# Grouping Loss Distributions by Tail Behavior Part I: Discrete Families

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Abstract: *This three part paper addresses the task of modelling the right hand tail of a severity distribution. In Part I the excess ratio function is used to define a discrete sequence of loss distributions with related moments and similar tail behavior. Part II extends this to continuous one-parameter families and provides some examples. Part III provides the main result: that under some reasonable conditions, each such family has a limiting distribution which is exponential. The paper then exploits this to 1) group loss distributions based on tail behavior and 2) promote the choice of (mixed) exponentials to model tail behavior.*

## 1 Background

Even a large claim database may not suffice to give an accurate picture of the (far) right hand tail of the severity distribution of the expected losses. Consider the approach in which a distribution is built from empirical data for the more common loss amounts but is then truncated and a theoretical distribution spliced on to model the tail, where there are few actual observations. This approach has considerable appeal because most of the "bumps" of the expected loss severity are (or are believed to be) at lower loss amounts where the behavior is revealed in the observed losses. Conceptually, the expected tail should not be subject to such bumps, but rather reflect a stable pattern (i.e., if there are still bumps, you have not gone far enough to enter the "tail"). A direct measure of "bumpiness" is the presence of local modal values, or points where the derivative of the density function changes sign. "Higher order bumps" are where higher order derivatives change sign. The mathematical concept of monotonality captures this. Ideally, the tail behavior should be less bumpy, i.e., more monotone, than the overall severity distribution. The task arises, then, given a severity distribution to find a related distribution with similar behavior but one that is more monotone. Here we describe such a related distribution, what we call the "coderived" distribution. The process of constructing the coderived distribution can be repeated, and this paper is especially interested in describing the resulting sequence of distributions. These sequences emerge as canonically related families of loss distributions. This suggests an organizational scheme for continuous loss distributions and provides an alternative to the more conventional organization of loss distributions into "families" according to the arithmetic form of the density function.

The ratio of losses in excess of a given loss limit $x$ to total losses defines a function $R(x)$ that formally resembles a survival function. The loss distribution defined by that survival function is the "coderived" distribution. Conceptually, the coderived distribution provides a "preview" into the tail. The coderived distribution is shown to exhibit (right hand) tail behavior and moments that are very closely related to those of the original loss distribution. However, the coderived distribution has a simpler, more "monotone", shape than the original, in a sense defined in the paper. There is no information lost, as the coderived distribution completely determines the original. Repeating this process of "coderiving" loss distributions yields (Part I) a discrete sequence of loss distributions that are observed (Part II) to fall within a continuous one-parameter collection of loss distributions. Such collections all have tails of the same ultimate settlement rate, again as defined later. We then (Part III) consider a simple approach to ordering loss distributions according to the "thickness" of their tails. Finally, we use these concepts to relate thickness with monotonality and ultimate settlement rate. A key finding is that the asymptotic behavior of the hazard rate function provides a natural bridge between these two perspectives. Another finding reveals a unique "fixed point" role played by the exponential class of loss distributions. Assuming a tail behavior that is sufficiently "simple", we show that the (mixed) exponential distribution has properties that favor it as a choice to fit the tail of the distribution.

## 2 Notation and Terminology

In this paper we consider "smooth loss distribution functions" or SLDFns, by which we mean:

**Definition 1** *A function $F\colon [0,\infty) \to [0,1]$ is a **loss distribution function**, or **LDFn,** provided that*

- $F(0) = 0$
- $\lim_{x\to\infty} F(x) = 1$
- *$F$ is nondecreasing.*

We often use the standard symbol "$\Rightarrow$" as an abbreviation for "implies" and more generally:

$$\begin{array}{cc} \Rightarrow & \text{implies} \\ \Leftrightarrow & \text{if and only if} \\ \Rightarrow\Leftarrow & \text{contradiction.} \end{array}$$

**Definition 2** *The* ***minimum loss of*** *$F$ denoted $\alpha_F \in \mathbb{R}$ is uniquely determined by*

$$x < \alpha_F \Rightarrow F(x) = 0 \text{ and } x > \alpha_F \Rightarrow F(x) > 0.$$

**Definition 3** *The* ***maximum loss of*** *$F$ denoted $\omega_F \in \mathbb{R} \cup \{\infty\}$ is uniquely determined by*

$$x < \omega_F \Rightarrow F(x) < 1 \text{ and } x > \omega_F \Rightarrow F(x) = 1.$$

**Definition 4** *$F$ is a* ***smooth loss distribution function****, or SLDFn, provided $F$ is infinitely differentiable on $(\alpha_F, \omega_F)$, continuous on $[0, \infty)$ and the limit*

$$\lim_{x \to \omega_F} -\frac{d\left(\ln\left(1-F\right)\right)}{dx} \in \mathbb{R} \cup \{\infty\}.$$

**Notation 5** *For any SLDFn $F$, we denote the corresponding density [PDF] as $f$ and have $f(x) = \frac{dF(x)}{dx}$ for $x \in (\alpha_F, \omega_F)$ and $f(x) = 0$ otherwise. We occasionally denote the corresponding expectation of a real valued function $g$ defined on $(\alpha_F, \omega_F)$ as*

$$E\left[g\left(X\right)\right] = \int_{\alpha_F}^{\omega_F} g\left(x\right) f(x) dx = \int_0^{\infty} g\left(y\right) f(y) dy.$$

*The survival function of $F$ is denoted $S = 1 - F$ and the mean as*

$$\mu = \int_0^{\infty} y dF(y) = \int_0^{\infty} y f(y) dy = E\left[X\right].$$

*We say $F$ has finite mean provided $\mu < \infty$. For any $c \in \mathbb{R}$, we set*

$$\mu^{(c)} = \int_0^{\infty} y^c f(y) dy = E\left[X^c\right].$$

*So $\mu^{(0)} = 1$ and $\mu^{(1)} = \mu$ and we call $\mu^{(c)}$ the c-th moment of F. Provided $0 < \mu < \infty$, the excess ratio function of $F$ is given by*

$$R(x) = \frac{\int_x^{\infty} (y - x) f(y) dy}{\mu}$$

*and we denote by $\widehat{S}$ the function*

$$\widehat{S}(x) = \frac{\int_x^{\infty} y f(y) dy}{\mu}$$

*for* $x \geq 0$. *We denote the hazard rate function by*

$$\lambda(x) = \frac{f(x)}{S(x)}$$

*for* $x \in (0, \omega_F)$. *We let*

$$L(t) = \int_0^\infty e^{-ty} f(y)dy = \int_{\alpha_F}^{\omega_F} e^{-ty} f(y)dy = E\left[e^{-tX}\right]$$

*denote the Laplace transform of* $F$ *and* $M(t) = L(-t)$ *the moment generating function. When* $F$ *has finite mean we denote the standard deviation as*

$$\sigma = \sqrt{\int_0^\infty (y-\mu)^2 f(y)dy} = \sqrt{\mu^{(2)} - \mu^2}$$

*and the coefficient of variation as* $CV = \frac{\sigma}{\mu}$. *We use subscripts on* $f_F$, $E_F$, $S_F$, $\mu_F$, $\mu_F^{(c)}$, $R_F$, $\widehat{S}_F$, $\sigma_F$, $CV_F$, $\lambda_F$, $L_F$, *and* $M_F$ *when necessary to indicate dependence on* $F$.

Note that for any SLDFn $F$, the requirement that $\lim\limits_{x\to\infty} F(x) = 1$ *forces* $f(a) > 0$ for some $a > 0$ and so $\mu^{(c)} > 0$ for every $c \in \mathbb{R}$.

**Definition 6** *For any SLDFn* $F$, *the* ***ultimate settlement rate*** *is*

$$\tau_F = \lim_{x\to\omega_F} \lambda_F(x).$$

Note that for any SLDFn $F$ we have for all $x \in (0, \omega_F)$ that $S(x) > 0$ and by the chain rule

$$-\frac{d\left(\ln\left(1-F\right)\right)}{dx} = -\frac{d(\ln S(x))}{dx} = -\frac{1}{S(x)}\frac{dS(x)}{dx} = \frac{1}{S(x)}\frac{dF(x)}{dx} = \frac{f(x)}{S(x)} = \lambda(x)$$

and so, by our definition of SLDFn, $\tau_F$ is well defined.

**Example 7** *The function*

$$F(x) = \left\{ \begin{array}{cc} 1 - e^{\frac{x(x-2)}{(x-1)^2}} & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{array} \right\}$$

*is an SLDFn that is infinitely differentiable on* $(0, \infty)$ *with* $\omega_F = 1$ *and* $\tau_F = \infty$.

We begin by noting that SLDFns are determined by their hazard rate functions:

**Proposition 8** *For any SLDFn* $F$*:*

$$S_F(x) = e^{-\int_0^x \lambda_F(t)dt} \text{ for every } x \in [0, \omega_F).$$

**Proof.** We have noted that for any $z \in (0, \omega_F)$

$$\frac{d(\ln S(x))}{dx} = -\lambda(x)$$

holds for all $x \in (0, z)$. We see that $\lambda(x)$ is integrable on $[0, z]$. But then $S(x)$ and $T(x) = e^{-\int_0^x \lambda(t)dt}$ are two continuous functions with the same logarithmic derivative on $(0, z)$. It follows that

$$\begin{aligned}
0 &= \frac{d(\ln T(x) - \ln S(x))}{dx} = \frac{d(\ln \frac{T(x)}{S(x)})}{dx} \\
&\Rightarrow \ln \frac{T(x)}{S(x)} = c \text{ is constant on } (0, z) \\
&\Rightarrow \frac{T(x)}{S(x)} = e^c \text{ is constant on } (0, z) \\
&\Rightarrow T(x) = e^c S(x) \text{ for every } x \in (0, z).
\end{aligned}$$

But then

$$\begin{aligned}
S(0) &= 1 = e^0 = e^{-\int_0^0 \lambda(t)dt} = T(0) \Rightarrow e^c = 1 \\
&\Rightarrow S(x) = T(x) = e^{-\int_0^x \lambda(t)dt} \text{ for every } x \in [0, z).
\end{aligned}$$

Since $z \in [0, \omega_F)$ was arbitrary,

$$\begin{aligned}
S_F(x) &= e^{-\int_0^x \lambda_F(t)dt} \\
\text{for every } x &\in \bigcup_{z \in [0,\omega_F)} [0, z) = [0, \omega_F)
\end{aligned}$$

as required. ∎

We will have occasion to consider the case when the hazard rate function is increasing or decreasing. This can often be readily determined, as in:

**Proposition 9** *For any SLDFn $F$ with $\lambda_F$ differentiable on $(\alpha_F, \omega_F) = (0, \infty)$ :*

$$\frac{d\lambda_F}{dx} = \lambda_F^2 + \frac{\frac{df_F}{dx}}{S_F} = \lambda_F \left( \lambda_F + \frac{d \ln f_F}{dx} \right).$$

**Proof.** From the definition of $\lambda = \lambda_F$

$$\begin{aligned}
\frac{d\lambda}{dx} &= \frac{d}{dx}\left(\frac{f}{S}\right) = \frac{S\frac{df}{dx} - f\frac{dS}{dx}}{S^2} \\
&= \frac{S\frac{df}{dx} - f(-f)}{S^2} = \frac{S\frac{df}{dx}}{S^2} + \frac{f^2}{S^2} \\
&= \left(\frac{f}{S}\right)^2 + \frac{\frac{df}{dx}}{S} = \lambda^2 + \frac{\frac{df}{dx}}{S} \\
&= \lambda^2 + \frac{\frac{df}{dx}f}{fS} = \lambda^2 + \lambda\frac{\frac{df}{dx}}{f} \\
&= \lambda^2 + \lambda\frac{d\ln f}{dx}
\end{aligned}$$

as required. ■

The following proposition expresses the excess ratio function in terms of $S$ and $\widehat{S}$.

**Proposition 10** *For any SLDFn $F$ with $\mu_F < \infty$,*

$$R_F(x) = \widehat{S_F}(x) - \frac{xS_F(x)}{\mu_F}, \text{ for every } x \geq 0.$$

**Proof.** From the definition of $R(x)$ we have

$$\begin{aligned} R(x) &= \frac{1}{\mu}\int_x^\infty (y-x)f(y)dy \\ &= \frac{1}{\mu}\left[\int_x^\infty yf(y)dy - x\int_x^\infty f(y)dy\right] \\ &= \frac{1}{\mu}\left[\int_x^\infty yf(y)dy - xS(x)\right] \\ &= \widehat{S}(x) - \frac{xS(x)}{\mu}. \end{aligned}$$

as required. ■

**Proposition 11** *For any SLDFn $F$ and $a, b, c \in \mathbb{R}$ with $a \geq b \geq 0$ and $\mu_F^{(c)} < \infty$, and further provided either $c \geq 0$ or $a > b$, we have (with the convention that $0^0 = 1$):*

$$c\int_a^\infty (y-b)^{c-1} S_F(y)dy = \int_a^\infty (y-b)^c f_F(y)dy - (a-b)^c S_F(a).$$

**Proof.** The case $c = 0$ reduces to the identity

$$0 = \int_a^\infty f(y)dy - (1)\,S(a) = S(a) - S(a).$$

So assume $c \neq 0$. The result follows from integration by parts

$$u = S(y) \qquad v = (y-b)^c$$

$$\begin{aligned} c\int_a^\infty (y-b)^{c-1} S(y)dy &= \int_a^\infty S(y)\left(c\,(y-b)^{c-1}\right)dy \\ &= \int_a^\infty udv = uv]_a^\infty - \int_a^\infty vdu \\ &= (y-b)^c\, S(y)]_a^\infty - \int_a^\infty (y-b)^c\,(-f(y))\,dy \end{aligned}$$

$$= \left(\lim_{y\to\infty} (y-b)^c\, S(y)\right) - (a-b)^c\, S(a) + \int_a^\infty (y-b)^c f(y)dy$$

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Now clearly

$$c < 0 \Rightarrow \lim_{y\to\infty} (y-b)^c S(y) \leq \lim_{y\to\infty} S(y) = 0$$

and for $y > b+1$ and $c > 0$

$$\begin{aligned} (y-b)^c S(y) &\leq y^c S(y) = y^c \int_y^\infty f(x)dx \\ &= \int_y^\infty y^c f(x)dx \leq \int_y^\infty x^c f(x)dx \end{aligned}$$

it follows that

$$\begin{aligned} 0 &\leq \lim_{y\to\infty} (y-b)^c S(y) \\ &\leq \lim_{y\to\infty} \int_y^\infty x^c f(x)dx = 0 \text{ since } \int_0^\infty x^c f(x)dx = E[X^c] < \infty \\ &\Rightarrow 0 = \lim_{y\to\infty} (y-b)^c S(y) \end{aligned}$$

and we conclude that

$$c\int_a^\infty (y-b)^{c-1} S(y)dy = -(a-b)^c S(a) + \int_a^\infty (y-b)^c f(y)dy$$

and the result follows. ■

**Corollary 12** *If either $a > b$ or $c > 0$, then:*

$$\int_a^\infty (y-b)^{c-1} S_F(y)dy < \infty \Leftrightarrow \int_a^\infty (y-b)^c f_F(y)dy < \infty.$$

**Proof.** Clear since under the conditions we must have $(a-b)^c S(a) < \infty$. ■

**Corollary 13** *For any SLDFn $F$ and $c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$ :*

$$\mu_F^{(c)} = \left\{ \begin{array}{cc} \lim\limits_{a\to 0, a>0} \left(a^c S_F(a) + c\int_a^\infty y^{c-1} S_F(y)dy\right) & c < 0 \\ 1 & c = 0 \\ c\int_0^\infty x^{c-1} S_F(x)dx & c > 0 \end{array} \right\}.$$

**Proof.** Suppose first that $c < 0$. Letting $a > b = 0$ in Proposition 11

$$\begin{aligned} \mu^{(c)} &= \int_0^\infty y^c f(y)dy \\ &= \lim_{a\to 0, a>0} \int_a^\infty y^c f(y)dy \\ &= \lim_{a\to 0, a>0} \left(a^c S(a) + c\int_a^\infty y^{c-1} S(y)dy\right) \end{aligned}$$

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as asserted. The result is apparent for $c = 0$. For $c > 0$ the result follows by letting $b = 0$ and $a > 0$ go to 0 in Proposition 11

$$\begin{aligned}
\mu^{(c)} &= \int_0^\infty y^c f(y)dy = \lim_{a\to 0}\int_a^\infty y^c f(y)dy = \lim_{a\to 0}\left(a^c S(a) + c\int_a^\infty y^{c-1}S(y)dy\right) \\
&= \lim_{a\to 0} a^c S(a) + \lim_{a\to 0}\left(c\int_a^\infty y^{c-1}S(y)dy\right) \\
&= \lim_{a\to 0} a^c + c\lim_{a\to 0}\int_a^\infty y^{c-1}S(y)dy \\
&= \left(\lim_{a\to 0} a\right)^c + c\int_0^\infty y^{c-1}S(y)dy \\
&= c\int_0^\infty y^{c-1}S(y)dy
\end{aligned}$$

as asserted. ■

The existence of $\mu_F^{(c)}$ for large positive $c$ is typically discussed in terms of the existence of $\mu_F^{(n)}$ for large $n \in \mathbb{N} = \{1, 2, \ldots\}$ and with $\mu_F^{(n)}$ termed a higher moment. And it is often noted that the existence of higher moments is suggestive of a thin right hand tail. We will see how to make that mathematically precise below. The above corollary suggests that the existence $\mu_F^{(c)}$ for negative $c$ is more subtle and we will see later that this relates with the analytic character of the distribution function, more specifically its degree of monotonality (alternating sign of higher order derivatives).

To any SLDFn $F$ we will associate other SLDFns whose moments are closely related to those of $F$. The simplest case comes from the observation that the function $\widehat{S}(x) = \frac{\int_x^\infty yf(y)dy}{\mu}$ resembles a survival function.

**Definition 14** *For any SLDFn $F$ we set $\widehat{F} = 1 - \widehat{S_F}$.*

**Proposition 15** *For any SLDFn $F$ with finite mean, $\widehat{F}$ is an SLDFn with*

$$\begin{aligned}
f_{\widehat{F}}(x) &= \frac{xf(x)}{\mu},\ \alpha_{\widehat{F}} = \alpha_F,\ \omega_{\widehat{F}} = \omega_F, \\
\tau_{\widehat{F}} &= \tau_F - \frac{1}{\omega_F} \text{ for finite } \omega_F \\
\tau_{\widehat{F}} &= \tau_F \text{ for } \omega_F = \infty \\
\text{and } \mu_{\widehat{F}}^{(c)} &= \frac{\mu_F^{(c+1)}}{\mu_F} \text{ for every } c \in \mathbb{R}.
\end{aligned}$$

**Proof.** We have

$$\begin{aligned}
\widehat{F}(0) &= 1 - \widehat{S}(0) = 1 - \frac{\int_0^\infty yf(y)dy}{\mu} = 1 - \frac{\mu}{\mu} = 1 - 1 = 0 \\
\text{and } \frac{d\widehat{F}}{dx} &= \frac{d\left(1 - \widehat{S_F}\right)}{dx} = -\frac{d\widehat{S_F}}{dx} = \frac{-1}{\mu}\frac{d\int_x^\infty yf(y)dy}{dx} = \frac{xf(x)}{\mu} \geq 0
\end{aligned}$$

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which clearly implies that $\widehat{F}$ is infinitely differentiable *on* $(\alpha_F, \omega_F)$ and continuous and nondecreasing on $[0, \infty)$. Also

$$\infty > \mu = \int_0^\infty yf(y)dy = \lim_{x\to\infty} \int_0^x yf(y)dy + \int_x^\infty yf(y)dy$$

$$= \lim_{x\to\infty} \int_0^x yf(y)dy + \mathit{lim}_{x\to\infty} \int_x^\infty yf(y)dy = \mu + \lim_{x\to\infty} \int_x^\infty yf(y)dy$$

$$\Rightarrow \lim_{x\to\infty} \int_x^\infty yf(y)dy = 0$$

whence

$$\begin{aligned} \lim_{x\to\infty} \widehat{F}(x) &= 1 - \lim_{x\to\infty} \widehat{S}(x) = 1 - \frac{1}{\mu} \lim_{x\to\infty} \int_x^\infty yf(y)dy \\ &= 1 - \frac{0}{\mu} = 1 \end{aligned}$$

and we see that $\widehat{F}$ is an SLDFn. It is clear that $\widehat{F}$ has PDF

$$f_{\widehat{F}}(x) = \frac{d\widehat{F}}{dx} = -\frac{d\widehat{S}}{dx} = -\frac{d}{dx}\left(\frac{\int_x^\infty yf(y)dy}{\mu}\right) = \frac{xf(x)}{\mu}$$

We will make frequent use of the observation that $F$ being an SLDFn implies that the PDF $f = f_F$ is continuous on $(0, \omega_F) \cup (\omega_F, \infty)$. In particular, we have

$$x < \omega_F \Rightarrow F(x) < 1$$

$$\Rightarrow \int_x^\infty f(y)dy = S(x) > 0$$

$$\Rightarrow \text{there exists some } z > x, \epsilon > 0 \ \text{ such that } \{|w - z| < \epsilon \Rightarrow f(w) > 0\}$$

$$\Rightarrow \widehat{S}(x) = \frac{\int_x^\infty yf(y)dy}{\mu} > 0$$

$$\Rightarrow \widehat{F}(x) < 1$$

moreover

$$x > \omega_F \Rightarrow F(x) = 1$$

$$\Rightarrow \int_x^\infty f(y)dy = S(x) = 0$$

$$\Rightarrow f(w) = 0 \text{ for every } w > x$$

$$\Rightarrow \widehat{S}(x) = \frac{\int_x^\infty yf(y)dy}{\mu} = 0$$

$$\Rightarrow \widehat{F}(x) = 1$$

which establishes $\omega_{\widehat{F}} = \omega_F$. Similarly

$$x < \alpha_F \Rightarrow f(x) = 0$$

$$\Rightarrow \widehat{F}(x) = \frac{\int_0^x yf(y)dy}{\mu} = \frac{\int_0^x y(0)dy}{\mu} = 0$$

moreover

$$x > \alpha_F \Rightarrow \int_0^x f(y)dy = F(x) > 0$$

$\Rightarrow$ there exist $z, \epsilon \in \mathbb{R}$ such that $0 < z < x, \epsilon > 0$ such that $\{|w - z| < \epsilon \Rightarrow f(w) > 0\}$

$$\Rightarrow \widehat{F}(x) = \frac{\int_0^x yf(y)dy}{\mu} > 0$$

which establishes $\alpha_{\widehat{F}} = \alpha_F$. Alternatively, since $f_{\widehat{F}}(x) > 0 \Leftrightarrow f(x) > 0$ it is clear that $\alpha_{\widehat{F}} = \alpha_F$ and $\omega_{\widehat{F}} = \omega_F$. Fist assume that $\omega_F$ is finite, then by l'H$\widehat{\text{o}}$pital:

$$\begin{aligned}
\tau_{\widehat{F}} &= \lim_{x\to\omega_F} \lambda_{\widehat{F}}(x) = \lim_{x\to\omega_F} \frac{xf(x)}{\mu\widehat{S}(x)} = \lim_{x\to\omega_F} \frac{x\frac{df}{dx} + f(x)}{\mu\frac{d\widehat{S}}{dx}} \\
&= -\lim_{x\to\omega_F} \frac{x\frac{df}{dx} + f(x)}{\mu f_{\widehat{F}}(x)} = -\lim_{x\to\omega_F} \frac{x\frac{df}{dx} + f(x)}{xf(x)} \\
&= -\lim_{x\to\omega_F} \left( \frac{\frac{df}{dx}}{f(x)} + \frac{1}{x} \right) = -\lim_{x\to\omega_F} \left( \frac{\frac{df}{dx}}{f(x)} \right) - \frac{1}{\omega_F} \\
&= -\lim_{x\to\omega_F} \left( \frac{f(x)}{-S(x)} \right) - \frac{1}{\omega_F} = \lim_{x\to\omega_F} (\lambda_F(x)) - \frac{1}{\omega_F} \\
&= \tau_F - \frac{1}{\omega_F}
\end{aligned}$$

as required. The same argument shows that $\tau_{\widehat{F}} = \tau_F$ for $\omega_F = \infty$. Finally

$$\mu_{\widehat{F}}^{(c)} = \int_0^\infty y^c f_{\widehat{F}}(y)dy = \int_0^\infty y^c \left( \frac{yf(y)}{\mu} \right) dy = \frac{1}{\mu} \int_0^\infty y^{c+1} f(y)dy = \frac{\mu_F^{(c+1)}}{\mu_F}$$

completing the proof. ■

**Remark 16** *The distribution of $\widehat{F}$ is sometimes referred to as the time-biased distribution. It has application to sampling theory when the probability of selection increases with time of exposure or attained age.*

It is easy to generalize the time-biased distribution:

**Definition 17** *For any SLDFn $F$ and $c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$, we denote by $\widehat{F}^{[c]}$ the SLDFn with PDF*

$$f_{\widehat{F}^{[c]}}(x) = \frac{x^c f(x)}{\mu^{(c)}}.$$

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**Proposition 18** *For any SLDFn $F$ and $c, d \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$:*

$$\omega_{\widehat{F}^{[c]}} = \omega_F \quad \text{and } \mu_{\widehat{F}^{[c]}}^{(d)} = \frac{\mu_F^{(c+d)}}{\mu_F^{(c)}}.$$

**Proof.** As before we see that

$$x < \omega_F \Rightarrow F(x) < 1$$

$$\begin{aligned}
&\Rightarrow \quad \int_x^\infty f(y)dy = S(x) > 0 \\
&\Rightarrow \quad \text{there exist } z > x, \epsilon > 0 \text{ such that } \{|w - z| < \epsilon \Rightarrow f(w) > 0\} \\
&\Rightarrow \quad S_{\widehat{F}^{[c]}}(x) = \frac{\int_x^\infty y^c f(y)dy}{\mu^{(c)}} > 0 \\
&\Rightarrow \quad \widehat{F}^{[c]}(x) < 1
\end{aligned}$$

and we have

$$x > \omega_F \Rightarrow F(x) = 1$$

$$\begin{aligned}
&\Rightarrow \quad \int_x^\infty f(y)dy = S(x) = 0 \\
&\Rightarrow \quad f(w) = 0 \text{ for every } w > x \\
&\Rightarrow \quad S_{\widehat{F}^{[c]}}(x) = \frac{\int_x^\infty y^c f(y)dy}{\mu^{(c)}} = 0 \\
&\Rightarrow \quad \widehat{F}^{[c]}(x) = 1
\end{aligned}$$

whence $\omega_{\widehat{F}^{[c]}} = \omega_F$ and also

$$\begin{aligned}
\mu_{\widehat{F}^{[c]}}^{(d)} &= \int_0^\infty y^d f_{\widehat{F}^{[c]}}(y)dy = \int_0^\infty y^d \left( \frac{y^c f(y)}{\mu^{(c)}} \right) dy \\
&= \frac{\mu^{(c+d)}}{\mu^{(c)}} \int_0^\infty \frac{y^{c+d} f(y)}{\mu^{(c+d)}} dy = \frac{\mu_F^{(c+d)}}{\mu_F^{(c)}}
\end{aligned}$$

as asserted. ■

Analogous to this construction (actually "dual" in a sense to be made precise below), we observe that the mean of any SLDFn $F$ with finite mean can be expressed in terms of its survival function as $\mu = \int_0^\infty S(x)dx$. Therefore the function $\widetilde{f}(x) = \frac{S(x)}{\mu}$ is the PDF of another related SLDFn, which we denote as $\widetilde{F}$.

**Definition 19** *For any SLDFn $F$ with finite mean, the* ***coderived distribution*** *of $F$, which we denote by $\widetilde{F}$, is the distribution function with PDF*

$$\widetilde{f}(x) = f_{\widetilde{F}}(x) = \frac{S(x)}{\mu}.$$

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**Remark 20** *Observe that*

$$\begin{aligned}\frac{d\widetilde{f}(x)}{dx} &= \frac{1}{\mu}\frac{dS(x)}{dx} = \frac{-f(x)}{\mu} \\ &\Rightarrow f(x) = -\mu\frac{d\widetilde{f}(x)}{dx}\end{aligned}$$

*and the PDF of the SLDFn $F$ is obtained by differentiation, or "derived", from that of $\widetilde{F}$. Back in the days of category theory, mathematicians liked to assign the "co-" prefix when reversing arrows. So $\widetilde{F}$ is "coderived" from $F$, which prompts the name assigned to $\widetilde{F}$.*

Klugman [5] relates the right hand tail behavior of the original distribution with that of the coderived distribution, which he terms the "equilibrium distribution". In particular, he considers the asymptotic behavior of the hazard rate functions of the two distributions. We will pursue that somewhat further in this paper. We begin with the observation that the excess ratio is the survival function of the coderived distribution:

**Proposition 21** *If $F$ is an SLDFn with finite mean, survival function $S$ and excess ratio function $R$, then:*

$$\alpha_{\widetilde{F}} = 0,\ \omega_{\widetilde{F}} = \omega_F \quad \text{and } R(x) = \frac{\int_x^\infty S(y)dy}{\int_0^\infty S(y)dy} = \int_x^\infty \widetilde{f}(y)dy = S_{\widetilde{F}}(x),\ \text{for } x \geq 0.$$

**Proof.** Let $F$ have PDF $f = f_F$, since $\widetilde{f}(0) = \frac{1}{\mu} > 0$ is continuous at 0, clearly $\alpha_{\widetilde{F}} = 0$. We also have

$$x < \omega_F \Rightarrow F(x) < 1$$

$$\begin{aligned}&\Rightarrow \mu\widetilde{f}(x) = \int_x^\infty f(y)dy = S(x) > 0 \\ &\Rightarrow \widetilde{S}(x) = \int_x^\infty \widetilde{f}(y)dy > 0 \\ &\Rightarrow \widetilde{F}(x) < 1\end{aligned}$$

and moreover

$$x > \omega_F \Rightarrow F(x) = 1$$

$$\begin{aligned}&\Rightarrow \int_x^\infty f(y)dy = S(x) = 0 \\ &\Rightarrow \widetilde{f}(w) = \frac{S(w)}{\mu} = 0 \text{ for every } w > x \\ &\Rightarrow \widetilde{S}(x) = \int_x^\infty \widetilde{f}(w)dw = 0 \\ &\Rightarrow \widetilde{F}(x) = 1\end{aligned}$$

which establishes $\omega_{\widetilde{F}} = \omega_F$. Now from Proposition 11 we have

$$\begin{aligned}\int_x^\infty S(y)dy &= \int_x^\infty yf(y)dy - xS(x) \\ &= -x\int_x^\infty f(y)dy + \int_x^\infty yf(y)dy \\ &= \int_x^\infty (y-x)f(y)dy,\end{aligned}$$

Thus

$$R(x) = \frac{\int_x^\infty (y-x)f(y)dy}{\mu} = \frac{\int_x^\infty S(y)dy}{\int_0^\infty S(y)dy} = \int_x^\infty \frac{S(y)}{\mu}dy = \int_x^\infty \widetilde{f}(y)dy.$$

as required. ■

**Corollary 22** *Under the assumptions of the Proposition:*

$$\frac{dR}{dx}(x) = \frac{-S(x)}{\mu} = -\widetilde{f}(x), \text{ for every } x \geq 0.$$

**Proof.** By the Fundamental Theorem of Calculus

$$\frac{dR}{dx} = \frac{d}{dx}\left(\frac{\int_x^\infty S(y)dy}{\mu}\right) = \frac{-S(x)}{\mu} = -\widetilde{f}(x).$$

as required. ■

Let $F$ be an SLDFn with finite mean. Observe that $\widetilde{F}$ is again an SLDFn and so provided $\mu_{\widetilde{F}} < \infty$ we can repeat the process to get $\widetilde{\widetilde{F}}$. More precisely, we can recursively construct the sequence of LDFns

$$\begin{aligned}\widetilde{F}^{[0]} &= F \\ \widetilde{F}^{[1]} &= \widetilde{F} \\ \widetilde{F}^{[n]} &= \widetilde{\widetilde{F}^{[n-1]}} \text{ for } n = 2,3,4,\ldots \text{provided } \mu_{\widetilde{F}^{[n-1]}} < \infty.\end{aligned}$$

and refer to $\widetilde{F}^{[n]}$ as the **n-th forward coderived LDFn** of $F$. It is clear that $\omega_{\widetilde{F}^{[n]}} = \omega_F$ for $n = 2,3,4,\ldots$provided $\mu_{\widetilde{F}^{[n-1]}} < \infty$.

We will soon see (Proposition 27) that quite generally the existence of an $n$-th forward coderived LDFn is equivalent to having a finite $n$-th moment

$$\widetilde{F}^{[n]} \text{ exists } \Leftrightarrow \mu^{(n)} < \infty.$$

The PDF of the coderived loss distribution is continuous and nonincreasing and so a mode of any such coderived distribution is at $x = 0$ where its PDF takes its maximum value of $\frac{1}{\mu}$. Conversely, if $F$ is an SLDFn with nonincreasing PDF $f$, then it is easy to verify that $G(x) = \frac{f(0)-f(x)}{f(0)}$ is an SLDFn with coderived distribution $\widetilde{G} = F$. It is also worth noting that because the survival curve completely determines the distribution, the coderived distribution completely determines the original distribution. And indeed for any $n$, the $n$-th forward coderived LDFn, should there be one, completely determines the original LDFn. We conclude this section with a rather general observation on the existence of moments.

**Proposition 23** *If $F$ is an SLDFn with finite mean, then there exist unique $a, c \in \mathbb{R} \cup \{\infty\}$ such that:*

$$(0,1) \subseteq (a,c) = \left\{ b \in \mathbb{R} - \{a,c\} \,|\mu_F^{(b)} < \infty \right\}.$$

**Proof.** Set $A = \left\{ b \in \mathbb{R} | \mu_F^{(b)} < \infty \right\}$. We claim that $A$ is a connected subset of $\mathbb{R}$. To see this, note that

$$\begin{aligned} a, c &\in A \Rightarrow \int_0^\infty y^a f(y)dy, \int_0^\infty y^c f(y)dy < \infty \\ &\Rightarrow \int_0^1 y^a f(y)dy, \int_1^\infty y^a f(y)dy, \int_0^1 y^c f(y)dy, \int_1^\infty y^c f(y)dy < \infty. \end{aligned}$$

So suppose $a < b < c$ with $a, c \in A$, then we have

$$\begin{aligned} 0 &< y < 1 \Rightarrow y^a > y^b > y^c \Rightarrow \infty > \int_0^1 y^a f(y)dy > \int_0^1 y^b f(y)dy \\ 1 &< y \Rightarrow y^a < y^b < y^c \Rightarrow \int_1^\infty y^b f(y)dy < \int_1^\infty y^c f(y)dy < \infty \\ &\Rightarrow \mu^{(b)} = \int_0^\infty y^b f(y)dy = \int_0^1 y^b f(y)dy + \int_1^\infty y^b f(y)dy < \infty \\ &\Rightarrow b \in A. \end{aligned}$$

And it follows that $A$ is connected, as claimed. Now since $F$ has finite mean, we clearly have

$$\begin{aligned} \mu^{(0)} &= \int_0^1 y^0 f(y)dy = \int_0^1 1 f(y)dy = 1 < \infty \Rightarrow 0 \in A \\ \mu^{(1)} &= \int_0^1 y^1 f(y)dy = \int_0^1 y f(y)dy = \mu < \infty \Rightarrow 1 \in A \end{aligned}$$

and since the connected subsets of $\mathbb{R}$ are exactly the intervals, the result follows. ■

**Example 24** *For the usual families of loss distributions (beta, Pareto, burr, Weibull, gamma,...) the set $\left\{c \in \mathbb{R} | \mu^{(c)} < \infty\right\}$ is an* ***open*** *interval. The following example, provided by Derek Schaff, shows that is not always the case for* *loss distributions. Define*

$$\begin{aligned} g(x) &= \left\{ \begin{array}{cc} 0 & x \leq 2 \\ \left(\frac{1}{x \ln x}\right)^2 & x > 2 \end{array} \right\} \\ &\Rightarrow \int_0^\infty y^c g(y) dy = \int_2^\infty y^c g(y) dy \\ &= \int_{\ln 2}^\infty \frac{e^{(c-1)u}}{u^2} du \left\{ \begin{array}{cc} < \infty & c \leq 1 \\ \infty & c > 1 \end{array} \right\} \end{aligned}$$

*where we used the change of variable $u = \ln y$, noting that for $c \leq 1$ the integral is dominated by the convergent integral $\int_0^\infty \frac{du}{u^2}$ while for $c > 1$ l'Hôpital shows that the integrand does not even go to 0 as $u \to \infty$. We see that*

$$f_F(x) = \frac{g(x)}{\int_0^\infty g(y) dy} \Rightarrow \{c \in \mathbb{R} | E[X^c] < \infty\} = (-\infty, 1].$$

# 3 Moments and the Coderived Distribution

The discussion leading to the definition of a coderived loss distribution together with Proposition 10 gives the first two Items of:

**Proposition 25** *For any SLDFn $F$ with finite mean $\mu$ and all $x \geq 0$:*

1. $f_{\widetilde{F}}(x) = \frac{S_F(x)}{\mu}$
2. $S_{\widetilde{F}}(x) = R_F(x) = \widehat{S}_F(x) - x f_{\widetilde{F}}(x)$
3. $S_F(x) > 0 \Rightarrow \lambda_{\widetilde{F}}(x) = \left(\mu \frac{\widehat{S}_F(x)}{S_F(x)} - x\right)^{-1} > 0$
4. $S_F(x) > 0 \Rightarrow \lambda_{\widetilde{F}}(x) = \frac{\int_0^\infty f(x+z)dz}{\int_0^\infty z f(x+z)dz}$.

**Proof.** Items 1 and 2 have been established. For Item 3, we have seen that

$$S_F(x) > 0$$

$$\Rightarrow S_{\widetilde{F}}(x) > 0 \text{ and } f_{\widetilde{F}}(x) = \frac{S_F(x)}{\mu} > 0$$

and so by Items 1 and 2

$$\begin{aligned} 0 &< \lambda_{\widetilde{F}}(x) = \frac{f_{\widetilde{F}}(x)}{S_{\widetilde{F}}(x)} = \left(\frac{S_{\widetilde{F}}(x)}{f_{\widetilde{F}}(x)}\right)^{-1} = \left(\frac{\widehat{S}_F(x) - x f_{\widetilde{F}}(x)}{f_{\widetilde{F}}(x)}\right)^{-1} \\ &= \left(\frac{\widehat{S}_F(x)}{f_{\widetilde{F}}(x)} - x\right)^{-1} = \left(\frac{\widehat{S}_F(x)}{\frac{S_F(x)}{\mu}} - x\right)^{-1} = \left(\mu \frac{\widehat{S}_F(x)}{S_F(x)} - x\right)^{-1} \end{aligned}$$

as required. And then Item 4 follows from definitions and the change of variable $z = y - x$

$$\lambda_{\widetilde{F}}(x) = \left(\frac{\mu \widehat{S}_F(x)}{S_F(x)} - x\right)^{-1} = \left(\frac{\int_x^\infty y f(y) dy}{\int_x^\infty f(y) dy} - x\right)^{-1}$$

$$= \left(\frac{\int_x^\infty y f(y) dy - x \int_x^\infty f(y) dy}{\int_x^\infty f(y) dy}\right)^{-1} = \frac{\int_x^\infty f(y) dy}{\int_x^\infty y f(y) dy - \int_x^\infty x f(y) dy}$$

$$= \frac{\int_x^\infty f(y) dy}{\int_x^\infty (y - x) f(y) dy} = \frac{\int_0^\infty f(x+z) dz}{\int_0^\infty z f(x+z) dz}$$

completing the proof. ■

As was noted, the coderived distribution determines the original:

**Proposition 26** *For any two SLDFns with finite means $F$ and $G$*

$$F = G \quad \Leftrightarrow \quad \widetilde{F} = \widetilde{G}.$$

**Proof.** Trivially, $F = G \Rightarrow \widetilde{F} = \widetilde{G}$. Conversely

$$\widetilde{F} = \widetilde{G} \Rightarrow \frac{S_F(x)}{\mu_F} = f_{\widetilde{F}}(x) = f_{\widetilde{G}}(x) = \frac{S_G(x)}{\mu_G}$$

and letting $x = 0$ we have

$$\begin{aligned} \frac{1}{\mu_F} &= \frac{S_F(0)}{\mu_F} = \frac{S_G(0)}{\mu_G} = \frac{1}{\mu_G} \\ &\Rightarrow \mu_F = \mu_G \\ &\Rightarrow 1 - F = S_F = S_G = 1 - G \\ &\Rightarrow F = G \end{aligned}$$

as asserted. ■

The moments of coderived distributions are readily obtained from those of the original distribution:

**Proposition 27** *If $F$ is an SLDFn and $n \in \mathbb{N}$, then:*

$$\mu_F^{(n)} < \infty \Rightarrow \mu_{\widetilde{F}}^{(k)} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F} < \infty \textit{ for } k = 0, 1, 2, ..., n - 1.$$

**Proof.** For $k = 0$ we have $\mu_{\widetilde{F}}^{(0)} = 1 = \frac{\mu}{\mu} = \frac{\mu_F^{(1)}}{(1)\mu_F}$. More generally, from Proposition 23 we know that

$$\mu_F^{(n)} < \infty \Rightarrow \mu_F^{(k)} < \infty \text{ for } k = 0, 1, 2, ..., n$$

and from Corollary 13

$$\mu_{\widetilde{F}}^{(k)} = \int_0^\infty x^k f_{\widetilde{F}}(x)dx = \int_0^\infty x^k \frac{S(x)}{\mu}dx = \frac{1}{(k+1)\,\mu}\int_0^\infty x^{k+1} f(x)dx = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F}$$

as required. ■

We see that

$$F \text{ has } n \text{ finite moments } \mu_F^{(k)},\ k = 1, 2, 3, ..., n$$

$$\Leftrightarrow \widetilde{F} \text{ has } n-1 \text{ finite moments } \mu_{\widetilde{F}}^{(k)},\ k = 1, 2, 3, ..., n-1.$$

Taking the coderived distribution can remove the existence of a higher moment.

The ultimate settlement rate $\tau_F$ is a useful measure of the tail behavior of a loss distribution. Our first significant result is that the tail behavior of the coderived loss variables has $\tau_F$ in common with the original, i.e., $\tau_F$ is a $\sim$invariant:

**Proposition 28** *If $F$ is an SLDFn with $\mu_F^{(n)} < \infty$ , then:*

$$\tau_F = \tau_{\widetilde{F}^{[k]}}, 0 \leq k \leq n.$$

**Proof.** Note that by Proposition 25 and Corollary 22

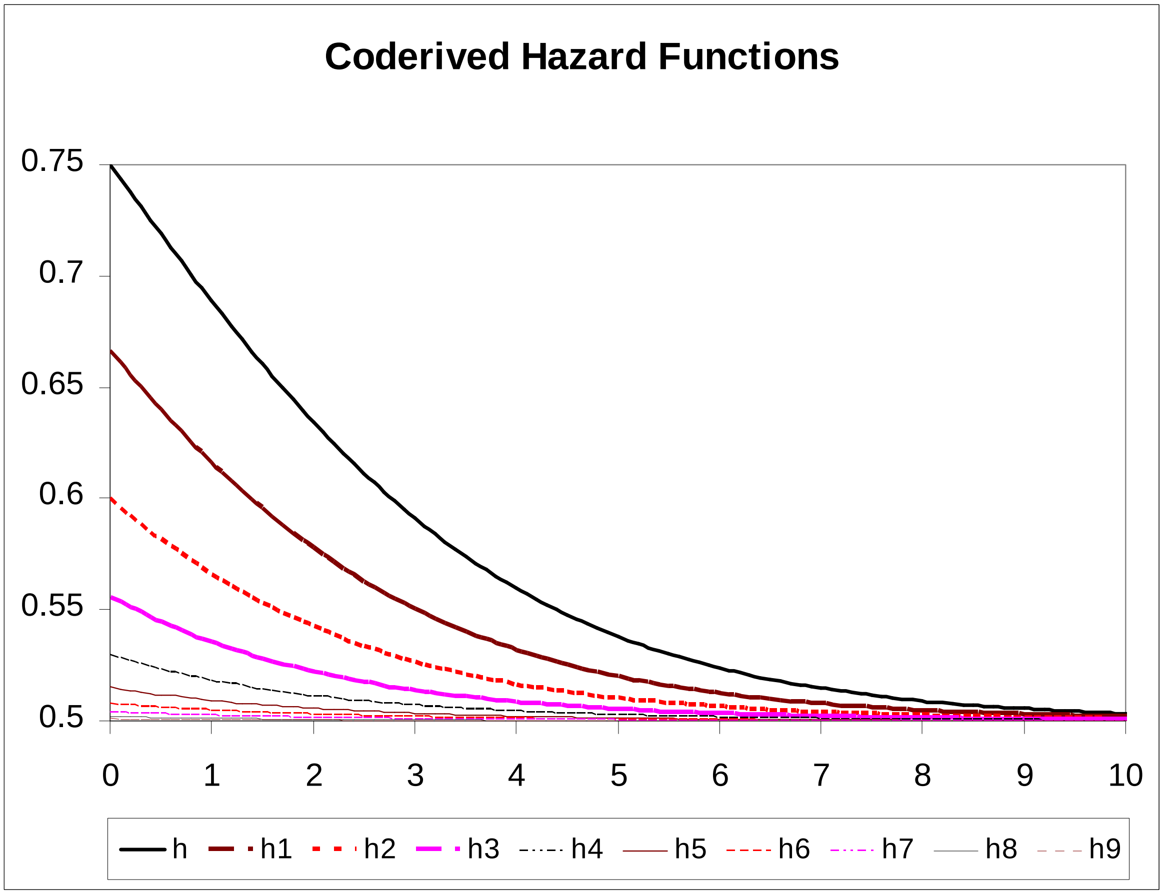
$$\begin{aligned} \tau_{\widetilde{F}} &= \lim_{x\to\omega_F} \lambda_{\widetilde{F}}(x) \\ &= \lim_{x\to\omega_F} \frac{f_{\widetilde{F}}(x)}{S_{\widetilde{F}}(x)} \\ &= \frac{1}{\mu}\lim_{x\to\omega_F} \frac{S(x)}{R(x)} \end{aligned}$$

and since $\lim_{x\to\omega_F} S(x) = 0 = \lim_{x\to\omega_F} R(x)$ we may invoke l'Hôpital

$$\begin{aligned} \tau_{\widetilde{F}} &= \frac{1}{\mu}\lim_{x\to\omega_F} \frac{\frac{dS}{dx}}{\frac{dR}{dx}} \\ &= \frac{1}{\mu}\lim_{x\to\omega_F} \frac{-f(x)}{\frac{-S(x)}{\mu}} \\ &= \frac{\mu}{\mu}\lim_{x\to\omega_F} \frac{f(x)}{S(x)} \\ &= \lim_{x\to\omega_F} \lambda_F(x) = \tau_F \end{aligned}$$

and since $\mu_F^{(n)} < \infty \Rightarrow \mu_F^{(k)} < \infty$, $1 \leq k \leq n$, the result for $n \geq 2$ follows by repeated application

$$\tau_F = \tau_{\widetilde{F}} = \tau_{\widetilde{F}^{[1]}} = \tau_{\widetilde{F}^{[2]}} = \cdots = \tau_{\widetilde{F}^{[n]}}$$




completing the proof. ■

Perhaps the easiest way to understand coderived distributions is to look at their hazard rate functions. In the chart below

$$h = \lambda_F \quad \text{with } \tau_F = \frac{1}{2} \quad \text{and } hx = \lambda_{\widetilde{F}[x]}, 1 \leq x \leq 10.$$

The chart illustrates how the higher coderived distributions "anticipate the tail", converging faster to the constant $\tau_F$.

Another way to see that the coderived variable shares tail behavior is to compare the survival curve of the coderived variable with that of conditional survival excess of a particular loss amount $c$. More precisely, we make the:

**Definition 29** *Let $F$ be an SLDFn and $c$ be a positive constant such that $F(c) < 1$. The* ***over c residual loss variable****, denoted $F^{>c}$ is the SLDFn*

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*determined from*

$$\begin{aligned} F^{>c}(x) &= 1 - \frac{S_F(x+c)}{S_F(c)} \text{ for every } x \geq 0 \\ \iff & S_{F^{>c}} = \frac{S_F(x+c)}{S_F(c)} \text{ for every } x \geq 0. \end{aligned}$$

The following shows that the tail behavior of a residual variable is also akin to that of the original loss variable and that there is a simple relationship between this residual and the coderived variables:

**Proposition 30** *If $F$ is an SLDFn and $c$ is a positive constant such that $S_F(c) > 0$ ,then:*

1. $\omega_{F^{>c}} = \omega_F - c$
2. $f_{F^{>c}}(x) = \frac{f_F(x+c)}{S_F(c)}$ for every $x \geq 0$
3. $d \geq 0$ such that $S_F(c+d) > 0 \Rightarrow \left(F^{>c}\right)^{>d} = F^{>c+d}$
4. $\lambda_{F^{>c}}(x) = \lambda_F(x+c)$ for every $x \geq 0$
5. $\tau_{F^{>c}} = \tau_F$
6. $\widetilde{F}^{>c} = \widetilde{F^{>c}}$
7. $\mu_F < \infty \Rightarrow \mu_{F^{>c}} = \mu_F \frac{S_{\widetilde{F}}(c)}{S_F(c)} = \frac{S_{\widetilde{F}}(c)}{f_{\widetilde{F}}(c)} = \frac{1}{\lambda_{\widetilde{F}}(c)}$
8. $\left(\widetilde{F}^{[n]}\right)^{>c} = \widetilde{F^{>c}}^{[n]}$ for every $n \in \mathbb{N}$.

**Proof.** Note that $S_F(c) > 0 \Rightarrow F(c) < 1 \Rightarrow c \leq \omega_F \Rightarrow \omega_F - c \geq 0$. Item 1 is obvious

$$\begin{aligned} x < \omega_F - c \Rightarrow x + c < \omega_F &\Rightarrow F(x+c) < 1 \\ &\Rightarrow S(x+c) > 0 \\ &\Rightarrow F^{>c}(x) = 1 - \frac{S_F(x+c)}{S_F(c)} < 1 \end{aligned}$$

and

$$\begin{aligned} x > \omega_F - c \Rightarrow x + c > \omega_F &\Rightarrow F(x+c) = 1 \\ &\Rightarrow S(x+c) = 0 \\ &\Rightarrow F^{>c}(x) = 1 - \frac{S_F(x+c)}{S_F(c)} = 1. \end{aligned}$$

Item 2 follows from the chain rule

$$\begin{aligned} f_{F^{>c}}(x) &= \frac{d}{dx}\left(F^{>c}(x)\right) = \frac{d}{dx}\left(\frac{-S_F(x+c)}{S_F(c)}\right) \\ &= \frac{-1}{S_F(c)}\frac{d}{dx}\left(S_F(x+c)\right) = \frac{-1}{S_F(c)}\frac{d\left(S_F(x+c)\right)}{d\left(x+c\right)}\frac{d\left(x+c\right)}{dx} \\ &= \frac{-1}{S_F(c)}\left(-f_F(x+c)\right) = \frac{f_F(x+c)}{S_F(c)}. \end{aligned}$$

For Item 3

$$\begin{aligned} S_{(F^{>c})^{>d}}(x) &= \frac{S_{F^{>c}}(x+d)}{S_{F^{>c}}(d)} = \frac{\frac{S_F((x+d)+c)}{S_F(c)}}{\frac{S_F(d+c)}{S_F(c)}} = \frac{S_F(x+(c+d))}{S_F(c+d)} = S_{F^{>c+d}}(x) \\ &\Rightarrow \left(F^{>c}\right)^{>d} = F^{>c+d} \end{aligned}$$

Item 4 is immediate from Item 2

$$\lambda_{F^{>c}}(x) = \frac{f_{F^{>c}}(x)}{S_{F^{>c}}(x)} = \frac{\frac{f_F(x+c)}{S_F(c)}}{\frac{S_F(x+c)}{S_F(c)}} = \frac{f_F(x+c)}{S_F(x+c)} = \lambda_F(x+c)$$

and Item 5 is immediate from Item 4

$$\tau_{F^{>c}} = \lim_{x\to\infty}\lambda_{F^{>c}}(x) = \lim_{x\to\infty}\lambda_F(x+c) = \lim_{x\to\infty}\lambda_F(x) = \tau_F.$$

Observe next that letting $G = F^{>c}$ we have the PDF

$$f_{\widetilde{G}}(x) = \frac{S_G(x)}{\mu_G} = \frac{S_{F^{>c}}(x)}{\mu_G} = \frac{\frac{S_F(x+c)}{S_F(c)}}{\mu_G} = \frac{S_F(x+c)}{\mu_G S_F(c)}$$

while by Item 2 we also have the PDF

$$f_{\widetilde{F}^{>c}}(x) = \frac{f_{\widetilde{F}}(x+c)}{S_{\widetilde{F}}(c)} = \frac{\frac{S_F(x+c)}{\mu_F}}{S_{\widetilde{F}}(c)} = \frac{S_F(x+c)}{\mu_F S_{\widetilde{F}}(c)}$$

which implies that the two PDFs are proportional, whence equal

$$\begin{aligned} \frac{S_F(x+c)}{\mu_G S_F(c)} &= f_{\widetilde{G}}(x) = f_{\widetilde{F}^{>c}}(x) = \frac{S_F(x+c)}{\mu_F S_{\widetilde{F}}(c)} \\ &\Rightarrow \widetilde{F^{>c}} = \widetilde{G} = \widetilde{F}^{>c} \end{aligned}$$

which proves Item 6. For Item 7 just note that from the above equation with $x = 0$

$$\begin{aligned} \frac{S_F(0+c)}{\mu_G S_F(c)} &= \frac{S_F(0+c)}{\mu_F S_{\widetilde{F}}(c)} \\ &\Rightarrow \mu_{F^{>c}} = \mu_G = \mu_F\frac{S_{\widetilde{F}}(c)}{S_F(c)} = \frac{S_{\widetilde{F}}(c)}{\frac{S_F(c)}{\mu_F}} \\ &= \frac{S_{\widetilde{F}}(c)}{f_{\widetilde{F}}(c)} = \frac{1}{\frac{f_{\widetilde{F}}(c)}{S_{\widetilde{F}}(c)}} = \frac{1}{\lambda_{\widetilde{F}}(c)}. \end{aligned}$$

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Finally, Item 8 is a straightforward induction on $n$ using Item 6; indeed, case $n = 1$ is Item 6, and then

$$\left(\widetilde{F}^{[n+1]}\right)^{>c} = \left(\widetilde{\widetilde{F}^{[n]}}\right)^{>c} = \widetilde{\left(\widetilde{F}^{[n]}\right)^{>c}} = \widetilde{\widetilde{F^{>c}}^{[n]}} = \widetilde{F^{>c}}^{[n+1]}$$

which completes the induction and the proof. ■

This provides a perspective on the coderived survival curve of an SLDFn, inasmuch as the coderived survival is to the original survival probability in the same proportion as the mean residual life is to the overall mean lifetime

$$\frac{S_{\widetilde{F}}(c)}{S_F(c)} = \frac{\mu_{F^{>c}}}{\mu_F}.$$

And the hazard rate function for the coderived distribution is the reciprocal of the mean residual life

$$\lambda_{\widetilde{F}}(c) = \frac{1}{\mu_{F^{>c}}}.$$

This perspective leads to a relationship between $\lambda_F$ and $\lambda_{\widetilde{F}}$ :

**Proposition 31** *If $F$ is an SLDFn with finite mean, then whenever $\lambda_F$ is increasing (nondecreasing, decreasing, nonincreasing) on $(0, \omega_F) = \left(0, \omega_{\widetilde{F}}\right)$, then so too is $\lambda_{\widetilde{F}}$.*

**Proof.** Suppose $\lambda$ is increasing and fix any $z > 0$, then for $y + z < \omega_F$

$$\frac{d}{dy}\left(\frac{S\left(y+z\right)}{S\left(y\right)}\right) = \frac{S\left(y\right)\frac{d}{dy}\left(S\left(y+z\right)\right) - S\left(y+z\right)\frac{d}{dy}\left(S\left(y\right)\right)}{S\left(y\right)^2}$$

$$\begin{aligned}
&= \frac{S\left(y\right)\left(-f\left(y+z\right)\right) + S\left(y+z\right)f(y)}{S\left(y\right)^2} \\
&= \frac{S\left(y+z\right)f(y) - S\left(y\right)\left(f\left(y+z\right)\right)}{S\left(y\right)^2} \\
&= \frac{S\left(y+z\right)}{S(y)}\lambda(y) - \frac{S\left(y+z\right)}{S(y)}\frac{f\left(y+z\right)}{S\left(y+z\right)} \\
&= \frac{S\left(y+z\right)}{S(y)}\lambda(y) - \frac{S\left(y+z\right)}{S(y)}\lambda(y+z) \\
&= \frac{S\left(y+z\right)}{S(y)}\left(\lambda(y) - \lambda(y+z)\right) < 0.
\end{aligned}$$

And so $\frac{S(y+z)}{S(y)}$ is a a decreasing function of $y$. It follows that

$$x < y \Rightarrow \frac{S\left(x+z\right)}{S\left(x\right)} > \frac{S\left(y+z\right)}{S\left(y\right)}$$

$$\Rightarrow \quad \mu_{F>x} = \int_0^\infty S_{F>x}(z)\,dz = \int_0^\infty \frac{S(x+z)}{S(x)}dz > \int_0^\infty \frac{S(y+z)}{S(y)}dz = \int_0^\infty S_{F>y}(z)\,dz = \mu_{F>y}$$

$$\Rightarrow \quad \lambda_{\widetilde{F}}(x) = \frac{1}{\mu_{F>x}} < \frac{1}{\mu_{F>y}} = \lambda_{\widetilde{F}}(y)$$

and $\lambda_{\widetilde{F}}$ is also increasing, as required. The case of $\lambda$ nondecreasing follows similarly, simply by changing strict inequalities to inequalities. The cases of $\lambda$ decreasing and nonincreasing follow by reversing inequalities. ■

**Proposition 32** *If $F$ is an SLDFn with finite mean and c ia a positive constant such that $\lambda_F(c) > 0$ ,then:*

$$\begin{aligned} \lambda_F \text{ nondecreasing} &\Rightarrow \lambda_{\widetilde{F}}(c) \geq \lambda_F(c) \\ \lambda_F \text{ increasing} &\Rightarrow \lambda_{\widetilde{F}}(c) > \lambda_F(c) \\ \lambda_F \text{ nonincreasing} &\Rightarrow \lambda_{\widetilde{F}}(c) \leq \lambda_F(c) \\ \lambda_F \text{ decreasing} &\Rightarrow \lambda_{\widetilde{F}}(c) < \lambda_F(c). \end{aligned}$$

**Proof.** Suppose $\lambda$ is nondecreasing. For any $z > 0$

$$\begin{aligned} S(c+z) &= e^{-\int_0^{c+z} \lambda(t)dt} \\ \frac{S(c+z)}{S(c)} &= e^{\int_0^{c} \lambda(t)dt - \int_0^{c+z} \lambda(t)dt} = e^{-\int_c^{c+z} \lambda(t)dt}. \end{aligned}$$

And since $\lambda$ is nondecreasing

$$t \in (c, c+z) \Rightarrow \lambda(t) \geq \lambda(c)$$

$$\begin{aligned} &\Rightarrow \int_c^{c+z} \lambda(t)dt \geq \int_c^{c+z} \lambda(c)dt = \lambda(c)\,z \\ &\Rightarrow -\int_c^{c+z} \lambda(t)dt \leq -\lambda(c)\,z \\ &\Rightarrow \frac{S(c+z)}{S(c)} = e^{-\int_c^{c+z} \lambda(t)dt} \leq e^{-\lambda(c)z}. \end{aligned}$$

And we have

$$\begin{aligned} 0 &< \frac{1}{\lambda_{\widetilde{F}}(c)} = \int_0^\infty \frac{S(c+z)}{S(c)}dz \leq \int_0^\infty e^{-\lambda(c)z}dz = \left[\frac{e^{-\lambda(c)z}}{-\lambda(c)}\right]_0^\infty = \frac{1}{\lambda(c)} \\ &\Rightarrow \lambda_{\widetilde{F}}(c) \geq \lambda(c) \end{aligned}$$

as required. The case of $\lambda$ increasing follows by making the inequalities strict. The case of $\lambda$ nonincreasing/decreasing follows similarly, reversing inequalities. ■

**Proposition 33** *If $F$ is an SLDFn with finite mean and $\tau_F > 0$, then $\mu_F^{(n)} < \infty$ for every $n \in \mathbb{N}$.*

**Proof.** We first show that $\tau_F > 0 \Rightarrow \mu_{\widetilde{F}} < \infty$. Observe that $f_{\widetilde{F}}(x) = \frac{S(x)}{\mu} > 0$ for every $x < \omega_F = \omega_{\widetilde{F}}$. By Proposition 28

$$\begin{aligned}
\lim_{x \to \omega_F} \frac{f_{\widetilde{F}}(x)}{S_{\widetilde{F}}(x)} &= \lim_{x \to \omega_F} \lambda_{\widetilde{F}}(x) = \tau_{\widetilde{F}} = \tau_F > 0 \\
&\Rightarrow \quad 0 < \lim_{x \to \omega_F} \frac{S_{\widetilde{F}}(x)}{f_{\widetilde{F}}(x)} = \frac{1}{\tau_F} < \infty.
\end{aligned}$$

This entails that there exist constants $M$ and $b > 0$ such that

$$\begin{aligned}
\frac{S_{\widetilde{F}}(x)}{f_{\widetilde{F}}(x)} &\leq b \text{ for every } x \in (M, \omega_F) \\
&\Rightarrow \quad S_{\widetilde{F}}(x) \leq b f_{\widetilde{F}}(x) \text{ for every } x \in (M, \omega_F).
\end{aligned}$$

Whence

$$\begin{aligned}
\mu_{\widetilde{F}} &= \int_0^\infty S_{\widetilde{F}}(x)dx \\
&= \int_0^M S_{\widetilde{F}}(x)dx + \int_M^{\omega_F} S_{\widetilde{F}}(x)dx \\
&\leq \int_0^M 1dx + \int_M^\infty b f_{\widetilde{F}}(x)dx \\
&\leq M + b < \infty
\end{aligned}$$

as claimed. But then, again by Proposition 28, we must have that $\mu_{\widetilde{F}[n]} < \infty$ for every $n \in \mathbb{N}$. The proof is completed by induction on $n$, the case $n = 1$ being clear. So assume the result holds for $k \leq n$. Note that $\tau_{\widetilde{F}} = \tau_F > 0$.Then we have, by induction applied to $\widetilde{F}$ and Proposition 27

$$\mu_F^{(n+1)} = (n+1)\,\mu_{\widetilde{F}}^{(n)} \mu_F < \infty$$

completing the induction and the proof. ■

**Remark 34** *The lognormal density shows that the converse does not hold.*

**Proposition 35** *If $F$ is an SLDFn with $0 < \tau_F < \infty$ , then $\lim_{c \to \omega_F} \mu_{F^{>c}} = \frac{1}{\tau_F}$.*

**Proof.** We have from l'Hôpital

$$\lim_{c \to \omega_F} \mu_{F^{>c}} = \lim_{c \to \omega_F} \int_0^\infty S_{F^{>c}}(x)dx = \lim_{c \to \omega_F} \int_0^\infty \frac{S_F(x+c)}{S_F(c)} dx$$

$$
\begin{aligned}
&= \lim_{c\to\omega_F} \frac{\int_0^\infty S_F(x+c)dx}{S_F(c)} = \lim_{c\to\omega_F} \frac{\int_c^\infty S_F(x)dx}{S_F(c)} \\
&= \lim_{c\to\omega_F} \frac{\frac{d}{dc}\int_c^\infty S_F(x)dx}{\frac{d}{dc}S_F(c)} = \lim_{c\to\omega_F} \frac{-S_F(c)}{-f_F(c)} \\
&= \lim_{c\to\omega_F} \frac{1}{\lambda_F(c)} = \frac{1}{\lim\limits_{c\to\omega_F}\lambda_F(c)} = \frac{1}{\tau_F}
\end{aligned}
$$

as claimed. ■

**Proposition 36** *If $F$ is an SLDFn with finite mean and $0 < \tau_F < \infty$ , then:*

$$
\lim_{x\to\omega_F} \frac{S_{\widetilde{F}}(x)}{S_F(x)} = \frac{1}{\mu_F \tau_F}.
$$

**Proof.** From the above propositions

$$
\lim_{x\to\omega_F} \frac{S_{\widetilde{F}}(x)}{S_F(x)} = \lim_{x\to\omega_F} \frac{\mu_{F>x}}{\mu_F} = \frac{\lim\limits_{x\to\omega_F}\mu_{F>x}}{\mu_F} = \frac{\frac{1}{\tau_F}}{\mu_F} = \frac{1}{\mu_F\tau_F}
$$

as claimed. ■

**Proposition 37** *If $F$ is an SLDFn with finite mean and $0 < \tau_F < \infty$ , then:*

$$
\lim_{x,c\to\omega_F} \frac{S_{\widetilde{F>c}}(x)}{S_{F>c}(x)} = 1.
$$

**Proof.** From the above

$$
\lim_{c,x\to\omega_F} \frac{S_{\widetilde{F>c}}(x)}{S_{F>c}(x)} = \lim_{c\to\omega_F}\left(\lim_{x\to\omega_F} \frac{S_{\widetilde{F>c}}(x)}{S_{F>c}(x)}\right)
$$

$$
\begin{aligned}
&= \lim_{c\to\omega_F} \frac{1}{\mu_{F>c}\tau_{F>c}} = \lim_{c\to\omega_F} \frac{1}{\mu_{F>c}\tau_F} \\
&= \frac{1}{\tau_F}\lim_{c\to\omega_F}\frac{1}{\mu_{F>c}} = \frac{1}{\tau_F}\frac{1}{\lim\limits_{c\to\omega_F}\mu_{F>c}} \\
&= \frac{1}{\tau_F}\frac{1}{\frac{1}{\tau_F}} = \frac{1}{1} = 1
\end{aligned}
$$

as claimed. ■

**Proposition 38** *For any SLDFn $F$ such that $\mu_F^{(n)} < \infty$ for every $n \in \mathbb{N}$:*

$$
L_{\widetilde{F}}(t) = \frac{1 - L_F(t)}{\mu t} \quad \text{for } t > 0
$$

*and if $F$ has a moment generating function, then so does $\widetilde{F}$ with*

$$
M_{\widetilde{F}}(t) = \frac{M_F(t) - 1}{\mu t} \quad \text{for } t > 0.
$$

**Proof.** We have, from Proposition 27, for every $t > 0$

$$
\begin{aligned}
L_{\widetilde{F}}(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k \mu_{\widetilde{F}}^{(k)} t^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{\mu_F^{(k+1)}}{(k+1)\mu} \right) t^k}{k!} \\
&= \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(k+1)} t^k}{(k+1)!} \\
&= \frac{1}{\mu t} \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(k+1)} t^{k+1}}{(k+1)!}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \quad -\mu t L_{\widetilde{F}}(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \mu_F^{(k+1)} t^{k+1}}{(k+1)!} \\
&= \quad \sum_{j=1}^{\infty} \frac{(-1)^j \mu_F^{(j)} t^j}{j!} = L_F(t) - 1 \\
&\Rightarrow \quad L_{\widetilde{F}}(t) = \frac{1 - L_F(t)}{\mu t}.
\end{aligned}
$$

And so if $M_F(t)$ exists, it follows that

$$
M_{\widetilde{F}}(t) = L_{\widetilde{F}}(-t) = \frac{1 - L_F(-t)}{-\mu t} = \frac{L_F(-t) - 1}{\mu t} = \frac{M_F(t) - 1}{\mu t}
$$

as required. ■

A straightforward integration by parts, however, provides the stronger result:

**Proposition 39** *For any SLDFn F:*

$$
L_{\widetilde{F}}(t) = \frac{1 - L_F(t)}{\mu t} \qquad \textit{for } t > 0.
$$

**Proof.** Fix $t > 0$, we have

$$\begin{aligned}
L_{\widetilde{F}}(t) &= \int_0^\infty e^{-tx} f_{\widetilde{F}}(x)dx \\
&= \int_0^\infty e^{-tx} \frac{S(x)}{\mu} dx \\
&= \frac{1}{\mu}\int_0^\infty udv \text{ where } u = S(x) \text{ and } v = -\frac{e^{-tx}}{t} \\
&= \frac{1}{\mu}\left([uv]_0^\infty - \int_0^\infty vdu\right) \\
&= \frac{1}{\mu}\left(\left[-\frac{e^{-tx}S(x)}{t}\right]_{x=0}^{x\to\infty} - \int_0^\infty \left(-\frac{e^{-tx}}{t}\right)(-f(x))\,dx\right) \\
&= \frac{1}{\mu t}\left(1 - \int_0^\infty e^{-tx} f(x)dx\right) \\
&= \frac{1}{\mu t}\left(1 - L_F(t)\right)
\end{aligned}$$

as required. ■

Another relationship between the moments of the original and the coderived distributions is:

**Proposition 40** *For any SLDFn F*

$$\mu_F^{(n)} < \infty \Rightarrow \mu_{\widetilde{F}[k]} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}} \textit{ for } k = 0, 1, 2, ..., n-1.$$

**Proof.** Note that by Proposition 23

$$\mu_F^{(n)} < \infty \Rightarrow \mu_F^{(k)} < \infty \text{ for } k = 0, 1, 2, ..., n$$

so our assumption is inductive. For $k = 0$ the assertion is

$$\mu_F = \mu_{\widetilde{F}[0]} = \frac{\mu_F^{(1)}}{(1)\,\mu_F^{(0)}} = \frac{\mu_F}{1 \cdot 1}$$

which is vacuous. For $k = 1$ the assertion is just

$$\mu_{\widetilde{F}} = \mu_{\widetilde{F}[1]} = \frac{\mu_F^{(2)}}{2\mu_F^{(1)}}$$

which holds by Proposition 27. We proceed by induction, invoking the case $n-1$ for $G = \widetilde{F}$ , which is indeed an SLDFn with $\mu_{\widetilde{F}}^{(n-1)} < \infty$. Invoking Proposition

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27 twice more

$$\begin{aligned}
\mu_{\widetilde{F}^{[k]}} &= \mu_{\widetilde{G}^{[k-1]}} \\
&= \frac{\mu_G^{(k)}}{k\mu_G^{(k-1)}} \\
&= \frac{\mu_{\widetilde{F}}^{(k)}}{k\mu_{\widetilde{F}}^{(k-1)}} \\
&= \left(\frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F}\right) \div \left(k\left(\frac{\mu_F^{(k)}}{k\mu_F}\right)\right) \\
&= \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}}
\end{aligned}$$

completing the induction and the proof. ■

**Corollary 41** *For any SLDFn $F$ and positive constant $c$ with $S_F(c) > 0$ :*

$$\mu_F^{(n)} < \infty \Rightarrow \frac{S_{\widetilde{F}^{[k]}}(c)}{S_F(c)} = \frac{\mu_{F>c}^{(k)}}{\mu_F^{(k)}} \text{ for } k = 0, 1, 2, ..., n.$$

**Proof.** The proof is by induction. Case $k = 1$ follows from Item 7 of Proposition 30. Combining Items 7 and 8 of that same Proposition, together with Proposition 40 and the induction hypothesis

$$\begin{aligned}
\frac{S_{\widetilde{F}^{[k+1]}}(c)}{S_F(c)} &= \frac{S_{\widetilde{\widetilde{F}^{[k]}}}(c)}{S_{\widetilde{F}^{[k]}}(c)} \frac{S_{\widetilde{F}^{[k]}}(c)}{S_F(c)} = \frac{\mu_{\left(\widetilde{F}^{[k]}\right)^{>c}}}{\mu_{\widetilde{F}^{[k]}}} \frac{\mu_{F>c}^{(k)}}{\mu_F^{(k)}} \\
&= \frac{\mu_{\widetilde{F>c}^{[k]}}}{\mu_{\widetilde{F}^{[k]}}} \frac{\mu_{F>c}^{(k)}}{\mu_F^{(k)}} = \frac{\frac{\mu_{F>c}^{(k+1)}}{k+1}}{\frac{\mu_F^{(k+1)}}{k+1}} = \frac{\mu_{F>c}^{(k+1)}}{\mu_F^{(k+1)}}
\end{aligned}$$

completing the induction and the proof. ■

The following result will come in handy later when we relate the concept of coderived distribution with ultimate settlement rates and tail "thickness".

**Proposition 42** *If $F$ is an SLDFn and $n \in \mathbb{N}$ with $\mu_F^{(n)} < \infty$ , then:*

$$\lim_{x \to \omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[m]}}(x)} = \frac{\tau_F^m \mu_F^{(m)}}{m!} \text{ for } m = 0, 1, 2, ..., n.$$

**Proof.** Note that

$$x < \omega_F = \omega_{\widetilde{F}^{[m-1]}} \Rightarrow f_{\widetilde{F}^{[m]}}(x) = \frac{S_{\widetilde{F}^{[m-1]}}(x)}{\mu_{\widetilde{F}^{[m-1]}}} > 0$$

assures that we are not dividing by 0. For $m = 0, 1$ we have

$$\begin{aligned}
\lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[0]}}(x)} &= \lim_{x\to\omega_F} \frac{f_F(x)}{f_F(x)} = \lim_{x\to\omega_F} 1 = 1 = \mu_F^{(0)} (\tau_F)^0 \\
\lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[1]}}(x)} &= \lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}}(x)} = \lim_{x\to\omega_F} \frac{f_F(x)}{\frac{S_F(x)}{\mu_F}} = \mu_F \lim_{x\to\omega_F} \frac{f_F(x)}{S_F(x)} = \mu_F \lim_{x\to\omega_F} \lambda_F(x) = \mu_F^{(1)} \tau_F
\end{aligned}$$

and the formula holds for $m = 0, 1$. Proceed by induction on $m$ noting that Proposition 23 assures that our hypothesis is inductive. Invoking the case $m = 1$ and the induction hypothesis

$$\begin{aligned}
\lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[m+1]}}(x)} &= \lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[m]}}(x)} \frac{f_{\widetilde{F}^{[m]}}(x)}{f_{\widetilde{\widetilde{F}^{[m]}}}(x)} \\
&= \lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{[m]}}(x)} \lim_{x\to\omega_F} \frac{f_{\widetilde{F}^{[m]}}(x)}{f_{\widetilde{\widetilde{F}^{[m]}}}(x)} \\
&= \frac{\tau_F^m \mu_F^{(m)}}{m!} \mu_{\widetilde{F}^{[m]}} \tau_{\widetilde{F}^{[m]}}
\end{aligned}$$

And then by Propositions 40 and 28

$$\lim_{x\to\omega_F} \frac{f_F(x)}{f_{\widetilde{F}^{(m+1)}}(x)} = \frac{\tau_F^m \mu_F^{(m)}}{m!} \frac{\mu_F^{(m+1)}}{(m+1)\,\mu_F^{(m)}} \tau_F = \frac{\tau_F^{m+1} \mu_F^{(m+1)}}{(m+1)!}$$

completing the induction and the proof. ∎

**Corollary 43** *If $F$ is a non-vanishing SLDFn with $0 < \tau_F < \infty$ , then:*

$$\textit{for every } m, n \in \mathbb{N}, \lim_{x\to\infty} \frac{f_{\widetilde{F}^{[n]}}(x)}{f_{\widetilde{F}^{[m]}}(x)} = \frac{\tau_F^{m-n} n! \mu_F^{(m)}}{m! \mu_F^{(n)}}.$$

**Proof.** By Proposition 33 $\mu_F^{(k)} < \infty$ for every $k \in \mathbb{N}$ and so the proposition gives

$$\begin{aligned}
\lim_{x\to\infty} \frac{f_{\widetilde{F}^{[n]}}(x)}{f_{\widetilde{F}^{[m]}}(x)} &= \lim_{x\to\infty} \frac{f_{\widetilde{F}^{[n]}}(x)}{f_F(x)} \lim_{x\to\infty} \frac{f_F(x)}{f_{\widetilde{F}^{[m]}}(x)} \\
&= \frac{n!}{\tau_F^n \mu_F^{(n)}} \frac{\tau_F^m \mu_F^{(m)}}{m!} = \frac{\tau_F^{m-n} n! \mu_F^{(m)}}{m! \mu_F^{(n)}}
\end{aligned}$$

as asserted. ∎

Proposition 42 suggests that the series of higher coderived SLDFns $F, \widetilde{F}, \widetilde{F}^{[2]}, \widetilde{F}^{[3]}, \ldots$ share a similar right hand tail behavior and that it may sometimes be viable to approximate the right hand tail of an SLDFn $F$ with that of a higher coderived loss distribution $\widetilde{F}^{[m]}$, adjusted by the scalar $\frac{\tau_F^m \mu_F^{(m)}}{m!}$. Generalizing Proposition 40 yet one step further, we see that all the moments of all coderived distributions are readily obtained from those of $F$:

**Proposition 44** *If $F$ is a non-vanishing SLDFn such that for every $n \in \mathbb{N}$ we have $\mu_F^{(n)} < \infty$ , then:*

$$\binom{j+k}{k}\mu_{\widetilde{F}^{[j]}}^{(k)} = \frac{\mu_F^{(j+k)}}{\mu_F^{(j)}} \quad \textit{for every } j,k \in \mathbb{N}\cup\{0\}.$$

**Proof.** The case $j = 0$ is just $\binom{k}{k}\mu_F^{(k)} = \mu_F^{(k)} = \frac{\mu_F^{(k)}}{\mu_F^{(0)}}$ which is certainly true for all integers $k \geq 0$. For the case $j = 1$, Proposition 27 gives

$$\binom{k+1}{k}\mu_{\widetilde{F}^{[1]}}^{(k)} = (k+1)\,\mu_{\widetilde{F}}^{(k)} = (k+1)\left(\frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F}\right) = \frac{\mu_F^{(k+1)}}{\mu_F^{(1)}}$$

and so the result again holds for all integers $k \geq 0$. The proof is by induction on $j$. Let $G = \widetilde{F}^{[j-1]}$. By Proposition 27

$$\begin{aligned}
\mu_{\widetilde{F}^{[j]}}^{(k)} &= \mu_{\widetilde{G}}^{(k)} \\
&= \frac{\mu_G^{(k+1)}}{(k+1)\,\mu_G} \\
&= \frac{\mu_{\widetilde{F}^{[j-1]}}^{(k+1)}}{(k+1)\,\mu_{\widetilde{F}^{[j-1]}}}.
\end{aligned}$$

Invoking induction on the numerator and Proposition 40 on the denominator, we have

$$\begin{aligned}
\mu_{\widetilde{F}^{[j]}}^{(k)} &= \frac{\frac{\mu_F^{(j-1+k+1)}}{\mu_F^{(j-1)}}}{\binom{j-1+k+1}{k+1}(k+1)\left(\frac{\mu_F^{(j-1+1)}}{(j-1+1)\mu_F^{(j-1)}}\right)} \\
&= \frac{\frac{\mu_F^{(j+k)}}{\mu_F^{(j-1)}}}{\binom{j+k}{k+1}(k+1)\left(\frac{\mu_F^{(j)}}{j\mu_F^{(j-1)}}\right)} \\
&= \frac{(k+1)!(j+k-(k+1))!j\mu_F^{(j+k)}}{(j+k)!\,(k+1)\,\mu_F^{(j)}} \\
&= \frac{(k+1)!\,(j-1)!j\mu_F^{(j+k)}}{(j+k)!(k+1)\mu_F^{(j)}} \\
&= \frac{k!j!\mu_F^{(j+k)}}{(j+k)!\mu_F^{(j)}} = \binom{j+k}{k}^{-1}\frac{\mu_F^{(j+k)}}{\mu_F^{(j)}}
\end{aligned}$$

completing the induction and the proof. ■

We next present a series of straightforward results on coderived loss distributions.

**Proposition 45** *If $F$ is an SLDFn such that for every $n \in \mathbb{N}$ we have $\mu_F^{(n)} < \infty$, then*

$$\text{for every } n \in \mathbb{N}, \text{ and } t > 0, \quad L_{\widetilde{F}^{[n]}}(t) = \frac{n!}{\mu_F^{(n)}} \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(n+k)} t^k}{(n+k)!}.$$

*And if $F$ has a moment generating function, then so does $\widetilde{F}^{[n]}$ with*

$$M_{\widetilde{F}^{[n]}}(t) = \frac{n!}{\mu_F^{(n)}} \sum_{k=0}^{\infty} \frac{\mu_F^{(n+k)} t^k}{(n+k)!}.$$

**Proof.** We need only verify the assertion for the Laplace transform, since that clearly implies the formula for the moment generating function. For $n = 1$ the assertion becomes

$$L_{\widetilde{F}}(t) = \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(k+1)} t^k}{(k+1)!} = \frac{-1}{\mu t} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \mu_F^{(k+1)} t^{k+1}}{(k+1)!}$$

$$= \frac{-1}{\mu t} \sum_{j=1}^{\infty} \frac{(-1)^j \mu_F^{(j)} t^j}{j!} = \frac{-1}{\mu t} \left( \sum_{j=0}^{\infty} \frac{(-1)^j \mu_F^{(j)} t^j}{j!} - 1 \right)$$

$$= \frac{-1}{\mu t} \left( L_F(t) - 1 \right) = \frac{1 - L_F(t)}{\mu t}$$

which is known to hold by Proposition 38. The proof is by induction on $n$. For $n > 1$ we again have by Proposition 38, induction, and Proposition 40

$$L_{\widetilde{F}^{[n]}}(t) = \frac{1 - L_{\widetilde{F}^{[n-1]}}(t)}{\mu_{\widetilde{F}^{[n-1]}} t}$$

$$= \frac{1}{\mu_{\widetilde{F}^{[n-1]}} t} \left( 1 - \frac{(n-1)!}{\mu_F^{(n-1)}} \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(n+k-1)} t^k}{(n+k-1)!} \right)$$

$$= \frac{1}{\frac{\mu_F^{(n-1+1)}}{(n-1+1)\mu_F^{(n-1)}} t} \left( - \frac{(n-1)!}{\mu_F^{(n-1)}} \sum_{k=1}^{\infty} \frac{(-1)^k \mu_F^{(n+k-1)} t^k}{(n+k-1)!} \right)$$

$$= \frac{1}{\frac{\mu_F^{(n)}}{n\mu_F^{(n-1)}} t} \left( - \frac{(n-1)!}{\mu_F^{(n-1)}} \sum_{k=1}^{\infty} \frac{(-1)^k \mu_F^{(n+k-1)} t^k}{(n+k-1)!} \right)$$

$$= \frac{n\mu_F^{(n-1)}}{\mu_F^{(n)}} \left( \frac{(n-1)!}{\mu_F^{(n-1)}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \mu_F^{(n+(k-1))} t^{k-1}}{(n+(k-1))!} \right)$$

$$= \frac{n!}{\mu_F^{(n)}} \left( \sum_{j=0}^{\infty} \frac{(-1)^j \mu_F^{(n+j)} t^j}{(n+j)!} \right)$$

completing the induction and the proof. ■

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**Proposition 46** *For any SLDFn $F$ and for every $n \in \mathbb{N}$:*

1. $\mu_F^{(n)} < \infty \Rightarrow f_{\widetilde{F}^{[n]}}(x) = \frac{n\int_x^\infty (y-x)^{n-1} f_F(y)dy}{\mu_F^{(n)}}$
2. $\mu_F^{(n)} < \infty \Rightarrow S_{\widetilde{F}^{[n]}}(x) = \frac{\int_x^\infty (y-x)^{n} f_F(y)dy}{\mu_F^{(n)}}$
3. $\mu_F^{(n)} < \infty \Rightarrow \lambda_{\widetilde{F}^{[n]}}(x) = \frac{n\int_x^\infty (y-x)^{n-1} f_F(y)dy}{\int_x^\infty (y-x)^{n} f_F(y)dy}$ for every $x < \omega_F$
4. $m, n \in \mathbb{N}, \mu_F^{(n)} < \infty, 0 \le m \le n \Rightarrow \frac{d^m S_{\widetilde{F}^{[n]}}}{dx^m} = \frac{(-1)^m n! \mu_F^{(n-m)} S_{\widetilde{F}^{[n-m]}}}{(n-m)!\mu_F^{(n)}}$
5. $(CV_F)^2 = 2\left(\frac{\mu_{\widetilde{F}}}{\mu_F}\right) - 1$
6. $CV_F = 1 \Leftrightarrow \mu_F = \mu_{\widetilde{F}}$
7. $CV_F < 1 \Leftrightarrow \mu_F > \mu_{\widetilde{F}}$
8. $CV_F > 1 \Leftrightarrow \mu_F < \mu_{\widetilde{F}}$

**Proof.** The proof of Item 1 is by induction on $n$. For $n = 1$ the assertion reduces to

$$f_{\widetilde{F}}(x) = \frac{\int_x^\infty (y-x)^0 f_F(y)dy}{\mu_F} = \frac{\int_x^\infty f_F(y)dy}{\mu} = \frac{S_F(x)}{\mu}$$

and for $n = 2$ the assertion is

$$f_{\widetilde{F}^{[2]}}(x) = \frac{2\int_x^\infty (y-x) f_F(y)dy}{\mu_F^{(2)}} = \frac{\frac{1}{\mu}\int_x^\infty (y-x) f_F(y)dy}{\frac{\mu_F^{(2)}}{2\mu}} = \frac{R_F(x)}{\frac{\mu_F^{(2)}}{2\mu}} = \frac{S_{\widetilde{F}}(x)}{\mu_{\widetilde{F}}}$$

and both hold by Proposition 25. Then we have, for $n > 1$, from Propositions 25 and 40, and induction

$$\begin{aligned} f_{\widetilde{F}^{[n]}}(x) &= \frac{S_{\widetilde{F}^{[n-1]}}(x)}{\mu_{\widetilde{F}^{[n-1]}}} \\ &= \frac{\int_x^\infty f_{\widetilde{F}^{[n-1]}}(z)dz}{\frac{\mu_F^{(n)}}{n\mu_F^{(n-1)}}} \end{aligned}$$

$$\begin{aligned} &= \frac{n\mu_F^{(n-1)}}{\mu_F^{(n)}} \int_x^\infty \frac{(n-1)\int_z^\infty (y-z)^{n-2} f_F(y)dy}{\mu_F^{(n-1)}} dz \\ &= \frac{n(n-1)}{\mu_F^{(n)}} \int_x^\infty \int_z^\infty (y-z)^{n-2} f_F(y)dydz \end{aligned}$$

$$
\begin{aligned}
&= \frac{n(n-1)}{\mu_F^{(n)}} \int_x^\infty \int_x^y (y-z)^{n-2} f_F(y)dzdy \\
&= \frac{n(n-1)}{\mu_F^{(n)}} \int_x^\infty f_F(y) \int_x^y (y-z)^{n-2} dzdy \\
&= \frac{n(n-1)}{\mu_F^{(n)}} \int_x^\infty f_F(y) \left[ -\frac{(y-z)^{n-1}}{n-1} \right]_{z=x}^{z=y} dy \\
&= \frac{n(n-1)}{\mu_F^{(n)}} \int_x^\infty f_F(y) \left( \frac{(y-x)^{n-1}}{n-1} \right) dy \\
&= \frac{n \int_x^\infty (y-x)^{n-1} f_F(y)dy}{\mu_F^{(n)}}
\end{aligned}
$$

completing the proof of Item 1. Item 2 follows from Item 1. Indeed, for $n = 1$ the assertion reduces to

$$
S_{\widetilde{F}}(x) = \frac{\int_x^\infty (y-x) f_F(y)dy}{\mu} = R_F(x)
$$

which holds by Proposition 25. Then we have from Item 1

$$
S_{\widetilde{F}^{[n]}}(x) = \int_x^\infty f_{\widetilde{F}^{[n]}}(z)dz
$$

$$
\begin{aligned}
&= \int_x^\infty \left( \frac{n \int_z^\infty (y-z)^{n-1} f_F(y)dy}{\mu_F^{(n)}} \right) dz \\
&= \frac{n}{\mu_F^{(n)}} \int_x^\infty \int_z^\infty (y-z)^{n-1} f_F(y)dydz \\
&= \frac{n}{\mu_F^{(n)}} \int_x^\infty \int_x^y (y-z)^{n-1} f_F(y)dzdy \\
&= \frac{n}{\mu_F^{(n)}} \int_x^\infty f_F(y) \int_x^y (y-z)^{n-1} dzdy \\
&= \frac{n}{\mu_F^{(n)}} \int_x^\infty f_F(y) \left[ -\frac{(y-z)^n}{n} \right]_{z=x}^{z=y} dzdy \\
&= \frac{1}{\mu_F^{(n)}} \int_x^\infty f_F(y) (y-x)^n dy
\end{aligned}
$$

completing the proof of Item 2. Item 3 follows from Items 1 and 2 and the fact that $\omega_{\widetilde{F}^{[n]}} = \omega_F$. Item 4 follows from Item 2 by differentiating under the integral

$$
\frac{d^m S_{\widetilde{F}^{[n]}}}{dx^m}(x) = \frac{d^m \left( \frac{\int_x^\infty (y-x)^n f_F(y)dy}{\mu_F^{(n)}} \right)}{dx^m}
$$

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$$
\begin{aligned}
&= \frac{1}{\mu_F^{(n)}} \left( \int_x^\infty \frac{d^m}{dx^m} \left( (y-x)^n f_F(y) \right) dy \right) \\
&= \frac{1}{\mu_F^{(n)}} \left( \int_x^\infty \frac{(-1)^m n!}{(n-m)!} (y-x)^{n-m} f_F(y) dy \right) \\
&= \frac{(-1)^m n!}{(n-m)! \mu_F^{(n)}} \left( \int_x^\infty (y-x)^{n-m} f_F(y) dy \right) \\
&= \frac{(-1)^m n! \mu_F^{(n-m)}}{(n-m)! \mu_F^{(n)}} \left( \frac{\int_x^\infty (y-x)^{n-m} f_F(y) dy}{\mu_F^{(n-m)}} \right) \\
&= \frac{(-1)^m n! \mu_F^{(n-m)} S_{\widetilde{F}^{[n-m]}}(x)}{(n-m)! \mu_F^{(n)}}
\end{aligned}
$$

which establishes Item 4. Note that

$$
\begin{aligned}
(CV_F)^2 &= \frac{\mu_F^{(2)} - \mu_F^2}{\mu_F^2} = \frac{\mu_F^{(2)}}{\mu_F^2} - 1 \\
&= \frac{2}{\mu_F} \left( \frac{\mu_F^{(2)}}{2\mu_F} \right) - 1 = \frac{2}{\mu_F} \left( \mu_{\widetilde{F}} \right) - 1
\end{aligned}
$$

which establishes Item 5. Since Items 6, 7 and 8 follow immediately from Item 5, this completes the proof. ■

A simple but useful observation is that taking the coderived distribution commutes with a change of scale:

**Definition 47** *Let $F$ be a SLDFn and $a > 0$ be any positive constant; the SLDFn $F_a$ is defined as*

$$F_a(x) = F(ax).$$

**Proposition 48** *For every $a, c > 0$ and for every $n \in \mathbb{N} \cup \{0\}$ :*

1. $\omega_{F_a} = a\omega_F$
2. $S_{F_a}(x) = S_F(ax)$
3. $f_{F_a}(x) = af_F(ax)$
4. $\lambda_{F_a}(x) = a\lambda_F(ax)$
5. $\mu_{F_a}^{(c)} = \frac{\mu_F^{(c)}}{a^c}$
6. $\widehat{(F_a)}^{[c]} = \left( \widehat{F}^{[c]} \right)_a$
7. $\widetilde{(F_a)}^{[n]} = \left( \widetilde{F}^{[n]} \right)_a$

8. $S(ac) > 0 \Rightarrow (F_a)^{>c} = (F^{>ac})_a$

**Proof.** Items 1 and 2 are obvious, Item 3 follows from the chain rule

$$f_{F_a}(x) = \frac{dF_a}{dx} = \left(\frac{dF}{dz}|_{z=ax}\right)\frac{dz}{dx} = (f_F(z)|_{z=ax})\,a = af_F(ax).$$

and Item 4 is then an immediate consequence

$$\lambda_{F_a}(x) = \frac{f_{F_a}(x)}{S_{F_a}(x)} = \frac{af_F(ax)}{S_F(ax)} = a\lambda_F(ax).$$

For Item 5, use Item 3 and the change of variable $z = ay$

$$\begin{aligned}
\mu_{F_a}^{(c)} &= \int_0^\infty y^c f_{F_a}(y)dy = \int_0^\infty y^c af_F(ay)dy \\
&= \int_0^\infty \left(\frac{z}{a}\right)^c f_F(z)dz = \frac{\int_0^\infty z^c f_F(z)dz}{a^c} \\
&= \frac{\mu_F^{(c)}}{a^c}.
\end{aligned}$$

Item 6 now follows from

$$\begin{aligned}
f_{\widehat{(F_a)}^{[c]}}(x) &= \frac{x^c f_{F_a}(x)}{\mu_{F_a}^{(c)}} = \frac{x^c af_F(ax)}{\frac{\mu_F^{(c)}}{a^c}} \\
&= a\left(\frac{(ax)^c f_F(ax)}{\mu_F^{(c)}}\right) = af_{\widehat{F}^{[c]}}(ax) \\
&= f_{\left(\widehat{F}^{[c]}\right)_a}(x).
\end{aligned}$$

For item 7, note first that this holds vacuously for $n = 0$

$$\widetilde{(F_a)}^{[0]} = F_a = \left(\widetilde{F}^{[0]}\right)_a$$

and for $n = 1$

$$\begin{aligned}
f_{\widetilde{F_a}}(x) &= \frac{S_{F_a}(x)}{\mu_{F_a}} = \frac{S_F(ax)}{\frac{\mu_F}{a}} = af_{\widetilde{F}}(ax) = f_{\widetilde{F}_a}(x) \\
&\Rightarrow \widetilde{(F_a)}^{[1]} = \widetilde{F_a} = \widetilde{F}_a = \left(\widetilde{F}^{[1]}\right)_a
\end{aligned}$$

and now Item 7 follows by induction

$$\begin{aligned}
\widetilde{(F_a)}^{[n+1]} &= \widetilde{\widetilde{(F_a)}^{[n]}} = \widetilde{\left(\widetilde{F}^{[n]}\right)_a} = \left(\widetilde{\widetilde{F}^{[n]}}\right)_a \\
&= \left(\widetilde{F}^{[n+1]}\right)_a
\end{aligned}$$

Finally, from Item 2 we have

$$\begin{aligned} S_{(F_a)^{>c}}(x) &= \frac{S_{F_a}(x+c)}{S_{F_a}(c)} = \frac{S_F(a(x+c))}{S_F(ac)} \\ &= \frac{S_F(ax+ac)}{S_F(ac)} = S_{F^{>ac}}(ax) = S_{(F^{>ac})_a}(x) \end{aligned}$$

completing the proof. ■

Since the coderived distribution relates with the excess ratio, the following result for mixed distributions is no surprise:

**Proposition 49** *Given $m \in \mathbb{N}$, SLDFns $F_1, ..., F_m$ all with finite means, and any real weights $w_i \geq 0$ with $1 = \sum_{i=1}^{m} w_i$, for the weighted mixture SLDFn $F = \sum_{i=1}^{m} w_i F_i$ with PDF $f_F = \sum_{i=1}^{m} w_i f_{F_i}$, we have:*

$$\begin{aligned} f_{\widetilde{F}} &= \sum_{i=1}^{m} u_i f_{\widetilde{F_i}} \quad \text{and } \widetilde{F} = \sum_{i=1}^{m} u_i \widetilde{F}_i, \\ \text{where } u_i &= \frac{w_i \mu_{F_i}}{\mu_F} \quad \text{and } 1 = \sum_{i=1}^{m} u_i. \end{aligned}$$

**Proof.** This is again a straightforward verification, from Proposition 25

$$\begin{aligned} f_{\widetilde{F}}(x) &= \frac{S_F(x)}{\mu_F} = \frac{\sum_{i=1}^{m} w_i S_{F_i}(x)}{\mu_F} \\ &= \frac{\sum_{i=1}^{m} w_i \mu_{F_i} \left(\frac{S_{F_i}(x)}{\mu_{F_i}}\right)}{\mu_F} = \sum_{i=1}^{m} u_i f_{\widetilde{F_i}}(x) \\ &\Rightarrow \quad S_{\widetilde{F}}(x) = \sum_{i=1}^{m} u_i S_{\widetilde{F_i}}(x) \end{aligned}$$

and since clearly

$$\begin{aligned} \mu_F &= \sum_{i=1}^{m} w_i \mu_{F_i} \\ &\Rightarrow \quad \sum_{i=1}^{m} u_i = \sum_{i=1}^{m} \frac{w_i \mu_{F_i}}{\mu_F} = \frac{\sum_{i=1}^{m} w_i \mu_{F_i}}{\mu_F} = \frac{\mu_F}{\mu_F} = 1 \end{aligned}$$

the result follows from

$$\begin{aligned} \widetilde{F}(x) &= 1 - S_{\widetilde{F}}(x) \\ &= 1 - \sum_{i=1}^{m} u_i S_{\widetilde{F_i}}(x) = \sum_{i=1}^{m} u_i - \sum_{i=1}^{m} u_i S_{\widetilde{F_i}}(x) \\ &= \sum_{i=1}^{m} u_i \left(1 - S_{\widetilde{F_i}}(x)\right) = \sum_{i=1}^{m} u_i \widetilde{F}_i(x). \end{aligned}$$

■

So while taking the coderived distribution does not "commute" with constructing a mixture, the coderived distribution of a mixture is nevertheless a mixture of the coderived distributions, but one in which the frequency weights of the original mix are replaced with loss weights for the coderived mix. We will find that this simple observation can prove surprisingly instructive. We will also require the:

**Corollary 50** *With the notation and assumptions of Proposition 49*

$$\mu_{F_i} \leq \mu_{\widetilde{F_i}} \;\; 1 \leq i \leq m \Rightarrow \mu_F \leq \mu_{\widetilde{F}} \; .$$

**Proof.** Proceed by induction on $m$. Withou tloss of generality we may order so that:

$$\mu_{F_1} \leq \mu_{F_2} \leq \mu_{F_3} \leq ... \leq \mu_{F_m}.$$

The case $m = 1$ is clear. Let $G$ be the mixture of $F_2, ..., F_m$ in which:

$$F_i \text{ has weight } \frac{w_i}{\sum_{i=2}^m w_i}.$$

Then $G$ has PDF

$$f_G = \frac{\sum_{i=2}^m w_i f_{F_i}}{\sum_{i=2}^m w_i} = \frac{\sum_{i=2}^m w_i f_{F_i}}{1 - w_1}$$

and we have:

$$\begin{aligned} \mu_{F_1} &\leq \mu_{F_2} \leq \mu_G \leq \mu_{F_m} \\ &\Rightarrow \mu_G \geq \mu_{F_1}. \end{aligned}$$

Then by induction $\mu_{\widetilde{G}} \geq \mu_G$ and so

$$\begin{aligned} \mu_F &= w_1 \mu_{F_1} + (1 - w_1)\, \mu_G \\ \mu_{\widetilde{F}} &= \frac{w_1 \mu_{F_1} \mu_{\widetilde{F_1}} + (1 - w_1)\, \mu_G \mu_{\widetilde{G}}}{\mu_F} \\ &\geq \frac{w_1 \mu_{F_1}^2 + (1 - w_1)\, \mu_G^2}{\mu_F} \\ &= \left( \frac{w_1 \mu_{F_1}}{\mu_F} \right) \mu_{F_1} + \left( \frac{(1 - w_1)\, \mu_G}{\mu_F} \right) \mu_G \\ &= (w_1 - \epsilon)\, \mu_{F_1} + (1 - w_1 + \epsilon)\, \mu_G \end{aligned}$$

in which

$$\frac{w_1 \mu_{F_1}}{\mu_F} = w_1 - \epsilon$$

and we find that

$$\begin{aligned}
\epsilon &= w_1 - \frac{w_1 \mu_{F_1}}{\mu_F} = w_1 \left(1 - \frac{\mu_{F_1}}{\mu_F}\right) \\
&= w_1 \left(\frac{\mu_F - \mu_{F_1}}{\mu_F}\right) = w_1 \left(\frac{w_1 \mu_{F_1} + (1 - w_1)\, \mu_G - \mu_{F_1}}{\mu_F}\right) \\
&= w_1 \left(\frac{(1 - w_1)\left(\mu_G - \mu_{F_1}\right)}{\mu_F}\right) \geq 0
\end{aligned}$$

$$\Rightarrow \epsilon \geq 0.$$

We see that

$$\begin{aligned}
\mu_{\widetilde{F}} &\geq (w_1 - \epsilon)\, \mu_{F_1} + (1 - w_1 + \epsilon)\, \mu_G \\
&= w_1 \mu_{F_1} + (1 - w_1)\, \mu_G + \epsilon \left(\mu_G - \mu_{F_1}\right) \\
&= \mu_F + \epsilon \left(\mu_G - \mu_{F_1}\right) \\
&\geq \mu_F
\end{aligned}$$

completing the proof ■

**Corollary 51** *With the notation and assumptions of Proposition 49*

$$CV_{F_i} \geq 1, \quad 1 \leq i \leq m \Rightarrow CV_F \geq 1.$$

**Proof.** Clear from Corollary 50 and Proposition 46. ■

We next show how the coderived distributions of an SLDFn $F$ "make up a part of the tail" of $F$. We begin with

**Lemma 52** *For any two SLDFns $F$ and $G$ with $w \in [0, 1]$ and $c \geq 0$ with $S_F(c) S_G(c) > 0$ :*

$$\begin{aligned}
(wF + (1 - w)\, G)^{>c} &= vF^{>c} + (1 - v)\, G^{>c} \\
\textit{where } v &= \frac{w S_F(c)}{w S_F(c) + (1 - w)\, S_G(c)}.
\end{aligned}$$

**Proof.** We have

$$1 - (wF + (1 - w)\, G)^{>c}(x) = \frac{S_{wF + (1-w)G}(x + c)}{S_{wF + (1-w)G}(c)}$$

$$\begin{aligned}
&= \frac{w S_F(x + c) + (1 - w)\, S_G(x + c)}{w S_F(c) + (1 - w)\, S_G(c)} \\
&= \frac{w S_F(c) \frac{S_F(x+c)}{S_F(c)} + (1 - w)\, S_G(c) \frac{S_G(x+c)}{S_G(c)}}{w S_F(c) + (1 - w)\, S_G(c)} \\
&= v \frac{S_F(x + c)}{S_F(c)} + (1 - v) \frac{S_G(x + c)}{S_G(c)} \\
&= v S_{F^{>c}}(x) + (1 - v) S_{G^{>c}}(x)
\end{aligned}$$

and it follows that

$$(wF + (1-w)\,G)^{>c} = 1 - \left(1 - (wF + (1-w)\,G)^{>c}\right)$$

$$\begin{aligned}
&= 1 - (vS_{F^{>c}} + (1-v)S_{G^{>c}}) \\
&= v - vS_{F^{>c}} + (1-v) - (1-v)S_{G^{>c}} \\
&= v\,(1 - S_{F^{>c}}) + (1-v)\,(1 - S_{G^{>c}}) \\
&= vF^{>c} + (1-v)\,G^{>c}
\end{aligned}$$

as asserted. ■

**Lemma 53** *For any SLDFn $F$ with finite mean and $0 < \tau_F$, there exists $c \geq 0$ and $w \in (0,1)$ and SLDFn $G$ with $\omega_G = \omega_{F^{>c}}$ and*

$$\begin{aligned}
F^{>c} &= w\left(\widetilde{F}\right)^{>c} + (1-w)\,G \\
&= w\widetilde{F^{>c}} + (1-w)\,G.
\end{aligned}$$

**Proof.** We have

$$0 < \tau_F = \lim_{x \to \omega_F} \lambda_F(x)$$

which implies that

$$\text{there exist } c, \epsilon \text{ with } \epsilon > 0, 0 \leq c < \omega_F$$

$$\text{and } \{\lambda_{F^{>c}}(x) = \lambda_F(x+c) | x \in (0, \omega_F - c)\} \subset (\epsilon, \infty).$$

Let $w = Min\left(\frac{1}{2}, \epsilon\mu_{F^{>c}}\right)$. Then we have $\mu_{F^{>c}} > 0$ and

$$\frac{w}{\mu_{F^{>c}}} \leq \epsilon$$

$$\Rightarrow \frac{w}{\mu_{F^{>c}}} < \lambda_{F^{>c}}(x) \text{ for every } x \in (0, \omega_F - c)$$

$$\Rightarrow \frac{w}{\mu_{F^{>c}}} < \frac{f_{F^{>c}}(x)}{S_{F^{>c}}(x)} \text{ for every } x \in (0, \omega_F - c)$$

$$\Rightarrow wf_{\widetilde{F^{>c}}}(x) = \frac{wS_{F^{>c}}(x)}{\mu_{F^{>c}}} < f_{F^{>c}}(x) \text{ for every } x \in (0, \omega_F - c)$$

Make the definition

$$g(x) = \frac{f_{F^{>c}}(x) - wf_{\widetilde{F^{>c}}}(x)}{1-w} > 0 \text{ for } x \in (0, \omega_F - c),$$

then

$$f_{F^{>c}}(x) = wf_{\widetilde{F^{>c}}}(x) + (1-w)\,g(x) \text{ for every } x \in (0, \omega_F - c)$$

whence $g$ is a $C^\infty$ PDF on $(0, \omega_F - c)$ and the result follows by setting $G(x) = \int_0^x g(z)dz$. ■

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**Proposition 54** *If $n \in \mathbb{N}$ and $F$ is an SLDFn with $\mu_F^{(n)} < \infty$ and $0 < \tau_F$ , then there exist $c \geq 0$, $w \in (0,1)$, and an SLDFn $G$ with*

$$\omega_G = \omega_{F^{>c}} \quad \text{and}$$

$$F^{>c} = w\left(\widetilde{F}^{[n]}\right)^{>c} + (1-w)\,G = w\widetilde{F^{>c}}^{[n]} + (1-w)\,G.$$

**Proof.** The proof is by induction on $n$, the case $n = 1$ being covered by the second lemma. By induction there exists $c_1 \geq 0$ and $w_1 \in (0,1)$ and SLDFn $G_1$ so that

$$F^{>c_1} = w_1\left(\widetilde{F}^{[n]}\right)^{>c_1} + (1-w_1)\,G_1.$$

Again by the second lemma there exists $c_2 \geq 0$ and $w_2 \in (0,1)$ and SLDFn $G_2$ so that

$$\left(\widetilde{F}^{[n]}\right)^{>c_2} = w_2\left(\widetilde{\widetilde{F}^{[n]}}\right)^{>c_2} + (1-w_2)\,G_2 = w_2\left(\widetilde{F}^{[n+1]}\right)^{>c_2} + (1-w_2)\,G_2.$$

It now follows from the first lemma that there exist $u, v \in (0,1)$

$$\begin{aligned}
F^{>c_1+c_2} &= \left(F^{>c_1}\right)^{>c_2} = \left(w_1\left(\widetilde{F}^{[n]}\right)^{>c_1} + (1-w_1)\,G_1\right)^{>c_2} = \\
&= u\left(\left(\widetilde{F}^{[n]}\right)^{>c_1}\right)^{>c_2} + (1-u)\,G_1^{>c_2} \\
&= u\left(\widetilde{F}^{[n]}\right)^{>c_1+c_2} + (1-u)\,G_1^{>c_2} \\
&= u\left(\left(\widetilde{F}^{[n]}\right)^{>c_2}\right)^{>c_1} + (1-u)\,G_1^{>c_2}
\end{aligned}$$

$$\begin{aligned}
&= u\left(w_2\left(\widetilde{F}^{[n+1]}\right)^{>c_2} + (1-w_2)\,G_2\right)^{>c_1} + (1-u)\,G_1^{>c_2} \\
&= u\left(v\left(\widetilde{F}^{[n+1]}\right)^{>c_1+c_2} + (1-v)\,G_2^{>c_1}\right) + (1-u)\,G_1^{>c_2} \\
&= uv\left(\widetilde{F}^{[n+1]}\right)^{>c_1+c_2} + (1-uv)G_3
\end{aligned}$$

and setting $w = uv \in (0,1)$ and $c = c_1 + c_2$ completes the induction and the proof. ■

We have seen, Proposition 26, that the coderived distribution determines the original. So it makes sense to ask, given an SLDFn $F$, is there an SLDFn $G$ (necessarily uniquely determined with finite mean) such that $\widetilde{G} = F$. This prompts:

**Definition 55** *Let $G$ be an SLDFn with finite mean and $F = \widetilde{G}$. The SLDFn $G$ is called the* ***backward coderived loss distribution function*** *of $F$. We set,* *recursively*

$$\begin{aligned}\widetilde{F}^{[-1]} &= G \\ \widetilde{F}^{[-n]} &= \widetilde{\widetilde{F}^{[-n+1]}}^{[-1]} \Leftrightarrow \widetilde{\widetilde{F}^{[-n]}} = \widetilde{F}^{[-n+1]} \text{for } n = 2, 3, 4, \ldots.\end{aligned}$$

*If the LDFn $\widetilde{F}^{[-n]}$ exists for some integer $n > 0$, then $\widetilde{F}^{[-n]}$ is called the **n-th backward coderived loss distribution** of $F$.*

Quite generally, for any loss distribution $F$ with differentiable PDF $f(x)$ such that $\frac{df}{dx} \leq 0$, we could define the **backward coderived loss distribution** to be the distribution with survival function equal to $T(y) = \frac{f(y)}{f(0)}$. Suppose $F$ and $G$ are loss variables with $G = \widetilde{F}^{[-1]}$the backward coderived distribution of $F$. Of course, the mean of $G$ is

$$\mu_G = \int_0^\infty S_G(y)dy = \int_0^\infty T(y)dy = \frac{\int_0^\infty f_F(y)dy}{f_F(0)} = \frac{1}{f_F(0)}$$

and for the PDF of $\widetilde{G}$ we have, as one would expect

$$\begin{aligned}f_{\widetilde{G}}(y) &= \frac{S_G(y)}{\mu_G} = \frac{\frac{f_F(y)}{f_F(0)}}{\frac{1}{f_F(0)}} = f_F(y) \\ &\Rightarrow \widetilde{G} = F.\end{aligned}$$

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