Risk Margins in Fair Value Loss Reserves:
Required Capital for Unpaid Losses and its Cost

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Abstract

There is a general consensus that, in the absence of a trading market for loss reserves, a reasonable estimate of the "fair value" of unpaid losses is the risk-free present value of an unbiased estimate of those losses plus a market-based risk margin. If the risk margin is defined as the risk-free present value of the market-clearing cost of the capital required to support the unpaid losses during the run-off period, the size of the risk margin depends on the amount of required capital. Existing literature shows how to calculate the risk margin in cases where the amount of required capital is specified exogenously. However, the European Solvency II directive defines the capital requirement as of any given date as an endogenous variable equal to the amount needed to ensure solvency over a one-year time horizon with 99.5% confidence. This paper derives and illustrates an integrated framework for quantifying the required capital, the implied cost-of-capital-based risk margin and the fair value reserve from the expected volatility, payment and other characteristics of an unpaid loss portfolio consistent with the Solvency II standard. The conceptual framework presented has application to both fair value reserving and economic capital modeling.

Keywords. Economic capital, fair value loss reserve, hindsight loss reserve estimate, risk margin, Solvency II, stochastic loss reserve modeling

1. INTRODUCTION

In a 2007 Casualty Actuarial Forum paper entitled "Consistent Measurement of Property-Casualty Risk-Based Capital Adequacy" [6], Wacek included formulas for calculation of the transfer value (or "fair value") of unbiased reserves for unpaid losses. His underlying premise was that the risk margin embedded in the fair value reserve is based on the market cost of the capital required to support the unpaid losses as they run off[1]. The risk margin formulas derived in that paper are easily applied in cases where the amount of required capital has already been explicitly specified. However, the paper provided no guidance on how to proceed in cases where required capital is defined indirectly as a function of potential loss reserve outcomes, e.g., in terms of the one-year Expected Policyholder Deficit (EPD), Value-at-Risk (VaR) or Tail Value-at-Risk (TVaR) at some specified target confidence level.

This paper partially fills the gap in [6] by presenting formulas and a procedure for determining the fair value of unpaid losses in the case where the capital requirement is based

[1] This is the Solvency II definition, which, according to PricewaterhouseCoopers, is intended to be "a market-consistent 'economic' approach to valuation of assets and liabilities." This approach is also "conceptually in line with proposals for a revised IFRS for insurance contracts." For more background on Solvency II and its implications, see PWC's 2007 paper, "Gearing up for Solvency II" [4]. The quotes included here are from that paper. IFRS refers to "International Financial Reporting Standards."

on a target VaR measure instead of on pre-specified capital-to-reserve ratios. In addition, it incorporates more realistic assumptions about interest rates, in particular, that rates can vary by maturity and over time. The focus here is on VaR, because the capital adequacy standard embedded in the European Solvency II directive is based on a Value-at-Risk measure at the 99.5% confidence level ($VaR_{99.5\%}$).

Other authors have explored the issue of risk margins in fair value reserves. In a 2004 research project partially funded by the Casualty Actuarial Society, Tillinghast actuaries Conger, Hurley and Lowe and PricewaterhouseCoopers actuaries Gutterman, Littmann, Tarrant and Thomas estimated market-based risk margins using various approaches [1] [5][2]. Conger, Hurley and Lowe estimated historical risk margins without reference to capital. Gutterman et al included a cost-of-capital method among the four approaches they modeled. In a 2006 paper Feldblum [3] discussed a cost-of-capital approach to determining risk margins. However, like Wacek [6], both Feldblum and Gutterman et al treated the amount of required capital in their cost-of-capital models as an exogenous variable. In contrast, in this paper we model required capital as an endogenous variable. We show how to use the characteristics of the unpaid loss portfolio itself to determine the amount of capital implied by the Solvency II $VaR_{99.5\%}$ standard, the risk margin based on the cost of that capital and, ultimately, the fair value of the unpaid losses.

The paper comprises four main sections including this introduction, plus two appendices. In Section 2, using the Solvency II conceptual framework, we derive the key formulas and a recursive procedure for the calculation of required capital, risk margins and fair value reserves for unpaid losses. While that section includes a number of formulas, some of which *look* daunting, the fact is that the mathematics does not go beyond basic algebra and probability concepts. In Section 3 we illustrate a detailed practical application of the formulas and procedure presented in Section 2. In Section 4 we briefly summarize the key points and implications of the paper, and identify some areas for further research. Appendix A describes how to determine forward interest rates from the standard yield curve. Appendix B presents the derivation of a formula used in Section 2. A complete list of abbreviations and notations appears after the appendices followed by a list of references.

The terminology and notation used in this paper are generally consistent with [6][3]. That paper used the term "transfer value" rather than "fair value" but the meaning of both terms

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[2] We provide separate references to these two self-contained papers published together in one volume.

[3] Familiarity with that paper is assumed, especially with the contents of sections 1, 2 (particularly 2.2 and 2.3) and Appendix B.

with respect to loss reserves is the same. Because "fair value" has become the more popular term, we adopt that usage in this paper.

2. DETERMINING THE FAIR VALUE OF UNPAID LOSSES

Conceptually, the fair value loss reserve is intended to be the price at which the loss reserve liability could be irrevocably transferred to a third party[4]. Because loss reserves are not normally traded, it is impossible to observe market prices directly. Instead, the fair value reserve must be determined indirectly as the risk-free present value of unpaid losses plus a risk margin reflecting the market-clearing cost of the capital required to minimize the risk of insolvency over a one-year time horizon due to loss reserve inadequacy.

In formula terms the fair value $T(L_n)$ [5] of unpaid losses L_n at time *n* is the sum:

$$T(L_n) = PV(L_n) + R'_n, \tag{2.1}$$

where $PV(L_n)$ is the risk-free present value sum of the future loss and R'_n is the loss reserve risk margin, both as of time *n*.

2.1 The Present Value $PV(L_n)$

The calculation of $PV(L_n)$ requires knowledge of the amounts and timing of the expected future loss payments $P_{n+1}, P_{n+2}, P_{n+3}, \ldots, P_{n+k}$ [6], where *k* represents the number of future loss payments, as well as an assumption about the risk-free yield curve. If we assume a flat yield curve, i.e., that the risk-free rate is the same irrespective of the time to maturity, we can use a single rate *r* in our present value analysis. In that case, if we assume that all loss payments are made at the midpoint of each payment year, then the value of $PV(L_n)$ is given by the formula:

$$\begin{aligned} PV(L_n) &= (1+\tfrac{1}{2}r)\cdot P_{n+1}\cdot v + (1+\tfrac{1}{2}r)\cdot P_{n+2}\cdot v^2 + (1+\tfrac{1}{2}r)\cdot P_{n+3}\cdot v^3 + \ldots + (1+\tfrac{1}{2}r)\cdot P_{n+k}\cdot v^k \\ &= (1+\tfrac{1}{2}r)\cdot(P_{n+1}\cdot v + P_{n+2}\cdot v^2 + P_{n+3}\cdot v^3 + \ldots + P_{n+k}\cdot v^k), \end{aligned} \tag{2.2}$$

[4] Our use of the term "loss" should be understood to include claim adjusting and defense costs as well as the administrative expenses associated with managing a portfolio of claims. Those costs would be assumed by a third party in the case of an irrevocable transfer.

[5] We retain the notation $T(L_n)$ to represent the fair value of unpaid losses in order to remain consistent with [6], where the equivalent term "transfer value" was used instead of "fair value".

[6] This definition of P_{n+i}, for $1 \le i \le k-1$, as an *expected value* as of time *n* of a future loss payment matches the one used in Appendix B of [6]. The reader should be aware that in the body of [6], P_{n+i} refers to the *actual* payment in the year ending at time *n+i*.

where $v = (1+r)^{-1}$ and $1+\frac{1}{2}r$ is the adjustment factor required to reflect our assumption that loss payments are made at the midpoint of each year. The flat yield curve assumption allows us to factor $1+\frac{1}{2}r$ out of each term.

However, because our intent is to develop a practical framework that can be applied in the real world, where interest rates typically vary by maturity, we assume that risk-free interest rates can display that kind of variation. That requires us to introduce notation that differentiates between rates for different maturities.

Let r_m represent the annual yield to maturity as of time *n* on the risk-free fixed income instrument maturing in *m* years[7]. $v_m = (1+r_m)^{-1}$ is the corresponding one-year discount factor. r_m is also known as the "spot" rate. The standard yield curve is sometimes called the "spot rate curve" to distinguish it from other curves, including the "forward rate" curve.

The forward rate $r_{f:m}$ as of time *n* is the annual yield, between time *n+f* and *n+f +m*, on risk-free fixed income instruments maturing at or after time *n+f +m*. For a discussion of how forward rates can be derived from the spot rate curve, see Appendix A.

Having introduced the necessary notation, we can now generalize Formula (2.2) to allow for risk-free rates that vary by maturity:

$$\begin{aligned} PV(L_n) = {} & (1+\tfrac{1}{2}r_{0.5:0.5})\cdot P_{n+1}\cdot v_1 + (1+\tfrac{1}{2}r_{1.5:0.5})\cdot P_{n+2}\cdot v_2^2 \\ & + (1+\tfrac{1}{2}r_{2.5:0.5})\cdot P_{n+3}\cdot v_3^3 + \ldots + (1+\tfrac{1}{2}r_{k-0.5:0.5})\cdot P_{n+k}\cdot v_k^k \end{aligned} \tag{2.3}$$

where $r_{0.5:0.5}$ is the six-month forward rate for six-month risk-free money, and, in general for integer $0 \le j \le k-1$, $r_{j+0.5:0.5}$ is the *j*-year + six-month forward rate for six-month risk-free money. The factor $1+\frac{1}{2}r_{j+0.5:0.5}$ in each term adjusts the loss payment assumption from year-end to mid-year.

Formula (2.3) accords with an insurer investment policy of buying a set of risk-free zero coupon securities at time *n* to fund the payment of losses plus interest at each year-end. In order to be in a position to meet each expected mid-year loss payment obligation, the insurer simultaneously enters into a set of forward sales of six-month risk-free zero coupon securities, whose par values match the par values of the purchased zeroes.

For example, in accordance with that policy, at time *n* the insurer would purchase a one-year risk-free zero-coupon security having par value $(1+\frac{1}{2}r_{0.5:0.5})\cdot P_{n+1}$ (and market value $(1+\frac{1}{2}r_{0.5:0.5})\cdot P_{n+1}\cdot v_1$) and at the same time enter into a six-month forward sale of a six-month risk-free zero-coupon security with par value $(1+\frac{1}{2}r_{0.5:0.5})\cdot P_{n+1}$ (and forward price of

[7] Note that *m* does not have to be an integer.

P_{n+1}). That combination of purchase and matching forward sale would guarantee proceeds of P_{n+1} at time $n+\frac{1}{2}$, which the insurer could use to make the expected loss payment due at that time[8]. The second year's loss payment would be funded by the purchase, also at time *n*, of two-year risk-free zeroes with par value $(1+\frac{1}{2}r_{1.5:0.5})\cdot P_{n+2}$ and the simultaneous eighteen-month forward sale of six-month risk-free zeroes with the same par value. The funding of the third and subsequent years' loss payments would be addressed in a similar way.

In general, the present value $PV(L_{n+i})$ of the expected unpaid loss L_{n+i} as of time *n*+*i* is given by:

$$\begin{aligned} PV(L_{n+i}) &= (1+\tfrac{1}{2}r_{i+0.5:0.5})\cdot P_{n+i+1}\cdot v_{i:1} + (1+\tfrac{1}{2}r_{i+1.5:0.5})\cdot P_{n+i+2}\cdot v_{i:2}^{2} \\ &+ (1+\tfrac{1}{2}r_{i+2.5:0.5})\cdot P_{n+i+3}\cdot v_{i:3}^{3} + \ldots + (1+\tfrac{1}{2}r_{k-0.5:0.5})\cdot P_{n+k}\cdot v_{i:k}^{k} \end{aligned} \tag{2.4}$$

for integers $0 \le i \le k-1$, where $v_{i:m}$ represents the *i*-year forward one-year discount factor as of time *n* for *m*-year risk-free money.

2.2 The Risk Margin R'_n

The risk margin R'_n is the second component of the fair value $T(L_n)$ of unpaid losses L_n at time *n*. It is the risk-free present value sum of expected future risk charges based on the market cost of the capital required at time *n* and beyond to support the unpaid losses as they run off.

Let C_n^R represent the amount of capital required at time *n* to support the unpaid losses L_n for the next year (to time *n*+1). It is expected that after the passage of a year the unpaid loss amount will be L_{n+1} and that the amount of capital required at time *n*+1 for the following year will be C_{n+1}^R. In general, based on the sequence of expected unpaid loss amounts $L_n, L_{n+1}, L_{n+2}, \cdots, L_{n+k-1}$[9] at times *n*, *n*+1, *n*+2, …, *n*+*k*-1, respectively, we can anticipate that a sequence of expected capital amounts $C_n^R, C_{n+1}^R, C_{n+1}^R, \cdots, C_{n+k-1}^R$ will be needed to support the unpaid losses as they run off.

We assume that the capital provider demands the market-clearing annualized after-tax return on equity of *roe* for each year the capital is exposed. Assuming a constant market

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[8] An equivalent alternative, of course, would be simply to buy a six-month zero-coupon instrument. Indeed, we could have expressed Formula (2.3) in terms of discount factors corresponding to an initial maturity six months out and at annual intervals thereafter. However, for the purposes of our presentation it is helpful to arrange for all cash flows to occur at the end of each year.

[9] This definition of L_{n+i}, for $1 \le i \le k-1$, as an *expected value* as of time *n* of a future unpaid loss amount matches the one used in Appendix B of [6]. The reader should be aware that in the body of [6], L_{n+i} refers to the *actual* unpaid loss amount at time *n*+*i*.

return on equity requirement and given a market-clearing tax rate of *tax*, the annual pre-tax return requirement on the required capital is $roe_{PT} = \dfrac{roe}{1-tax}$, a portion of which will be provided by the risk-free interest earned on the capital itself. For example, if r_1 is the one-year risk-free rate as of time *n*, then the portion of the required rate of return on capital C_n^R between time *n* and time *n*+1 that must come from the underwriting assets set aside to fund unpaid losses is $roe_{PT} - r_1$. The cost of the capital required to support the unpaid losses L_n for this first year of the run-off is $(roe_{PT} - r_1) \cdot C_n^R$.

The comparable expected rate of return on C_{n+1}^R between times *n*+1 and *n*+2 is $roe_{PT} - r_{1:1}$, where $r_{1:1}$ is the one-year forward rate as of time *n* for one-year money. We use the forward rate $r_{1:1}$ in order to match the expected deployment of C_{n+1}^R at time *n*+1. The use of the forward rate mimics the effect of entering into a one-year forward contract at time *n* to invest C_{n+1}^R in a one-year zero-coupon security at time *n*+1. The cost of the capital required to support the expected remaining unpaid losses L_{n+1} for this second year of the run-off is $(roe_{PT} - r_{1:1}) \cdot C_{n+1}^R$.

The annual costs, expected as of time *n*, related to the capital to support unpaid losses over the entire the run-off period are represented by the sequence $(roe_{PT} - r_1) \cdot C_n^R$, $(roe_{PT} - r_{1:1}) \cdot C_{n+1}^R$, $(roe_{PT} - r_{2:1}) \cdot C_{n+2}^R$, …, $(roe_{PT} - r_{k-1:1}) \cdot C_{n+k-1}^R$, where $r_{i:1}$ is the *i*-year forward rate as of time *n* for one-year money and $0 \le i \le k-1$[10].

We are now in a position to express R_n' as the following present value sum:

$$\begin{aligned} R_n' &= (roe_{PT} - r_1) \cdot C_n^R \cdot v_1 + (roe_{PT} - r_{1:1}) \cdot C_{n+1}^R \cdot v_2^2 + (roe_{PT} - r_{2:1}) \cdot C_{n+2}^R \cdot v_3^3 \\ &\quad + \ldots + (roe_{PT} - r_{k-1:1}) \cdot C_{n+k-1}^R \cdot v_k^k \end{aligned} \tag{2.5}$$

where v_1, v_2, v_3,…, v_k are the one-year risk-free discount factors implied by the yields at time *n* on fixed income instruments maturing in one, two, three, …, and *k* years, respectively[11].

R_n' can also be expressed recursively in terms of the risk margin R_{n+1}' associated with the expected unpaid losses L_{n+1} at time *n*+1:

$$R_n' = v_1 \cdot ((roe_{PT} - r_1) \cdot C_n^R + R_{n+1}'), \tag{2.6}$$

where

[10] Because the zero-year forward rate as of time *n* for one-year money is the same as the "spot" rate, we can use the notation $r_{0:1}$ and r_1 interchangeably.

[11] $v_{i+1} = (1 + r_{i+1})^{-1}$ for integer $0 \le i \le k-1$.

$$R'_{n+1} = (roe_{PT} - r_{1:1}) \cdot C^R_{n+1} \cdot v_{1:1} + (roe_{PT} - r_{2:1}) \cdot C^R_{n+2} \cdot v_{1:1} \cdot v_{2:1}$$
$$+ (roe_{PT} - r_{3:1}) \cdot C^R_{n+3} \cdot v_{1:1} \cdot v_{2:1} \cdot v_{3:1} + \ldots + (roe_{PT} - r_{k-1:1}) \cdot C^R_{n+k-1} \cdot v_{1:1} \cdot v_{2:1} \cdot v_{3:1} \cdots v_{k-1:1}$$

See Appendix B for a derivation of Formula (2.6).

If the sequence of expected required capital $C^R_n, C^R_{n+1}, C^R_{n+1}, \cdots, C^R_{n+k-1}$ is known, whether from the application of prescribed capital factors or through some other means, we can use Formula (2.5) or (2.6) to first determine R'_n and then Formula (2.1) to determine $T(L_n)$[12].

In general, the risk margin R'_{n+i} associated with the expected unpaid losses L_{n+i} at time $n+i$ can be expressed in terms of the risk margin R'_{n+i+1} associated with the expected unpaid losses L_{n+i+1} one year later at time $n+i+1$:

$$R'_{n+i} = v_{i:1} \cdot ((roe_{PT} - r_{i:1}) \cdot C^R_{n+i} + R'_{n+i+1}), \tag{2.7}$$

where $0 \le i \le k-1$.

2.3 Funding Assets S_{n+1} and Funding Need t_{n+1}

Let's assume that the required capital sequence has not been directly specified, and that instead the capital requirement has been described in the form of the objective to ensure that the one-year probability of insolvency due to fair value loss reserve inadequacy is no more than $1-\alpha$. In Value-at-Risk terms that implies capital calibration at the α confidence level and a time horizon of one year[13].

Specifically, that objective establishes the required capital C^R_n at time n as the amount needed in addition to assets equal to the fair value $T(L_n)$ of the unpaid losses L_n to ensure (with a probability of α) adequate funding of those unpaid losses L_n one year out (at time n+1). The total *funding assets* available at time n+1, including accumulated interest at the risk-free rate r_1, will be the amount defined by $S_{n+1} = (T(L_n) + C^R_n) \cdot (1 + r_1)$.

The *funding need* at time n+1 will be the amount equal to the fair value of the one-year hindsight estimate of L_n. The term *one-year hindsight estimate of* L_n is a succinct way of referring to the unpaid losses remaining at time n+1 plus the losses paid during the preceding year. It can be represented at time n by the random variable $h_{n+1} = l_{n+1} + p_{n+1}$, where l_{n+1} and p_{n+1} are also random variables defined as of time n that correspond to the unpaid and paid loss components, respectively, of the hindsight estimate.

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12 We assume that all other parameters needed for Formulas (2.1) and (2.5) or (2.6) are known.

13 Under Solvency II, α =99.5%.

Let $t_{n+1} = T(h_{n+1})$ represent the random variable, defined as of time *n*, for the fair value of the one-year hindsight estimate of L_n at time *n*+1. The fair value of the one-year hindsight estimate is the sum of the fair values of the unpaid and paid loss components[14]. The fair value of the time *n*+1 unpaid loss estimate is the sum of its present value and the associated risk margin. Because paid losses require no capital support, the fair value of the paid component as of time *n*+1 simply reflects an interest adjustment. Putting all of this together allows us to express t_{n+1} as:

$$\begin{aligned} t_{n+1} &= T(l_{n+1}) + T(p_{n+1}) \\ &= PV(l_{n+1}) + R'_{n+1}(l_{n+1}) + p_{n+1} \cdot (1 + \tfrac{1}{2} r_{0.5:0.5}) \end{aligned} \tag{2.8}$$

where $R'_{n+1}(l_{n+1})$ is the random variable for the required risk margin associated with the unpaid loss component l_{n+1}. We can recombine the terms in (2.8) involving l_{n+1} and p_{n+1} to express t_{n+1} more succinctly as:

$$t_{n+1} = PV(h_{n+1}) + R'_{n+1}(l_{n+1}), \tag{2.9}$$

where $PV(h_{n+1}) = PV(l_{n+1}) + p_{n+1} \cdot (1 + \tfrac{1}{2} r_{0.5:0.5})$ represents the random variable for the present value of the one-year hindsight estimate of L_n as of time *n*+1.

If we assume that the relationship between the risk margin R'_{n+1} associated with the expected unpaid loss L_{n+1} at time *n*+1 and the present value $PV(L_{n+1})$ of that unpaid loss, embodied in the ratio $\dfrac{R'_{n+1}}{PV(L_{n+1})}$, is representative of the general relationship between the risk margin and the present value of the associated unpaid loss at time *n*+1, then we can express $R'_{n+1}(l_{n+1})$ as follows:

$$R'_{n+1}(l_{n+1}) = PV(l_{n+1}) \cdot \frac{R'_{n+1}}{PV(L_{n+1})} \tag{2.10}$$

and we can then rewrite Formula (2.9) as:

$$t_{n+1} = PV(h_{n+1}) + PV(l_{n+1}) \cdot \frac{R'_{n+1}}{PV(L_{n+1})} \tag{2.11}$$

[14] See Section 2.2 or 2.3 of [6].

2.4 Solving for C_n^R

We can express the one-year solvency objective in terms of the relationship between the funding need and the funding assets at time n+1 in the following alternative, but equivalent, ways:

$$\Pr ob(t_{n+1} \geq S_{n+1}) \leq 1 - \alpha \,, \tag{2.12}$$

$$\int_0^{S_{n+1}} t_{n+1} dt_{n+1} \geq \alpha \tag{2.13}$$

$$VaR_{\alpha}(t_{n+1}) = T_{n+1}^{-1}(\alpha) \leq S_{n+1} \,, \tag{2.14}$$

where $VaR_{\alpha}(t_{n+1})$ refers to the Value-at-Risk with respect to t_{n+1} at the α confidence level and T_{n+1}^{-1} represents the inverse distribution function of t_{n+1}, both of which define the funding need at the α confidence level.

The value of C_n^R that satisfies the following equilibrium relationship between the funding need at the α confidence level at time n+1 and the available funding assets at that time represents the amount of capital required at time n to support unpaid losses of L_n:

$$VaR_{\alpha}(t_{n+1}) = (T(L_n) + C_n^R) \cdot (1 + r_1) \tag{2.15}$$

Solving for C_n^R, we arrive at:

$$C_n^R = v_1 \cdot VaR_{\alpha}(t_{n+1}) - T(L_n) \,, \tag{2.16}$$

where $v_1 = (1 + r_1)^{-1}$ represents the one-year risk-free discount factor as of time n.

Using Formula (2.6) to expand Formula (2.1) we obtain the following formula for $T(L_n)$ in terms of C_n^R and R'_{n+1}:

$$T(L_n) = PV(L_n) + v_1 \cdot ((roe_{PT} - r_1) \cdot C_n^R + R'_{n+1}) \,, \tag{2.17}$$

Substituting the Formula (2.17) expression for $T(L_n)$ into (2.16) and isolating C_n^R, we obtain a revised formula for required capital at time n:

$$\begin{aligned} C_n^R &= \frac{v_1 \cdot VaR_{\alpha}(t_{n+1}) - (PV(L_n) + v_1 \cdot R'_{n+1})}{1 + v_1 \cdot (roe_{PT} - r_1)} \\ &= \frac{VaR_{\alpha}(t_{n+1}) - (PV(L_n) \cdot (1 + r_1) + R'_{n+1})}{1 + roe_{PT}} \end{aligned} \tag{2.18}$$

As a fair value quantity the α-quantile funding need $VaR_{\alpha}(t_{n+1})$ includes an embedded risk margin, which we can isolate using Formula (2.11):

$$VaR_{\alpha}(t_{n+1}) = VaR_{\alpha}(PV(h_{n+1})) + PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1}))) \cdot \frac{R'_{n+1}}{PV(L_{n+1})} \quad (2.19)$$

where $VaR_{\alpha}(PV(h_{n+1}))$ represents the time n+1 present value of the one-year hindsight loss estimate at the α confidence level and $PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1}))$ represents the time n+1 present value of the *unpaid loss* component of $VaR_{\alpha}(PV(h_{n+1}))$ [15].

We can see from Formula (2.19) that the α-quantile funding need at time n+1 contemplates funding for not only the run-off of the unpaid losses but also for the risk margin needed to cover the cost of the capital required to support the unpaid losses during the run-off period.

If we substitute the Formula (2.19) expression of $VaR_{\alpha}(t_{n+1})$ into Formula (2.18) and rearrange the terms in the numerator, we obtain the following formula for C_n^R:

$$C_n^R = \frac{VaR_{\alpha}(PV(h_{n+1})) + PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1}))) \cdot \frac{R'_{n+1}}{PV(L_{n+1})} - PV(L_n) \cdot (1+r_1) - R'_{n+1}}{1 + roe_{PT}}$$

$$= \frac{VaR_{\alpha}(PV(h_{n+1})) - PV(L_n) \cdot (1+r_1) + R'_{n+1} \cdot \left(\frac{PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1})))}{PV(L_{n+1})} - 1 \right)}{1 + roe_{PT}}$$

$$= \frac{F_{n+1} + f_{n+1} \cdot R'_{n+1}}{1 + roe_{PT}} \quad (2.20)$$

where

$$F_{n+1} = VaR_{\alpha}(PV(h_{n+1})) - PV(L_n) \cdot (1 + r_1) \quad (2.21)$$

defines the additional amount needed at time n+1 to bring present value loss funding up to the α confidence level, and

$$f_{n+1} = \frac{PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1})))}{PV(L_{n+1})} - 1 \quad (2.22)$$

[15] The notation $PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1})))$ is intended to convey the idea that the random variable $PV(l_{n+1})$ collapses to a single present value unpaid loss amount when conditioned on the specific present value hindsight estimate $VaR_{\alpha}(PV((h_{n+1}))$ and that that present value unpaid loss amount is the one included within $VaR_{\alpha}(PV((h_{n+1}))$.

is the percentage by which the time n+1 present value unpaid losses embedded in the one-year hindsight estimate at the α confidence level exceed the expected time n+1 present value unpaid loss amount.

The general formula for the anticipated future capital C^{R}_{n+i} required to support the expected unpaid losses L_{n+i} at time n+i, for $0 \le i \le k-1$, is given by:

$$C^{R}_{n+i} = \frac{F_{n+i+1} + f_{n+i+1} \cdot R'_{n+i+1}}{1 + roe_{PT}} \tag{2.23}$$

where

$$F_{n+i+1} = VaR_{\alpha}(PV(h_{n+i+1})) - PV(L_{n+i}) \cdot (1 + r_{i:1}) \tag{2.24}$$

$$f_{n+i+1} = \frac{PV(L_{n+i+1} \mid VaR_{\alpha}(PV(h_{n+i+1})))}{PV(L_{n+i+1})} - 1 \tag{2.25}$$

2.5 Recursive Procedure for C^{R}_{n} and R'_{n}

The expected unpaid loss amount L_{n+k} at time n+k is zero. At that point and beyond, the capital requirement C^{R}_{n+k} and the risk margin R'_{n+k} also are zero. At time n+k-1, because the terms depending on R'_{n+k} drop out, Formulas (2.23) and (2.6) simplify to:

$$C^{R}_{n+k-1} = \frac{F_{n+k}}{1 + roe_{PT}} \tag{2.26}$$

and

$$R'_{n+k-1} = v_{n+k-1:1} \cdot (roe_{PT} - r_{n+k-1:1}) \cdot C^{R}_{n+k-1} \tag{2.27}$$

By working recursively backward from time n+k-1, it is possible to determine the required capital and risk charges at any time from n through n+k-1. This can be achieved by the executing the following procedure, the first four steps of which do not rely on recursive relationships:

1) Tabulate risk-free spot rates r_m for $0 \le m \le k$ and the implied forward rates for one-year maturities based on Formula (A.2)[16].
2) Calculate and tabulate $PV(L_{n+i})$ for $0 \le i \le k-1$ using Formula (2.4).

[16] We suggest U.S. Treasury rates, but we acknowledge that others may prefer a different risk-free benchmark.

3) Model $PV(h_{n+i+1}) = PV(l_{n+i+1}) + p_{n+i+1} \cdot (1 + \frac{1}{2} r_{i:1})$ for $0 \le i \le k-1$ and tabulate $VaR_{\alpha}(PV(h_{n+i+1}))$ and $PV(L_{n+i+1} \mid VaR_{\alpha}(PV(h_{n+i+1}))$[17].

4) Calculate and tabulate F_{n+i+1} for $0 \le i \le k-1$ using Formula (2.24).

5) Calculate and tabulate f_{n+i+1} for $0 \le i \le k-1$ using Formula (2.25).

6) Calculate C^{R}_{n+k-1} and R'_{n+k-1} using the following recursive procedure:

   a) First calculate C^{R}_{n+k-1} using Formula (2.26) and then R'_{n+k-1} (a function of C^{R}_{n+k-1}) using (2.27).

   b) Calculate C^{R}_{n+k-2} (a function of R'_{n+k-1}) and R'_{n+k-2} (a function of R'_{n+k-1} and C^{R}_{n+k-2}), in that order, using Formulas (2.23) and (2.7), respectively, with the formula subscript *i* replaced in every case by *k*-2.

   c) Similarly, calculate C^{R}_{n+k-3} and R'_{n+k-3}, in that order, using Formulas (2.23) and (2.7), respectively, with the formula subscript *i* replaced in every case by *k*-3.

   d) Continue stepwise in this fashion by decrementing the subscript by one and calculating the values of C^{R}_{n+i} and R'_{n+i}, in that order, using Formulas (2.23) and (2.7), respectively, with the formula subscript *i* chosen to reflect the decremented subscript for the step. Repeat until C^{R}_{n} and R'_{n} have been calculated, and then stop. The required capital and the required risk margin as of time *n* have been determined.

7) Calculate and tabulate required capital ratios to unpaid losses $c_{n+i} = C^{R}_{n+i} / L_{n+i}$ for $0 \le i \le k-1$. (Optional)

8) Use Formula (2.1) to calculate the fair value of unpaid losses $T(L_n)$ as of time *n*.

3. ILLUSTRATION

In this section we present a realistic illustration of the procedure described in Section 2.5 using unpaid loss and volatility patterns based mainly on Schedule P data reported by a diversified U.S. insurer as of December 31, 2007[18].

For purposes of illustration we make the following assumptions:

1) *n*=2007 corresponds to the valuation date of December 31, 2007.

2) The unbiased unpaid loss estimate as of December 31, 2007 is $L_{2007} = \$10{,}000$.

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[17] Discussion about how to perform this modeling is beyond the scope of this paper. For one approach, see Appendix C of [6]. Another alternative is to fit distributions to historical one-year hindsight loss relationships.
[18] The derivation and discussion of those patterns is beyond the scope of this paper.

3) The applicable risk-free rates are the U.S. Treasury rates as of December 31, 2007 as shown in Table 1[19].

4) The unpaid losses as of December 31, 2007 run off over ten years (k=10) as shown in Table 2.

5) The market-clearing pre-tax return on equity is a constant $roe_{PT} = 18.75\%$, based on market-clearing $roe = 15\%$ and $tax = 20\%$[20].

6) Required capital is calibrated to VaR_α with $\alpha = 99.5\%$ over a one-year time horizon.

7) Forward interest rates for six-month and one-year money maturing on the same date are equal: $r_{j+0.5:0.5} = r_{j:1}$ for $0 \le j \le k-1$[21].

We illustrate the eight steps of the procedure by constructing a series of eight corresponding tables, each of which contains the key inputs and outputs of the respective step. In addition, we provide two additional tables which illustrate the cash flows associated with the fair value reserve run-off (Table 9) and the adequacy of the required capital to ensure fair value funding at the 99.5% confidence level (Table 10).

Table 1 summarizes the risk-free interest rates used in this illustration. The spot rates comprising the U.S. Treasury yield curve as of December 31, 2007 have been tabulated in Column (2) by the number of years m to maturity. For example, the one-year spot rate was 3.34% and the spot rate for the two-year maturity was 3.05%. The m-1 year forward rates for one-year money, derived from the December 2007 spot rates using Formula (A.2) from Appendix A, appear in Column (3). For example the one-year forward rate for one-year money was 2.76%, which was calculated using Formula (A.2) with f=1 and m=1 as follows:

$$r_{1:1} = \frac{(1+r_2)^2}{1+r_1} - 1 = \frac{(1.0305)^2}{1.0334} - 1 = 2.76\%$$

The one-year forward discount factors for one-year money are shown in Column (4). They were calculated from the forward rates in Column (3) using the formula $v_{m-1:1} = \dfrac{1}{1+r_{m-1:1}}$, where $1 \le m \le 10$. For example, the one-year forward one-year discount factor is 97.31%, which was calculated as $v_{1:1} = \dfrac{1}{1+r_{1:1}} = \dfrac{1}{1+2.76\%}$.

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[19] We assume that U.S. Treasury rates are reasonable proxies for risk-free rates. Note that some researchers dispute that notion. That debate is beyond the scope of this paper.

[20] These return on equity and tax rate assumptions are merely illustrative, but are not unrealistic.

[21] The purpose of this assumption is merely to avoid having to introduce of an additional set of forward rates.

TABLE 1

Risk-Free Interest Rate Summary
As of December 31, 2007

(1)	(2)	(3)	(4)
Number of Years	Spot Rate	*m*-1 Year Forward One-Year Rate	*m*-1 Year Forward One-Year Discount Factor
m	r_m [1]	$r_{m-1:1}$ [2]	$v_{m-1:1}$ [3]
1	3.34%	3.34%	96.77%
2	3.05%	2.76%	97.31%
3	3.07%	3.11%	96.98%
4	3.25%	3.79%	96.35%
5	3.45%	4.25%	95.92%
6	3.57%	4.17%	96.00%
7	3.70%	4.48%	95.71%
8	3.81%	4.58%	95.62%
9	3.92%	4.80%	95.42%
10	4.04%	5.13%	95.12%

1 Source: U.S. Treasury website; spot rates shown for 4-year, 8-year and 9-year maturities are interpolated values.
2 Formula (A.2) (Appendix A)
3 Formula (A.3) (Appendix A)

Table 2 shows the expected run-off pattern of the unpaid losses as of December 31, 2007. Column (2) shows initial unpaid losses of \$10,000 as of December 2007 and the expected remaining unpaid losses at successive December valuation dates through 2017, at which time all losses are expected to have been paid. The expected loss payments are shown in Column (3). The first payment of \$2,839 is expected to be made during 2008, and the last payment of \$160 is expected to be made in 2017. Column (4) shows the present value of the expected unpaid losses as of December 31, 2007 and the expected present values of the expected unpaid losses at successive December valuation dates. The December 2007 present value $PV(L_{2007}) = \$9{,}148$ was calculated from the expected losses payments in Column (3) using Formula (2.3) and the risk-free rates tabulated in Table 1[22]. Following the expected loss payment of $2,839 during 2008, the remaining unpaid losses as of December 31, 2008 are expected to be $7,161. The expected present value $PV(L_{2008})$ as of December 2008 of that expected unpaid loss amount is $6,566, which was calculated using Formula (2.4) and forward rates derived from Table 1[23].

TABLE 2

Unpaid Loss Reserve and Expected Run-off
As of December 31, 2007

(1)	(2)	(3)	(4)
Year Ending 12/07+ i	Expected Unpaid Losses	Expected Paid Losses	Expected PV of Unpaid Losses
$n+i$ [1]	L_{n+i}	P_{n+i}	$PV(L_{n+i})$ [2]
2007	$10,000	n/a	$9,148
2008	$7,161	$2,839	$6,566
2009	$5,105	$2,055	$4,664
2010	$3,568	$1,538	$3,247
2011	$2,467	$1,100	$2,249
2012	$1,717	$750	$1,579
2013	$1,206	$511	$1,123
2014	$752	$454	$709
2015	$442	$310	$425
2016	$160	$283	$156
2017	$0	$160	$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Formula 2.4, using Column (3) and forward discount rates based on Table 1

[22] For the mid-year loss payment adjustment we used the simplifying assumption that the forward rates for six-month and one-year money having the same maturity date are the same: $r_{j+0.5:.05} = r_{j:1}$ for $0 \le j \le k-1$.

[23] The actual present value of unpaid losses as of December 31, 2008 can vary from the expected due to a change in interest rates and/or a change in the unpaid loss estimate.

TABLE 3

Present Value Hindsight Statistics One Year Out
99.5% Confidence Level
At Successive Annual Valuation Dates through 2017
Expected as of December 31, 2007

(1)	(2)	(3)	(4)	(5)
Year Ending 12/07+ i	Expected PV of Unpaid Losses	PV of Expected Hindsight One Year Out	PV of 99.5% Hindsight One Year Out	PV of Unpaid in 99.5% Hindsight One Year Out
$n+i$ [1]	$PV(L_{n+i})$ [2]	$PV(L_{n+i}) \cdot (1+r_{i:1})$	$VaR_{99.5\%}(PV(h_{n+i+1}))$ [3]	[4]
2007	\$9,148	\$9,453	\$11,067	\$7,840
2008	\$6,566	\$6,748	\$8,186	\$5,847
2009	\$4,664	\$4,809	\$6,189	\$4,294
2010	\$3,247	\$3,370	\$4,857	\$3,359
2011	\$2,249	\$2,345	\$3,427	\$2,395
2012	\$1,579	\$1,645	\$2,591	\$1,797
2013	\$1,123	\$1,173	\$1,819	\$1,125
2014	\$709	\$742	\$1,257	\$668
2015	\$425	\$445	\$803	\$241
2016	\$156	\$164	\$292	\$0
2017	\$0	\$0	\$0	\$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Table 2, Column (4)
3 From stochastic hindsight loss analysis
4 $PV(L_{n+i+1} \mid VaR_{\alpha}(PV(h_{n+i+1}))$ from stochastic hindsight loss analysis

Table 3 summarizes the key results needed from the modeling of the one-year hindsight loss estimate represented by the random variable $h_{2007+i+1}$ for $0 \le i \le k = 10$. The details underlying that analysis are beyond the scope of this paper, but let us assume that we know the values of $VaR_{99.5\%}(PV(h_{2007+i+1}))$ and $PV(L_{2007+i+1} \mid VaR_{99.5\%}(PV(h_{2007+i+1}))$, which we have tabulated in Columns (4) and (5) respectively. Column (3) shows the expected present value one-year hindsight estimate $PV(L_{2007+i})\cdot(1+r_{i:1})$ as of December 2007+*i*+1, which provides a useful baseline comparison for the 99.5% quantile hindsight estimate in Column (4). Column (2) shows the present value $PV(L_{2007+i})$ as of December 2007+*i* in order to provide context for the entries in Column (3).

For example, as of December 31, 2007 the present value of unpaid losses is $PV(L_{2007}) = \$9{,}148$, as shown in Column (2). Reflecting interest at a rate of $r_1 = 3.34\%$, the expected value of that \$9,148 one year out on December 31, 2008 is \$9,453. That amount, shown in Column (3), is also the present value of the expected hindsight estimate as of that date. The present value as of December 31, 2008 of the one-year hindsight estimate at the 99.5% confidence level $VaR_{99.5\%}(PV(h_{2008}))$ is shown in Column (4) as \$11,067, which is 17% higher than the baseline value of \$9,453. As the loss portfolio runs off, the gap between the 99.5% quantile present value hindsight estimate and the baseline estimate is expected to increase. For example, the expected 99.5% level present value hindsight estimate of December 2011 unpaid losses of $VaR_{99.5\%}(PV(h_{2012})) = \$3{,}427$ is 46% higher than the baseline of \$2,345. By December 2016 the gap is expected to widen further to 78% (\$292 vs. \$164). This pattern is a manifestation of the expectation of increasing one-year volatility in the unpaid loss estimates as the portfolio ages.

Column (5) shows the expected present value one year out from each valuation date of the portion of the one-year hindsight estimate that is expected to remain unpaid as of that date. For example, as of December 31, 2007 the expected December 31, 2008 present value of the unpaid portion of the one-year hindsight estimate of \$11,067 is \$7,840, expressed formally as: $PV(L_{2008} \mid VaR_{99.5\%}(PV(h_{2008}) = \$11{,}067) = \$7{,}840$.

Table 4 illustrates the calculation of $F_{2007+i+1}$ for $0 \le i \le k = 10$, which represents the additional amount needed one year out from each valuation date December 2007+*i* to bring present value loss funding up to the 99.5% confidence level. Columns (2) and (3), both taken from Table 3, represent the expected and the 99.5% confidence level present value hindsight estimates one year out, respectively. For example, as of December 31, 2007 the expected present value of the one-year hindsight estimate one year out is \$9,453. That amount, shown in Column (2), meets the present value loss funding requirement as of December 31, 2008, if the loss payments in 2008 and beyond follow the expected pattern. However, at the 99.5% quantile, the present value one-year hindsight estimate one year out is \$11,067, shown in Column (3), which implies that an additional amount of $F_{2008} = \$1{,}614$ is needed to ensure full present value loss funding one year out at the 99.5% quantile. The additional required funding amounts one year out, shown in Column (4), generally decline as the portfolio runs

TABLE 4

Additional Loss Funding Need One Year Out
99.5% Confidence Level

At Successive Annual Valuation Dates through 2017
Expected as of December 31, 2007

(1)	(2)	(3)	(4)
Year Ending 12/07+ i	PV of Expected Hindsight One Year Out	PV of 99.5% Hindsight One Year Out	Additional Loss Funding Need One Year Out at 99.5%
$n+i$ [1]	$PV(L_{n+i}) \cdot (1+r_{i:1})$ [2]	$VaR_{99.5\%}(PV(h_{n+i+1}))$ [3]	F_{n+i+1} [4]
2007	$9,453	$11,067	$1,614
2008	$6,748	$8,186	$1,439
2009	$4,809	$6,189	$1,381
2010	$3,370	$4,857	$1,487
2011	$2,345	$3,427	$1,082
2012	$1,645	$2,591	$946
2013	$1,173	$1,819	$646
2014	$742	$1,257	$515
2015	$445	$803	$358
2016	$164	$292	$128
2017	$0	$0	$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Table 3, Column (3)
3 Table 3, Column (4)
4 Formula 2.24: (4)-(3)

off, reaching $F_{2017} = \$128$ as of December 31, 2016. The additional funding requirement one year out from December 31, 2017 is $F_{2018} = \$0$, because the final loss payment is expected to occur during 2017.

Table 5 illustrates the calculation of $f_{2007+i+1}$ for $0 \le i \le k-1 = 9$. $f_{2007+i+1}$ is the amount by which the present value of the unpaid loss component of the 99.5% quantile one-year hindsight estimate of the unpaid loss L_{2007+i} as of December 2007+*i*+1 exceeds the expected present value of the unpaid loss at that same valuation date, expressed as a ratio to the latter. The expected present values of unpaid losses one year out $PV(L_{2007+i+1})$ appear in Column (2). The values of $PV(L_{2007+i+1} \mid VaR_{99.5\%}(PV(h_{2007+i+1}))$, representing the present value unpaid loss components of the 99.5% quantile hindsight estimates, are shown in Column (3). Column (4) shows the values of $f_{2007+i+1}$, which are calculated from the entries in Columns (2) and (3) using Formula (2.25). For example, in the row corresponding to the December 31, 2007 valuation date, the entry for f_{2008} in Column (4) of 19.4% is the ratio of the Column (2) entry of \$7,840 to the Column (3) entry of \$6,566, less one. The value $f_{2008} = 19.4\%$ tells us that the expected present value unpaid loss amount one year out $PV(L_{2008} \mid VaR_{99.5\%}(PV(h_{2008})) = \$7{,}840$ embedded in the 99.5% quantile present value one-year hindsight estimate $VaR_{99.5\%}(PV(h_{2008})) = \$11{,}067$ of the December 31, 2007 unpaid loss $L_{2007} = 10{,}000$ is 19.4% higher than the expected present value loss $PV(L_{2008}) = \$6{,}566$ as of December 31, 2008[24]. That in turn implies a 19.4% higher risk margin requirement as of December 31, 2008 at the 99.5% confidence loss level than at the expected loss level. As of December 31, 2016 the risk margin top-up factor one year out is treated as $f_{2017} = 0\%$. Both the expected and 99.5% quantile present value hindsight estimates one year out from December 31, 2016 are zero, which implies that the risk margin $R'_{2017} = \$0$.

Table 6 summarizes the recursive calculation of C^R_{2007+i} and R'_{2007+i} for $0 \le i \le k = 10$. Columns (2) and (4) are retabulations of F_{n+i+1} and f_{n+i+1} from Tables 4 and 5, respectively. Column (3) shows the expected risk margin one year out. This is a retabulation of the risk margins shown in Column (7), shifted by one row. For example, as of December 2007 the expected risk margin *one year out* shown in Column (3) is $R'_{2008} = \$1{,}009$, which is also the amount shown in Column (7) as the expected risk margin as of December 2008. The expected risk margin one year out as of December 2016 is $R'_{2017} = \$0$, because the unpaid loss amount as of December 2017 is zero, which implies no further capital or risk margin requirement.

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[24] See Table 3, Column (4) for $VaR_{99.5\%}(PV(h_{2008})) = \$11{,}067$ and Table 2, Column (2) for $L_{2007} = 10{,}000$.

TABLE 5

Growth in Risk Margin Need One Year Out
99.5% Confidence Level
At Successive Annual Valuation Dates through 2017
Expected as of December 31, 2007

(1)	(2)	(3)	(4)
Year Ending 12/07+ *i*	Expected PV of Unpaid Losses One Year Out	PV of Unpaid in 99.5% Hindsight One Year Out	Additional Risk Margin Need One Year Out at 99.5%
$n+i$ [1]	$PV(L_{n+i+1})$ [2]	[3]	f_{n+i+1} [4]
2007	$6,566	$7,840	19.4%
2008	$4,664	$5,847	25.4%
2009	$3,247	$4,294	32.2%
2010	$2,249	$3,359	49.3%
2011	$1,579	$2,395	51.7%
2012	$1,123	$1,797	60.1%
2013	$709	$1,125	58.6%
2014	$425	$668	57.2%
2015	$156	$241	55.1%
2016	$0	$0	0.0%

1 n=2007, value of $0 \le i \le k-1=9$ implied by valuation year
2 Table 3, Column (2) one row down
3 Table 3, Column (5)
4 Formula 2.25: (3)/(2)-1

TABLE 6

Required Capital and Risk Margins
Calibrated to 99.5% Confidence Level
At Successive Annual Valuation Dates through 2017
Expected as of December 31, 2007

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Year Ending 12/07+ i	Additional Loss Funding Need One Year Out at 99.5%	Expected Risk Margin One Year Out	Additional Risk Margin Need One Year Out at 99.5%	Required Capital	Annual Pre-Tax Cost of Capital (Paid One Year Out)	Expected Risk Margin
$n+i$ [1]	F_{n+i+1} [2]	R'_{n+i+1} [3]	f_{n+i+1} [4]	C^R_{n+i} [5]	[6]	R'_{n+i} [7]
2007	$1,614	$1,009	19.4%	$1,524	$235	$1,204
2008	$1,439	$815	25.4%	$1,386	$222	$1,009
2009	$1,381	$632	32.2%	$1,334	$209	$815
2010	$1,487	$441	49.3%	$1,435	$215	$632
2011	$1,082	$309	51.7%	$1,045	$152	$441
2012	$946	$191	60.1%	$893	$130	$309
2013	$646	$114	58.6%	$601	$86	$191
2014	$515	$54	57.2%	$460	$65	$114
2015	$358	$14	55.1%	$308	$43	$54
2016	$128	$0	0.0%	$108	$15	$14
2017	$0	$0	0.0%	$0	$0	$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Table 4, Column (4)
3 Column (7) one row down
4 Table 5, Column (4)
5 Formula 2.23: [(2)+(4)×(3)]$/(1+roe_{PT})$; $roe_{PT} = roe/(1-tax)$; with $roe =$ 15% and $tax = 20\%$
6 $(roe_{PT} - r_{i1}) \cdot C^R_{n+i}$; $r_{i:1}$ from Table 1, Column (3) with i=m-1; C^R_{n+i} from Column (5)
7 Formula 2.7: $v_{i:1}$ × [(6) × (5)+(7) one row down]$/(1+roe_{PT})$; $v_{i:1}$ from Table 1, Column (4) with i=m-1

In accordance with step 6(a) of the procedure described in Section 2.5, we start with the last year-end valuation date before the expected final loss payment in 2017, which is December 31, 2016. Because $R'_{2017} = \$0$, Formula (2.23) simplifies to Formula (2.26) and the required capital C^R_{2016} at that date is a function only of F_{2017} and roe_{PT}. Given $roe_{PT} = 15\% / (1 - 20\%) = 18.75\%$ and the value of $F_{2017} = \$128$ shown in Column (2), application of Formula (2.26) results in $C^R_{2016} = \$128 / 1.1875 = \$108$, which appears in Column (5). Next, because $R'_{2017} = \$0$, Formula (2.7) simplifies to Formula (2.27), which defines R'_{2016} simply as the cost of capital $(roe_{PT} - r_{9:1}) \cdot C^R_{2016}$ payable on December 31, 2017 (tabulated in Column (6)), discounted back to December 31, 2016 at the forward rate $r_{9:1}$. Using $r_{9:1} = 5.13\%$ and $v_{9:1} = 95.12\%$ from Table 1 together with $roe_{PT} = 18.75\%$ and $C^R_{2016} = \$108$ in Formula (2.27), $R'_{2016} = 95.12\% \cdot (18.75\% - 5.13\%) \cdot \$108 = \$14$, which appears in Column (7). This completes step 6(a).

Continuing with step 6(b), we back up one year to December 31, 2015. Formulas (2.23) and (2.7) yield requirements $C^R_{2015} = (\$358 + 55.1\% \cdot \$14) / 1.1875 = \$308$ (Column (5)) and $R'_{2015} = 95.42\% \cdot [(18.75\% - 4.80\%) \cdot \$308 + \$14] = 95.42\% \cdot (\$43 + \$14) = \$54$ (Column (7)).

In step 6(c), again using Formulas (2.23) and (2.7), now with $n + i = 2014$, the implied requirements are $C^R_{2014} = (\$515 + 57.2\% \cdot \$54) / 1.1875 = \$460$, shown in Column (5), and $R'_{2014} = 95.62\% \cdot [(18.75\% - 4.58\%) \cdot \$460 + \$54] = 95.62\% \cdot (\$65 + \$54) = \$114$, shown in Column (7).

In accordance with step 6(d), we continue in this fashion to populate Table 6 by working backward one year at a time until reaching the December 31, 2007 valuation date, at which point C^R_{2007} and R'_{2007} are calculated as $C^R_{2007} = \$1{,}614 + 19.4\% \cdot \$1{,}009 / 1.1875 = \$1{,}524$ and $R'_{2007} = 96.77\% \cdot [(18.75\% - 3.34\%) \cdot \$1{,}524 + \$1{,}009] = 96.77\% \cdot (\$235 + \$1{,}009) = \$1{,}204$.

While the ultimate objective of steps 6(a-d) is to determine the risk margin R'_{2007} as of December 31, 2007, valuable byproducts of the recursive procedure summarized in Table 6 are the expected required capital C^R_{2007+i} and risk margin R'_{2007+i} at each successive December valuation date during the run-off period.

Table 7 summarizes the required capital as a ratio to the expected unpaid losses as of December 2007 and at successive December valuation dates through 2016. It shows that the unpaid loss run-off and volatility patterns used in this illustration imply a required capital ratio that starts at 15% of unpaid losses at December 2007 and can be expected to rise during the run-off period, peaking at 70% as of December 2015. We do not know whether that pattern of generally increasing required capital ratios as a run-off portfolio ages is a general phenomenon or a unique result arising from the data used in this illustration. It seems plausible that the one-year volatility of unpaid loss estimates generally increases as a loss portfolio ages, and it seems likely that, in turn, that would lead to a higher capital requirement for a loss portfolio in run-off. However, further study would be required to determine a definitive answer to that question.

TABLE 7

Ratios of Required Capital to Unpaid Loss
Calibrated to 99.5% Confidence Level
At Successive Annual Valuation Dates through 2017
Expected as of December 31, 2007

(1)	(2)	(3)	(4)
Year Ending 12/07+ i	Expected Unpaid Losses	Required Capital	Required Capital Ratio
$n+i$ [1]	L_{n+i} [2]	C^R_{n+i} [3]	c_{n+i} [4]
2007	$10,000	$1,524	15%
2008	$7,161	$1,386	19%
2009	$5,105	$1,334	26%
2010	$3,568	$1,435	40%
2011	$2,467	$1,045	42%
2012	$1,717	$893	52%
2013	$1,206	$601	50%
2014	$752	$460	61%
2015	$442	$308	70%
2016	$160	$108	68%
2017	$0	$0	n/a

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Table 2, Column (2)
3 Table 6, Column (5)
4 $c_{n+i} = L_{n+i} / C^R_{n+i}$: (3)/(2)

TABLE 8 Fair Value Reserves Capital Calibration at 99.5% Confidence Level At Successive Annual Valuation Dates through 2017 Expected as of December 31, 2007						
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Year Ending 12/07+ i	Expected Unpaid Losses	Expected PV of Unpaid Losses	Expected Risk Margin	Fair Value Reserve	Risk Margin Ratio to PV of Unpaid Losses	Fair Value Reserve Ratio to Unpaid Losses
$n+i$ [1]	L_{n+i} [2]	$PV(L_{n+i})$ [3]	R'_{n+i} [4]	$T(L_{n+i})$ [5]	$\frac{R'_{n+i}}{PV(L_{n+i})}$	$\frac{T(L_{n+i})}{L_{n+i}}$
2007	\$10,000	\$9,148	\$1,204	\$10,351	13.2%	1.04
2008	\$7,161	\$6,566	\$1,009	\$7,575	15.4%	1.06
2009	\$5,105	\$4,664	\$815	\$5,479	17.5%	1.07
2010	\$3,568	\$3,247	\$632	\$3,879	19.5%	1.09
2011	\$2,467	\$2,249	\$441	\$2,691	19.6%	1.09
2012	\$1,717	\$1,579	\$309	\$1,888	19.6%	1.10
2013	\$1,206	\$1,123	\$191	\$1,314	17.0%	1.09
2014	\$752	\$709	\$114	\$823	16.1%	1.09
2015	\$442	\$425	\$54	\$479	12.8%	1.08
2016	\$160	\$156	\$14	\$170	9.0%	1.06
2017	\$0	\$0	\$0	\$0	0.0%	n/a

1 n=2007, value of $0 \leq i \leq k = 10$ implied by valuation year
2 Table 2, Column (2)
3 Table 2, Column (4)
4 Table 6, Column (7)
5 Formula (2.1) generalized for $n+i$: (3)+(4)

Table 8 summarizes the calculation of the fair value of unpaid losses as of December 31, 2007 and subsequent December valuation dates. The fair value reserves are tabulated by valuation date in Column (5). These fair value estimates are based on capital calibration to the 99.5% confidence level combined with a market-clearing return on equity of 15% and market-clearing tax rate of 20% (corresponding to a pre-tax return roe_{PT} of 18.75%). The fair value of the unpaid losses as of December 31, 2007 is \$10,351, which is 4% higher than the unpaid loss estimate as of that date. As the loss portfolio runs off, the ratio of the fair value reserve to unpaid losses can be expected to rise from 1.04 as of December 2007 to a peak of 1.10 as of December 2012 and then gradually decline to 1.06 in December 2016. These ratios are shown in Column (7). Column (6) shows the ratio of the risk margin component of the fair value reserve to the present value of the unpaid losses at each valuation date.

In our illustration the fair value reserve is much more sensitive to changes in interest rates than it is to changes in the pre-tax return requirement roe_{PT}. If the spot rate curve as of December 31, 2007 had been one hundred basis points lower at each point, the fair value reserve would have been \$10,658 rather than \$10,351, which corresponds to shift in the ratio of the fair value reserve to unpaid losses from 1.04 to 1.07. On the other hand, if roe_{PT} had been 17.75% instead of 18.75%, a decline of one hundred basis points, the fair value reserve would have declined from \$10,351 to \$10,273, which corresponds to a decline in the ratio of the fair value reserve to unpaid losses from 1.04 to 1.03. The change in fair value due to a one hundred basis point change in the risk free rate is about four times the change in fair value resulting from a one hundred basis point change in the required pre-tax return roe_{PT} [25]! Note also that a reduction in the risk-free rate increases the fair value reserve, while a reduction in roe_{PT} reduces it.

Table 9 shows the expected cash flows associated with the runoff of the fair value reserve of \$10,351 as of December 31, 2007. Column (2) shows the underwriting assets corresponding to the fair value reserves as of December 31, 2007 and at successive December 31 valuation dates. Implicit in the fair value reserve calculations is the assumption that the fair value reserve amount will be invested in interest bearing assets consistent with the valuation formulas. Accordingly, the entries in Column (2) should be interpreted as invested asset amounts equal to the fair value reserves at each valuation date. Columns (3), (4) and (5) show the expected paid losses, net interest earned and cost of capital incurred during the one-year period following each valuation date. Column (6) shows the assets remaining at the end of each one-year period. Those ending amounts match the ending fair value reserve amounts shown in Column (7).

[25] (10,658-10,351)/(10,273-10,351)=-3.94

TABLE 9

Fair Value Reserve Expected Run-off Cash Flows
As of December 31, 2007

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Year Ending 12/07+ i	Expected Beginning U/W Assets	Expected Paid Losses in Next Year	Expected Net Interest Earned in Next Year	Expected Cost of Capital in Next Year	Expected Ending U/W Assets	Expected Ending Fair Value Reserve
$n+i$ [1]	$T(L_{n+i})$ [2]	P_{n+i+1} [3]	[4]	[5]	[6]	$T(L_{n+i+1})$ [7]
2007	$10,351	($2,839)	$298	($235)	$7,575	$7,575
2008	$7,575	($2,055)	$181	($222)	$5,479	$5,479
2009	$5,479	($1,538)	$146	($209)	$3,879	$3,879
2010	$3,879	($1,100)	$126	($215)	$2,691	$2,691
2011	$2,691	($750)	$99	($152)	$1,888	$1,888
2012	$1,888	($511)	$68	($130)	$1,314	$1,314
2013	$1,314	($454)	$49	($86)	$823	$823
2014	$823	($310)	$31	($65)	$479	$479
2015	$479	($283)	$16	($43)	$170	$170
2016	$170	($160)	$5	($15)	$0	$0
2017	$0	$0	$0	$0	$0	$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Equal to fair value reserve: Table 8, Column (5)
3 Table 2, Column (3) one row down, expressed as negative number
4 ((2)+0.5×(3))× $r_{i:1}$, $r_{i:1}$ from Table 1, Column (3)
5 Table 6, Column (6), expressed as negative number
6 (2)+(3)+(4)+(5)
7 Table 8, Column (5) one row down

For example, the December 31, 2007 underwriting assets of $10,351 corresponding to the fair value reserve of the same amount are expected to be reduced over the following year by paid losses of $2,839 (Column (3)) and cost of capital $235 (Column (4)) and increased by $298 of net interest earned (Column (5)), resulting in a balance of $7,575 after one year

(Column (6)). That balance matches the fair value reserve amount as of December 31, 2008 of $7,575 shown in Column (7).

Table 10 illustrates the adequacy of the required capital to ensure fair value funding of unpaid losses at the 99.5% confidence level over each successive one-year time horizon. Columns (2) through (5) are analogous to the same columns of Table 9, and, in fact, for the year beginning December 31, 2007 the entries in Columns (2) and (5) are identical. However, the paid loss amount shown in Column (3) is the paid loss portion of the 99.5% confidence level hindsight estimate one year out (rather than the expected value amount shown in Table 9) and the net interest earned shown in Column (4) reflects that higher paid loss amount. Column (6) shows the accumulated value of the capital assets after one year. Column (7) shows the year-end value of the combined underwriting and capital assets. Column (8) shows the fair value of the unpaid losses embedded in the 99.5% confidence level hindsight estimate.

For example, in the year beginning December 31, 2007 the paid loss portion of the 99.5% confidence level one-year hindsight estimate is $3,174 (vs. the $2,839 in the expected case). Interest earned is slightly lower due to the higher loss payment ($293 vs. $298). The cost of capital is the same $235 as in the expected case. The value of the capital assets at the end of the year is $1,809 ($1,524 × 1.1875). The ending value of the combined underwriting and capital assets after one year is $9,045, which matches the fair value of the unpaid loss portion of the 99.5% confidence level hindsight estimate as of December 31, 2008, which is shown in Column (8).

Table 10 shows that at each successive valuation date through December 31, 2016, the combined underwriting and capital assets are adequate to meet the fair value funding requirement at the 99.5% confidence level. In practical terms that means that sufficient assets are available to fund both the 99.5% confidence level loss obligations as they become payable and the cost of the capital required to support the unpaid losses at that level throughout the run-off period. Because the fair value reserve includes a risk margin sufficient to pay the market cost of capital, the insurer should be able to raise additional capital, if necessary, or, alternatively, a regulator should be able to arrange for a transfer of the unpaid losses to a third party reinsurer with spare capital.

TABLE 10

Adequacy of Capital to Ensure Fair Value Reserve Funding

99.5% Confidence Level

Expected as of December 31, 2007

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Year Ending 12/07+ i	Expected Beginning U/W Assets	Expected Paid Losses in 99.5% Level Hindsight	Expected Net Interest Earned in Next Year	Expected Pre-Tax Cost of Capital in Next Year	Expected Ending Capital Assets	Expected Ending U/W + Capital Assets	99.5% Level Ending Fair Value Reserve
$n+i$ [1]	$T(L_{n+i})$ [2]	P_{n+i+1} [3]	[4]	[5]	[6]	[7]	[8]
2007	$10,351	($3,174)	$293	($235)	$1,809	$9,045	$9,045
2008	$7,575	($2,307)	$177	($222)	$1,646	$6,870	$6,870
2009	$5,479	($1,866)	$141	($209)	$1,585	$5,130	$5,130
2010	$3,879	($1,470)	$119	($215)	$1,704	$4,018	$4,018
2011	$2,691	($1,011)	$93	($152)	$1,241	$2,863	$2,863
2012	$1,888	($778)	$63	($130)	$1,061	$2,103	$2,103
2013	$1,314	($679)	$44	($86)	$713	$1,306	$1,306
2014	$823	($575)	$25	($65)	$546	$753	$753
2015	$479	($549)	$10	($43)	$366	$263	$263
2016	$170	($284)	$1	($15)	$128	$0	$0

1 n=2007, value of $0 \le i \le k = 10$ implied by valuation year
2 Equal to fair value reserve: Table 8, Column (5)
3 Paid loss component of 99.5% confidence level one-year hindsight estimate:
[Table 4, Column (3) – Table 5, Column (3)] / (1 + 0.5 × Table 1, Column (3)), expressed as negative number
4 ((2)+0.5×(3))× $r_{i:1}$, $r_{i:1}$ from Table 1, Column (3)
5 Table 6, Column (6), expressed as negative number
6 Table 7, Column (3) × 1.1875: ($C^{R}_{n+i} \times (1 + roe_{PT})$)
7 (2)+(3)+(4)+(5)+(6)
8 Table 5, Column (3) × (1 + Table 8, Column(6) one row down)

4. SUMMARY AND CONCLUSIONS

In this paper we have derived and illustrated a comprehensive framework for the determination of the fair value reserve for unpaid losses that is consistent with a capital requirement established with the objective of ensuring adequate loss and cost-of-capital funding at the α confidence level for each successive year of the run-off period. That framework supports the consistent quantification of the required capital, the implied cost-of-capital-based risk margin and the fair value reserve from the expected volatility, payment and other characteristics of an unpaid loss portfolio.

Because the fair value reserve at time n is a function of capital, which in turn is a function of the sequence of expected fair value reserves in the run-off period, which are functions of future required capital, and so on, it is necessary to determine the required capital and the fair value reserve using an integrated recursive procedure. The key ingredients required for execution of that procedure are 1) the market-clearing cost of capital, and 2) α-quantile estimates from the distribution of the one-year hindsight loss estimate at each run-off period annual valuation date, as well as knowledge of the time n risk-free yield curve and the expected unpaid loss run-off pattern.

In our illustration we used a market-clearing pre-tax cost of capital roe_{PT} of 18.75%, reflecting an after-tax return on equity assumption of 15% and a tax rate of 20%. Further research is needed on the question of the true market-clearing cost of capital in this context. Conceptually, it is appealing to seek to infer the required after-tax return on equity from observed market returns. However, there are at least two issues which complicate such an analysis.

First, in response to demands by reinsurance buyers for high quality security, active reinsurers have historically held capital far beyond the regulatory minimum level. We suspect that the reinsurance market does not compensate reinsurers for holding that additional capital at the same rate as for the base capital tranche corresponding to the regulatory requirement. If that is true, then unadjusted cost of capital estimates inferred from market returns on held capital will understate the actual cost of capital on the basic Solvency II capital tranche, unless a way can be found to determine and correct for differential market returns by capital tranche.

A second complication relates to the market-clearing tax rate. U.S. reinsurers face a 35% statutory rate, while off-shore reinsurers face much lower statutory rates. However, U.S. reinsurers often pay less than the statutory rate and off-shore reinsurers often pay more. For example, Bermuda reinsurers, subject to a statutory rate of zero at home, typically pay income and excise taxes on some of their U.S. business. The key issue is the *effective* tax rate. The Economist magazine has reported U.S. and OECD-average effective corporate tax rates of 24% and 20%, respectively [2]. However, the Economist-cited study did not examine the effective rates specifically applicable to reinsurers, and those rates might differ from the corporate average. Clearly, further research on the market-clearing tax rate is warranted.

Discussion of how to model the behavior of the successive one-year hindsight loss estimates of unpaid losses throughout the run-off period is beyond the scope of this paper. Clearly, results from such modeling are critical to the application of the framework we have presented and further research in that area would be welcome.

For our illustration, in order to estimate plausible one-year hindsight loss estimate distributions, we analyzed the historical volatility and correlation of one-year loss development by Annual Statement Schedule P line of business and by age reported by the insurer selected for this example. After selecting volatility and correlation parameters, we modeled the one-year development behavior of the illustrative insurer's reserves for all lines. There are other and perhaps better ways of estimating one-year loss reserve development distributions[26].

Fair value reserves are an essential component of insurance company economic capital modeling. As we have shown in this paper, economic capital is also an essential component of fair value reserving. The two are inextricably linked.

An insurer's available economic capital is the difference between its actual fair value assets and its fair value liabilities. Its required economic capital is the amount consistent with a target such as that embedded in the Solvency II directive, where the total capital requirement addresses the risks arising not only from unpaid losses but all other balance sheet and underwriting risks as well. While the focus of this paper has been on the amount of capital required to support fair value loss reserves in isolation, the concepts presented here clearly have application to the economic capital requirements arising from those other risks and indeed the entire insurance enterprise.

APPENDIX A

Deriving Forward Rates from the Spot Rate Curve

We can identify the set of required forward rates by decomposing the yield curve into forward rate components. For example, the two-year spot rate r_2 as of time n is an average

[26] For example, see Appendix C of [6].

rate for the two-year period to maturity, comprising a rate of r_1 for the first year and a rate of $r_{1:1} = \frac{(1+r_2)^2}{1+r_1} - 1$ for the second year. $r_{1:1}$ is the one-year forward rate implied by the spot rate curve as of time n for the one-year maturity. Likewise, the three-year spot rate r_3 as of time n can be decomposed into three discrete one-year rates r_1, $r_{1:1}$, and $r_{2:1} = \frac{(1+r_3)^3}{(1+r_2)^2} - 1$ corresponding to the first, second and third years, respectively, of the three year term to maturity. In general, $r_{f:1} = \frac{(1+r_{f+1})^{f+1}}{(1+r_f)^f} - 1$ is the *f*-year forward rate implied by the time n yield curve for the one-year maturity.

The discount factor $v_m^m = (1+r_m)^{-1}$ implied by the *m*-year maturity (*m* an integer) rate r_m can also be expressed in terms of forward discount factors for the one-year maturity:

$$v_m^m = v_1 \cdot v_{1:1} \cdot v_{2:1} \cdots v_{m-1:1} \tag{A.1}$$

Generally, we can determine any *f*-year forward rate implied by the time n yield curve for any *m*-year maturity (including non-integer values of *f* and *m*) as follows:

$$r_{f:m} = \left(\frac{(1+r_{f+m})^{f+m}}{(1+r_f)^f} \right)^{1/m} - 1 \tag{A.2}$$

For example, using Formula (3.2) we can decompose the one-year rate r_1 into the six-month rate $r_{0.5}$ and the six-month forward six-month rate $r_{0.5:0.5} = \left(\frac{1+r_1}{(1+r_{0.5})^{0.5}} \right)^2 - 1$. In similar fashion we can also determine forward rates $r_{1.5:0.5}$, $r_{2.5:0.5}$, $r_{3.5:0.5}, \ldots, r_{k-0.5:0.5}$. Note that, in practice, we don't always have rates for maturities at odd intervals such as $r_{1.5}$, $r_{2.5}$, $r_{3.5}, \ldots,$ $r_{k-0.5}$ and, in such cases, interpolation is necessary to obtain estimates of such rates.

We can also determine forward rates for multi-year maturities from the forward rates for one-year maturities. For example, given the one-year and two-year forward rates $r_{1:1}$ and $r_{2:1}$ for the one-year maturity, we can determine the one-year forward rate for the two-year maturity as $r_{1:2} = \left((1+r_{1:1}) \cdot (1+r_{2:1})\right)^{1/2}$. Similarly, the one-year forward discount factor can be expressed in terms of the one-year and two-year forward discount factors for the one-year maturity: $v_{1:2}^2 = v_{1:1} \cdot v_{2:1}$. In general, the formula for the *f*-year forward discount factor for the *m*-year maturity (*m* an integer) can be expressed as:

$$v_{f:m}^m = v_{f:1} \cdot v_{f+1:1} \cdot v_{f+2:1} \cdots v_{f+m-1:1}$$

$$= \prod_{i=0}^{m-1} v_{f+i:1} \tag{A.3}$$

APPENDIX B

Proof of Formula (2.6): $\boldsymbol{R'_n = v_1 \cdot ((\frac{roe}{1-tax} - r_1) \cdot C_n^R + R'_{n+1})}$

Formula (2.5) expresses $T(L_n)$ as follows:

$$\begin{aligned} R'_n &= (\frac{roe}{1-tax} - r_1) \cdot C_n^R \cdot v_1 + (\frac{roe}{1-tax} - r_{1:1}) \cdot C_{n+1}^R \cdot v_2^2 + (\frac{roe}{1-tax} - r_{2:1}) \cdot C_{n+2}^R \cdot v_3^3 \\ &+ \ldots + (\frac{roe}{1-tax} - r_{k-1:1}) \cdot C_{n+k-1}^R \cdot v_k^k \end{aligned} \tag{2.5}$$

If we replace the multi-year risk-free discount factors $v_2^2, v_3^3,\ v_4^4, \ldots, v_k^k$ with equivalent factors based on forward rates for one-year money, we can rewrite Formula (2.5) as:

$$\begin{aligned} R'_n = &\left(\frac{roe}{1-tax} - r_1\right) \cdot C_n^R \cdot v_1 + \left(\frac{roe}{1-tax} - r_{1:1}\right) \cdot C_{n+1}^R \cdot v_1 \cdot v_{1:1} \\ &+ \left(\frac{roe}{1-tax} - r_{2:1}\right) \cdot C_{n+2}^R \cdot v_1 \cdot v_{1:1} \cdot v_{2:1} + \ldots + \left(\frac{roe}{1-tax} - r_{k-1:1}\right) \cdot C_{n+k-1}^R \cdot v_1 \cdot v_{1:1} \cdot v_{2:1} \cdots v_{k-1:1} \end{aligned}$$

Factoring out the one-year discount factor v_1 from all of the terms, we obtain:

$$\begin{aligned} R'_n = v_1 \cdot \Bigg(&\left(\frac{roe}{1-tax} - r\right)_1 \cdot C_n^R + \left(\frac{roe}{1-tax} - r_{1:1}\right) \cdot C_{n+1}^R \cdot v_{1:1} \\ &+ \left(\frac{roe}{1-tax} - r_{2:1}\right) \cdot C_{n+2}^R \cdot v_{1:1} \cdot v_{2:1} + \ldots + \left(\frac{roe}{1-tax} - r_{k-1:1}\right) \cdot C_{n+k-1}^R \cdot v_{1:1} \cdot v_{2:1} \cdots v_{k-1:1} \Bigg) \end{aligned}$$

and, finally:

$$R'_n = v_1 \cdot \left(\left(\frac{roe}{1-tax} - r_1\right) \cdot C_n^R + R'_{n+1} \right), \tag{2.6}$$

where

$$\begin{aligned} R'_{n+1} = &\left(\frac{roe}{1-tax} - r_{1:1}\right) \cdot C_{n+1}^R \cdot v_{1:1} + \left(\frac{roe}{1-tax} - r_{2:1}\right) \cdot C_{n+2}^R \cdot v_{1:1} \cdot v_{2:1} \\ &+ \left(\frac{roe}{1-tax} - r_{3:1}\right) \cdot C_{n+3}^R \cdot v_{1:1} \cdot v_{2:1} \cdot v_{3:1} + \ldots + \left(\frac{roe}{1-tax} - r_{k-1:1}\right) \cdot C_{n+k-1}^R \cdot v_{1:1} \cdot v_{2:1} \cdot v_{3:1} \cdots v_{k-1:1} \end{aligned}$$

R'_{n+1} can be characterized as the time *n* estimate of the present value risk charge required at time *n*+1. In general, R'_{n+i}, the time *n* estimate of the present value risk charge required at time $n+i$, can be expressed for $1 \le i \le k-1$ as:

$$
\begin{aligned}
R'_{n+i} = {} & \left(\frac{roe}{1-tax} - r_{i:1}\right) \cdot C^{R}_{n+i} \cdot v_{i:1} + \left(\frac{roe}{1-tax} - r_{i+1:1}\right) \cdot C^{R}_{n+i+1} \cdot v_{i:1} \cdot v_{i+1:1} \\
& + \left(\frac{roe}{1-tax} - r_{i+2:1}\right) \cdot C^{R}_{n+i+2} \cdot v_{i:1} \cdot v_{i+1:1} \cdot v_{i+2:1} + \ldots \\
& + \left(\frac{roe}{1-tax} - r_{k-1:1}\right) \cdot C^{R}_{n+k-1} \cdot v_{i:1} \cdot v_{i+1:1} \cdot v_{i+2:1} \cdots v_{k-1:1}
\end{aligned}
$$

or, more succinctly, as

$$
R'_{n+i} = v_{i:1} \cdot \left(\left(\frac{roe}{1-tax} - r_{i:1} \right) \cdot C^{R}_{n+i} + \cdot R_{n+i+1} \right) \tag{2.7}
$$

Abbreviations and Notations

α	= confidence level (probability) that insolvency can be avoided
c_n	= ratio of required capital to unpaid losses at time n : C_n^R / L_n
C_n^R	= $c_n \cdot L_n$ = required capital at time n
f	= subscript denoting the time (years) to a forward contract delivery date
f_{n+1}	= fraction by which the time n+1 unpaid losses embedded in the one-year hindsight estimate at α confidence level exceeds the expected time n+1 one-year hindsight estimate h_{n+1} = l_{n+1} + p_{n+1} = random variable, at time n, for one-year hindsight losses as of time n+1, given L_n
i	= integer subscript denoting a number of years beyond the initial valuation date at time n, $0 \le i \le k-1$
k	= integer number of years of loss payments beyond time n
L_n	= unpaid losses at time n
l_{n+1}	= random variable, at time n, for unpaid losses as of time n+1, given L_n
$L_{n+1} + P_{n+1}$	= one year hindsight estimate of L_n at time n+1
m	= integer subscript denoting the time (years) to maturity of a bond
n	= integer subscript denoting the first of a sequence of annual loss reserve valuation dates (time n+i is i years later)
P_{n+1}	= paid losses between time n and n+1
p_{n+1}	= random variable, at time n, for paid losses between time n and n+1, given L_n
$Prob(\cdot)$	= probability operator
$PV(\cdot)$	= risk-free present value operator
$PV(h_{n+1})$	$= PV(L_{n+1}) + p_{n+1} \cdot (1+\frac{1}{2}r)$ = random variable, at time n, for the present value of h_{n+1} as of time n+1, given L_n
$PV(L_{n+1} \mid VaR_\alpha(PV(h_{n+1}))$	= present value of the unpaid loss component of the one-year hindsight loss estimate a the α confidence level
R_n'	= risk-free present value of future risk charges associated with unpaid losses L_n at time n
r	= risk-free annual interest rate assuming a flat yield curve

$r_{f:m}$ = risk-free annual *f*-year forward interest rate on the *m*-year maturity bond for the period from time *n*+*f* to *n*+*f*+*m*

r_m = risk-free annual interest rate for the *m*-year maturity bond for the period from time *n* to *n*+*m*

r'_n = $c_n \cdot (\frac{roe}{1-tax} - r)$ = annual risk charge expressed as a rate of return on L_n

roe = annualized required after-tax return on equity (capital)

roe_{PT} = annualized required pre-tax return on equity (capital)

S_{n+1} = $(T(L_n) + C_n^R) \cdot (1+r)$ = accumulated value at time *n*+1 of initial assets equal to time *n* capital and loss reserve fair value plus interest

tax = income tax rate

$T(\cdot)$ = fair value at time *n* of unpaid losses L_n

$T(L_n)$ = fair value at time *n* of unpaid losses L_n

$T(l_{n+1})$ = random variable, at time *n*, for fair value at time *n*+1 of unpaid losses, given L_n

$T(L_{n+1} + P_{n+1})$ = $T(L_{n+1}) + P_{n+1} \cdot (1 + \frac{1}{2} r)$ = fair value at time *n*+1 of one-year hindsight estimate of L_n

T_{n+1}^{-1} = inverse distribution function of t_{n+1}

$T(P_{n+1})$ = $P_{n+1} \cdot (1 + \frac{1}{2} r)$ = fair value at time *n*+1 of paid losses P_{n+1}

$T(p_{n+1})$ = random variable, at time *n*, for fair value at time *n*+1 of paid losses between time *n* and *n*+1, given L_n

t_{n+1} = $T(l_{n+1} + p_{n+1})$ = $T(l_{n+1}) + p_{n+1} \cdot (1 + \frac{1}{2} r)$ = random variable, at time *n*, for fair value at time *n*+1 of one-year hindsight estimate of L_n

v = $(1+r)^{-1}$ = one-year risk-free discount factor assuming a flat yield curve

$v_{f:m}$ = $(1+r_{f:m})^{-1}$ = one-year risk-free discount factor corresponding to $r_{f:m}$

v_m = $(1+r_m)^{-1}$ = one-year risk-free discount factor r_m

$VaR_\alpha(t_{n+1})$ = Value-at-Risk with respect to t_{n+1} at the α confidence level

$VaR_\alpha(PV(h_{n+1}))$ = Value-at-Risk with respect to $PV(h_{n+1})$ at the α confidence level

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Note: An Excel spreadsheet supporting the calculation of the values of Tables 1 through 10 is available at http://www.casact.org/library/index.cfm?fa=caveat.