The Report of the Research Working Party on Correlations and Dependencies Among All Risk Sources

Part 2

Aggregating Bivariate Claim Severities With Numerical Fourier Inversion

David L. Homer Member, CAS Research Working Party on Correlations and Dependencies Among All Risk Sources

AGGREGATING BIVARIATE CLAIM SEVERITIES WITH NUMERICAL FOURIER INVERSION

DAVID L. HOMER

Abstract

This chapter will apply continuous Fourier transforms to compute the bivariate aggregate claims distribution arising from a bivariate severity distribution and a univariate claim count distribution.

1. INTRODUCTION

This chapter will apply continuous Fourier transforms to compute the bivariate aggregate claims distribution arising from a bivariate severity distribution and a univariate claim count distribution.

Section 1 provides a general description of univariate aggregate claims methods followed by a general description of bivariate aggregate claims methods.

Section 2 provides a brief summary of the univariate Fourier transform method applied by Heckman and Meyers [3] since this will provide the foundation for the bivariate method presented in section 3. The abbreviation "HM" will be used for "Heckman and Meyers". Section 4 presents examples.

1.1. Univariate Methods

There are several methods described in the actuarial literature for computing the *univariate* aggregate loss distribution arising from a univariate severity distribution and a univariate claim count distribution. These methods include HM's numerical Fourier inversion [3], discrete Fourier transforms as discussed by Wang [10] and Robertson [8], and Panjer's recursive techniques [7].

Heckman and Meyers' numerical Fourier inversion method uses a severity distribution with claim size intervals of constant density and a possible point mass at the maximum claim size. The claim count model is Binomial, Poisson, or Negative Binomial. This method works best when the expected claim counts are large because the numeric integral computed by this method coverges more quickly when the claim counts are large.

The basic discrete Fourier transform method requires a discrete claim size distribution with claim sizes at equally spaced intervals. It works best when the expected claim counts are small because of computer memory constraints. The interval size must be small enough to accurately represent the claim size distribution while the largest claim size represented must be large enough to capture the aggregate distribution. This generally means a large number of intervals are required and limited computer memory can make computations for large claim counts impractical.

Robertson's method is a clever adaptation of the basic discrete Fourier transform for application with claim size distributions with equally spaced intervals of constant density. This is nearly the same claim size model used by HM, but with a few additional limitations. There is no point mass allowed at the maximal claim size and the intervals of constant density must have uniform width. The claim count model is a finite list of probabilities. This method works best when the expected claim counts are small because of computer memory constraints.

Additional calculations are required to correct the basic discrete Fourier transform for the non-discrete severity density. In practice, the cost of the additional calculations may outweigh the benefit, if any, of using severities with intervals of constant density. However, since Robertson's method is exact it is extremely useful for checking methods like the HM method which has an error term. The testing must be done with examples with a moderate number of expected claims since the HM method works best with a large number and Robertson's method works best with a small number. In this paper we will use a two-dimensional application of Robertson's method to compute the error of the two-dimensional extension of HM.

The recursive technique uses a discrete severity distribution with uniformly spaced claim sizes. The claim count model includes the Binomial, Poisson and Negative Binomial distributions.[1]. This method works well when the expected claim counts are small for reasons similar to those given for discrete Fourier transform methods.

In the methods described above, a pair of risk collections—each with its own severity and claim count distribution—would be aggregated assuming the collections were independent. Heckman and Meyers also allow a *mixing parameter* that reflects parameter risk in the scale of the aggregate distribution and induces a correlation between collections. Wang [10] and Meyers [6] discuss the univariate aggregation of correlated collections.

1.2. Bivariate Methods

The actuarial literature also describes the computation of *bivariate* aggregate distributions. Homer and Clark [4] describe bivariate examples using two-dimensional discrete Fourier transforms. Sundt [9] extends Panjer recursions to multiple dimensions. Walhin [11] describes an application of two-dimensional Panjer recursions. Like their univariate counterparts, these methods work best when the expected claim counts are small due to computer memory constraints.

This chapter extends the HM method to bivariate aggregate distributions. As with the univariate method, this extension works best when the expected claim counts are large because the numeric integrals computed converge more quickly with large claim counts.

The following sections will provide a brief review of the HM univariate method, develop the bivariate method, and present some examples.

[1]The claim count model for recursion technique includes a larger group of distributions which are the members of the $(a, b, 0)$ or $(a, b, 1)$ classes as described by Klugman et al [3]. The HM method can be modified to use (a, b, x) members.

2. UNIVARIATE NUMERICAL FOURIER INVERSION

2.1. *Univariate Collective Risk Model*

The collective risk model describes aggregate claims for a collection of risks with a claim count or frequency distribution and a claim size or severity distribution. The individual claims sizes X_k are independent and identically distributed (iid). The individual claim sizes are also independent of the claim count N. The aggregate losses are

$$Z = X_1 + ... + X_N. \tag{2.1}$$

This model may be used to describe the aggregate losses for a single line or book of business.

2.2. *Univariate Aggregate Characteristic Function*

The aggregate loss distribution is conveniently described through its characteristic function in terms of the characteristic function of the claim size distribution and the probability generating function of the claim count distribution.

Recall that the characteristic function (cf) for a distribution is defined as

$$\phi_X(t) = E(e^{itX}), \tag{2.2}$$

and that the probability generating function (pgf) for a discrete distribution is defined as

$$PGF_N(t) = E(t^N). \tag{2.3}$$

The aggregate loss characteristic function $\phi_Z(t)$ is equal to the composition of the claim count probability generating function $PGF_N(t)$ with the claim size characteristic function $\phi_X(t)$,

$$\begin{aligned} \phi_Z(t) &= E(e^{itZ}) \\ &= E(e^{X_1+...+X_N}) \\ &= E_N(\phi_X(t)^N | N) \\ &= PGF_N(\phi_X(t)). \end{aligned} \tag{2.4}$$

The cdf $F(z)$ of Z can be obtained from $\phi_Z(t)$ when it is continuous

$$F(z) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} |\phi_Z(t)| \frac{\sin(zt - \arg(\phi_Z(t)))}{t} dt. \tag{2.5}$$

Although $\phi(t)$ is complex, Equation 2.5 is real valued; $|\phi|$ is the *modulus* of ϕ and $\arg(\phi)$ is its *argument*. The right hand side of 2.5 yields $F(z) - Pr(Z = z)/2$ at steps when $F(z)$ is not continuous. Given $\phi(t)$, $F(z)$ is obtained via numeric integration.

Equation 2.5 is equivalent to HM equation 6.5. By applying a scale change of variable $t \rightarrow t/\sigma$ and substituting $f(t) = |\phi(t/\sigma)|$ and $g(t) = \arg(\phi(t/\sigma)))$ into equation 2.5 we get HM equation 6.5,

$$F(z) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{f(t)}{t} \sin(tz/\sigma - g(t))dt. \tag{2.6}$$

2.3. Univariate Severity Model

The severity density is approximated to make the calculation of $\phi_x(t)$ easy. It is approximated with n intervals (a_k, a_{k+1}) of constant density d_k $(k = 1, ...n)$ and an optional point mass p at the maximal claim size a_{n+1} such that

$$\sum_{k=1}^{n} d_k(a_{k+1} - a_k) + p = 1. \tag{2.7}$$

Figure 2.1 shows a sample severity density with two intervals (a_1, a_2) and (a_2, a_3) and a point mass at a_3. With this severity model we easily obtain

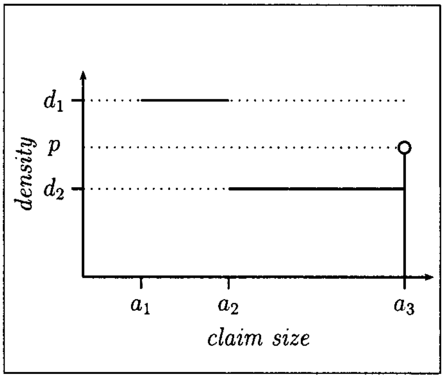
$$\begin{aligned} \phi_X(t) &= E_X(e^{itX}) \\ &= \sum_{k=1}^{n} d_k \frac{e^{ita_{k+1}} - e^{ita_k}}{it} + pe^{ita_{n+1}}. \end{aligned} \tag{2.8}$$

2.4. Univariate Numerical Inversion

Heckman and Meyers integrate 2.5 using five point Gaussian quadrature with special treatment of the portion of the integral near zero. We will extend this to two dimensions using five point quadrature first along one dimension and again along the second dimension.

FIGURE 2.1

UNIVARIATE SEVERITY DENSITY — INTERVALS OF CONSTANT DENSITY



3. BIVARIATE NUMERICAL FOURIER INVERSION

3.1. *Bivariate Collective Risk Model*

The collective risk model can be extended to model two collections of risks and their dependencies. There are two forms for this extension.

The first form is the *bivariate severity form.* It is useful for modeling aggregate losses together with the corresponding aggregate adjustment expenses. This form uses a single claim count distribution and a bivariate claim size distribution. While the bivariate pair (X_k, Y_k) may have any dependency structure, the pairs arising from different claims are assumed to be iid. The claim size pairs are also independent from the claim count N. The aggregate loss pair is

$$(Z_x, Z_y) = (X_1 + ... + X_N, Y_1 + ... + Y_N). \tag{3.1}$$

The second form is the *bivariate count form.* It is useful for modeling two risk collections with different but related claim counts. The claim size severities X_k and Y_j are separately iid and also independent from each other. The claim counts for each risk collection arise from a bivariate claim count distribution. The claim count pair (M, N) is independent from each of the claim sizes. The aggregate pair is

$$(Z_x, Z_y) = (X_1 + ... + X_M, Y_1 + ... + Y_N). \tag{3.2}$$

This chapter will focus on the bivariate severity form, but the methods presented here can also be applied to the bivariate count form.

3.2. *The Bivariate Aggregate Characteristic Function*

The aggregate characteristic function for the bivariate severity form of the collective risk model is a composition of the claim count pgf with the bivariate severity characteristic function.

$$\begin{aligned}
\phi_{Z_x,Z_y}(s,t) &= E(e^{isZ_x+itZ_y}) \\
&= E(e^{is(X_1+...+X_N)+it(Y_1+...+Y_N)}) \\
&= E(e^{isX_1+itY_1}...e^{isX_N+itY_N}) \\
&= E_N(\phi_{X,Y}(s,t)^N|N) \\
&= PGF_N(\phi_{X,Y}(s,t))
\end{aligned} \tag{3.3}$$

For the bivariate count form, Wang [10] gives the aggregate characteristic function.

$$\phi_{Z_x,Z_y}(s,t) = PGF_{M,N}(\phi_X(s), \phi_Y(t)). \tag{3.4}$$

Where $PGF_{M,N}(s,t)$ is the bivariate claim count pgf.

Appendices A and B develop an expression for $F(z_x, z_y)$ in terms of $\phi_{Z_x,Z_y}(s,t)$ when F is continuous,

$$F(x,y) = \frac{1}{2}(F(x) + F(y)) - \frac{1}{4} + \frac{1}{4\pi^2}I, \tag{3.5}$$

where

$$\begin{aligned}
I &= 2\int_0^\infty \int_0^\infty \big(|\phi(s,t)|\cos(sx+ty-\arg(\phi(s,t))) - \\
&\quad |\phi(s,-t)|\cos(sx-ty-\arg(\phi(s,-t)))\big)\frac{dsdt}{(is)(it)}.
\end{aligned} \tag{3.6}$$

When F is not continuous, the right hand side of 3.5 yields $F(z_x, z_y) + m/4$, where m is a correction for probability mass that lies along the lines $Z_x = z_x$ and $Z_y = z_y$, and

$$
\begin{aligned}
m &= Pr(Z_x > z_x \cap Z_y = z_y) && (3.7)\\
&\quad +Pr(Z_y > z_y \cap Z_x = z_x) && (3.8)\\
&\quad -Pr(Z_x \leq z_x \cap Z_y = z_y) && (3.9)\\
&\quad -Pr(Z_y < z_y \cap Z_x = z_x). && (3.10)
\end{aligned}
$$

3.3. Bivariate Severity Model

In an extension of the univariate severity model, the bivariate severity density will be approximated with rectangles of constant density. That is, the severity domain will be divided into mn rectangles $(a_j, a_{j+1}) \times (b_k, b_{k+1})$ of constant density $d_{j,k}$ $(j = 1...m)(k = 1...n)$. Like the one dimensional case, this simplifies the calculation of $\phi_{X,Y}(s,t)$,

$$
\begin{aligned}
\phi_{X,Y}(s,t) &= E(e^{isX+itY})\\
&= \sum_{j=1}^{m}\sum_{k=1}^{n}\int_{b_k}^{b_{k+1}}\int_{a_j}^{a_{j+1}} d_{j,k}e^{isx+ity}dxdy\\
&= \sum_{j=1}^{m}\sum_{k=1}^{n} d_{j,k}\frac{e^{isa_{j+1}} - e^{isa_j}}{is}\frac{e^{itb_{k+1}} - e^{isb_k}}{it}. \qquad (3.11)
\end{aligned}
$$

Figure 3.1 shows a sample bivariate density.

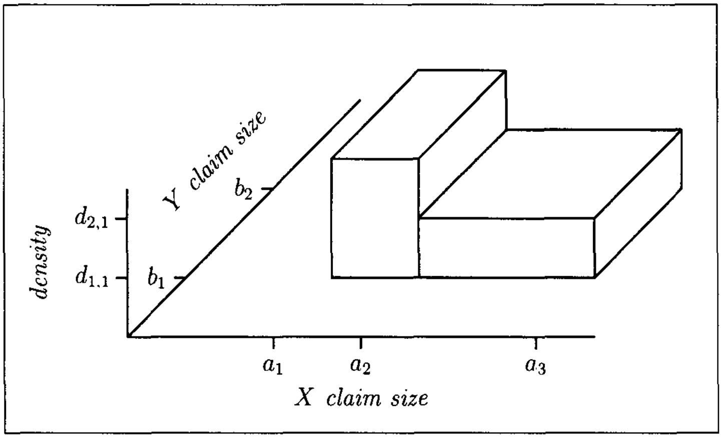
Here we have not included mass points or mass lines, but it is possible to do so.

3.4. Bivariate Numerical Fourier Inversion

We will make use of two-dimensional five point Gaussian quadrature. Appendix C provides additional descriptions of two-dimensional quadrature. Sample code will also be provided in a spreadsheet that can be downloaded from the CAS Web site. It will follow key elements of the HM code fairly closely.

FIGURE 3.1

BIVARIATE SEVERITY DENSITY—RECTANGLES OF CONSTANT DENSITY



In particular, HM split the line into 256 intervals of width $h = 2\pi\sigma/x_{max}$. We will split the grid into rectangles of widths $h_x = \pi/x_{max}$ and $h_y = \pi/y_{max}$ respectively. We are using half of the HM interval and trying to economize on the total number of rectangles. We leave out the additional factor of σ which is the standard deviation of the aggregate distribution and is not required. Heckman and Meyers additionally split the first interval into 5 smaller intervals $(0, h/16), (h/16, h/8), (h/8, h/4), (h/4, h/2), (h/2, h)$. This is helpful because the integrand changes rapidly near zero.

As suggested by HM it is speculated that the key source of error in this method is truncation error, since the integrals are from zero to infinity, but our algorithm must stop at a finite values. Errors in our sample calculations will be computed with comparisons to known values.

4. BIVARIATE EXAMPLES

This section presents two examples. The first example applies the 2d inversion technique to a bivariate severity and a claim count distribution allowing only a single claim. Thus, the aggregate distribution is the same as the bivariate severity and the error is readily computed.

The second example applies the 2d inversion to the same bivariate severity with a moderate number of expected counts. This result is compared to an exact calculation produced by a two dimensional version of Robertson's method [8].

4.1. Example 1—Exactly One Claim

Table 4.1 shows a sample bivariate severity distribution. If we also assume the claim count distribution has a 100% probability of 1 claim the resulting aggregate distribution computed by our method is shown in Table 4.2. This method should reproduce Table 4.1. The error is shown in Table 4.3.

TABLE 4.1

SAMPLE BIVARIATE SEVERITY CUMULATIVE DISTRIBUTION FUNCTION

$F(x,y)$		y 0	200	600	800	1,200
	0	0.0000	0.0000	0.0000	0.0000	0.0000
x	200	0.0000	0.4705	0.7557	0.7845	0.8120
	400	0.0000	0.4858	0.8243	0.8621	0.8990
	600	0.0000	0.4917	0.8540	0.8964	0.9380
	1,000	0.0000	0.4953	0.8735	0.9190	0.9640
	2,000	0.0000	0.4991	0.8949	0.9440	0.9930
	3,000	0.0000	0.4996	0.8978	0.9474	0.9970
	5,000	0.0000	0.5000	0.9000	0.9500	1.0000

TABLE 4.2

AGGREGATION OF SAMPLE BIVARIATE SEVERITY CDF WITH 100% PROBABILITY OF 1 CLAIM

$F(z_x, z_y)$		z_y 0	200	600	800	1,200
	0	0.0000	0.0047	0.0076	0.0079	0.0082
z_x	200	0.0011	0.4652	0.7485	0.7774	0.8046
	400	0.0012	0.4850	0.8236	0.8617	0.8985
	600	0.0012	0.4909	0.8535	0.8961	0.9377
	1,000	0.0012	0.4946	0.8731	0.9189	0.9639
	2,000	0.0012	0.4984	0.8945	0.9439	0.9929
	3,000	0.0012	0.4989	0.8975	0.9474	0.9969
	5,000	0.0012	0.4993	0.8996	0.9499	0.9999

TABLE 4.3

ERROR FOR EXAMPLE 1 AGGREGATE CDF

Error		z_y 0	200	600	800	1,200
	0	0.0000	0.0047	0.0076	0.0079	0.0082
z_x	200	0.0011	(0.0054)	(0.0072)	(0.0072)	(0.0074)
	400	0.0012	(0.0008)	(0.0007)	(0.0005)	(0.0005)
	600	0.0012	(0.0008)	(0.0005)	(0.0002)	(0.0003)
	1,000	0.0012	(0.0007)	(0.0004)	(0.0001)	(0.0001)
	2,000	0.0012	(0.0007)	(0.0004)	(0.0001)	(0.0001)
	3,000	0.0012	(0.0007)	(0.0004)	(0.0001)	(0.0001)
	5,000	0.0012	(0.0007)	(0.0004)	(0.0001)	(0.0001)

4.2. Example 2—Variable Claim Counts

In this example we use the claim size distribution from Example 1 and a claim count distribution with a maximum claim size. This allows us to compute the exact answer using an alternative method based on Robertson's one-dimensional method. Appendix D provides a brief discussion of a 2d Robertson method. In addition, sample R code showing the 2d Robertson calculation will be made available for downloading. Table 4.4 shows the count distribution. Table 4.5 shows the exact calculation based on the Robertson method. Table 4.6 shows the result from numerical Fourier inversion. The error is shown in Table 4.7. The errors are substantially smaller that those from Example 1 and this is attributed to the larger claim counts forcing the integrand to converge to zero more quickly.

5. CONCLUSION

Numerical Fourier inversion is a viable technique for exploring claim dependencies. When the claim counts are large, it may be more efficient than other techniques such as discrete Fourier transforms, recursion, or simulation.

Additional development is possible for alternate severity structures such as a bivariate distribution for primary and excess claim portions. Given the aggregate characteristic function, conditional expected values can also be computed. These calculations could have potential applications in reserving and surplus allocation.

6. ACKNOWLEDGEMENTS

The author would like to thank the members of the Correlation Working Party for encouragment and helpful suggestions. Particular thanks go to Glenn Meyers, Stephen Mildenhall, Shawna Ackerman, and Chuck Thayer.

TABLE 4.4

EXAMPLE 2—CLAIM COUNT DISTRIBUTION

Count	Probability	Cumulative
0	0.000	0.000
1	0.000	0.000
2	0.000	0.000
3	0.000	0.000
4	0.000	0.000
5	0.000	0.000
6	0.000	0.000
7	0.000	0.000
8	0.100	0.100
9	0.100	0.200
10	0.100	0.300
11	0.100	0.400
12	0.100	0.500
13	0.100	0.600
14	0.100	0.700
15	0.100	0.800
16	0.100	0.900
17	0.100	1.000
Mean	12.500	
Std	2.872	
Var	8.250	

TABLE 4.5

EXAMPLE 2—EXACT SOLUTION FROM 2D ROBERTSON METHOD

$F(z_x, z_y)$		z_y 1,000	2,000	4.000	6,000	10,000
	1,000	0.0009	0.0195	0.0550	0.0566	0.0567
z_x	2,000	0.0017	0.0577	0.3249	0.3924	0.3951
	3,000	0.0019	0.0688	0.4850	0.6649	0.6782
	4,000	0.0019	0.0724	0.5613	0.8193	0.8431
	5,000	0.0019	0.0737	0.5925	0.8928	0.9239
	6,000	0.0019	0.0744	0.6073	0.9287	0.9639
	8,000	0.0019	0.0747	0.6170	0.9547	0.9935
	15,000	0.0019	0.0747	0.6185	0.9601	1.0000

TABLE 4.6

EXAMPLE 2—AGGREGATE CDF FROM NUMERICAL FOURIER INVERSION

$F(z_x, z_y)$	0	z_y 1,000	2,000	4.000	6,000	10,000
	1,000	0.0009	0.0195	0.0550	0.0566	0.0567
z_x	2,000	0.0017	0.0577	0.3249	0.3924	0.3951
	3,000	0.0019	0.0688	0.4850	0.6649	0.6782
	4,000	0.0019	0.0724	0.5613	0.8193	0.8431
	5,000	0.0019	0.0737	0.5925	0.8928	0.9239
	6,000	0.0019	0.0744	0.6073	0.9287	0.9639
	8,000	0.0019	0.0747	0.6170	0.9547	0.9935
	15,000	0.0019	0.0747	0.6185	0.9601	1.0000

TABLE 4.7

EXAMPLE 2—ERROR

Error		z_y 1,000	2,000	4,000	6,000	10,000
	1,000	0.00000	0.00000	0.00000	0.00000	0.00000
z_x	2,000	0.00000	0.00000	0.00000	0.00000	0.00000
	3,000	0.00000	(0.00000)	(0.00000)	(0.00000)	(0.00000)
	4,000	0.00000	0.00000	0.00000	0.00000	0.00000
	5,000	0.00000	0.00000	0.00000	0.00000	0.00000
	6,000	0.00000	0.00000	0.00000	0.00000	0.00000
	8,000	0.00000	0.00000	0.00000	0.00000	0.00000
	15,000	0.00000	0.00000	0.00000	0.00000	0.00000

7. BIOGRAPHY

David Homer is a Principal and Vice President of the Reinsurance business of Towers Perrin. He holds a bachelor's degree in mathematics and physics from Swarthmore College and a master's degree in mathematics from the University of Pennsylvania.

David's current work involves pricing and modeling reinsurance solutions. He is a past winner of the Dorweiller Prize.

REFERENCES

[1] Abramowitz, Milton, and Irene A. Stegun, "Handbook of Mathematical Functions," New York, Dover Publications, Inc., 1970.

[2] Burden, Richard L., and J. Douglas Faires, "Numerical Analysis," 3rd ed., Boston, Prindle, Weber and Schmidt, 1985.

[3] Heckman, Philip E. and Glenn G. Meyers, "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," *PCAS* LXX, 1983, pp. 22–61.

[4] Homer, David L. and David R. Clark, "Insurance Applications of Bivariate Distributions," *PCAS* XC, 2003, pp. 274–307.

[5] Klugman, Stuart A., Harry H. Panjer, and E. Willmot Gordon, *Loss Models: From Data to Decisions*, New York, John Wiley & Sons, Inc., 1998.

[6] Meyers, Glenn G., "Discussion of Shaun Wang's Aggregation of Correlated Risk Portfolios: Models and Algorithms," *PCAS* LXXXV, 1999, pp. 781–805.

[7] Panjer, Harry H., "Recursive Evaluation of a Family of Compound Distributions," *ASTIN Bulletin*, Vol. 12, No. 1, 1981, pp. 22–26.

[8] Robertson, John P., "The Computation of Aggregate Loss Distributions," *PCAS* LXXIX, 1992, pp. 57–133.

[9] Sundt, Bjørn, "On Multivariate Panjer Recursions," *ASTIN Bulletin*, Vol. 29, No. 1, 1999, pp. 29–45.

[10] Wang, Shaun S., "Aggregation of Correlated Risk Portfolios: Models and Algorithms," *PCAS* LXXXV, 1998, pp. 848–939.

[11] Walhin, Jean-François, "On the Optimality of Multiline Excess of Loss Covers," *CAS Forum*, Spring 2003, pp. 231-243.

APPENDIX A

Two-Dimensional Integration Formula

Consider the integral

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{isx}(e^{ity}\phi(-s,-t) - e^{-ity}\phi(-s,t)) \\ &\quad -e^{-isx}(e^{ity}\phi(s,-t) - e^{-ity}\phi(s,t))\frac{dsdt}{isit}. \end{aligned} \tag{A.1}$$

Substitute the integral form for ϕ and apply Fubini's theorem to change the order of integration. Then,

$$\begin{aligned} I &= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty e^{is(x-u)}(e^{it(y-v)} - e^{it(v-y)}) \\ &\quad -e^{is(u-x)}(e^{it(y-v)} - e^{it(v-y)})\frac{dsdtdF(u,v)}{isit}. \end{aligned} \tag{A.2}$$

Since

$$\int_0^\infty \frac{e^{isx} - e^{-isx}}{is} ds = \pi \mathrm{sgn}(x), \tag{A.3}$$

where

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}, \tag{A.4}$$

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\pi\operatorname{sgn}(x-u)(e^{it(y-v)}-e^{it(v-y)})\frac{dt}{it}dF(u,v)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\pi^2\operatorname{sgn}(x-u)\operatorname{sgn}(y-v)dF(u,v)\\
&= \int_{-\infty}^{x}\int_{-\infty}^{y}\pi^2 dF(u,v) - Pr(Z_x \le z_x \cap Z_y = z_y)\\
&\quad -Pr(Z_y < z_y \cap Z_x = z_x)\\
&\quad -\int_{-\infty}^{x}\int_{y}^{\infty}\pi^2 dF(u,v) + Pr(Z_y > z_y \cap Z_x = z_x)\\
&\quad -\int_{x}^{\infty}\int_{-\infty}^{y}\pi^2 dF(u,v) + Pr(Z_x > z_x \cap Z_y = z_y)\\
&\quad +\int_{x}^{\infty}\int_{y}^{\infty}\pi^2 dF(u,v)\\
&= \pi^2(F(x,y)-(F(x,\infty)-F(x,y))-(F(\infty,y)-F(x,y))\\
&\quad +(1-F(\infty,y)-F(x,\infty)+F(x,y))+m). \qquad \text{(A.5)}
\end{aligned}
$$

Where m is a correction for probability mass that lies along the lines $Z_x = z_x$ and $Z_y = z_y$ when $F(z_x, z_y)$ is not continuous, since $sgn(0) = 0$.

$$
\begin{aligned}
m &= Pr(Z_x > z_x \cap Z_y = z_y) && \text{(A.6)}\\
&\quad +Pr(Z_y > z_y \cap Z_x = z_x) && \text{(A.7)}\\
&\quad -Pr(Z_x \le z_x \cap Z_y = z_y) && \text{(A.8)}\\
&\quad -Pr(Z_y < z_y \cap Z_x = z_x). && \text{(A.9)}
\end{aligned}
$$

So,

$$I = \pi^2(4F(x,y) - 2(F(x)+F(y)) + 1 + m). \tag{A.10}$$

Finally,

$$F(x,y) = \frac{1}{2}\big(F(x)+F(y)\big) - \frac{1}{4} + \frac{1}{4\pi^2}I - \frac{m}{4}. \tag{A.11}$$

APPENDIX B

Expansion of I for Numerical Integration

Appendix A provides an expression for the bivariate cdf $F(x, y)$.

$$F(x,y) = \frac{1}{2}(F(x) + F(y)) - \frac{1}{4} + \frac{1}{4\pi^2}I, \tag{B.1}$$

where,

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \left(e^{isx}\left(e^{ity}\phi(-s,-t) - e^{-ity}\phi(-s,t)\right)\right. \\ &\quad \left. -e^{-isx}\left(e^{ity}\phi(s,-t) - e^{-ity}\phi(s,t)\right)\right) \frac{dsdt}{(is)(it)}. \end{aligned} \tag{B.2}$$

It will be helpful to write ϕ in polar form and make use of a few symmetries. Let

$$R(s,t) = |\phi(s,t)| \tag{B.3}$$

$$\theta(s,t) = \arg(\phi(s,t)), \tag{B.4}$$

then,

$$\phi(s,t) = R(s,t)e^{i\theta(s,t)} = E(e^{isx+ity}). \tag{B.5}$$

We have the complex conjugate of ϕ

$$\begin{aligned} \overline{\phi(s,t)} &= R(s,t)e^{-i\theta(s,t)} & \text{(B.6)} \\ &= \overline{E(e^{isx+ity})} & \text{(B.7)} \\ &= E(e^{-isx-ity}) & \text{(B.8)} \\ &= R(-s,-t)e^{i\theta(-s,-t)}. & \text{(B.9)} \end{aligned}$$

Thus,

$$\begin{aligned} \theta(s,t) &= -\theta(-s,-t) & \text{(B.10)} \\ \theta(s,-t) &= -\theta(-s,t) & \text{(B.11)} \\ R(s,t) &= R(-s,-t) & \text{(B.12)} \\ R(s,-t) &= R(-s,t). & \text{(B.13)} \end{aligned}$$

Now writing I with ϕ in polar form

$$I = \int_0^\infty \int_0^\infty \left(e^{isx}\left(e^{ity}R(-s,-t)e^{i\theta(-s,-t)} - e^{-ity}R(-s,t)e^{i\theta(-s,t)}\right) - e^{-isx}\left(e^{ity}R(s,-t)e^{i\theta(s,-t)} - e^{-ity}R(s,t)e^{i\theta(s,t)}\right)\right)\frac{dsdt}{(is)(it)} \tag{B.14}$$

and simplifying using equations B.10-B.13,

$$I = \int_0^\infty \int_0^\infty \left(R(s,t)\left(e^{isx+ity-i\theta(s,t)} + e^{-isx-ity+i\theta(s,t)}\right) - R(s,-t)\left(e^{isx-ity-i\theta(s,-t)} + e^{-isx+ity+i\theta(s,-t)}\right)\right)\frac{dsdt}{(is)(it)} \tag{B.15}$$

we can now write

$$I = \int_0^\infty \int_0^\infty \left(R(s,t)2cos(sx+ty-\theta(s,t)) - R(s,-t)2cos(sx-ty-\theta(s,-t))\right)\frac{dsdt}{(is)(it)}. \tag{B.16}$$

In terms of ϕ we have

$$I = \int_0^\infty \int_0^\infty \left(|\phi(s,t)|2\cos(sx+ty-\arg(\phi(s,t))) - |\phi(s,-t)|2\cos(sx-ty-\arg\phi(s,-t))\right)\frac{dsdt}{(is)(it)}. \tag{B.17}$$

APPENDIX C

Gaussian Quadrature for Two Dimensions

This section will develop the formulae for two-dimensional Gaussian quadrature. The basic form to be approximated is

$$I = \int_a^b \int_c^d f(x,y)dxdy. \tag{C.1}$$

Using a change of variables, change the integral domain from the rectangle $[a,b] \times [c,d]$ to $[-1,1] \times [-1,1]$.

$$u = \frac{1}{b-a}(2x - a - b). \tag{C.2}$$

$$v = \frac{1}{d-c}(2y - c - d). \tag{C.3}$$

Equation C.1 becomes

$$I = \int_{-1}^1 \int_{-1}^1 f\left(\frac{u(b-a)+a+b}{2}, \frac{v(d-c)+c+d}{2}\right)\frac{(b-a)(d-c)}{4}dudv. \tag{C.4}$$

The integral C.4 is now computed as a double sum,

$$I = \sum_{i=1}^5 \sum_{j=1}^5 w_i w_j f\left(\frac{x_i(b-a)+a+b}{2}, \frac{x_j(d-c)+c+d}{2}\right)\frac{(b-a)(d-c)}{4} + \epsilon. \tag{C.5}$$

The error term ϵ depends on how well $f(x,y)$ can be approximated by polynomials of finite degree (nine or less for five point Gaussian quadrature). By choosing sufficiently small intervals ϵ can be made small. See [2] for additional details.

The quadrature values x_i and w_i are taken from Abramowitz and Stegun [1].

TABLE C.1

ABSCISSAS AND WEIGHTS FOR FIVE POINT GAUSSIAN QUADRATURE

k	x_k	w_k
1	-0.90617 98459 38664	0.23692 68850 56189
2	-0.53846 93101 05683	0.47862 86704 99366
3	0.00000 00000 00000	0.56888 88888 88889
4	+0.53846 93101 05683	0.47862 86704 99366
5	+0.90617 98459 38664	0.23692 68850 56189

APPENDIX D

TWO-DIMENSIONAL ROBERTSON METHOD

This appendix provides a brief discussion of extending Robertson's method [8] to two dimensions. It begins with a summary of the one dimensional method.

Robertson's method computes the aggregate distribution for a finite claim count distribution and a claim size distribution with equal width and constant density intervals. The method is exact and it uses discrete Fourier transforms.

A more basic application of the discrete Fourier transform requires a discrete claim size distribution with claim sizes at integral intervals.

Robertsons's method uses the usual discrete Fourier technique to compute convolutions, but adds a correction to reflect the constant density claim size intervals. The method is quite clever and it is not hard to develop an intuition to see why it works.

Consider a discrete random variable X with integral size intervals of width I. Now add a random variable U that is uniform on the interval I. The result $X + U$ is a random variable with claim size intervals of constant density.

This observation can be applied to develop the aggregate distribution with claim size distribution F_{X+U} and claim count distribution P. Note that the sum of n independent copies of $X + U$ has the same distribution as the sum of n independent copies of X plus n independent copies of U. The aggregate cumulative distribution function is then

$$F(z) = \sum_{n=0}^{n=n_{max}} P(n)F_{X+U}^{(n)}(z) \tag{D.1}$$

$$= \sum_{n=0}^{n=n_{max}} P(n)(F_X^{(n)}(z) * F_U^{(n)}(z)). \tag{D.2}$$

The quantity $F_X^{(n)}(z)$ can be computed with the discrete Fourier transform and Robertson explains how $F_U^{(n)}(z)$ can be obtained. For integral values of z the convolution of the two is

$$F_{X+U}^{(n)}(z) = \sum_{j=0}^{j=z}(f_X^{(n)}(j)F_U^{(n)}(z-j)). \tag{D.3}$$

Now consider $F_{X+U}^{(n)}$ for integral values of z,

$$F_{X+U}^{(n)}(z) - F_{X+U}^{(n)}(z-1) = \sum_{j=0}^{j=z}(f_X^{(n)}(j)(F_U^{(n)}(z-j) - F_U^{(n)}(z-j-1)) \tag{D.4}$$

Robertson explains that the differences $(F_U^{(n)}(z-j) - F_U^{(n)}(z-j-1))$ are the factors a_{z-j}^n where,

$$a_0^n = 1/n! \qquad n \geq 1, \tag{D.5}$$

$$a_j^1 = 0 \qquad j \geq 1, \tag{D.6}$$

$$a_j^n = (1/n)((n-j)a_{j-1}^{n-1} + (j+1)a_j^{n-1} \quad n \geq 2,\ j \geq 1. \tag{D.7}$$

The right hand side of equation D.4 is the convolution of $f^{(n)}$ with a_j^n and can be computed using discrete Fourier transforms.

The two-dimensional extension works in exactly the same way by considering the discrete random pair (X, Y) with integral size intervals of widths I and J. By adding an independent pair (U, V) where U is uniform on I and V is uniform on J, we get the random pair $(X + U, Y + V)$, which has claim size rectangles of constant density. The two-dimensional correction factors for the nth convolution are outer products of the one-dimensional correction factors, since U and V are independent.

$$a_{(i,j)}^n = a_i^n a_j^n \tag{D.8}$$

Sample R-code will be submitted with this chapter for downloading.