is the expected loss amount to the layer $(D, D+L)$ *given that a loss to that layer has occurred.* As it is well known (see, e.g., [10]), $E(N^{D,L}) = E(N)\cdot \text{Prob}(D \le X < D+L)$ and $E(X^{D,L}) = \dfrac{E(X \wedge (D+L)) - E(X \wedge D)}{\text{Prob}(D \le X < D+L)}$, where $X \wedge a := \min(X, a)$; $E(N)$ is the expected number of losses above μ; $E(X)$ is the expected amount of those losses above μ. One can then write $E(S^{D,L}) = E(N)\cdot\big(E(X \wedge (D+L)) - E(X \wedge D)\big)$. The risk premium for layer $(D, D+L)$ can then be written as $\varphi^{D,L} = \lambda \cdot U^{D,L}$, where $\lambda = E(N)/w$ is the expected frequency per unit of exposure above $\mu \le D$, and

$$U^{D,L} = E(X \wedge D+L) - E(X \wedge D) = \int_D^{D+L} S(x)dx \tag{4.3}$$

Since the severity curve above $\mu \le D$ is the same for the client and the market, the risk premium of the client and the market can be expressed respectively as $\varphi_c^{D,L} = \lambda_c \cdot U^{D,L}$ and $\varphi_m^{D,L} = \lambda_m \cdot U^{D,L}$. The estimated risk premium for the client (the market) can be expressed as $\hat{\varphi}_c^{D,L} = \hat{\lambda}_c \cdot U^{D,L}$ ($\hat{\varphi}_m^{D,L} = \hat{\lambda}_m \cdot U^{D,L}$) respectively, where $\hat{\lambda}_c$ ($\hat{\lambda}_m$) is the estimated client (market) frequency.

The severity distribution is known with infinite accuracy. Therefore,

$$\begin{aligned}
\left(\sigma_m^{D,L}\right)^2 &= \text{Var}(\hat{\varphi}_m^{D,L}) = \text{Var}(U^{D,L}\cdot\hat{\lambda}_m) = \left(U^{D,L}\right)^2\cdot\left(\sigma_m^{\lambda}\right)^2 \\
\left(\sigma_c^{D,L}\right)^2 &= \text{Var}(\hat{\varphi}_c^{D,L}) = \text{Var}(U^{D,L}\cdot\hat{\lambda}_c) = \left(U^{D,L}\right)^2\cdot\left(\sigma_c^{\lambda}\right)^2 \\
\left(\sigma_h^{D,L}\right)^2 &= \text{Var}(\varphi_c^{D,L}) = \text{Var}(U^{D,L}\cdot\lambda_c) = \left(U^{D,L}\right)^2\cdot\left(\sigma_h^{\lambda}\right)^2 \\
\rho_{m,c}^{D,L} &= \text{Cov}(\hat{\varphi}_m^{D,L},\hat{\varphi}_c^{D,L})\Big/\sqrt{\text{Var}(U^{D,L}\cdot\hat{\lambda}_m)\cdot\text{Var}(U^{D,L}\cdot\hat{\lambda}_c)} = \rho_{m,c}^{\lambda}
\end{aligned} \tag{4.4}$$

and $\left(U^{D,L}\right)^2$ can be removed from both the numerator and the denominator of (4.1), yielding (4.2).

$\square$

Discussion of the assumptions. The collective risk model is a standard assumption. The assumption (i) of a single severity curve for all clients above a certain threshold underlies common reinsurance practice for the pricing of high layers. This does not mean that it is always reasonable, and should be tested against available data when possible. In Section 5 the validity of this assumption will be illustrated in the case of UK motor reinsurance.

Assumption (ii) is not realistic as the severity curve of the market is always estimated based on a finite set of data and therefore it is affected by model, data and parameter uncertainty. However, Assumption (ii) is often a useful approximation when the uncertainty for the market severity is small – i.e., for all but the top layers. Even if this assumption only holds approximately, it shows that when all clients follow the same single severity curve, the credibility factor is broadly independent of the specific layer being priced.

The following proposition explores what happens when Assumption (ii) is dropped and the inaccuracy of the market severity curve is taken into account.

Proposition 3 – Single severity model with "inaccurate" severity distribution. *Let* $\varphi_c^{D,L}$, $\hat{\varphi}_c^{D,L}$, $\varphi_m^{D,L}$, $\hat{\varphi}_m^{D,L}$, $\hat{\varphi}^{D,L}$, $\hat{\lambda}_c$, $\hat{\lambda}_m$ *and* $U^{D,L}$ *be as in Proposition 2 and in its proof. Also, let* $\hat{U}^{D,L}$ *bet the estimate of* $U^{D,L}$. *Assume the validity of the collective risk model, and assume that the severity curve is the same for all clients (i.e., the market severity curve) above a threshold* $\mu \le D$.

Then the credibility factor $Z^{D,L}$ *is equal to:*

$$Z = \frac{\left(\sigma_h^\lambda\right)^2 + \left(\sigma_m^\lambda\right)^2 - \rho_{m,c}^\lambda \sigma_c^\lambda \sigma_m^\lambda + \dfrac{\mathrm{Var}(\hat{U}^{D,L})}{\left(U^{D,L}\right)^2}\left(\lambda_m^2 - \lambda_m\lambda_c + \left(\sigma_m^\lambda\right)^2 - \rho_{m,c}^\lambda \sigma_h^\lambda \sigma_m^\lambda\right)}{\left(\sigma_h^\lambda\right)^2 + \left(\sigma_m^\lambda\right)^2 + \left(\sigma_c^\lambda\right)^2 - 2\rho_{m,c}^\lambda \sigma_c^\lambda \sigma_m^\lambda + \dfrac{\mathrm{Var}(\hat{U}^{D,L})}{\left(U^{D,L}\right)^2}\left(\lambda_m^2 + \lambda_c^2 - \lambda_m\lambda_c + \left(\sigma_m^\lambda\right)^2 + \left(\sigma_c^\lambda\right)^2 - \rho_{m,c}^\lambda \sigma_h^\lambda \sigma_m^\lambda\right)} \quad (4.5)$$

where $\mathrm{Var}(\hat{U}^{D,L})$ *is the variance of the estimator* $\hat{U}^{D,L}$ *for* $U^{D,L}$, and σ_h^λ, σ_m^λ, σ_c^λ, $\rho_{m,c}^\lambda$ *are as in Proposition 2*.

Proof (outline). The proof goes as for Proposition 2, but remembering the relationship $1 + CV^2(XY) = (1 + CV^2(X))(1 + CV^2(Y))$ (*CV*(*X*) being the coefficient of variation of *X*) when expanding $\left(\sigma_m^{D,L}\right)^2, \left(\sigma_c^{D,L}\right)^2, \left(\sigma_h^{D,L}\right)^2$: e.g., $\dfrac{\left(\sigma_m^{D,L}\right)^2}{\left(\varphi_m^{D,L}\right)^2} = \dfrac{\mathrm{Var}(\hat{U}^{D,L})}{(U^{D,L})^2} + \dfrac{\left(\sigma_m^\lambda\right)^2}{\lambda_m^2} + \dfrac{\mathrm{Var}(\hat{U}^{D,L})}{(U^{D,L})^2}\dfrac{\left(\sigma_m^\lambda\right)^2}{\lambda_m^2}$.

Furthermore, it should be noticed that

$$\mathrm{Cov}(\hat{\varphi}_m^{D,L}, \hat{\varphi}_c^{D,L}) = \mathrm{Var}(\hat{U}^{D,L})\left(\mathrm{Cov}(\hat{\lambda}_m, \hat{\lambda}_c) + \lambda_m\lambda_c\right) + (U^{D,L})^2 \cdot \mathrm{Cov}(\hat{\lambda}_m, \hat{\lambda}_c),$$

which is different from zero even when $\rho_{m,c}^\lambda = 0$. □

Comments on Proposition 3.

- When $\frac{\mathrm{Var}(\hat{U}^{D,L})}{\left(U^{D,L}\right)^2} << 1$ (bottom layers) the credibility factor will be roughly as predicted by Proposition 2.
- However, when the standard error on the estimator of the risk premium for a layer is comparable with the risk premium itself for that layer, the credibility factor is distorted. In the limit for which $\frac{\mathrm{Var}(\hat{U}^{D,L})}{\left(U^{D,L}\right)^2} \to \infty$, the credibility factor will tend to a limit independent of frequency heterogeneity: $Z^{D,L} \to \frac{\lambda_m^2 - \lambda_m \lambda_c + \left(\sigma_m^\lambda\right)^2 - \rho_{m,c}^\lambda \sigma_h^\lambda \sigma_m^\lambda}{\lambda_m^2 + \lambda_c^2 - \lambda_m \lambda_c + \left(\sigma_m^\lambda\right)^2 + \left(\sigma_c^\lambda\right)^2 - \rho_{m,c}^\lambda \sigma_h^\lambda \sigma_m^\lambda}$. The exact value of this limit is of little significance. What *is* important about this is the practical message that beyond a certain value D^* of the excess, which might arbitrarily be set to that for which $Var(\hat{U}^{D^*,0}) = \left(U^{D^*,0}\right)^2$ (we call this the *pricing horizon*), the uncertainty of both the client and the market estimates becomes overwhelming and the credibility estimate is of little relevance.
- As for Proposition 1, the credibility factors of Proposition 2 and 3 are theoretical credibility factors, and in practice the values of σ_h^λ, σ_m^λ, σ_c^λ, $\rho_{m,c}^\lambda$ λ_m, λ_c, $U^{D,L}$ will have to be replaced by their estimated counterparts: s_h^λ, s_m^λ, s_c^λ, $r_{m,c}^\lambda$ $\hat{\lambda}_m$, $\hat{\lambda}_c$, $\hat{U}^{D,L}$. The estimation of s_h^λ, s_m^λ, s_c^λ, $r_{m,c}^\lambda$ proceeds as explained in Section 3.4.1 to 3.4.3. Specifically, the correlation can be written as $r_{m,c}^\lambda = \sqrt{\frac{\hat{\lambda}_c \sum_j w_c^j}{\hat{\lambda}_m \sum_j w_m^j}}$: notice how the term related to the average severity has disappeared.

4.2 Hybrid models

In many cases, a hybrid model will be needed, which uses a full uncertainty model (as per Proposition 1) for the bottom layers and a single-severity model (Proposition 2 and 3) for the higher layers. There is no conceptual difficulty in doing this, but it is crucial to deal adequately with how the transition from one method to the other affects the uncertainties.

Specifically, assume the severity distribution of a client is given by

$$F(x) = \begin{cases} F_c(x) & \text{for } \mu \le x \le \mu' \\ F_c(\mu') + \left(1 - F_c(\mu')\right) \cdot F_m(x) & \text{for } x > \mu' \end{cases} \tag{4.6}$$

where $F_c(x)$ is the cumulative distribution which is specific to the client and $F_m(x)$ is the market severity curve, defined above μ' and such that $F_m(\mu')=0$, $F_m(\infty)=1$.

When this is the case, the risk premium for the layer $(D, D+L)$ with $D \geq \mu'$ is given by

$$\begin{aligned}\varphi_c^{D,L} &= \lambda_{\geq\mu} \cdot \int_D^{D+L} (1-F\ (x))dx = \\ &= \lambda_{\geq\mu} \cdot \left(1-F_x(\mu')\right) \cdot \int_D^{D+L} (1-F_m(x))dx = \\ &= \lambda_{\geq\mu'} \cdot \int_D^{D+L} (1-F_m(x))dx\end{aligned} \tag{4.7}$$

where $\lambda_{\geq\mu}$ is the frequency above μ and $\lambda_{\geq\mu'}$ is the frequency above μ' (both per unit of exposure). As a consequence, the uncertainty on the risk premium $\varphi_c^{D,L}$ depends on the uncertainty on $\lambda_{\geq\mu}$ and on the uncertainty on the parameters of both $F_c(x)$ and $F_m(x)$. The compound effect of these uncertainties is best estimated by stochastic simulation, except in the most trivial cases.

5. A REAL-WORLD APPLICATION: PRICING MOTOR REINSURANCE IN THE UK

We have applied the uncertainty-based credibility methodology to pricing motor reinsurance in the UK, based on a sample of 25 clients (about 70% of the UK market share in terms of premium).

5.1 The pricing process

Losses were first revalued according to an appropriate claim inflation rate (see, e.g., the study by Swiss Re [18]). Pricing was then carried out by considering a collective risk model where the frequency is a Poisson process and the severity is a Generalised Pareto distribution (GPD), with a cumulative distribution function equal to $F(x) = 1-\left(1+\xi(x-\mu)/\sigma\right)^{-1/\xi}$. The choice of the GPD as the distribution for modelling severity is justified by the Pickands-Balkema-de Haan theorem ([14], [1]), according to which under broad conditions the losses above a certain threshold converge in the distribution sense to a GPD.

Using this model, the risk premium for layer $(D, D+L)$ is

$$\varphi_c^{D,L} = \frac{\lambda\sigma}{1-\xi}\left(\left(1+\xi/\sigma\,(D-\mu)\right)^{1-1/\xi} - \left(1+\xi/\sigma\,(D+L-\mu)\right)^{1-1/\xi}\right) \tag{5.1}$$

where $\lambda = E(N)/w$ is the expected frequency per unit of exposure above $\mu \le D$. This can be easily proven by writing $\varphi_c^{D,L} = \lambda \cdot \int_D^{D+L} (1 - F(x))dx$ as in the proof of Proposition 3 and setting $1 - F(x) = \left(1 + \xi(x-\mu)/\sigma\right)^{-1/\xi}$.

5.2 Calculating the uncertainties

Determining an estimate $\hat{\lambda}$ of the Poisson rate, λ, is quite complex as motor insurance has a long-tail component (bodily injury claims) and the number of claims above a certain threshold for a given year is known accurately only after all claims for that year have been settled. As a consequence, claim count projection techniques such as chain ladder or Bornhuetter-Fergusson must be used. The uncertainty on λ, σ_λ, depends on the errors of the chain ladder estimates (Mack [11]; Renshaw & Verrall [15]) for each individual year and on the errors involved in the regression analysis to fit the results for the different years. The distribution of $\hat{\lambda}$ can be roughly considered normal ($\hat{\lambda} \sim N(\lambda, \sigma_\lambda)$) although positive-definite distributions such as Gamma may be more appropriate.

The values of ξ, σ for the GPD can be estimated using maximum-likelihood based on the revalued losses database. The uncertainties are (asymptotically) normally distributed: $(\hat{\xi}, \hat{\sigma}) \sim N((\xi, \sigma), \Sigma)$, where the covariance matrix can be estimated as

$$\Sigma = \frac{(1+\xi)}{n}\begin{pmatrix} (1+\xi) & -\sigma \\ -\sigma & 2\sigma^2 \end{pmatrix} \approx \frac{(1+\hat{\xi})}{n}\begin{pmatrix} (1+\hat{\xi}) & -\hat{\sigma} \\ -\hat{\sigma} & 2\hat{\sigma}^2 \end{pmatrix} \tag{5.2}$$

n being the size of the loss database (Smith [16]; Embrechts et al. [8]). Notice that the uncertainties on ξ and σ are (negatively) correlated.

By drawing random instances of $\hat{\lambda}, \hat{\xi}, \hat{\sigma}$ from the distributions above, we obtain the (indirect) sampling distribution for $\hat{\varphi}_c^{D,L}$ and we can estimate the standard deviation $\sigma_c^{D,L}$ of the risk premium estimator. (There are other uncertainties, such as that on claim inflation, that are not client-specific and are best addressed by sensitivity analysis.)

In practice, one finds that both the frequency estimation and the severity estimation are subject to very large parameter uncertainty and contribute significantly to the overall value of $\sigma_c^{D,L}$. One also finds that the distribution of $\hat{\varphi}_c^{D,L}$ is approximately normal for the bottom layers and significantly skewed for the higher layers.

5.3 Credibility pricing

A hybrid model for credibility pricing was adopted, which:

- uses the client severity distribution up to £2m, modelled as a GPD (using the GPD model is not critical in that region, and simpler models such as the single-parameter Pareto distribution can be also used);
- uses the market severity distribution above £2m, again modelled as a (different) GPD.

The rationale behind this model is described below. Notice that these results reflect the situation in 2007, with loss data from the latest 10 years.

The hypothesis that there is a single severity curve (the market severity curve) that fits the empirical data of all clients was tested for data above different thresholds: £1m, £2m, £3m. Goodness of fit was tested using the two-sample Kolmogorov-Smirnov test for each client. This test calculates the K-S distance between the empirical distribution of a client and the empirical distribution of the whole market after removing the client's data, and compares this distance with the critical value for a chosen confidence value (see, e.g., Gibbons & Chakraborti [9]). The results of this test are summarised in Table 1.

Analysis Level (£M)	No. of datasets failing test	No. of datasets in test
1	5	18
2	2	21
3	2	23

Table 1 – The number of data sets failing the two-sample KS test as compared to the total number of samples in the set. Notice that the total number of samples varies with the analysis level as for some of the clients the reporting level is too high for an analysis level of, say, £1m to be possible.

The results indicate that while the severity curve of different clients differ significantly above £1m, *the single severity curve hypothesis is valid for a threshold of £2m or above*.

This hybrid approach recognises that in the UK motor reinsurance market there are, broadly speaking, three regions of behaviour:

I. A "bottom" region (from the lowest excess up to £2m) where clients are quite different as to frequency and severity, and credibility generally decreases with the layer excess (assuming infinitesimal layers).

II. A "middle" region (from £2m to the market pricing horizon) where clients are assumed to have different frequencies but the same severity distribution. In this region, which extends up to the market pricing horizon (~£20-30m), credibility is broadly independent of excess.

III. A "top" region that lies beyond the market pricing horizon. In this region, little can be said about the client price, except perhaps providing a broad upper bound to it, and credibility is of little help because both the client price and the market price are far too inaccurate to gain much accuracy by their combination.

5.4 Practical issues

In practice, since motor liability is a long-tail business for which bodily injury claims are reported and settled with considerable delay, the risk premium will usually be amended to take into account the time value of money. Specifically,

- losses are usually discounted to take into account the return on investment on the technical reserves between the accident date and the payment/settlement of claims;
- layers' excesses and limits are usually indexed by earnings inflation. This mechanism is commonly used by reinsurers to avoid excessive gearing effects due to claims inflation.

The effect of these modifications will usually have to be assessed by running a stochastic model, as an exact formula such as (5.1) will not be available. However, the modified risk premium $\tilde{\varphi}_c^{D,L}$ can be well approximated (errors of 2-5% using standard values of earnings inflation and investment discount rates for a typical reinsurance structure) by the following analytical formula:

$$\tilde{\varphi}_c^{D,L} \approx \frac{\lambda\sigma}{1-\xi} \frac{\left(\left(1+\xi/\sigma\left(IC(D)-\mu\right)\right)^{1-1/\xi} - \left(1+\xi/\sigma\left(IC(D+L)-\mu\right)\right)^{1-1/\xi}\right)}{\left(1+r_{INV}\right)^{\tau-\tau_0}} \tag{5.3}$$

where: λ is the Poisson rate, ξ, σ, μ are the GPD parameters, $IC(X) = \left(1+i_{FI}\right)^{\tau-\tau_1} X$ is the layer level after full indexation with future inflation, i_{FI} is the expected future (earnings) inflation, r_{INV} is the investment discount rate, τ *is the mean settlement time*, whereas τ_0 and τ_1 are offset values that depend on specific assumptions of the algorithm. This formula is an approximation in the sense that it assumes that all claims that happen at time *t* will be settled with a single payment at time $t+\tau$. This approximation is useful because it allows calculating the standard deviation of the estimator for the risk premium without running a stochastic simulation for every draw of the parameters, thereby reducing the computational complexity of determining credibility. A similar approximation can be used for the market risk premium, $\tilde{\varphi}_m^{D,L}$. Notice that the mean settlement time of the market and of the client may differ, which has some (minor) effect on the behaviour of the credibility factor as a function of *D*, *L*.

Except for using $\tilde{\varphi}_c^{D,L}$ and $\tilde{\varphi}_m^{D,L}$ instead of $\varphi_c^{D,L}$ and $\varphi_m^{D,L}$, the calculation of credibility for the UK motor reinsurance market is a standard application of the methods described in Section 4 – specifically, it is a hybrid model which uses the market severity curve above £2m. Other adjustments (aggregate deductible/limit, reinstatements) are not usually implemented in the UK motor reinsurance market and were therefore ignored in our study.

6. RELATIONSHIP WITH PREVIOUS WORK

We are now in a position to discuss at more length the considerations already touched upon in Section 1.1 on the relationship with other research.

The method is formally similar to other methods, in particular to the classical Bühlmann and Bühlmann-Straub methods [4][5][6]. By rearranging the formula for the Bühlmann credibility factor, $Z = \frac{n}{n + v/a}$ ($a = Var(\mu(\theta))$ is the variance of the means of the different clients; $v = E(v(\theta))$ is the mean of the variances for each client; *n* is the number of years of experience), one obtains $Z = \frac{a}{a + v/n}$, which has the same form as formula (3.5), by interpreting v/n as a measure of the standard deviation σ_c of the estimator of the risk premium, and by assuming that the corresponding quantity for the market, σ_m, is zero.

Analogous considerations apply to the Bühlmann-Straub methodology [5][6]. The key difference between the Bühlmann method and the Bühlmann-Straub method is that the latter takes exposure into account – it gives more weight to years with greater exposure. In our case, this is taken into account implicitly, as the standard deviation of the estimator depends crucially on the overall exposure over all years of past experience.

Another similarity to the methods above is the use of a collectively unbiased estimator for the credibility premium (see discussion in Section 3.2).

A work that is closer in spirit to ours is that by Boor [3], where the two estimators X_1, X_2 of the same random variable Y representing losses are credibility-weighted according to their accuracy and to the difference between them ($X_1 - X_2$), to produce the credibility estimate $Z \cdot X_1 + (1-Z) \cdot X_2$. The general formula for the credibility factor is then $Z = (E((X_2 - Y)^2) - E((X_1 - Y)^2) + E((X_2 - X_1)^2))/2E((X_2 - X_1)^2)$. When applied to the case of producing a rate for a subgroup α (*n* elements) of a large group $\Gamma = \alpha \cup \beta$ (*n*+*m* elements –

ideally, the whole market), this produces the following credibility estimate: $\varphi = Z \cdot \mu_\alpha + (1 - Z) \cdot \mu_\beta$, where

$$Z = \frac{\sigma_m'^2 / m + (\mu_\alpha - \mu_\beta)^2}{\sigma_n'^2 / n + \sigma_m'^2 / m + (\mu_\alpha - \mu_\beta)^2} \tag{5.4}$$

and μ_α, μ_β are the estimated means for α and β. This has the same structure as formula (3.6), which holds when the client is compared to the *rest* of the market:

- $\sigma_m'^2 / m$ and $\sigma_n'^2 / n$ in (5.4) are central-limit theorem approximations for the quantities σ_m^2 and σ_c^2 in (3.6);
- $(\mu_\alpha - \mu_\beta)^2$ is used rather than the spread of the market, σ_h^2. Obviously, σ_h^2 can be seen as the average value of $(\mu_\alpha - \mu_\beta)^2$ over all clients except. α

Apart from this, the two models are different: [3] uses a two-samples model, whereas we use a collective model where a single measure of the market spread is used for all clients and the correlation between each client and the market is explicitly used.

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Credibility for excess-of-loss reinsurance was first examined by Straub [17]. This was extended by Patrik & Mashitz [13]. An implementation of this approach has been carried out by Bonche [2] for the UK motor reinsurance market. All these efforts have focused on the Poisson/Gamma credibility model applied to claim frequency. The key idea in [17] is that the credibility factor for the Poisson/Gamma model, which is $Z = \frac{k}{k+b}$ in the Bühlmann case (no excess of loss), becomes $Z(D) = \frac{k}{k + b/P(X > D)}$ when applied to excess layers, with $P(X > D)$ being the exceedance probability, which depends on the severity distribution. The best linear estimate of λ (the Poisson rate) in this context is therefore $\lambda_{CRED}^{(1)}(> D) = Z(D)\frac{\sum_{j=1}^{k} n_j(D)}{k} + (1 - Z(D))\frac{a}{b}P(X > D)$ where $n_j(D)$ is the number of claims above *D* in year *j* for the client. Notice that *Z*(*D*) decreases as *D* increases – a property which conforms to the intuition that the client's experience can be trusted to a lower degree for the higher layers.

Patrik & Mashitz [13] extended this work to the case where $P(X > D)$ is not assumed to be known with certainty, thus recognising the need to take account of severity uncertainty (see Section 2.3.2 in their paper). This brings to a modified estimate of the credibility frequency, with the credibility factor becoming

$$Z(D) = \frac{k}{k + \dfrac{b}{E(P(X > D))(1 + (a+1)(CV(P(X > D))^2)}}.$$

In this formula, $E(P(X > D))$ and $CV(P(X > D))$ are the expected value and the coefficient of variation of $P(X > D)$. In [13], $CV(P(X > D))$ is selected so as to incorporate both parameter uncertainty and the subjective beliefs in the *a priori* estimates of the parameters of the severity distribution. Interestingly, in this case $Z(D)$ is not guaranteed to decrease in D. Whether or not this is the case depends on the degree to which the increase in the term containing the coefficient of variation compensates the decrease in the expected value of the survival probability.

The main difference between our work and that by Straub [17] and Patrik & Mashitz [13] is that these authors have restricted their attention to claim counts rather than considering the uncertainties on aggregate loss, which is (to borrow an expression from Patrik and Mashitz) the real item of interest when pricing a reinsurance excess cover.

The other obvious difference emerges in the special case where we assume that all clients have the same severity distribution, that of the market. In our single-severity model the credibility factor is broadly constant across the layers, whereas the credibility factor decreases as a function of layer excess in the work of Straub, Patrik and Mashitz (ignoring for the moment the problem of the inaccuracy of the severity distribution). The underlying reason for this difference is that in the Straub-Patrik-Mashitz approach the client frequency above threshold D is taken as the empirical mean above that threshold: $\dfrac{\sum_{j=1}^{k} n_j(D)}{k}$, and as such it is less credible if D increases; in our approach the credibility factor is constant, but the client rate above D is based not on the empirical frequency measured separately for each excess, but on the empirical frequency λ_c at the lowest excess level (μ) projected according to the severity distribution: $\lambda_c \cdot P(X > D)$.

This explains the difference in the behaviour of Z. Our preference is for an approach that gives an approximately constant credibility factor because, if we really believe that the severity distribution is known with certainty, it is more accurate to use as an initial estimate of the number of losses above $D > \mu$ the quantity $\lambda_c(> D) = \lambda_c \cdot P(X > D)$ rather than $\frac{\sum_{j=1}^{k} n_j(D)}{k}$, as the latter approach deliberately disregards the information below *D*.

The comparison becomes of course more complicated when the picture is completed considering errors in the severity curve. It is interesting to notice that both our work and [13], despite using different models, reach the conclusion that the uncertainty on the severity distribution ultimately corrupts the behaviour of the credibility estimate and does not guarantee *a priori* that the client will have decreasing credibility.

Cockroft [7] has extended the ideas in [17] and [13] to provide a credibility formula for claim aggregate loss rather than claim frequency. The formula for the credibility factor is still in the form $Z = \frac{k}{k+b}$, with *b* calculated analytically in terms of infinite series summations. Overall, Cockroft's solution is at this stage quite complex and relies on the assumption that the number of claims is Poi(λ) with a Gamma prior distribution for λ, and that the claim amounts are distributed according to a Pareto ($F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$) with a Gamma prior distribution for α.

7. LIMITATIONS AND FUTURE RESEARCH

We now look into the limitations of this work and areas for improvement.

- The credibility estimate relies on second-order statistics only. This may not always be appropriate when errors on the parameters are large and the standard deviation may not in itself characterise the distortions on the risk premium in a sufficiently accurate way. More general estimates can be obtained by replacing the mean-squared error minimisation criterion used in Proposition 1 with more sophisticated criteria, perhaps based on the quantiles or the higher moments of the aggregate loss distribution. Further research is needed to explore these different criteria.
- In order to get sound results for the credibility factor a good knowledge of the pricing process and its uncertainties is required. Consider, however, that it is part of the actuary's job to acquire a sufficiently thorough knowledge of the uncertainties of the pricing process anyway. If this knowledge is available, the credibility estimate is simply a byproduct.

- For the method to work it is critical that the process by which the uncertainties are computed be fully automated and that its computational complexity be kept at bay, identifying the variables that have real financial significance. This is especially important if an analytical formula for the price is not available.

- Specifically for reinsurance:
    - the credibility premium is calculated for each different layer *in isolation,* as if the reinsurance of each layer were bought/sold separately for each layer (this may or may not be the case). As a consequence, the credibility premium – the sum of the credibility premiums of the different layers – is in general not additive, in the sense that $\varphi^{D,L_1+L_2} \neq \varphi^{D,L_1} + \varphi^{D+L_1,D+L_1+L_2}$. The overall credibility premium paid for a reinsurance programme may in general depend on the details of the proposed layer structure. Further research is needed to understand what happens when additivity or other regularity conditions are imposed on the credibility premium. Notice that this problem only arises under the general uncertainty-based model, whereas additivity is automatically satisfied for the exact single-severity model (Proposition 2) and approximately satisfied for the single-severity model with inaccurate severity curve (Proposition 3);
    - the credibility estimate does not give sensible results beyond the *pricing horizon* of the market. This, however, is not strictly a limitation of the method – it is rather the natural consequence of the intrinsic lack of adequate market experience about very high layers;
    - the empirical calculation of market heterogeneity for the higher layers is quite difficult, due to the large errors involved (see introductory part in Section 4). This reduces the reliability of the credibility premium for those layers. One solution is to produce a realistic model for the behaviour of market heterogeneity as a function of layer excess, rather than relying on the empirical estimate only, much in the same way as we replace the empirical severity distribution with a continuous parametric distribution. We have carried out some preliminary work on this, which has shown that in the case where market heterogeneity becomes negligible in the limit $D \to \infty$, then – under quite general conditions – the credibility factor goes to zero. However, further evidence and research is needed to verify whether this "vanishing heterogeneity" hypothesis is realistic and supported by empirical evidence for some insurance classes. Incidentally, this hypothesis is at odds with the single severity hypothesis, which is strongly supported by empirical evidence in the case of motor XL reinsurance and is quite promising for other lines of

business, too.

8. CONCLUSIONS

This paper has presented a novel approach to calculating the credibility premium, called uncertainty-based credibility because it uses the standard deviation of the estimator of the risk premium (for both the client and the market) as the key to calculating the credibility factors.

This approach is especially useful for pricing XL reinsurance, where the balance of client uncertainty, market uncertainty and market heterogeneity is different for each layer of reinsurance. It has been used for pricing motor reinsurance in the UK market.

The methodology is in itself quite general and can be applied to many different problems, essentially to all situations where it is possible to compute the uncertainties of the pricing process and the heterogeneity of the market. Other examples include experience rating in direct insurance (possibly with different excesses) and combining exposure rating (as calculated by using exposure curves) and experience rating in property and liability reinsurance.

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Abbreviations and notations
GPD, Generalised Pareto distribution
XL, excess of loss

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