A Stochastic Framework for Incremental Average Reserve Models

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Abstract

Motivation. Chain ladder forecasts are notoriously volatile for immature exposure periods. The Bornhuetter-Ferguson method is one commonly used alternative but needs a priori estimates of ultimate losses. Berquist and Sherman presented another alternative that used claim counts as an exposure base and used trended incremental severities to "square the triangle." A significant advantage of the Berquist and Sherman method is the simultaneous estimate of underlying inflation. Though not the first to do so, this paper looks to extend the incremental severity method to a stochastic environment. Rather than using logarithmic transforms or (generalized) linear models, used in many other approaches, we use maximum likelihood estimators, bringing to bear the strength of that approach avoiding limiting assumptions necessitated when taking logarithms.

Method. Given that incremental severities can be looked at as averages over a number of claims, the law of large numbers would suggest those averages follow an approximately normal distribution. We then assume the variance of the incremental payments in a cell are proportional to a power of the mean (with the constant of proportionality and power constant over the triangle). We then use maximum likelihood estimators (MLEs) to estimate the incremental severities, along with the inherent claims inflation to "square the triangle." We also use properties of MLEs to estimate the variance-covariance matrix of the parameters, giving not only estimates of process but also of parameter uncertainty for this method. Not only do we consider the model described by Berquist and Sherman, but we also set the presentation in a more general framework that can be applied to a wide range of potential underlying models.

Results. A reasonably common and powerful method now presented in a stochastic framework allowing for assessment of variability in the forecasts of the method.

Availability. The R script for these estimates appear on the CAS Web Site.

Keywords. Stochastic reserving, maximum likelihood, normal-*p*, incremental severity method, PPCI

1. INTRODUCTION

The chain ladder method has long been recognized as leading to potentially volatile forecasts for immature exposure periods. As a result, other methods that depended on information in addition to the amounts to date were soon used to augment the chain ladder method for less mature ages. These methods include the Bornhuetter-Ferguson method [1], incremental severity methods shown in Berquist and Sherman [2], and the operational time models from Wright [3], among others. In effect, these approaches replace the multiplicative model inherent in the chain ladder with additive increments. The Bornhuetter-Ferguson method looks to historical development and an a priori estimate of ultimate losses to derive these additive increments, while the incremental severity method considers incremental average costs per ultimate claim (or other unit of exposure) and a measure of inflationary trend to derive these increments. In the discussion by Berquist and Sherman, the trend itself is estimated from the data.

Thus by adding a single parameter trend to be estimated from the data, Berquist and Sherman avoided assumptions about the relative adequacy of pricing or the need of deriving a priori ultimate loss estimates by exposure year. Of course, they do require a measure of relative exposure, usually claim counts.

There has been much published about stochastic generalizations of the chain ladder method. Verall and England [4] presents a very nice summary. We will not touch on those here, but rather attempt to re-cast the incremental severity method in a stochastic light.

In the present paper we first consider the incremental severity method in a stochastic framework. We note that the incremental severities are themselves averages over a number of observations and, as a result of the law of large numbers, would likely have a distribution that is asymptotically normal. This is a very significant observation and was made by Stelljes [5] and provides a bit of support to at least one answer to the question of what statistical model to use. Stelljes assumes that the development pattern follows a mixed exponential over time and does not measure the trend inherent in the data.

We however, start with the classic incremental severity model (allowing for different averages at each age) but measure the inflation inherent in the loss experience. Not only does this allow for a broader range of runoff curves, it also allows for systematic negative incremental amounts, making it possible to model not only paid amounts (net of recoverable) but also incurred amounts. In addition, rather than making somewhat restrictive assumptions about the underlying variance structure as present in Stelljes that allows the use of non-linear regression, we will take a somewhat more general approach of maximum likelihood estimators allowing more flexible assumptions regarding the underlying variance structure.

In this paper we not only derive parameter estimates for our model, including inherent trend, but also estimates of the standard deviation of those parameter estimates, often called the standard error of the parameters. The standard error can be used to measure the significance of the parameter as well as the parameter uncertainty inherent in the forecasts of this model. We also derive estimates of the distribution of outcomes for this model, not to be confused with the distribution of potential outcomes for the liabilities under review.

1.1 Research Context

In the context of reserves for a book of liabilities at a point in time, there is a wide range of possible outcomes, some of which may be more likely than others. We call this entire range of outcomes along with their likelihoods the "distribution of outcomes" for the liabilities under consideration. This observation seems to have pervaded the analysis of reserves for decades. Traditional reserving approaches, although relying on deterministic methods, usually had the actuary applying a variety of those methods with the unstated goal of providing at least a subjective view of the distribution of outcomes, or at least the portion of that distribution that contained "reasonable estimates."

More recently, though, questions of just how "good" the "reasonable estimates" were led to consideration of stochastic methods to rigorously quantify that uncertainty. Statements such as "My selection for unpaid liabilities is $a million. In my view it is just as likely that the ultimate unpaid liabilities will be between $x million and $y million as outside that range and in addition, it is very unlikely that the ultimate unpaid liabilities will be below $w million or above $z million" provide much more useful information to a principal than "My best estimate is $a million and I believe a range of reasonable estimates is between $b million and $c million." Because of this there has been increased focus on models that will assist the actuary in estimating the distribution of outcomes.

Just as no traditional reserve method completely captures all the complexities possible for all lines of business, it is not likely that any current stochastic model can capture all those complexities. Because of this, results presented here should <u>not</u> be interpreted as estimates of the distribution of outcomes, but rather the distribution of possibilities <u>under the specific assumptions of the single model we present</u>.

1.2 Objective

The incremental average cost method has long been a very powerful alternative to the chain ladder method that can be quite volatile for more immature exposure periods. The Cape Cod and Bornhuetter-Ferguson methods are often used as alternatives that try to overcome this problem. There has been research setting all of these methods in stochastic frameworks. Our objective is to take another powerful alternative to the chain ladder method, the incremental average loss method presented by Berquist and Sherman [2], and set it into a stochastic framework.

One substantial contribution of the Berquist and Sherman approach is the estimation of trend in the averages from the averages themselves. This is in contrast to the necessary external trend usually necessary in stochastic versions of both the Bornhuetter-Ferguson and Cape Cod methods.

Another weakness of many stochastic generalizations of traditional methods is the necessity of assumptions about the form of the distributions used. Because of the central limit theorem, averages of independent samples from a distribution are asymptotically normal, thus suggesting a form for the distributions in the stochastic model.

Another inherent limitation of most stochastic generalizations is the necessity of assuming all incremental amounts are positive. This limits the generalization of those methods in the case of incurred losses, or in the case of consistent downward paid development. The use of the normal distribution allows more flexibility in handling consistent negative incremental averages.

The goal of this paper is to set the traditional incremental average method in a stochastic framework taking advantage of the ease of computation afforded by the normal distribution and ability to handle negative values. In addition to moving the average cost method into a stochastic framework, this paper also shows the relative ease of moving to a completely non-linear environment, thereby avoiding the constraints inherent in linear or generalized linear methods, echoing the comments of Venter in several venues, including [7].

1.3 Outline

In Section 2 we set out our stochastic generalization of the incremental average method presented in Berquist and Sherman [2]. Section 3 discusses the results of applying these methods to the adjusted paid automobile bodily injury liability data in that paper. We present our conclusions in Section 4 with Appendix A showing the derivatives used in the estimation along with the R script that we used in the calculations.

2. BACKGROUND AND METHODS

Klugman, Panjer, and Willmot [6] present a very clear and concise discussion of maximum likelihood estimates (MLEs). We will make use of that approach in this paper.

For this paper C_{ij} denotes payments made or the change in incurred losses (defined as payments plus case reserve estimates) for exposure (policy, accident, underwriting, etc.) period (year, quarter, month, etc.) i during development period j. For convenience here we will assume the same frequency for both i and j, and hence the resulting development triangle will have the same number of rows as columns, denoted as n here. Without loss of generality, we will talk in terms of accident and development years.

For each accident year we have some measure of loss exposure, either an exposure count or an estimate of ultimate claim counts. Exposure count, such as earned car years for automobile coverages, generally does not require estimation. The same cannot be said for claim counts that must be estimated and hence should be treated as random variables. We will not make that generalization here but rather leave it as a future project.

We do note that, just as there are a number of models that can be used to estimate ultimate loss amounts, there are a number of approaches that can be used to estimate the ultimate number of claims. If the number of reported counts is deemed to be a reliable and stable base, that is, if there has been no change in the definition or nature of reported claims during the experience period under consideration, they often provide a measure of exposure that matures more quickly than losses and hence those estimates will likely have less inherent uncertainty, i.e., lower standard error, than losses. It might well be that consideration of both chain ladder estimates and those of an incremental average frequency method, such as presented here applied to claim counts, using earned exposures as an exposure base, could provide reasonable estimate of ultimate reported counts for use here.

In any event, we will denote this measure of relative exposure as E_i for accident year i. We will thus focus on the incremental averages A_{ij} defined by equation (2.1).

$$A_{ij} = \frac{C_{ij}}{E_i}. \tag{2.1}$$

The traditional incremental severity method then "squares the triangle" with trended averages as in equation (2.2).

$$A_{ij} = \alpha_j \tau^i,\ i = 2, 3, \ldots, n;\ j = n - i + 2, \ldots, n. \tag{2.2}$$

We will effectively take this same approach to frame a stochastic model based on this method. It is not unusual, see for example Venter [6], to assume that the variance of the incremental amounts is a power of their expected value. We will take this same approach. However, since we will allow the expected values to be negative we will, without loss of generality, we take the variance to be a power of the square of the mean. Also we are taking the constant of proportionality among the variances as an exponential, thereby allowing the parameter to take on any value. However, we note that the variance of the average of n items is inversely proportional to the number of items so we further adjust our assumed variances to reflect the potential for a different number of exposures or claims in the various accident years. For this we let e denote the number of exposures or claims for the year. Following the notation in [6] we will assume the relationships in (2.3), suppressing subscripts for the moment.

$$\begin{aligned}\mathrm{E}(A) &= \mu. \\ \mathrm{Var}(A) &= e^{\kappa - e}\mu^{2p}.\end{aligned} \tag{2.3}$$

Now, since the A_{ij} are averages, the law of large numbers implies that they are asymptotically normal with parameters given in (2.4), again suppressing subscripts for the moment.

$$A \sim \mathrm{N}\left(\mu, e^{\kappa - e}\mu^{2p}\right). \tag{2.4}$$

Since we are concerned with maximum likelihood estimates, the negative log likelihood for this distribution will be key to our analysis. Since we have a normal distribution the likelihood function is relatively simple and given by (2.5).

$$\mathrm{f}(x;\mu,\kappa,p) = \frac{1}{\sqrt{2\pi e^{\kappa - e}\mu^{2p}}} e^{-\frac{(x-\mu)^2}{2e^{\kappa-e}\mu^{2p}}}. \tag{2.5}$$

This gives a negative log likelihood for a single variable given in (2.6).

$$\begin{aligned}\mathrm{l}(x;\mu,\kappa,p) &= -\ln\left(\mathrm{f}(x;\mu,\kappa,p)\right) \\ &= -\ln\left(\frac{1}{\sqrt{2\pi e^{\kappa - e}\mu^{2p}}} e^{-\frac{(x-\mu)^2}{2e^{\kappa-e}\mu^{2p}}}\right) \\ &= \frac{1}{2}\left(\kappa - e + \ln\left(2\pi\mu^{2p}\right)\right) + \frac{(x-\mu)^2}{2e^{\kappa-e}\mu^{2p}}.\end{aligned} \tag{2.6}$$

We note the incremental amounts A_{ij} under consideration are averages of a number of observations. If we assume the observations are themselves independent, then the central limit theorem would imply that they have asymptotically normal distributions. For this reason we will assume that the A_{ij} variables are all independent and have normal distributions. We generalize the incremental severity model with the parametric model shown in (2.7).

$$A_{ij} \sim \mathrm{N}\left(\alpha_j\tau^i, e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}\right). \tag{2.7}$$

With observations in a typical loss triangle we get the negative log likelihood function given in (2.8).

$$\mathrm{l}\left(A_{11}, A_{12}, \ldots, A_{n1}; \alpha_1, \alpha_2, \ldots, \alpha_n, \tau, \kappa, p\right) = \sum_{(i,j)\in S} \frac{\kappa - e_i + \ln\left(2\pi\left(\alpha_j\tau^i\right)^{2p}\right)}{2} + \frac{\left(A_{ij} - \alpha_j\tau^i\right)^2}{2e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}}. \tag{2.8}$$

The set *S* in (2.8) denotes the set of all index pairs for which data are available. If the data were available in a full triangle, with *n* rows and *n* columns then *S* would follow the form given in (2.9).

$$S = \left\{ (i, j) \middle| i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n - i + 1 \right\}. \tag{2.9}$$

However, we will not restrict ourselves to this regular case. We also note in formula (2.8) the e_i values are known constants (the natural logs of the number of exposures for accident year *i*, not parameters to be estimated.

Once parameters that minimize the negative log likelihood function are determined, then it is straight-forward to obtain estimates of the distribution of outcomes under the assumption that this model and the resulting parameters completely describe the loss emergence phenomenon. Let us denote these estimates by $\hat{\alpha}_k$, $\hat{\kappa}$, $\hat{\tau}$, and $\hat{p}$. Under our assumptions we can now conclude that the distribution of average future payments for each year is given by (2.10).

$$R_i \sim \mathrm{N}\left(\sum_{j=n-i+2}^{n} \hat{\alpha}_j \hat{\tau}^i, \sum_{j=n-i+2}^{n} e^{\hat{\kappa}-e_i} \left(\hat{\alpha}_j \hat{\tau}^i \right)^{2\hat{p}} \right). \tag{2.10}$$

This then gives the effect of process uncertainty on the total forecast incremental severity by accident year. This does not, however, address the issue of parameter uncertainty. Just as the standard error provides insight into parameter uncertainty in usual regression applications, the information matrix can be helpful in estimating the variance-covariance matrix of the parameters. For this, we first define the Fisher Information Matrix as the matrix of expected values of the Hessian of the negative log likelihood function. That is, the matrix whose element in i^{th} row and j^{th} column is the second derivative of the negative log likelihood function, once with respect to the i^{th} variable and once with respect to the j^{th}. We show these expectations, along with both the elements of the gradient and Hessian of the negative log likelihood function in the appendix to this paper. The inverse of the information matrix is then an approximation for the variance-covariance matrix for the parameters.

Since the mean and variance for individual incremental averages are functions of the parameters, we elected to estimate the distribution of future amounts both by exposure period and in total using simulation. For this we first selected the parameters from a multivariate normal distribution with expected values equal to the MLE estimates and variance-covariance matrix equal to the inverse of the information matrix. Given those parameters, we then randomly selected future incremental averages in each cell using the relationship in (2.7). We added up the indications by exposure year and multiplied by the denominator (claim count or exposure count) to obtain a single observation for an exposure year and then added all those simulations together to get a single observation of the total future amount.

At this juncture if we wished to assume that claim counts, instead of being deterministic, were themselves stochastic, but independent of the incremental severities, we could simulate the ultimate number of claims by exposure year at this juncture to add a provision for uncertainty in those estimates in the final forecast.

3. RESULTS AND DISCUSSION

As an example of this model, the top portion of Exhibit 1 shows the incremental averages based on automobile bodily injury liability data from Berquist and Sherman [2]. The last column is the forecast ultimate claim counts from Exhibit J of that paper. The incremental severities are based on adjusted paid losses in Exhibit N divided by these claim count estimates.

The bottom portion of Exhibit 1 shows the parameter estimates derived by minimizing the negative log likelihood function shown in (2.8). Shown in the "standard error" row is the square root of the diagonal of the approximate parameter variance-covariance matrix.

Exhibit 2 shows scatter plots of the standardized residuals from the fitted model, calculated as the ratio of the difference between the historical average minus the expected average from the model, divided by the estimated standard deviation by cell. The first three charts show the residuals first by calendar year, then by accident year, and finally by development lag. The last histogram shows the simulated range of forecasts from 25,000 simulations. The line on that histogram presents the distribution assuming independence and the mean and variance by cell implied by the parameter estimates.

Exhibit 3 shows the expected averages and related variances by cell indicated by the estimated parameters and the model shown in (2.7). Exhibit 4 shows the indicated mean forecast and standard deviation by accident year and for all years combined. Exhibit 4 also shows the forecasts for the next calendar year, both with and without parameter uncertainty. These estimates can be used to assess how well emerging experience fits with what is forecast by the model, a critical test for the on-going application of just about any model.

Since the model in (2.7) assumes the incremental averages are independent, the future average forecast is simply the sum of the future indications by accident year, as is the variance for the future forecast, assuming process uncertainty only. The resulting means and standard deviations, after multiplication by the number of claims are shown under the "Process Only" columns.

The remaining columns summarize the results of the simulation. We first randomly simulated a selection of parameters given the parameter estimates and the approximate variance-covariance matrix, using a multivariate normal distribution. Given those parameters, we then randomly simulated individual incremental averages by cell using a normal distribution with the mean and standard deviation shown in (2.7). We then totaled the results for one simulation to derive both the simulated future average estimates by accident year and then, after multiplying by claim counts, the total indicated future amounts. The averages and standard deviations in the right portion of that exhibit represent the mean and standard deviation of the simulated amounts as are the fifth percentile and 95th percentile (the 90% probability interval) for the simulations. These last columns thus present an estimate of the distribution of possible forecasts from this model, given the loss data in the top of Exhibit 1.

As can be seen, parameter uncertainty clearly contributes substantially to the uncertainty in the forecasts for this model. The standard deviation including parameter uncertainty is nearly three times that for process uncertainty only. In addition, as one would expect there are correlations in the forecasts among accident years, particularly since the forecast for an accident year depends not only on the losses for that year but also on the losses and forecasts for previous years. If the accident years were independent, then the standard deviation for the total would equal the square root of the sum of the squares of the standard deviations for the various years. That calculation yields approximately 1.1 million, compared with the final 1.5 million shown in Exhibit 4.

Although we do not show the results of the calculations, the model and estimation process reacts as one should expect with negative values. A simple test would simply replace the incrementals in a column with their negatives. When doing this all values of the parameters and variance-covariance matrix remain unchanged, except with a sign change in the parameter estimates and covariances related to the affected column.

The R script used to derive these estimates are also shown in Appendix A. Generally the approach is quite straight forward. Key to deriving the estimates is the function R nlminb. As with many optimization routines, this function requires a starting value. In this case, we first selected a starting value for τ as the trend in the averages for the first development period (unless that trend generates an error, in which case we selected 1.03). We then estimated the initial α_j values as the averages of the averages, discounted at the initial τ estimate, and selected the initial values for κ and p as the natural logarithm of the largest exposure number and 1.5, respectively (somewhat arbitrarily).

This R function also allows for different iteration increments for the various variables to be optimized. Users should consult the documentation that accompanies R for this function. We selected relative scaling among variables inversely proportional to the initial averages for the α_j variables and five for the remaining three.

4. CONCLUSIONS

Although we focused on a very simple model of incremental averages, nothing in what we have done relies on the specific structure of the underlying model. This is in contrast to many stochastic approaches that require non-negative incrementals, and the necessity of making additional assumptions about the distributions of the incremental amounts. The framework we chose, along with the central limit theorem, suggests the normal distribution for the incremental averages.

As shown in (2.8), this distribution leads to a rather convenient form for the negative log likelihood function. Together with the ability to differentiate the assumed model for the average and resulting standard deviation makes this approach easily expandable to other models for the incremental averages. Coupled with powerful, reasonably easy-to-use, and affordable statistical software such as the language R, actuaries now have quite flexible tools to use to expand the models used in estimating future losses, even beyond the simple model presented here.

Supplementary Material

The R script used for these calculations is stored electronically on the CAS Web Site.

Appendix A

In order to derive estimates of parameter uncertainty we need the matrix of second derivatives of the negative log likelihood function. In this appendix we list those derivatives.

Recall from (2.8) the negative log likelihood function is given by

$$1\left(A_{11}, A_{12}, \ldots, A_{n1}; \alpha_1, \alpha_2, \ldots, \alpha_n, \tau, \kappa, p\right) = \sum_{(i,j)\in S} \frac{\kappa - e_i + \ln\left(2\pi\left(\alpha_j \tau^i\right)^{2p}\right)}{2} + \frac{\left(A_{ij} - \alpha_j \tau^i\right)^2}{2e^{\kappa - e_i}\left(\alpha_j \tau^i\right)^{2p}}.$$

Suppressing arguments and parameters we thus have the following first partial derivatives:

$$\frac{\partial \mathrm{l}}{\partial \alpha_k} = \sum_{i \in \{i | (i,k) \in S\}} \frac{p}{\alpha_k} - \frac{\alpha_k \tau^i \left(A_{ik} - \alpha_k \tau^i\right) + p\left(A_{ik} - \alpha_k \tau^i\right)^2}{e^{\kappa - e_i} \alpha_k \left(\alpha_k \tau^i\right)^{2p}}.$$

$$\frac{\partial \mathrm{l}}{\partial \kappa} = \sum_{(i,j) \in S} \frac{1}{2}\left(1 - \frac{\left(A_{ij} - \alpha_j \tau^i\right)^2}{e^{\kappa - e_i} \left(\alpha_j \tau^i\right)^{2p}}\right).$$

$$\frac{\partial \mathrm{l}}{\partial \tau} = \sum_{(i,j) \in S} \frac{pi}{\tau} - \frac{i\alpha_j \tau^i \left(A_{ij} - \alpha_j \tau^i\right) + pi\left(A_{ij} - \alpha_j \tau^i\right)^2}{\tau e^{\kappa - e_i} \left(\alpha_j \tau^i\right)^{2p}}.$$

$$\frac{\partial \mathrm{l}}{\partial p} = \sum_{(i,j) \in S} \ln\left(\alpha_j \tau^i\right)\left(1 - \frac{\left(A_{ij} - \alpha_j \tau^i\right)^2}{e^{\kappa - e_i} \left(\alpha_j \tau^i\right)^{2p}}\right).$$

These then give the following second derivatives:

$$\frac{\partial^2 \mathrm{l}}{\partial \alpha_k^2} = \sum_{i \in \{i | (i,k) \in S\}} \left(\frac{\alpha_k^2 \tau^{2i} + 4p\alpha_k \tau^i \left(A_{ik} - \alpha_k \tau^i\right) + p(2p+1)\left(A_{ik} - \alpha_k \tau^i\right)^2}{\alpha_k^2 e^{\kappa - e_i} \left(\alpha_k \tau^i\right)^{2p}} - \frac{p}{\alpha_k^2}\right).$$

$$\frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \alpha_m} = \frac{\partial^2 \mathrm{l}}{\partial \alpha_m \partial \alpha_k} = 0, m \neq k.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \kappa \partial \alpha_k} = \frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \kappa} = \sum_{i \in \{i | (i,k) \in S\}} \frac{\alpha_k \tau^i \left(A_{ik} - \alpha_k \tau^i\right) + p\left(A_{ik} - \alpha_k \tau^i\right)^2}{e^{\kappa - e_i} \alpha_k \left(\alpha_k \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \tau \partial \alpha_k} = \frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \tau} = \sum_{i \in \{i | (i,k) \in S\}} \frac{i\alpha_k^2 \tau^{2i} + (4p-1)i\alpha_k \tau^i \left(A_{ik} - \alpha_k \tau^i\right) + 2p^2 i\left(A_{ik} - \alpha_k \tau^i\right)^2}{e^{\kappa - e_i} \tau \alpha_k \left(\alpha_k \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial p \partial \alpha_k} = \frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial p} = \sum_{i \in \{i | (i,k) \in S\}} \frac{1}{\alpha_k} + \frac{2\ln\left(\alpha_k \tau^i\right)\alpha_k \tau^i \left(A_{ik} - \alpha_k \tau^i\right) + \left(2p\ln\left(\alpha_k \tau^i\right) - 1\right)\left(A_{ik} - \alpha_k \tau^i\right)^2}{e^{\kappa - e_i} \alpha_k \left(\alpha_k \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \kappa^2} = \sum_{(i,j) \in S} \frac{\left(A_{ij} - \alpha_j \tau^i\right)^2}{e^{\kappa - e_i} \left(\alpha_j \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \tau \partial \kappa} = \frac{\partial^2 \mathrm{l}}{\partial \kappa \partial \tau} = \sum_{(i,j) \in S} \frac{i\alpha_j \tau^i \left(A_{ij} - \alpha_j \tau^i\right) + pi\left(A_{ij} - \alpha_j \tau^i\right)^2}{\tau e^{\kappa - e_i} \left(\alpha_j \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial p \partial \kappa} = \frac{\partial^2 \mathrm{l}}{\partial \kappa \partial p} = \sum_{(i,j)\in S} \frac{\ln\left(\alpha_j \tau^i\right)\left(A_{ij} - \alpha_j \tau^i\right)^2}{e^{\kappa - e_i}\left(\alpha_j \tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \tau^2} = \sum_{(i,j)\in S} \frac{i^2\alpha_j^2\tau^{2i} + \left(4ip - i + 1\right)i\alpha_j\tau^i\left(A_{ij} - \alpha_j\tau^i\right) + (2ip+1)pi\left(A_{ij} - \alpha_j\tau^i\right)^2}{\tau^2 e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}} - \frac{pi}{\tau^2}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial \tau \partial p} = \frac{\partial^2 \mathrm{l}}{\partial p \partial \tau} = \sum_{(i,j)\in S} \frac{i}{\tau} + \frac{2i\alpha_j\tau^i \ln\left(\alpha_j\tau^i\right)\left(A_{ij} - \alpha_j\tau^i\right) + \left(2pi\ln\left(\alpha_j\tau^i\right) - 1\right)\left(A_{ij} - \alpha_j\tau^i\right)^2}{\tau e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}}.$$

$$\frac{\partial^2 \mathrm{l}}{\partial p^2} = \sum_{(i,j)\in S} \frac{2\ln^2\left(\alpha_j\tau^i\right)\left(A_{ij} - \alpha_j\tau^i\right)^2}{e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}}.$$

The information matrix then requires the expected values of these derivatives. To this end recall that because of (2.7) we have the following relationships:

$$\mathrm{E}\left(A_{ij}\right) = \alpha_j\tau^i, \text{and}$$
$$\mathrm{Var}\left(A_{ij}\right) = \mathrm{E}\left(\left(A_{ij} - \alpha_j\tau^i\right)^2\right) = e^{\kappa - e_i}\left(\alpha_j\tau^i\right)^{2p}.$$

We can then derive the entries of the information matrix as follows:

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \alpha_k^2}\right) = \frac{2p^2}{\alpha_k^2}\sum_{i\in\{i|(i,k)\in S\}} 1 + \sum_{i\in\{i|(i,k)\in S\}} \frac{1}{e^{\kappa - e_i}\left(\alpha_k^2\right)^p \tau^{2i(p-1)}}.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \alpha_m}\right) = 0, m \neq k.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \kappa \partial \alpha_k}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \kappa}\right) = \frac{p}{\alpha_k}\sum_{i\in\{i|(i,k)\in S\}} 1.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \tau \partial \alpha_k}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial \tau}\right) = \frac{2p^2}{\tau\alpha_k}\sum_{i\in\{i|(i,k)\in S\}} i + \frac{1}{\tau\alpha_k}\sum_{i\in\{i|(i,k)\in S\}} \frac{i}{e^{\kappa - e_i}\left(\alpha_k^2\right)^{p-1}\tau^{2i(p-1)}}.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial p \partial \alpha_k}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \alpha_k \partial p}\right) = \frac{p\ln\left(\alpha_k^2\right)}{\alpha_k}\sum_{i\in\{i|(i,k)\in S\}} 1 + \frac{2p\ln(\tau)}{\alpha_k}\sum_{i\in\{i|(i,k)\in S\}} i.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \kappa^2}\right) = \sum_{(i,j)\in S} 1.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \tau \partial \kappa}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \kappa \partial \tau}\right) = \frac{p}{\tau}\sum_{(i,j)\in S} i.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial p \partial \kappa}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \kappa \partial p}\right) = \sum_{(i,j)\in S} \frac{\ln\left(\alpha_j^2 \tau^{2i}\right)}{2}.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \tau^2}\right) = \frac{2p^2}{\tau^2}\sum_{(i,j)\in S} i^2 + \sum_{(i,j)\in S} \frac{i^2}{e^{\kappa - e_i}\tau^{2+2i(p-1)}\left(\alpha_i^2\right)^{p-1}}.$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial \tau \partial p}\right) = \mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial p \partial \tau}\right) = \sum_{(i,j)\in S} \frac{i-1}{\tau} + \frac{p}{\tau}\left(\sum_{(i,j)\in S} i \ln\left(\alpha_j^2\right) + \sum_{(i,j)\in S} i \ln\left(\tau^{2i}\right)\right).$$

$$\mathrm{E}\left(\frac{\partial^2 \mathrm{l}}{\partial p^2}\right) = \sum_{(i,j)\in S} \frac{\ln^2\left(\alpha_j^2 \tau^{2i}\right)}{2}.$$

The calculations in this paper made use of the following R script:

```
library(mvtnorm)
library(MASS)

A0=matrix(c(178.73,361.03,283.69,264.00,137.94,61.49,15.47,8.82,
  196.56,393.24,314.62,266.89,132.46,49.57,33.66,NA,
  194.77,425.13,342.91,269.45,131.66,66.73,NA,NA,
  226.11,509.39,403.20,289.89,158.93,NA,NA,NA,
  263.09,559.85,422.42,347.76,NA,NA,NA,NA,
  286.81,633.67,586.68,NA,NA,NA,NA,NA,
  329.96,804.75,NA,NA,NA,NA,NA,NA,
  368.84,NA,NA,NA,NA,NA,NA,NA),8,8,byrow=TRUE)
dnom=c(7822,8674,9950,9690,9590,7810,8092,7594)

# Input (A0) is a development array of incremental averages with a the
# exposures (claims) used in the denominator appended as the last column.
# Assumption is for the same development increments as exposure
# increments and that all development lags with no development have #
# been removed.  Data elements that are not available are indicated as
# such.  This should work (but not tested for) just about any subset of
# an upper triangular data matrix.  Another requirement of this code is
# that the matrix contain no columns that are all zero.

# Matrix shape, m rows, n columns
m=(nrow(A0))[1]
n=(ncol(A0))[1]

# Generate a matrix to reflect exposure count in the variance structure
logd=log(matrix(dnom,m,n))

# Set up matrix of rows and columns, makes later calculations simpler
```

```
r=row(A0)
c=col(A0)


# msk is a mask matrix of allowable data, upper triangular assuming same
# development increments as exposure increments, msn picks off the first
# forecast diagonal
msk=(m-r)>=c-1
msn=(m-r)==c-2

# Negative loglikelihood function, to be minimized
l.obj=function(a,A) {
    e=outer(a[n+2]^(1:m),a[1:n])
    v=exp(a[n+1]-logd)*(e^2)^a[n+3]
    t1=log(2*pi*v)/2
    t2=(A-e)^2/(2*v)
  sum(t1+t2,na.rm=TRUE)}

# Gradient of the objective function
l.grad=function(a,A) {
    e=outer(a[n+2]^(1:m),a[1:n])
    v=exp(a[n+1]-logd)*(e^2)^a[n+3]
    da=colSums(a[n+3]-(e*(A-e)+a[n+3]*(A-e)^2)/
      v,na.rm=TRUE)/a[1:n]
    yy=1-(A-e)^2/v
    dk=sum(yy/2,na.rm=TRUE)
    dp=sum(yy*log(e^2)/2,na.rm=TRUE)
    du=sum((a[n+3]*r/a[n+2])-
      (r*e*(A-e)+a[n+3]*r*(A-e)^2)/(a[n+2]*v),na.rm=TRUE)
  c(da,dk,du,dp)}

 # Hessian of the objective function
l.hess=function(a,A) {
    e=outer(a[n+2]^(1:m),a[1:n])
    v=exp(a[n+1]-logd)*(e^2)^a[n+3]
    daa=diag(
          colSums((e^2+4*a[n+3]*e*(A-e)+
            a[n+3]*(2*a[n+3]+1)*(A-e)^2)/v-a[n+3],
          na.rm=TRUE)/a[1:n]^2)
    dak=colSums((e*(A-e)+a[n+3]*(A-e)^2)/v,na.rm=TRUE)/a[1:n]
    dat=colSums((r*e^2+(4*a[n+3]-1)*r*e*(A-e)+
          2*a[n+3]^2*r*(A-e)^2)/v,
          na.rm=TRUE)/(a[1:n]*a[n+2])
    dap=colSums(msk+(log(e^2)*e*(A-e)+
          (a[n+3]*log(e^2)-1)*(A-e)^2)/v,na.rm=TRUE)/a[1:n]
    dkk=sum((A-e)^2/v,na.rm=TRUE)
    dkt=sum((r*e*(A-e)+a[n+3]*r*(A-e)^2)/(a[n+2]*v),na.rm=TRUE)
    dkp=sum(log(e^2)*(A-e)^2/(2*v),na.rm=TRUE)
    dtt=sum((r^2*e^2+(4*r*a[n+3]-r+1)*r*e*(A-e)+
            (2*r*a[n+3]+1)*a[n+3]*r*(A-e)^2)/v-a[n+3]*r,
            na.rm=TRUE)/a[n+2]^2
    dtp=sum(r+(r*e*log(e^2)*(A-e)+
            (a[n+3]*r*log(e^2)-1)*(A-e)^2)/v,na.rm=TRUE)/a[n+2]
    dpp=sum(log(e^2)^2*(A-e)^2/(2*v),na.rm=TRUE)
    dm1=matrix(c(dak,dat,dap),n,3)
    dm2=matrix(c(dkk,dkt,dkp,dkt,dtt,dtp,dkp,dtp,dpp),3,3)
```

```
    rbind(cbind(daa,dm1),cbind(t(dm1),dm2))}

  # Set up starting values, take trend from first column, unless it errors
  # out (because of 0 or negatives) in which case take 3% as a default
  tmp=na.omit(data.frame(x=1:m,y=log(A0[,1])))
  trd=1.03
  trd=exp(coef(lm(tmp$y~tmp$x))[2])
  a0=c(colSums(A0/(trd^c),na.rm=TRUE)/colSums(msk+0*A0,na.rm=TRUE),log(max(dnom))
,trd,1.5)

  max=list(10000,10000)
  names(max)=c("iter.max","eval.max")

  # Actual minimization
  mle= nlminb(a0,l.obj,gradient=l.grad,hessian=l.hess,
    scale=c(abs(1/a0[1:n]),rep(5,3)),A=A0,control=max)

  # mean and var are model fitted values, stres standardized residuals
  mean=outer(mle$par[n+2]^(1:m),mle$par[1:n])
  var=exp(mle$par[n+1]-logd)*(mean^2)^mle$par[n+3]
  stres=(A0-mean)/sqrt(var)

  # Calculate the information matrix using second derivatives of the
  # log likelihood function

  # Second with respect to alpha parameters
  aa=diag(
    (2*mle$par[n+3]^2*
      colSums(msk+0*A0,na.rm=TRUE)/
        mle$par[1:n]^2)+
      colSums((msk+0*A0)/
        outer(exp(mle$par[n+1]-log(dnom))*mle$par[n+2]^(2*(1:m)*(mle$par[n+3]-
1)),
          (mle$par[1:n]^2)^mle$par[n+3])
        ,na.rm=TRUE)
      )

  # Second with respect to alpha and kappa
  ak=(mle$par[n+3]/mle$par[1:n])*
    colSums(msk+0*A0,na.rm=TRUE)

  # Second with respect to alpha and tau
  at=(2*mle$par[n+3]^2/(mle$par[n+2]*mle$par[1:n]))*
    colSums((msk+0*A0)*r,na.rm=TRUE)+
      colSums((msk+0*A0)*outer((1:m)/(exp(mle$par[n+1]-log(dnom))*
        mle$par[n+2]^(2*(1:m)*(mle$par[n+3]-1))),
          1/(mle$par[1:n]^2)^(mle$par[n+3]-1)),
      na.rm=TRUE)/(mle$par[n+2]*mle$par[1:n])

  # Second with respect to alpha and p
  ap=(mle$par[n+3]*log(mle$par[1:n]^2)/mle$par[1:n])*
      colSums((msk+0*A0),na.rm=TRUE)+
    (mle$par[n+3]*log(mle$par[n+2]^2)/mle$par[1:n])*
      colSums((msk+0*A0)*r,na.rm=TRUE)

  # Second with respect to kappa
```

```
kk=sum((msk+0*A0),na.rm=TRUE)

# Second with respect to kappa and tau
kt=mle$par[n+3]*sum((msk+0*A0)*r,na.rm=TRUE)/mle$par[n+2]

# Second with respect to kappa and p
kp=sum((msk+0*A0)*log(outer(mle$par[n+2]^(2*(1:m)),
  mle$par[1:n]^2)),na.rm=TRUE)/2

# Second with respect to tau
tt=2*mle$par[n+3]^2*sum((msk+0*A0)*r^2,na.rm=TRUE)/mle$par[n+2]^2+
  sum((msk+0*A0)*
    outer((1:m)^2/(exp(mle$par[n+1]-log(dnom))*mle$par[n+2]^
        (2+2*((1:m)*(mle$par[n+3]-1)))),
      1/(mle$par[1:n]^2)^(mle$par[n+3]-1)),
    na.rm=TRUE)

# Second with respect to tau and p
tp=sum((msk+0*A0)*(r-1),na.rm=TRUE)/mle$par[n+2]+mle$par[n+3]*(
  sum((msk+0*A0)*outer(1:m,
    log(mle$par[1:n]^2)),
    na.rm=TRUE)+
  sum((msk+0*A0)*r*log(mle$par[n+2]^(2*r)),na.rm=TRUE))/
  mle$par[n+2]

# Second with respect to p
pp=sum((msk+0*A0)*log(outer(mle$par[n+2]^(2*(1:m)),
  mle$par[1:n]^2))^2,na.rm=TRUE)/2

# Create information matrix in blocks
m1=matrix(c(ak,at,ap),n,3)
m2=matrix(c(kk,kt,kp,kt,tt,tp,kp,tp,pp),3,3)
inf=rbind(cbind(aa,m1),cbind(t(m1),m2))

# Variance-covariance matrix for parameters, inverse of information
# matrix
vcov=solve(inf)

# Initialize simulation array to keep simulation results
sim=matrix(0,0,m+1)
smn=matrix(0,0,m+1)

# Simulation for distribution of future amounts
# Want 10,000 simulations, but exceeds R capacity, so do
# in batches of 5,000
nsim=5000
smsk=aperm(array(c(msk),c(m,n,nsim)),c(3,1,2))
smsn=aperm(array(c(msn),c(m,n,nsim)),c(3,1,2))

for (i in 1:5) {

# Randomly generate parameters from multivariate normal
spar=rmvnorm(nsim,mle$par,vcov)

# Arrays to calculate simulated means
ttoi=array(c(outer(spar[,n+2],1:m,"^")),c(nsim,m,n))
```

```
alph=aperm(array(c(spar[,1:n]),c(nsim,n,m)),c(1,3,2))
esim=alph*ttoi

# Arrays to calculate simulated variances
ksim=array(exp(outer(spar[,n+1],log(dnom),"-")),c(nsim,m,n))
psim=array(spar[,n+3],c(nsim,m,n))
vsim=ksim*(esim^2)^psim

# Randomly simulate future averages
temp=array(rnorm(nsim*m*n,c(esim),sqrt(c(vsim))),c(nsim,m,n))

# Combine to total by exposure period and in aggregate
# notice separate array with name ending in "n" to capture
# forecast for next accounting period
sdnm=t(matrix(dnom,m,nsim))
fore=sdnm*rowSums(temp*!smsk,dims=2)
forn=sdnm*rowSums(temp*smsn,dims=2)

# Cumulate and return for another 5,000
sim=rbind(sim,cbind(fore,rowSums(fore)))
smn=rbind(smn,cbind(forn,rowSums(forn)))
}

summary(sim)
summary(smn)

# Scatter plots of residuals & Distribution of Forecasts
windows()
par(mfrow=c(2,2))
plot(na.omit(cbind(c(r+c-1),c(stres))),
  main="Standardized Residuals by CY",xlab="CY",
  ylab="Standardized Residual",pch=18)
plot(na.omit(cbind(c(r),c(stres))),
  main="Standardized Residuals by AY",xlab="AY",
  ylab="Standardized Residual",pch=18)
plot(na.omit(cbind(c(c),c(stres))),
  main="Standardized Residuals by Lag",xlab="Lag",
  ylab="Standardized Residual",pch=18)
proc=list(x=(density(sim[,m+1]))$x,
    y=dnorm((density(sim[,m+1]))$x,
      sum(matrix(c(dnom),m,n)*mean*!msk),
      sqrt(sum(matrix(c(dnom),m,n)^2*var*!msk))))
truehist(sim[,m+1],ymax=max(proc$y),
  main="All Years Combined Future Amounts",xlab="Aggregate")
lines(proc)

# Summary of mean, standard deviation, and 90% confidence interval from
# simulation, similar for one-period forecast
sumr=matrix(0,0,4)
sumn=matrix(0,0,4)

for (i in 1:(m+1)) {
sumr=rbind(sumr,c(mean(sim[,i]),sd(sim[,i]),quantile(sim[,i],c(.05,.95))))
sumn=rbind(sumn,c(mean(smn[,i]),sd(smn[,i]),quantile(smn[,i],c(.05,.95))))
}
```

5. REFERENCES

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Abbreviations and notations
MLE, maximum likelihood estimator

Biography of the Author

Roger Hayne is a Consulting Actuary in the Pasadena, CA office of Milliman, Inc. He is a Fellow of the Casualty Actuarial Society, a Member of the American Academy of Actuaries and holds a Ph.D. in mathematics from the University of California. Roger is an active volunteer in the CAS, serving on several CAS committees and task forces as chair of several, and also as Vice President – Research and Development for the CAS. He has published numerous papers in the *CAS Forum*, the *Proceedings of the Casualty Actuarial Society (PCAS)*, and *Variance*. One of his *PCAS* papers was awarded the 1995 Dorweiller Prize.

Exhibit 1

Incremental Adjusted Average Paid Losses Per Ultimate Claim
Berquist & Sherman Automobile Liability Data

Accident Year	Months of Development 12	24	36	48	60	72	84	96	Forecast Counts
1969	178.73	361.03	283.69	264.00	137.94	61.49	15.47	8.82	7,822
1970	196.56	393.24	314.62	266.89	132.46	49.57	33.66		8,674
1971	194.77	425.13	342.91	269.45	131.66	66.73			9,950
1972	226.11	509.39	403.20	289.89	158.93				9,690
1973	263.09	559.85	422.42	347.76					9,590
1974	286.81	633.67	586.68						7,810
1975	329.96	804.75							8,092
1976	368.84								7,594

Estimates

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
Parameter	143.78	316.77	251.78	197.68	102.53	46.23	21.36	7.36
Std. Error	6.20	11.54	9.16	7.62	5.25	3.75	3.07	2.41

	z	τ	p
Parameter	8.5871	1.1265	0.5782
Std. Error	0.2321	0.0077	0.0303

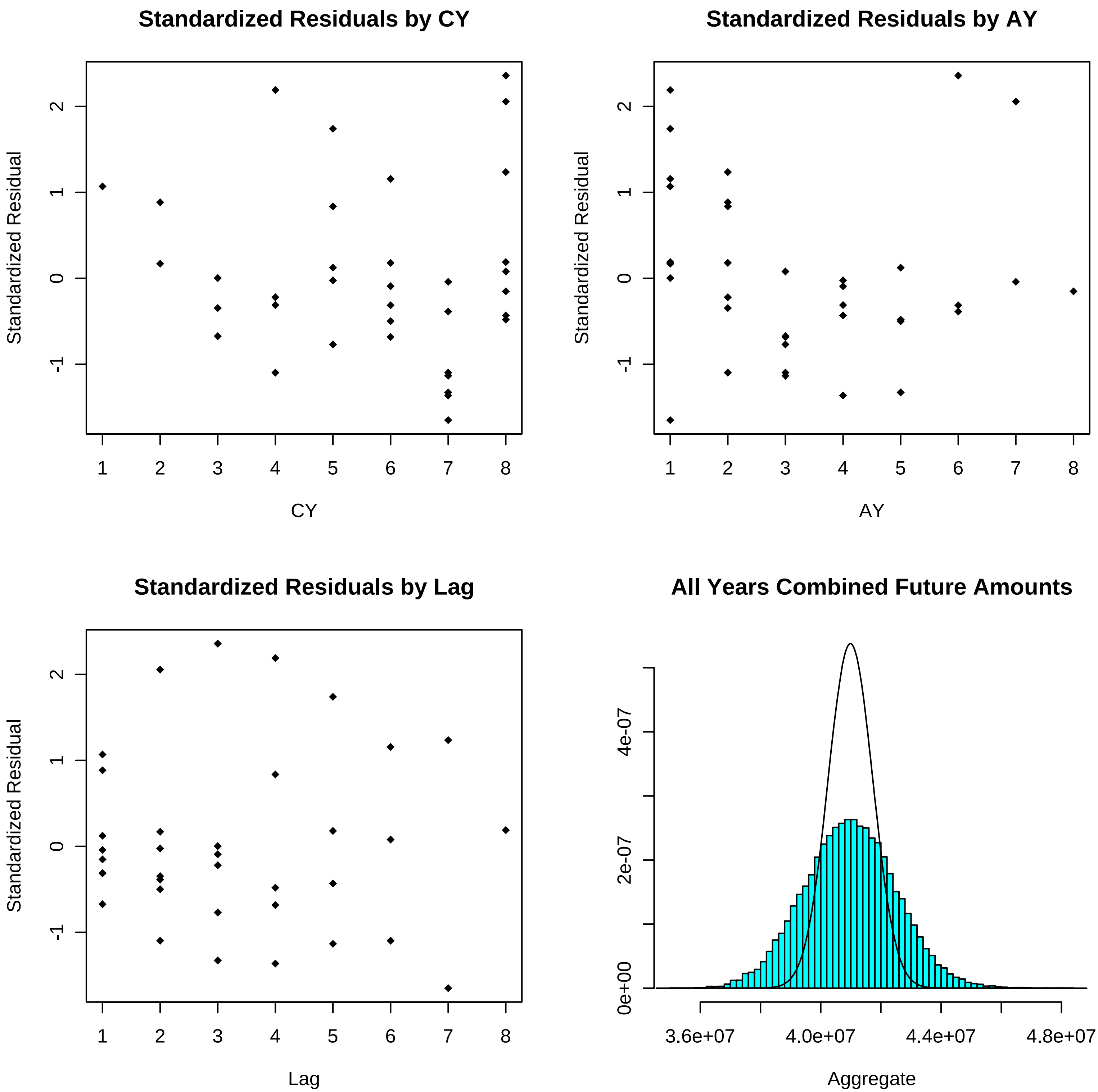
Standardized Residuals by CY
Standardized Residual
CY
Standardized Residuals by AY
Standardized Residual
AY
Standardized Residuals by Lag
Standardized Residual
Lag
All Years Combined Future Amounts
0e+00
2e-07
4e-07
3.6e+07
4.0e+07
4.4e+07
4.8e+07
Aggregate

Exhibit 3

Incremental Adjusted Average Paid Losses Per Ultimate Claim
Berquist & Sherman Automobile Liability Data

Forecast Expected

Accident Year	Months of Development 24	36	48	60	72	84	96	Total
1969								
1970							9.34	9.34
1971						30.54	10.52	41.06
1972					74.43	34.40	11.85	120.68
1973				185.96	83.84	38.75	13.34	321.90
1974			403.89	209.48	94.45	43.65	15.03	766.50
1975		579.48	454.96	235.97	106.39	49.17	16.93	1,442.91
1976	821.26	652.77	512.50	265.81	119.84	55.39	19.07	2,446.64

Forecast Variance

Accident Year	Months of Development 24	36	48	60	72	84	96	Total
1969								
1970							8.19	8.19
1971						28.10	8.19	36.29
1972					80.84	33.11	9.65	123.60
1973				235.51	93.74	38.40	11.19	378.84
1974			709.12	331.88	132.10	54.11	15.77	1,242.97
1975		1,039.02	785.45	367.61	146.32	59.93	17.47	2,415.80
1976	1,657.07	1,270.62	960.54	449.55	178.93	73.29	21.36	4,611.37

Exhibit 4

Incremental Adjusted Average Paid Losses Per Ultimate Claim
Berquist & Sherman Automobile Liability Data

Estimates of Accident Year Future Loss Forecasts

	Process Only		Including Parameter Uncertainty			
Accident		Standard		Standard	Percentile	
Year	Mean	Deviation	Mean	Deviation	5%	95%
1969	0	0	0	0	0	0
1970	80,981	26,503	80,551	36,442	24,148	144,035
1971	408,500	63,754	407,019	82,070	274,928	545,616
1972	1,169,365	106,448	1,169,765	137,850	945,662	1,399,015
1973	3,087,023	172,060	3,086,394	233,709	2,702,457	3,476,160
1974	5,986,335	216,225	5,984,922	344,212	5,425,005	6,551,203
1975	11,676,044	307,380	11,671,230	549,685	10,783,705	12,583,860
1976	18,579,788	375,626	18,581,701	808,465	17,258,898	19,916,569
Total	40,988,036	572,742	40,981,581	1,513,557	38,528,696	43,485,373

Forecasts for Next Calendar Year

	Process Only		Including Parameter Uncertainty			
Accident		Standard		Standard	Percentile	
Year	Mean	Deviation	Mean	Deviation	5%	95%
1969	0	0	0	0	0	0
1970	80,981	24,817	80,551	36,442	24,148	144,035
1971	303,859	52,742	302,553	68,934	192,431	418,164
1972	721,230	87,122	721,793	105,826	551,032	898,662
1973	1,783,372	147,171	1,783,236	172,967	1,502,286	2,075,631
1974	3,154,365	207,974	3,154,597	240,834	2,764,684	3,559,245
1975	4,689,180	260,836	4,686,348	309,909	4,179,644	5,204,351
1976	6,236,615	309,130	6,236,267	372,667	5,629,261	6,854,599
Total	16,969,602	489,384	16,965,345	652,968	15,893,889	18,045,385