Trending Entry Ratio Tables

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September, 2005

1 Background

Entry ratio tables are often a convenient mechanism for capturing information that is subject only to scale transforms. For example, the National Council on Compensation Insurance, Inc. (NCCI) stores excess loss factors (ELFs) in entry ratio tables. To determine an ELF at an attachment point, you simply divide the attachment point by the mean loss, and use that "entry ratio" value to look up the ELF in the table. A key assumption is that the underlying size of loss distribution changes only by a uniform scale transform over time (or by a transform that is close enough to a scale transform; c.f. Venter [3] for a discussion of scale adjustments and excess losses).

In fact, there can be forces at work that change the shape of size of loss distributions in ways that are not captured by scale transforms. For example, large claims might have greater trend factors than small claims (differential severity trend). Also, the frequency of small claims might decrease more than the frequency of large claims over some period of time (differential frequency trend). Not surprisingly, both of these possible effects act to "stiffen" the size of loss distribution, that is, increase the probability that a claim is "large," given that a claim occurs. A surprising result of our analysis is that the adjustments to entry ratio tables to take these phenomena into account, when they occur, often work in opposite directions. When large claims have

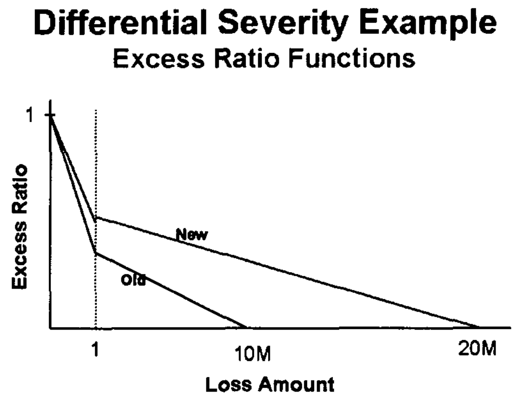
*Much thanks goes to Greg Engl and John Robertson, also of NCCI. Greg reviewed numerous drafts and his input improved the work throughout. John was key in promoting the topic within NCCI's actuarial research agenda. Both made direct and significant contributions to the paper.

greater trend factors than small claims, it might be necessary to increase the entry ratio table ELFs for large entry ratios. But when small claim frequency declines more rapidly than large claim frequency over a period of time, it might be necessary to *reduce* the tabular ELFs for large entry ratios.

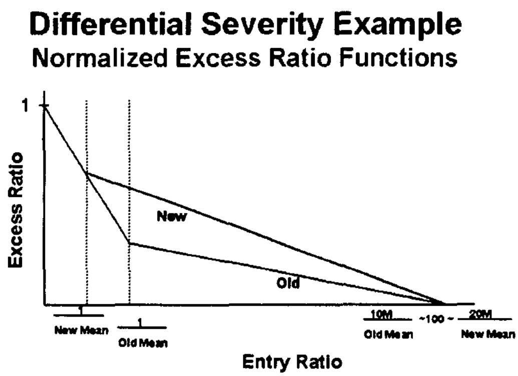
In this note we specify a generic, spreadsheet-friendly, format for an entry ratio table and consider the effects of differential trend and differential frequency changes. Each is illustrated by a real world Workers Compensation (WC) case study. We then describe general techniques for modifying an entry ratio table to account for not only a change in scale but also a change in the relativity between the mean and the median loss (or any fixed percentile loss) or a proportional shift in the hazard rate function of the loss distribution. The findings suggest that entry ratio tables work surprisingly well even for non-uniform trend and that in some important instances just a small adjustment can extend the shelf life of an entry ratio table.

2 Background

Before we get into the details of the paper, we present a thought experiment to illustrate some of the issues. Suppose we have 100 claims, 99 of which are for \$1 and the other is a \$10M claim. Consider what happens if over the next year inflation is expected to double the cost of the \$10M claim, but leave the other 99 unchanged. Observe that the mean cost per claim is expected to roughly double, going from about \$100K to about \$200K. Recall that the **excess ratio** is simply the ratio of the sum of losses in excess of a per claim loss limitation to the total of all first dollar and up losses. The following is a sketch of the graph of the old and new excess ratios, expressed as functions of the loss limitation amount:

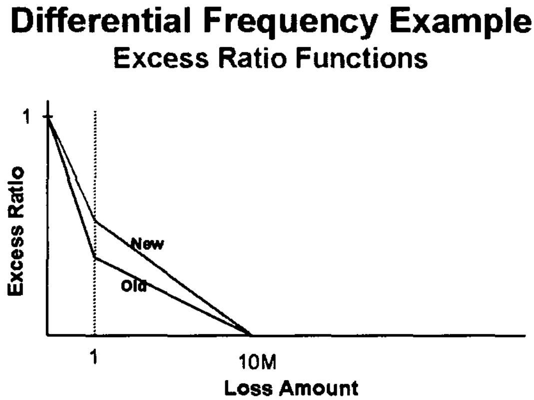


In practice, excess ratios are often captured in "entry ratio" tables, i.e. tables in which losses have been normalized to a mean value of 1. In this case, when we normailize the old and new losses by dividing by their respective means, the graph of the tabular excess ratios looks something like:

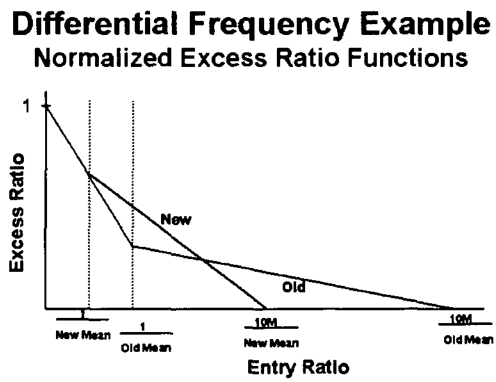


Observe that the new tabular values all lie at or above the old, which makes intuitive sense. Indeed, the inflation targeted the big claim, thereby "thickening" the tail of the loss distribution and necessitating the use of higher excess ratios next year. Because inflation changed the cost of claims selectively by size, this is a case of what the paper calls "Differential Severity".

Now suppose we begin with those same old 100 claims, but this time we consider what happens when, due to a safety initiative, half the \$1 claims do not emerge the next year. Because the change impacts claim frequency selectively by size, this is a case of what the paper calls "Differential Frequency". Notice that this experience change again roughly doubles the mean cost per case. Here the chart of the old and new excess ratio as a function of the loss limitation amount looks like:



and when normalized to entry ratio tabular values becomes:



Because the safety initiative is expected to be successful only for small claims, intuition again suggests a thickening of the tail. Observe, however, that the new tabular excess ratio values start out equal, then lie above, and eventually fall below the old. This suggests that, despite the similar impact on the mean cost per claim, something genuinely different is happening in the two scenarios. Actuaries should take heed that intuition can be a fallible guide to updating entry ratio tables.

3 Notation and Terminology

We start with a definition and, to keep the discussion self-contained, we derive some straightforward and familiar formulas:

Definition 1 *A random variable X is a* ***loss variable*** *if it has finite mean $\mu = E[X] > 0$ and has a density [PDF] f that is continuous when restricted to $[0, +\infty)$ and whose support is contained in $[0, +\infty)$. We denote the distribution function of X by $F(x) = \int_0^x f(y)dy$, whence $\frac{dF}{dx} = f(x)$ on $[0, +\infty)$. The survival function of X is $S = 1 - F$. The excess ratio function of X is given by $R(x) = \frac{E[Max(X-x,0)]}{\mu} = \frac{\int_x^\infty (y-x)f(y)dy}{\mu}$ for $x \geq 0$. We denote by $\widehat{F}$ the function given by $\widehat{F}(x) = \frac{\int_0^x yf(y)dy}{\mu}$ for $x \geq 0$. We use subscripts on μ_X, f_X, F_X, S_X, R_X, and $\widehat{F}_X$ when necessary to indicate dependence on X.*

The following proposition expresses the excess ratio function in terms of F and $\widehat{F}$.

Proposition 2 $R(x) = 1 - \widehat{F}(x) - \frac{x}{\mu}[1 - F(x)]$, *for all $x \geq 0$.*

Proof. From the definition of $R(x)$ we have

$$\begin{aligned} R(x) &= \frac{1}{\mu}\int_x^\infty (y-x)f(y)dy \\ &= \frac{1}{\mu}\left[\int_x^\infty yf(y)dy - x\int_x^\infty f(y)dy\right] \\ &= \frac{1}{\mu}\left[\mu - \int_0^x yf(y)dy - xS(x)\right] \\ &= 1 - \frac{1}{\mu}\int_0^x yf(y)dy - \frac{x}{\mu}S(x) \\ &= 1 - \widehat{F}(x) - \frac{x}{\mu}[1 - F(x)]. \end{aligned}$$

as required. This completes the proof. ∎

It is well known that the mean of a nonnegative random variable, X, can be expressed in terms of its survival function as $E[X] = \int_0^\infty S(x)dx$. It is easy to see that a similar result also holds for excess ratio functions.

Proposition 3 *Let X be a loss variable with survival function S and excess ratio function R, then*

$$R(x) = \frac{\int_x^\infty S(y)dy}{\int_0^\infty S(y)dy}, \text{ for all } x \geq 0.$$

Proof. Let X have density f, then noting that $\frac{dS}{dy} = -f(y)$ and using integration by parts, we have

$$\begin{aligned} \int_x^\infty S(y)dy &= yS(y)]_x^\infty + \int_x^\infty yf(y)dy \\ &= -xS(x) + \int_x^\infty yf(y)dy \\ &= -x\int_x^\infty f(y)dy + \int_x^\infty yf(y)dy \\ &= \int_x^\infty (y-x)f(y)dy, \end{aligned}$$

where the second equality follows from:

$$\begin{aligned} \mu &= E[X] < \infty \Rightarrow \text{[read "implies"]} \\ xS(x) &= x\int_x^\infty f(y)dy \leq \int_x^\infty yf(y)dy \to 0 \text{ as } x \to \infty. \end{aligned}$$

Thus

$$R(x) = \frac{\int_x^\infty (y-x)f(y)dy}{E[X]} = \frac{\int_x^\infty S(y)dy}{\int_0^\infty S(y)dy}.$$

as required. ∎

Corollary 4 $\frac{dR}{dx}(x) = \frac{-S(x)}{\mu}$, *for all $x \geq 0$.*

Proof. By the Fundamental Theorem of Calculus:

$$\frac{dR}{dx}(x) = \frac{d}{dx}\left(\frac{\int_x^\infty S(y)dy}{\mu}\right) = \frac{-S(x)}{\mu}.$$

as required. ∎

Proposition 5 *Let X be a loss variable with density f_X and distribution function F_X and let $\alpha, \beta > 0$ be any two positive constants. Set:*

$$Y = \alpha X^\beta$$

then for every $x, y > 0$:

1. $f_Y(y) = \frac{1}{\alpha^{\frac{1}{\beta}}\beta} y^{\frac{1-\beta}{\beta}} f_X\left(\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}\right)$

2. $F_Y(y) = F_X\left(\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}\right)$

3. $\widehat{F}_Y(y) = \frac{\int_0^{\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}} w^\beta f_X(w)dw}{\mu_{X^\beta}}$

4. $R_X(x) = R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}} x)$

Proof. We note that

$$F_Y(y) = \Pr(Y \le y) = \Pr(\alpha X^\beta \le y) = \Pr(X \le \left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}) = F_X\left(\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}\right)$$

proving 2.

For 1, just differentiate 2, using the change of variable $z = \left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}} \Rightarrow \frac{dz}{dy} = \frac{1}{\beta}\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}\frac{1}{\alpha} = \frac{1}{\alpha^{\frac{1}{\beta}}\beta} y^{\frac{1-\beta}{\beta}}$:

$$f_Y(y) = \frac{dF_Y}{dy} = \frac{dF_X\left(\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}\right)}{dy} = \frac{dF_X(z)}{dz}\frac{dz}{dy} = f_X\left(\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}\right)\frac{1}{\alpha^{\frac{1}{\beta}}\beta} y^{\frac{1-\beta}{\beta}}$$

And for 3 just integrate using the change of variable

$$w = \left(\frac{z}{\alpha}\right)^{\frac{1}{\beta}} \Leftrightarrow \alpha w^\beta = z \Rightarrow \frac{dw}{dz} = \frac{1}{\alpha^{\frac{1}{\beta}}\beta} z^{\frac{1-\beta}{\beta}}$$

we have:

$$\begin{aligned}
\widehat{F}_Y(y) &= \frac{\int_0^y z f_Y(z)dz}{\mu_Y} \\
&= \frac{\int_0^{z=y} \alpha w^\beta \frac{1}{\alpha^{\frac{1}{\beta}}\beta} z^{\frac{1-\beta}{\beta}} f_X\left(\left(\frac{z}{\alpha}\right)^{\frac{1}{\beta}}\right) dz}{\alpha\mu_{X^\beta}} \\
&= \frac{\int_0^{z=y} w^\beta f_X\left(w\right) \frac{dw}{dz} dz}{\mu_{X^\beta}} \\
&= \frac{\int_0^{\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}} w^\beta f_X(w)dw}{\mu_{X^\beta}}
\end{aligned}$$

Finally:

$$\begin{aligned}
Y &= \alpha X^\beta \Rightarrow Y^{\frac{1}{\beta}} = \alpha^{\frac{1}{\beta}} X \Rightarrow \mu_{Y^{\frac{1}{\beta}}} = \alpha^{\frac{1}{\beta}}\mu_X \\
\alpha^{\frac{1}{\beta}}\mu_X R_X(x) &= \alpha^{\frac{1}{\beta}} E[Max(X-x,0)] \\
&= \alpha^{\frac{1}{\beta}} E[Max(\left(\frac{Y}{\alpha}\right)^{\frac{1}{\beta}} - x, 0)] \\
&= E[Max(Y^{\frac{1}{\beta}} - \alpha^{\frac{1}{\beta}}x, 0)] \\
&= \mu_{Y^{\frac{1}{\beta}}} R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}}x) \\
&= \alpha^{\frac{1}{\beta}}\mu_X R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}}x) \\
&\Rightarrow R_X(x) = R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}}x)
\end{aligned}$$

completing the proof. ∎

The special case $\beta = 1$ applies when normalizing losses, in particular when dividing by the mean loss to get entry ratios:

Corollary 6 *Let X be a loss variable with density f_X and distribution function F_X, and let $\alpha > 0$, then*

1. $f_{\alpha X}(y) = \frac{f_X(\frac{y}{\alpha})}{\alpha}$

2. $F_{\alpha X}(y) = F_X(\frac{y}{\alpha})$
3. $\widehat{F}_{\alpha X}(y) = \widehat{F}_X(\frac{y}{\alpha})$
4. $R_{\alpha X}(y) = R_X(\frac{y}{\alpha})$

Proof. All but number 3 are clear from Proposition 5, and 3 is very nearly so:

$$\widehat{F}_{\alpha X}(y) = \frac{\int_0^{\frac{y}{\alpha}} w f_X(w)dw}{\mu_X} = \widehat{F}_X(\frac{y}{\alpha})$$

as required. ∎

We associate to a loss variable X with (finite) mean $\mu = \mu_X = E[X]$ an entry ratio table, which we term the $rAB = rAB_X$ table. The table consists of the two functions:

$$\begin{aligned} A_X(r) &= F_{X/\mu}(r) = \mu \int_0^r f(\mu x)dx = F_X(\mu r) \\ B_X(r) &= \widehat{F}_{X/\mu}(r) = \mu \int_0^r x f(\mu x)dx = \widehat{F}_X(\mu r) \end{aligned}$$

Clearly, for any positive scalar $\alpha > 0$ if $Y = \alpha X$, then

$$\frac{Y}{\mu_Y} = \frac{\alpha X}{\alpha \mu_X} = \frac{X}{\mu} \Rightarrow A_X = A_Y \text{ and } B_X = B_Y \Rightarrow rAB_Y = rAB_X$$

and indeed the entry ratio table is invariant under such a transformation of scale.

The dependent variable r is termed an "entry ratio" and corresponds to losses (but has applications to any positive real valued distribution, e.g. a wage distribution) normalized to a mean of 1. We often speak of these two functions as determining the A and B "columns" of the entry ratio table. Note that:

$$\begin{aligned} A_X(\infty) &= \lim_{r\to\infty} A_X(r) = 1 \\ B_X(\infty) &= \lim_{r\to\infty} B_X(r) = 1 \end{aligned}$$

Column A is sometimes described as the percent of claims at or below the corresponding entry ratio (r), while column B is described as the percent of losses corresponding to the claims in column A. This rAB setup is employed in WC benefit on-level calculations, and is especially practical for spreadsheets that deal with calculations that involve normalized loss variables.

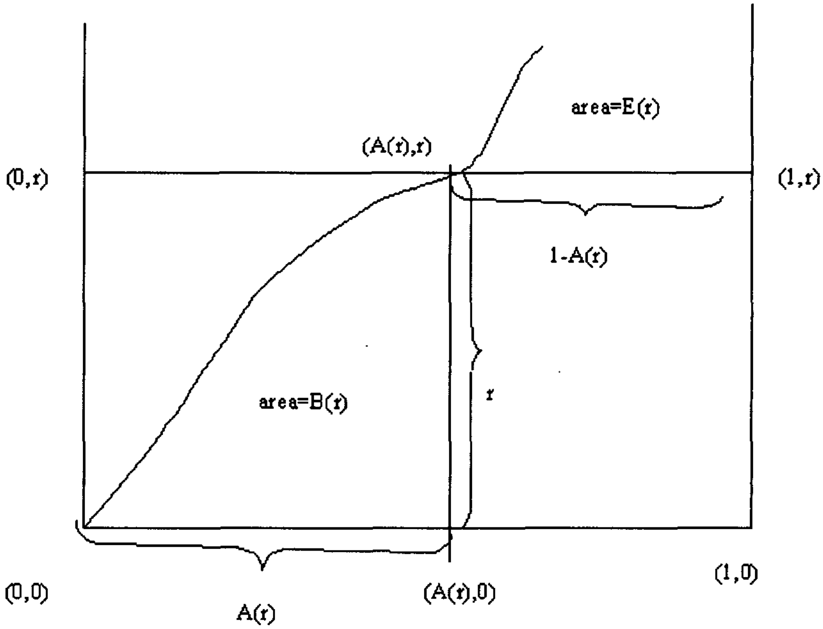
We are particularly interested in determining how $E_X(r)$, which we also refer to as the normalized excess ratio, behaves subject to a non-scale "trend" adjustment. For convenience we often expand the entry ratio table to include a third column E, readily derived from the others by applying Proposition 2 and Corollaries 4 and 6 to X/μ:

$$\begin{aligned} E_X(r) &= R_{X/\mu}(r) = 1 - B_X(r) - r(1 - A_X(r)) \\ \frac{dE_X}{dr}(r) &= \frac{dR_{X/\mu}}{dr}(r) = \frac{-S_{X/\mu}(r)}{1} = -S_X(\mu r) = F_X(\mu r) - 1 \end{aligned}$$

The following picture, reminicent of the area interpretation of integration by parts (c.f. Lee [2]), illustrates the usual way of visualing the rAB table and illustrates the formula for the normalized excess ratio:

$$E(r) = 1 - B(r) - r(1 - A(r))$$

in terms of r, A, and B:



We will let Y denote a loss variable that captures the effect of applying "trend" to X. We also set:

$$\begin{aligned} G &= F_Y \\ g &= f_Y \\ \nu &= E[Y]. \end{aligned}$$

Our goal is to determine rAB_Y from rAB_X. We are particularly interested in the absolute and relative impacts on the normalized excess ratio:

$$\begin{aligned} \delta(r) &= \delta_{XY}(r) = E_Y(r) - E_X(r) \\ \rho(r) &= \frac{\delta(r)}{E_X(r)}. \end{aligned}$$

We clearly have:

$$\begin{aligned} \delta(0) &= \rho(0) = 0 \\ \lim_{r\to\infty} \delta(r) &= 0 \end{aligned}$$

Taking derivatives and applying L'Hôpital (twice), we have:

$$\begin{aligned}
\frac{d\delta}{dr} &= G(\nu r) - F(\mu r) \\
\frac{d^2\delta}{dr^2} &= \nu g(\nu r) - \mu f(\mu r) \\
\frac{d\rho}{dr} &= \frac{E_X(r)G(\nu r) - E_Y(r)F(\mu r) + \delta(r)}{E_X(r)^2} \\
1 + \lim_{r\to\infty} \rho(r) &= 1 + \lim_{r\to\infty} \frac{\delta(r)}{E_X(r)} = 1 + \lim_{r\to\infty} \frac{G(\nu r) - F(\mu r)}{F(\mu r) - 1} \\
&= 1 + \lim_{r\to\infty} \frac{G(\nu r) - 1 - (F(\mu r) - 1)}{F(\mu r) - 1} \\
&= 1 + \lim_{r\to\infty} \left(\frac{G(\nu r) - 1}{F(\mu r) - 1} - 1 \right) \\
&= \lim_{r\to\infty} \frac{G(\nu r) - 1}{F(\mu r) - 1} = \lim_{r\to\infty} \frac{\nu g(\nu r)}{\mu f(\mu r)} \\
&= \frac{\nu}{\mu} \lim_{s\to\infty} \frac{g(\frac{\nu}{\mu} s)}{f(s)} \text{ (since } r \to \infty \Leftrightarrow s = \mu r \to \infty \text{).}
\end{aligned}$$

For large entry ratios, the impact of trend on the normalized excess ratio column, $E_X(r)$ vs. $E_Y(r)$, is dictated by the impact of trend on the mean and on the largest losses. For any loss variable X let M_X denote the maximum loss (in the case of no finite maximum loss amount, we set $M_X = \infty$).

Proposition 7 *Suppose X and Y are two loss variables with $M_X, M_Y < \infty$ and $\frac{M_X}{\mu_X} > \frac{M_Y}{\mu_Y}$, then there exists $b > 0$ such that $E_Y(b) < E_X(b)$ and $0 = E_Y(r) \leq E_X(r)$ for $r \geq b$.*

Proof. Setting $b = \frac{M_Y}{\mu_Y} < \frac{M_X}{\mu_X}$ we have

$$\begin{aligned}
b\mu_X &< M_X \Rightarrow \\
E_Y(b) &= R_{Y/\mu_Y}(b) = R_Y(\mu_Y b) \\
&= R_Y(M_Y) = 0 < R_X(b\mu_X) = R_{X/\mu_X}(b) = E_X(b) \\
\text{and } r &\geq b \Rightarrow \\
r\mu_Y &\geq b\mu_Y = M_Y \Rightarrow E_Y(r) = R_{Y/\mu_Y}(r) \\
&= R_Y(\mu_Y r) = 0 \leq E_X(r)
\end{aligned}$$

as required. ■

We will find a use for the following later in Section 4:

Proposition 8 *Suppose X and Y are two loss variables with the same maximum loss $M_X = M_Y < \infty$ and with $\mu_Y > \mu_X$, then there exists $a > 0$ such that $R_Y(r) > R_X(r)$ for $0 < r \le a$ and there exists $b > 0$ such that $E_Y(b) < E_X(b)$ and $0 = E_Y(r) \le E_X(r)$ for $r \ge b$.*

Proof. Since $\mu_Y > \mu_X$, the existence of b follows from Proposition 7. For the existence of a, we have from Corollary 4:

$$\frac{dR_X}{dx}(0) = \frac{-1}{\mu_X} < \frac{-1}{\mu_Y} = \frac{dR_Y}{dy}(0)$$

Now clearly $R_Y(0) = R_X(0) = 1$ and since R_Y and R_X are continuously differentiable there exists $a > 0$ with

$$\begin{aligned} \frac{R_X(x)-1}{x} &= \frac{R_X(x)-R_X(0)}{x-0} \\ &< \frac{R_Y(y)-R_Y(0)}{y-0} = \frac{R_Y(y)-1}{y} \text{ for every } x, y \in (0, a]. \end{aligned}$$

In particular:

$$\begin{aligned} 0 &< r \le a \Rightarrow \frac{R_X(r)-1}{r} < \frac{R_Y(r)-1}{r} \\ &\Rightarrow R_X(r) - 1 < R_Y(r) - 1 \Rightarrow R_X(r) < R_Y(r). \end{aligned}$$

This completes the proof. ■

4 Differential Severity Trend

Let the function $h(x)$ defined on $[0, \infty)$ be such that $h(x) \ge 1$ and $\frac{dh}{dx} > 0$ on $[0, \infty)$. In this section we assume $f(x) > 0$ for $x > 0$. Think of $h(x)$ as a severity trend factor that increases with the size of loss x. The random variable of the trended loss is $Y = \psi(X)$, where the transformation $\psi(x) = h(x)x$ has $\frac{d\psi}{dx} = h(x) + x\frac{dh}{dx} > 1$ for $x > 0$ and is order preserving and invertible (and expands distances). Thus:

$$G(\psi(x)) = \Pr(Y \le \psi(x)) = \Pr(\psi(X) \le \psi(x)) = \Pr(X \le x) = F(x).$$

We clearly have $\psi(x) \geq x \Rightarrow \nu = E[Y] = E[\psi(X)] \geq E[X] = \mu$ and $F(x) = G(\psi(x) \geq G(x)$ Observe that:

$$\begin{aligned}
x &\geq a \Leftrightarrow \psi(x) \geq \psi(a) \\
x &\geq a \Rightarrow \psi(x) - \psi(a) = h(x)x - h(a)a \geq h(a)x - h(a)a \\
&= h(a)\,(x-a) \geq x - a \\
&\Rightarrow \nu R_Y(\psi(x)) = E[Max(Y - \psi(x), 0)] \\
&= E[Max(\psi(X) - \psi(x), 0)] \geq E[Max(X - x, 0)] = \mu R_X(x) \\
&\Rightarrow R_Y(\psi(x)) \geq \left(\frac{\mu}{\nu}\right) R_X(x)
\end{aligned}$$

But the relationship between the normalized excess ratios $E_Y(r)$ and $E_X(r)$ is more subtle.

Let $h_M = \lim_{x\to\infty} h(x)$ and $h_m = h(0)$, then $1 \leq h_m < h_M \leq \infty$ and we have:

$$\begin{aligned}
h_m\mu &= h_m E[X] < E[h(X)X] = E[\psi(X)] = E[Y] = \nu < h_M E[X] = h_M\mu \\
&\Rightarrow h_m < \frac{\nu}{\mu} < h_M \\
&\Rightarrow \text{there exists exactly one } a > 0 \text{ such that } h(a) = \frac{\nu}{\mu}.
\end{aligned}$$

However, we see that since F and ψ are both monotonic increasing, whence invertible, and so too is $G = F \circ \psi^{-1}$. Whence for $r > 0$ we have the equivalence:

$$\begin{aligned}
0 &= \frac{d\delta}{dr} = G(\nu r) - F(\mu r) \Leftrightarrow G(\nu r) = F(\mu r) \Leftrightarrow \nu r = \psi(\mu r) \\
&\Leftrightarrow h(\mu r)\mu r = \nu r \\
&\Leftrightarrow h(\mu r)\mu = \nu \Leftrightarrow h(\mu r) = \frac{\nu}{\mu} = h(a) \Leftrightarrow a = \mu r \Leftrightarrow r = \frac{a}{\mu}.
\end{aligned}$$

Now $0 = \delta(0) = \lim_{r\to\infty} \delta(r)$ and so it follows that, unless $\delta(r) = 0$ for every $r \geq 0$, the function $\delta(r)$ has either a unique minimum or a unique maximum on $(0, \infty)$, and consequently $\delta(r)$ is either always ≥ 0 or always ≤ 0, for all $r \geq 0$. We claim that $\delta(r) \geq 0$ for all $r \geq 0$. To verify this, select β such that

$h_m < \beta < \frac{\nu}{\mu}$ and let $b = \mu s = h^{-1}(\beta) > 0$; then:

$$\begin{aligned} a &= r\mu, b = s\mu, 1 < \beta = h(b) < \frac{\nu}{\mu} = h(a) \Rightarrow b < a \\ \psi(s\mu) &= \psi(b) = h(b)b = \beta b < \frac{\nu}{\mu}b = \frac{\nu}{\mu}s\mu = \nu s \\ &\Rightarrow F(s\mu) = G(\psi(s\mu)) < G(\nu s) \\ &\Rightarrow \frac{d\delta}{dr}(s) = G(\nu s) - F(s\mu) > 0 \end{aligned}$$

It follows that $\delta(r)$ is increasing at $s = \frac{b}{\mu}$ and therefore on the entire interval $(0, \frac{a}{\mu})$. Since $\delta(0) = 0$, this clearly forces $\delta(\frac{a}{\mu}) > 0$ and consequently $\delta(r) \geq 0$ for all $r \geq 0$, as claimed.

We see that the graph of $\delta(r)$ is $\cap$-shaped, i.e. is concave with $0 = \delta(0) = \lim_{r\to\infty} \delta(r)$, with a unique maximum at $r = \frac{a}{\mu}$. We have established:

Proposition 9 *In the case of the differential severity trend model $G(\psi(x)) = F(x)$ and $f(x) > 0$ for $x > 0$, as defined above, $E_Y(r) - E_X(r) > 0$ for all $r > 0$.*

Let $r_0 = 0 < r_1 < r_2 < \cdots < r_M$ be a sequence of entry ratios and set

$$A_i = A_X(r_i), B_i = B_X(r_i), 0 \leq i \leq M.$$

Suppose that $A_i = A_X(r_i) > A_X(r_{i-1}), 1 \leq i \leq M$ and $A_M = 1$. Set $\Delta A_i = A_i - A_{i-1}, \Delta B_i = B_i - B_{i-1}$. Note that $\mu\frac{\Delta B_i}{\Delta A_i}, 1 \leq i \leq M$, is the mean value of the untrended loss over the interval $[\mu r_{i-1}, \mu r_i]$. For $1 \leq i \leq M$, set

$$\begin{aligned} \Delta\widetilde{B}_i &= \Delta A_i \left(\psi\left(\mu\frac{\Delta B_i}{\Delta A_i}\right)\right) \\ \widetilde{B}_i &= \sum_{k=1}^{i} \Delta\widetilde{B}_k. \end{aligned}$$

Since ψ is order preserving, it is reasonable to assume that $\psi\left(\mu\frac{\Delta B_i}{\Delta A_i}\right)$ is a good estimate of the mean value of the trended losses on the interval

$[\psi(\mu r_{i-1}), \psi(\mu r_i)]$, (the smaller the interval, the more accurate the estimate).

$$\begin{aligned}
\widetilde{B}_M &= \sum_{k=1}^{M} \Delta \widetilde{B}_k = \sum_{k=1}^{M} \Delta A_i \left(\psi\left(\mu \frac{\Delta B_i}{\Delta A_i}\right)\right) \\
&= \sum_{k=1}^{M} \left(G(\psi(\mu r_i) - G(\psi(\mu r_{i-1}))\right) \left(\psi\left(\mu \frac{\Delta B_i}{\Delta A_i}\right)\right) \\
&= \sum_{k=1}^{M} \Pr\left(\psi(\mu r_{i-1}) < Y \leq \psi(\mu r_i)\right) \left(\psi\left(\mu \frac{\Delta B_i}{\Delta A_i}\right)\right) \\
&\approx \sum_{k=1}^{M} \Pr\left(\psi(\mu r_{i-1}) < Y \leq \psi(\mu r_i)\right) E[Y \mid \psi(\mu r_{i-1}) < Y \leq \psi(\mu r_i)] \\
&= E[Y] = \nu
\end{aligned}$$

And we have the estimate $\widetilde{B}_M \approx \nu$. The sequence $\{A_i\}$ can be viewed as the cumulative percentage of cases over the intervals of the trended losses and thus approximates the A column of the entry ratio table of the trended losses. The sequence $\left\{\widetilde{B}_i\right\}$ approximates the cumulative losses for the trended loss cases from the corresponding intervals. So the sequence $\left\{\widetilde{B}_i\right\}$ is proportional to the B column of the entry ratio table of the trended losses. Also, we have observed that the sequence $\{\psi(\mu r_i)\}$ provides the endpoints of the corresponding intervals of the trended losses which have overall mean $= \nu \approx \widetilde{B}_M$. So setting:

$$\widehat{r}_i = \frac{\psi(\mu r_i)}{\widetilde{B}_M}, \ \widehat{A}_i = A_i, \ \widehat{B}_i = \frac{\widetilde{B}_i}{\widetilde{B}_M}, \ 0 \leq i \leq M$$

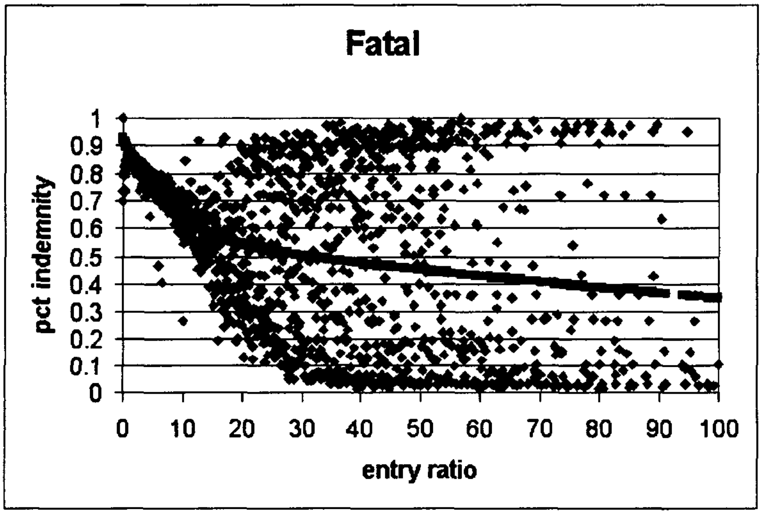
we have approximated the rAB table for the trended losses $rAB_Y \approx \widehat{rAB}$ in the case of differential severity trend. This differential severity trend adjustment to the rAB table is a simple three-step process (1-fix A, 2-estimate B, 3-normalize r and B). In practice, this approximation can yield small negative values for $\delta(r)$ which by Proposition 9 should be set equal to 0.

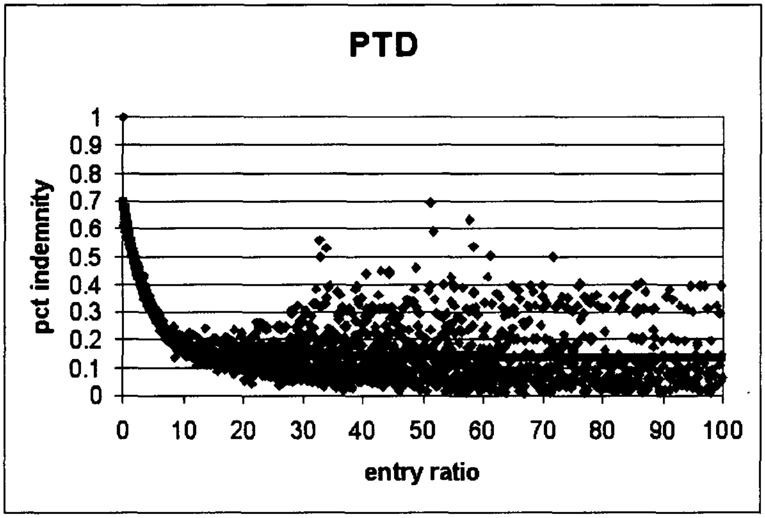
4.1 WC Case Study of Differential Severity Trend

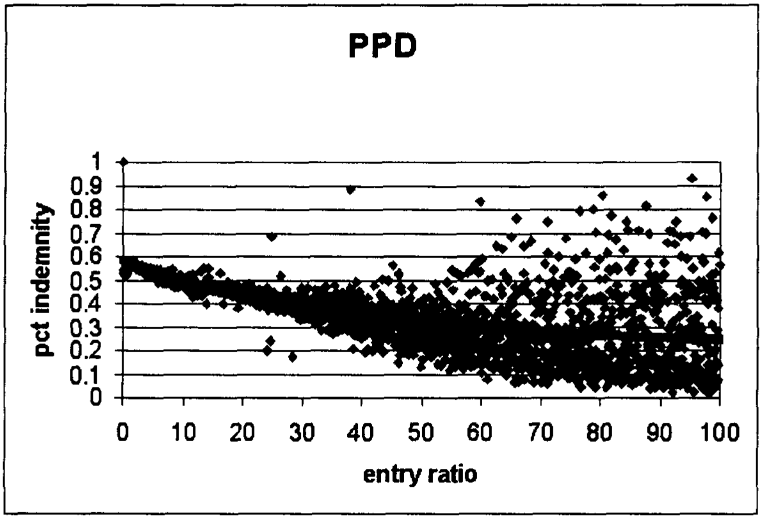
The tables for excess ratios in WC are specific to the five types of WC injury: Fatal, Permanent Total Disability [PTD], Permanent Partial Disability

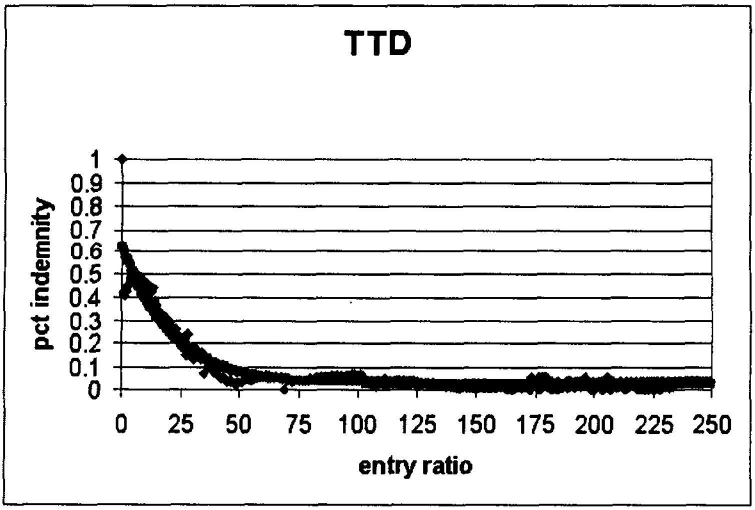
[PPD], Temporary Total Disaility [TTD], and medical only. It is standard to itemize WC losses into medical and indemnity (or wage replacement) components. While indemnity benefits are limited, either implicitly or by statutory maximum aggregates, the medical portion is unlimited and subject to broadly inclusive statutes as regards the medical procedures covered. In any event, it has been noted that as the claim size rises, the percentage of the benefit that goes for medical also rises. This is generally observed within all the injury types (except medical only). A series of charts below provide a more detailed picture of this phenomenon. Combine that observation with the fact that medical losses are subject to greater upward inflationary pressure than wages, and you have a scenario in which to apply the differential severity trend model of the previous section.

In this case study we assume constant annual trend factors of $t_0 = 1.075$ for indemnity and $t_1 = 1.095$ for medical, applicable to all injuries and all loss sizes. Normalized WC loss data by injury type was itemized into medical and indemnity components and used to produce the following charts, by injury type, that show the percentage of the total [=medical + indemnity] loss by entry ratio (the role of the fitted curve will be described later). It is worth noting that the percentages shown in the charts are determined over a common interval width of entry ratio. Since there are typically more claims at lower entry ratios, one consequence is more claims per plotted point at the lower entry ratios, whence the greater spread of the plotted points at the higher entry ratios.


Fatal
pct indemnity
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100
entry ratio


PTD
pct indemnity
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100
entry ratio





For each injury type = i, a simple curve (akin to a mixed exponential survival curve, and shown on the charts) was fit to the patterns of decreasing indemnity proportion $\pi_i(r)$ by entry ratio r as the loss size increases:

$$\pi_i(r) = a_i \left(b_i e^{\alpha_i r} + c_i e^{\beta_i r} + (1 - b_i - c_i) e^{\gamma_i r} \right)$$

Injury	i	a_i	b_i	c_i	α_i	β_i	γ_i
Fatal	1	0.9280	0.6240	0.3761	−0.0051	−0.1416	−0.4599
PTD	2	0.6928	0.7905	0.2095	−0.2542	−0.0007	−0.4599
PPD	3	0.5811	0.3827	0.6173	0	−0.0281	0
TTD	4	0.6237	0.0397	0.9603	0	−0.0475	−0.4599

We set $h_i(r) = \pi_i(r)t_0 + (1 - \pi_i(r))\,t_1$, then:

$$1 < t_0 < t_1, \frac{d\pi_i}{dr} < 0 \Rightarrow 1 < h_i(r) \text{ and } \frac{dh_i}{dr} = \frac{d\pi_i}{dr}(t_0 - t_1) > 0.$$

and so each injury type other than medical only provides a differential severity trend model.

Letting X_i denote the random variable of losses by injury type and N_{X_i} the corresponding claim counts, the usual formula (readily obtained from Definition 1; see Gillam [1]) for the combined excess ratio over the injury types at attachment A is:

$$XSratio(A) = XSratio(X_1, X_2, X_3, X_4, X_5; A) = \frac{\sum_i N_{X_i}\mu_{X_i}E_{X_i}\left(\frac{A}{\mu_{X_i}}\right)}{\sum_i N_{X_i}\mu_{X_i}}$$

Of course, to accomodate differential severity trend one could produce new rAB tables as detailed above. A simpler alternative is to determine the difference:

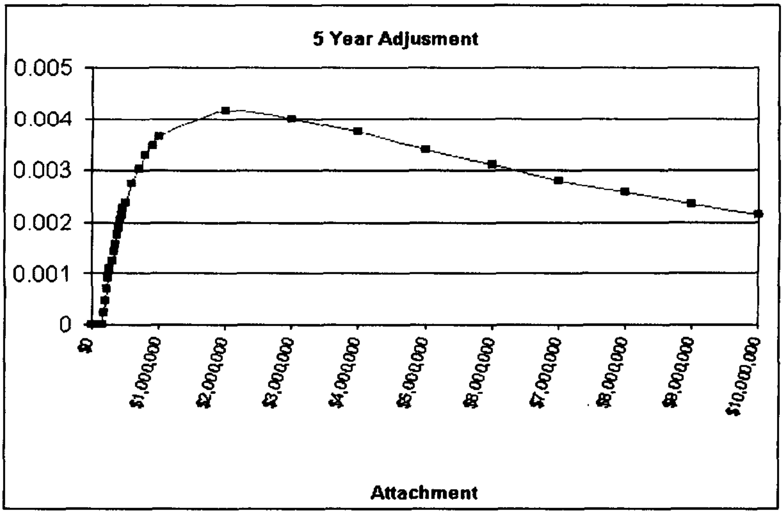
$$\begin{aligned}
& \Delta XSratio(A) \\
= \; & XSratio(Y_1, Y_2, Y_3, Y_4, Y_5; A) - XSratio^*(X_1, X_2, X_3, X_4, X_5; A) \\
= \; & \frac{\sum_i N_{Y_i}\mu_{Y_i}E_{Y_i}\left(\frac{A}{\mu_{Y_i}}\right)}{\sum_i N_{Y_i}\mu_{Y_i}} - \frac{\sum_i N_{Y_i}\mu_{Y_i}E_{X_i}\left(\frac{A}{\mu_{Y_i}}\right)}{\sum_i N_{Y_i}\mu_{Y_i}} \\
= \; & \frac{\sum_i N_{Y_i}\mu_{Y_i}\left(E_{Y_i}\left(\frac{A}{\mu_{Y_i}}\right) - E_{X_i}\left(\frac{A}{\mu_{Y_i}}\right)\right)}{\sum_i N_{Y_i}\mu_{Y_i}} \\
= \; & \frac{\sum_i N_{Y_i}\mu_{Y_i}\delta_{X_iY_i}\left(\frac{A}{\mu_{Y_i}}\right)}{\sum_i N_{Y_i}\mu_{Y_i}}
\end{aligned}$$

expressed in terms of the $\delta_{X_iY_i}$ and where the $*$ attached to $XSratio^*$ is meant to emphasize that one would consistently use the newer claim counts N_{Y_i} and means μ_{Y_i} in doing the calculation. While in principle you would need updated rAB tables to precisely determine the $\delta_{X_iY_i}$ terms, if there were a simplified form to approximate that term based on inflation data or other cost trend considerations, this would provide the ability to refine the excess ratio calculation:

$$\begin{aligned} & XSratio(Y_1, Y_2, Y_3, Y_4, Y_5; A) \\ = \; & XSratio^*(X_1, X_2, X_3, X_4, X_5; A) + \Delta XSratio(A) \end{aligned}$$

without recourse to new rAB tables.

The use of entry ratio tables is a very good way to account for inflation when calculating excess ratios. Indeed, even compounded over a five year time interval, the $\Delta XSratio$ adjustment in this case study is very small. The following chart is indicative of what the calculation described here produces. Of course, a bigger difference between medical and indemnity trend or a longer time interval will produce bigger adjustments. Because excess ratios decline with increasing attachment points, as the attachment point increases the adjustment will typically increase as a percentage of the excess ratio.



5 Differential Frequency Trend

Let the function $h(x)$ defined on $[0, \infty)$ be such that $0 \leq h(x) \leq 1$, with h piecewise continuous and non-decreasing on $[0, \infty)$. So as to relate h with the 'untrended' loss variable X, we also assume that there exist $a, b > 0$ such that $h(a) < h(b)$ with h continuous at a and at b and with $f(x) > 0$ for every $x \in (a, b)$. Observe that this clearly forces $a < b$, and so there exist $b_k \in (a, b)$ such that $\lim_{k\to\infty} b_k = b$. But then, since h continuous at b:

$$\begin{aligned} h(a) &< h(b) \Rightarrow h(a) < h(b) = h\left(\lim_{k\to\infty} b_k\right) = \lim_{k\to\infty} h(b_k) \\ &\Rightarrow \text{ there exists } M \in \mathbb{N} \text{ such that } h(b_k) > h(a) \text{ for every } k \geq M. \end{aligned}$$

In particular, letting $c = b_M$ we have:

$$\begin{aligned} c &= b_M \in (a, b) \\ &\Rightarrow f(c) > 0, h(c) > h(a) \Rightarrow h(c)f(c) > 0 \Rightarrow 0 < E[h(X)] < E[1] = 1. \end{aligned}$$

We consider the 'trended' loss model defined by the PDF:

$$g(x) = \frac{h(x)f(x)}{E[h(X)]} = \widehat{h}(x)f(x).$$

Think of $h(x)$ as a proportional decline in the incidence rate that decreases with the size of loss x. For the trended loss variable Y, we have:

$$\Pr(Y \leq a) = \int_0^a g(x)dx = \int_0^a \widehat{h}(x)f(x)dx = \Pr(\widehat{h}(X) \leq a).$$

And accordingly, for the differential frequency trend model we take $Y = \widehat{h}(X)$. Also, if h is differentiable (except at perhaps finitely many points), integration by parts gives:

$$\begin{aligned}
G(y) &= \int_0^y g(x)dx = \int_0^y \widehat{h}(x)f(x)dx \\
&= \widehat{h}(x)F(x)]_0^y - \int_0^y F(x)\frac{d\widehat{h}}{dx}dx \\
&= \widehat{h}(y)F(y) - \int_0^y F(x)\frac{d\widehat{h}}{dx}dx \\
&\geq \widehat{h}(y)F(y).
\end{aligned}$$

For the differential frequency trend model we cannot have $F(x) \geq G(x)$ for all $x \geq 0$, since by the above that would force the contradiction

$$\begin{aligned}
G(x) &\geq \widehat{h}(x)F(x) \geq \widehat{h}(x)G(x) \\
\Rightarrow \quad & 1 \geq \widehat{h}(x) \text{ for all } x \geq 0 \text{ such that } f(x) > 0 \text{ with } 1 > \widehat{h}(a) \text{ for some } a \geq 0 \text{ such that } f(\\
\Rightarrow \quad & 1 < E\left[\widehat{h}(X)\right] = E\left[\frac{h(X)}{E\left[h(X)\right]}\right] = 1 \Rightarrow\Leftarrow
\end{aligned}$$

In particular, differential trend models and differential frequency models are disjoint from one another

Remark 10 *The reader should note that unless we make the stronger assumption that h is continuous on $[0, \infty)$, we cannot be assured that this Y is a loss variable, as that term is defined here. The weaker assumption on h is to include the case in which h is a step function. The reader may prefer to demand that h be continuous, in which case some of the arguments can be simplified.*

Proposition 11 *In the case of the differential frequency trend model $g(x) = \widehat{h}(x)f(x)$, as above, $\nu > \mu$.*

Proof. Note that the function $\widehat{h}(x)$ is piecewise continuous and non-decreasing on $[0, \infty)$. We claim that $\widehat{h}(0) < 1$, since otherwise:

$$\widehat{h}(x) \geq 1 \text{ for every } x \geq 0 \Rightarrow g(x) = \widehat{h}(x)f(x) \geq f(x) \text{ for every } x \geq 0.$$

But then $g(x)$ and $f(x)$ are two piecewise continuous funtions on $[0, \infty)$ with the same finite integral $= 1$. So the relation $g(x) \geq f(x)$ entails that $g(x) = f(x)$ except possibly at points of discontinuity of g. So $\widehat{h}(x) = 1$ except for a discrete set of values or where $f(x) = 0$. By our model assumptions, there exist $\alpha, \beta > 0$ such that $h(\alpha) < h(\beta)$ with h continuous at α and at β and with $f(x) > 0$ for every $x \in (\alpha, \beta)$. It follows that $\widehat{h}(x) = 1$ on (α, β), except for perhaps a discrete set of points:

$$\begin{aligned}
&\Rightarrow \text{ there exist } a_i, b_i \in (\alpha, \beta) \text{ such that} \\
\alpha &= \lim_{i\to\infty} a_i, \beta = \lim_{i\to\infty} b_i \text{ and } \widehat{h}(a_i) = \widehat{h}(b_i) = 1 \\
&\Rightarrow \widehat{h}(\alpha) = \widehat{h}\left(\lim_{i\to\infty} a_i\right) = \lim_{i\to\infty} \widehat{h}(a_i) = \lim_{i\to\infty} 1 = 1 \\
&= \lim_{i\to\infty} \widehat{h}(b_i) = \widehat{h}\left(\lim_{i\to\infty} b_i\right) = \widehat{h}(\beta) \\
&\Rightarrow h(\alpha) = \widehat{h}(\alpha)E[h(X)] = \widehat{h}(\beta)E[h(X)] = h(\beta) \\
&\Rightarrow h(\alpha) = h(\beta) > h(\alpha) \quad \Rightarrow\Leftarrow \text{[read “contradiction”]}.
\end{aligned}$$

This contradiction shows that $\widehat{h}(0) < 1$. Similarly, we claim that $\widehat{h}(a) > 1$ for some $a > 0$, since otherwise:

$$\widehat{h}(x) \leq 1 \text{ for all } x \geq 0 \Rightarrow g(x) = \widehat{h}(x)f(x) \leq f(x) \text{ for all } x \geq 0$$

and again $g(x)$ and $f(x)$ are two piecewise continuous funtions on $[0, \infty)$ with the same finite integral. This again entails that they are equal except possibly at points of discontinuity. Then again $\widehat{h}(x) = 1$ except for a discrete set of values or where $f(x) = 0$ and just as before we arrive at a contradiction. So we have

$$\begin{aligned}
\widehat{h}(0) &< 1 < \widehat{h}(a) \\
&\Rightarrow \text{ there exists } b > 0 \text{ such that } \widehat{h}(x) \leq 1 \text{ on } [0, b) \\
\text{and } \widehat{h}(x) &\geq 1 \text{ on } (b, \infty).
\end{aligned}$$

Next we claim that there exists $c > 0$ such that $\widehat{h}(c) \neq 1$ and $f(c) > 0$ since otherwise

$$x > 0, f(x) > 0 \Rightarrow \widehat{h}(x) = 1 \Rightarrow h(x) = E[h(X)]$$

But by our model assumptions, there exist $\alpha, \beta > 0$ such that $h(\alpha) < h(\beta)$

with h continuous at α and at β and with $f(x) > 0$ for every $x \in (\alpha, \beta)$:

$$\begin{aligned} &\Rightarrow \text{ there exists } c \in \left(\alpha, \frac{\alpha+\beta}{2}\right), d \in \left(\frac{\alpha+\beta}{2}, \beta\right) \\ \text{such that } h(c) &\neq h(d),\ f(c) > 0, f(d) > 0 \\ &\Rightarrow E[h(X)] = h(c) \neq h(d) = E[h(X)]. \end{aligned}$$

It follows that there exists $c > 0$ such that $\widehat{h}(c) \neq 1$ and $f(c) > 0$ and we have:

$$\begin{aligned} \nu - \mu &= \int_0^\infty xg(x)dx - \int_0^\infty xf(x)dx = \int_0^\infty x\,(g(x) - f(x))\,dx \\ &= \int_0^\infty x\left(\widehat{h}(x) - 1\right)f(x)dx \\ &= \int_0^b x\left(\widehat{h}(x) - 1\right)f(x)dx + \int_b^\infty x\left(\widehat{h}(x) - 1\right)f(x)dx \\ &> b\int_0^b \left(\widehat{h}(x) - 1\right)f(x)dx + b\int_b^\infty \left(\widehat{h}(x) - 1\right)f(x)dx \\ &= b\int_0^\infty \left(\widehat{h}(x) - 1\right)f(x)dx = b\int_0^\infty (g(x) - f(x))\,dx \\ &= b\left(\int_0^\infty g(x)dx - \int_0^\infty f(x)dx\right) = b(1-1) = 0 \\ &\Rightarrow \nu > \mu \end{aligned}$$

as required. ■

As in the case of differential severity trend in the preceding section, we again are considering a change that increases the mean severity. Suppose we use a fixed entry ratio table to calculate excess ratios. Then for a fixed attachment point A, we have declining entry ratios $\frac{A}{\mu} > \frac{A}{\nu}$ and the lookup into the same entry ratio table leads to excess ratios that increase from $E_X\left(\frac{A}{\mu}\right)$

to $E_X\left(\frac{A}{\nu}\right)$. In the case of differential severity trend, we observed in Proposition 9 of the previous section that the increase is consistently understated. In the case of differential frequency trend, however, we will show that the increase may be either overstated or understated. This may at first seem somewhat counterintuitive for the two "trends" to move the mean upward but the normalized excess ratio tabular amounts in perhaps opposite directions. However, the entry ratio lookup is dominated by the change in the mean. For differential severity trend the overall trend factor consistently understates the impact of trend on the largest loss amounts, which helps explain why the calculation consistently understates the excess ratio. But the case of differential frequency trend is quite different: selectively removing smaller sized losses will have a leveraged upward impact on the overall mean severity while leaving the size of the largest claims unchanged.

With differential frequency trend we have, from the proof of Proposition 11:

$$\begin{aligned} x &\geq b \Rightarrow \nu R_Y(x) - \mu R_X(x) = \int_x^\infty (y-x)\,(g(y)-f(y))\,dy \\ &= \int_x^\infty (y-x)\left(\widehat{h}(y)-1\right) f(y)dy \geq 0 \\ &\Rightarrow R_Y(x) \geq \frac{\mu}{\nu} R_X(x). \end{aligned}$$

But again the relationship between $E_Y(r)$ and $E_X(r)$ is more subtle.

In the case that X has a maximum loss $M = M_X < \infty$, since $\widehat{h}(x)$ is non-decreasing on $[0,\infty)$ and there exists $c > 0$ such that $\widehat{h}(c) > 0$ and $f(c) > 0$, and we have $c \leq M$ and $\widehat{h}(d) > 0$ for every $d \geq c$, whence:

$$M_Y = \sup\{x|g(x) > 0\} = \sup\{x|\widehat{h}(x)f(x) > 0\} = \sup\{x|f(x) > 0\} = M.$$

So too must Y have maximum loss M and Proposition 8 assures us that $E_Y(r) \leq E_X(r)$ for large enough r. More precisely, we have:

Proposition 12 *In the case of the differential frequency trend model $g(x) = \widehat{h}(x)f(x)$, as defined above, in which X has a maximum loss $M_X < \infty$, there exists $b > 0$ such that $E_Y(b) < E_X(b)$ and $E_Y(r) \leq E_X(r)$ for all $r \geq b$.*

Before stating a result that deals with the relationship between $E_Y(r)$ and $E_X(r)$ in the case $M_X = \infty$, it is instructive to make a few observations.

Note that since the non-decreasing function h is bounded above by 1, it is reasonable (but not necessary) to have the decline in frequency flatten out for large losses, say in the sense that the derivative $\frac{dh}{dx} \to 0$ as $x \to \infty$. We also observe that:

Proposition 13 *In the case of the differential frequency trend model* $g(x) = \widehat{h}(x)f(x)$*, as above, the limit* $\lim_{x\to\infty} \widehat{h}(x) = \lambda$ *exists and* $\frac{\nu}{\mu} \leq \lambda$.

Proof. Since h is non decreasing and bounded above by 1, existence of the limit is apparent. We evidently have:

$$\widehat{h}(x) \leq \lambda \text{ for all } x \geq 0 \Rightarrow \nu = E[X\widehat{h}(X)] \leq E[X\lambda] = \lambda E[X] = \lambda\mu \Rightarrow \frac{\nu}{\mu} \leq \lambda$$

as required. ∎

Proposition 14 *Assume* $M_X = \infty$*, then for any* $\rho > 1$ *for which the limit* $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ *exists:*

$$\lim_{x\to\infty} \frac{S(\rho x)}{S(x)} \leq \frac{1}{\rho}.$$

Proof. Note that $M_X = \infty$ is equivalent to $S(x) > 0$ for every $x > 0$ and so $\frac{S(\rho x)}{S(x)}$ is always well defined. Thus the expression $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ makes sense and further our assumption is that the limit exists for some $\rho > 1$. Note that the integral $\int_0^\infty S(x)dx = \mu < \infty$. Suppose, by way of contradiction, that $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)} > \frac{1}{\rho}$. Then, using the change of variable $z = \rho x$, we would have:

$$\begin{aligned}
\text{there exists } c &> 0 \text{ such that } \rho S(\rho x) > S(x) \text{ for every } x > c \\
&\Rightarrow \int_c^\infty \rho S(\rho x)dx > \int_c^\infty S(x)dx \\
&\Rightarrow \frac{1}{\rho}\int_c^\infty S(x)dx < \int_c^\infty S(\rho x)dx \\
\int_c^\infty S(\rho x)dx &= \frac{1}{\rho}\int_c^\infty S(\rho x)\rho dx \\
&= \frac{1}{\rho}\int_{\rho c}^\infty S(z)dz \\
\rho &> 1 \Rightarrow \frac{1}{\rho}\int_{\rho c}^\infty S(z)dz < \frac{1}{\rho}\int_c^\infty S(z)dz \\
&\Rightarrow \frac{1}{\rho}\int_c^\infty S(x)dx < \frac{1}{\rho}\int_c^\infty S(z)dz \Rightarrow\Leftarrow
\end{aligned}$$

This contradiction completes the proof. ■

Remark 15 *Appendix A considers the implications of the existence of the limit* $\lim_{x\to\infty}\frac{S(\rho x)}{S(x)}$. *The discussion shows that if you assume that the limit* $\lim_{x\to\infty}\frac{S(\rho x)}{S(x)}$ *exists for all* $\rho > 1$ *and is not identically* 0 *for all* $\rho > 1$, *then the tail behavior is essentially determined up to just a single parameter. More precisely, consider the one-parameter survival function:*

$$T(\beta;x)=\left\{\begin{array}{cc}1 & x\leq 1\\ x^{-\beta} & x>1\end{array}\right\}.$$

For $T(\beta;x)$ *such limits exist and are particularly manageable as we clearly have*

$$\rho,\beta,x\geq 1\Rightarrow\frac{T(\beta;\rho x)}{T(\beta;x)}=\frac{(\rho x)^{-\beta}}{x^{-\beta}}=\rho^{-\beta}=\lim_{y\to\infty}\frac{T(\beta;\rho y)}{T(\beta;y)}.$$

It turns out that for a loss variable X *with* $S=S_X$ *and for which there exist* $\rho_k>1, k\in\mathbb{N}$ *such that* $\lim_{k\to\infty}\rho_k=1$ *and* $\lim_{x\to\infty}\frac{S(\rho_k x)}{S(x)}$ *exists for every* $k\in\mathbb{N}$, *then for all* $\rho>1$:

$$\begin{aligned}\textit{either}\ \lim_{x\to\infty}\frac{S(\rho x)}{S(x)} &= 0\\ \textit{or}\ \lim_{x\to\infty}\frac{S(\rho x)}{S(x)} &= \rho^{-\beta}\ \textit{where}\ \beta=-\ln\left(\lim_{x\to\infty}\frac{S(ex)}{S(x)}\right)\geq 1.\end{aligned}$$

We see that under these assumptions, the conditional probability of survival $\frac{S(y)}{S(x)}$ *for* $y>x$ *and* x *large is asymptotically the same as that of* $T(\beta;x)$ *for some unique* β, *with* $1\leq\beta\leq\infty$.

Example 16 *For the "thin-tailed" exponential density* $S(x)=e^{-\frac{x}{\theta}}$ *we have, for any constant* $\rho>1$, *that*

$$\lim_{x\to\infty}\frac{S(\rho x)}{S(x)}=\lim_{x\to\infty}\frac{e^{-\frac{\rho x}{\theta}}}{e^{-\frac{x}{\theta}}}=\lim_{x\to\infty}e^{-\frac{(\rho-1)x}{\theta}}=0.$$

Example 17 *For the "thicker tailed" Pareto density* $S(x)=\left(\frac{\theta}{\theta+x}\right)^{\alpha}$ *we have, for any constant* $\rho>1$, *that*

$$\lim_{x\to\infty}\frac{S(\rho x)}{S(x)}=\lim_{x\to\infty}\frac{\left(\frac{\theta}{\theta+\rho x}\right)^{\alpha}}{\left(\frac{\theta}{\theta+x}\right)^{\alpha}}=\lim_{x\to\infty}\left(\frac{\theta+x}{\theta+\rho x}\right)^{\alpha}=\rho^{-\alpha}.$$

Example 18 *This example shows that the inequality in Proposition 14 cannot, in general, be improved. Consider the survival function:*

$$\begin{aligned}
S(x) &= \frac{e}{(x+e)(\ln(x+e))^2} \\
\mu &= \int_0^\infty S(x)dx = \int_e^\infty \frac{e}{u(\ln(u))^2}du \text{ where } u = x+e \\
&= e\int_1^\infty \frac{1}{w^2}dw \text{ where } w = \ln(u) \\
&= e\left[-\frac{1}{w}\right]_1^\infty = e < \infty
\end{aligned}$$

with finite mean. We have, with several applications of L'Hôpital:

$$\begin{aligned}
\lim_{x\to\infty}\frac{S(\rho x)}{S(x)} &= \lim_{x\to\infty}\left(\frac{e}{(\rho x+e)(\ln(\rho x+e))^2}\frac{(x+e)(\ln(x+e))^2}{e}\right) \\
&= \lim_{x\to\infty}\left(\left(\frac{x+e}{\rho x+e}\right)\frac{(\ln(x+e))^2}{(\ln(\rho x+e))^2}\right) \\
&= \lim_{x\to\infty}\left(\frac{x+e}{\rho x+e}\right)\lim_{x\to\infty}\left(\frac{\ln(x+e)}{\ln(\rho x+e)}\right)^2 \\
&= \frac{1}{\rho}\left(\lim_{x\to\infty}\frac{\ln(x+e)}{\ln(\rho x+e)}\right)^2 \\
&= \frac{1}{\rho}\left(\lim_{x\to\infty}\frac{\frac{1}{x+e}}{\frac{\rho}{\rho x+e}}\right)^2 \\
&= \frac{1}{\rho}\left(\lim_{x\to\infty}\frac{\rho x+e}{\rho x+\rho e}\right)^2 \\
&= \frac{1}{\rho}.
\end{aligned}$$

Example 19 *Define the function:*

$$h(x) = \left\{ \begin{array}{cc} x & 0 \le x < 1 \\ 2x-1 & 1 \le x < 2 \\ 3 & 2 \le x < 4 \\ 1+\frac{x}{2} & 4 \le x < 8 \\ \vdots & \vdots \\ k+1 & 2^{k-1} \le x < 2^k \text{ and } k > 1 \text{ even} \\ k-2+\frac{x}{2^{k-2}} & 2^{k-1} \le x < 2^k \text{ and } k > 1 \text{ odd} \end{array} \right\}$$

then the reader can readily verify that h is continuous and non-decreasing with $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. It follows that $S(x) = e^{-h(x)}$ is a survival function. Let X be a nonnegative random variable with $S = S_X$. The reader can verify the following:

$$\begin{aligned} h(4x) &= h(x) + 2 \text{ for } x > 2 \\ h(2^k) &= \left\{ \begin{array}{cc} k+2 & k > 1 \text{ odd} \\ k+1 & k > 1 \text{ even} \end{array} \right\} \end{aligned}$$

And we find that for $x > 2$:

$$\begin{aligned} \frac{S(4x)}{S(x)} &= \frac{e^{-h(4x)}}{e^{-h(x)}} = e^{h(x)-h(4x)} = e^{-2} \\ &\Rightarrow \lambda(4) = \lim_{x\to\infty} \frac{S(4x)}{S(x)} = \frac{1}{e^2}. \end{aligned}$$

Since $\lambda(4) = \frac{1}{e^2} < \frac{1}{4}$ it is at least possible for this distribution to have a finite mean; and indeed, the reader can readily verify that:

$$\begin{aligned} x &> 2 \Rightarrow h(x) \ge \frac{\ln x}{\ln 2} - 1 \\ &\Rightarrow S(x) \le e x^{-\frac{1}{\ln 2}} \\ &\Rightarrow \int_2^\infty S(x)dx \le \int_2^\infty e x^{-\frac{1}{\ln 2}} dx = e \left(\frac{2^{1-\frac{1}{\ln 2}}}{\frac{1}{\ln 2} - 1} \right) < \infty \\ &\Rightarrow \mu_X = \int_0^\infty S(x)dx < \infty \end{aligned}$$

and we see that X is a loss variable. Observe that:

$$\begin{aligned}
h(2^k) &= \left\{ \begin{array}{cc} h(2^{k-1})+2 & k>1 \text{ odd} \\ h(2^{k-1}) & k>1 \text{ even} \end{array} \right\} \\
\Rightarrow & \frac{S(2\cdot 2^{k-1})}{S(2^{k-1})} = \frac{S(2^k)}{S(2^{k-1})} = \frac{e^{-h(2^k)}}{e^{-h(2^{k-1})}} \\
&= e^{h(2^{k-1})-h(2^k)} = \left\{ \begin{array}{cc} e^{-2} & k>1 \text{ odd} \\ 1 & k>1 \text{ even} \end{array} \right\} \\
\Rightarrow & \lim_{x\to\infty} \frac{S(2x)}{S(x)} \text{ fails to exist.}
\end{aligned}$$

Finally, observe that should $\lim_{x\to\infty}\frac{S(4x)}{S(x)}$ exist, that is not sufficient to guarantee that $\lim_{x\to\infty}\frac{S(\beta x)}{S(x)}$ exists for $\beta > 4$. Indeed, setting $x_k = \frac{2^k}{5}$ we have:

$$\begin{aligned}
h(5x_k) - h(x_k) &= \left\{ \begin{array}{cc} \frac{14}{5} & k>3 \text{ odd} \\ 2 & k>3 \text{ even} \end{array} \right\} \\
\Rightarrow & \frac{S(5x_k)}{S(x_k)} = e^{-h(5x_k)+h(x_k)} = \left\{ \begin{array}{cc} e^{-\frac{14}{5}} & k>3 \text{ odd} \\ e^{-2} & k>3 \text{ even} \end{array} \right\} \\
\Rightarrow & \lim_{x\to\infty} \frac{S(5x)}{S(x)} \text{ fails to exist.}
\end{aligned}$$

This example is meant to provide some additional insight into the nature of the assumption made in the very special case considered in the above remark, namely that $\lim_{x\to\infty}\frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$.

The two limits $\lim_{x\to\infty}\widehat{h}(x) \geq \frac{\nu}{\mu} \geq 1$ (Propositions 11 and 13) and $\lim_{x\to\infty}\frac{S(\frac{\nu}{\mu}x)}{S(x)} = \lim_{x\to\infty}\frac{S(\nu x)}{S(\mu x)} \leq \frac{\mu}{\nu} \leq 1$ play a key role in determining the sign of $\delta(r)$ for large enough entry ratio r, as demonstrated in the following:

Proposition 20 *In the case of the differential frequency trend model $g(x) = \widehat{h}(x)f(x)$, as defined above, assume that $M_X = \infty$, that h is differentiable on $(0,\infty)$ (except at perhaps finitely many points) and that there exists $c > 0$ with $\frac{dh}{dx} = 0$ for all $x \geq c$. Let $\rho = \frac{\nu}{\mu}$ and assume that the limit $\lambda = \lim_{x\to\infty}\frac{S(\rho x)}{S(x)}$ exists. Then*

$$\begin{aligned}
\lambda\widehat{h}(c) &> 1 \Rightarrow \text{there exists } b>0 \text{ such that } E_Y(r) > E_X(r) \text{ for all } r \geq b \\
\lambda\widehat{h}(c) &< 1 \Rightarrow \text{there exists } b>0 \text{ such that } E_Y(r) < E_X(r) \text{ for all } r \geq b.
\end{aligned}$$

Proof. To compare $E_Y(r)$ and $E_X(r)$ for large entry ratios, we again investigate the derivative of $\delta(r)$:

$$\begin{aligned}\frac{d\delta}{dr} &= G(\nu r) - F(\mu r)\\ &= \int_0^{\nu r} g(x)dx - \int_0^{\mu r} f(x)dx\\ &= \int_{\mu r}^{\nu r} \widehat{h}(x)f(x)dx + \int_0^{\mu r} \left(\widehat{h}(x) - 1\right) f(x)dx\end{aligned}$$

Observe that the first integral is always ≥ 0 and converges to 0 as $r \to \infty$ and that the second integral is an increasing function of r for r large enough to force $\widehat{h}(\mu r) > 1$ and the second integral also converges to 0 as $r \to \infty$. Let $r > \frac{c}{\mu}$, our assumptions together with $\frac{dh}{dx} \geq 0$, give us:

$$\begin{aligned}G(\nu r) &= \widehat{h}(\nu r)F(\nu r) - \int_0^{\nu r} F(x)\frac{d\widehat{h}}{dx}dx\\ &= \widehat{h}(c)F(\nu r) - \int_0^{c} F(x)\frac{d\widehat{h}}{dx}dx\\ &= \widehat{h}(c)F(\nu r) - \gamma \text{ for some constant } \gamma \geq 0.\end{aligned}$$

Taking the limit as $r \to \infty$:

$$\begin{aligned}1 &= \widehat{h}(c) - \gamma\\ 1 - \widehat{h}(c) &= -\gamma\\ G(\nu r) &= \widehat{h}(c)F(\nu r) - \gamma\\ &= \widehat{h}(c)F(\nu r) + 1 - \widehat{h}(c)\\ &= -\widehat{h}(c)\left(1 - F(\nu r)\right) + 1\\ &\Rightarrow \frac{d\delta}{dr} = -\widehat{h}(c)\left(1 - F(\nu r)\right) + 1 - F(\mu r)\\ &= -\widehat{h}(c)S(\nu r) + S(\mu r).\end{aligned}$$

Now suppose $\lambda\widehat{h}(c) > 1$:

$$\begin{aligned}
\lambda\widehat{h}(c) &> 1 \Rightarrow \lim_{x\to\infty}\frac{\widehat{h}(c)S(\rho x)}{S(x)} > 1 \\
\text{there exists } b &> 0 \text{ such that } \widehat{h}(c)S(\rho x) > S(x) \text{ for every } x \geq \mu b \\
\rho &= \frac{\nu}{\mu} \Rightarrow \widehat{h}(c)S(\nu r) > S(\mu r) \text{ for every } \mu r \geq \mu b \\
&\Rightarrow \widehat{h}(c)S(\nu r) > S(\mu r) \text{ for every } r \geq b \\
&\Rightarrow -\widehat{h}(c)S(\nu r) < -S(\mu r) \text{ for every } r \geq b \\
&\Rightarrow \frac{d\delta}{dr} = -\widehat{h}(c)S(\nu r) + S(\mu r) < 0 \text{ for every } r \geq b.
\end{aligned}$$

And it follows that $\delta(r)$ is decreasing for $r \geq b$. Since $\delta(r) \to 0$ as $r \to \infty$ it follows that $E_Y(r) - E_X(r) = \delta(r) > 0$ for $r \geq b$. We have established:

$$\begin{aligned}
\lambda\widehat{h}(c) &> 1 \Rightarrow \text{there exists } b > 0 \text{ such that} \\
E_Y(r) - E_X(r) &= \delta(r) > 0 \Rightarrow E_Y(r) > E_X(r) \text{ for all } r \geq b.
\end{aligned}$$

Reversing inequalities in the above argument shows:

$$\begin{aligned}
\lambda\widehat{h}(c) &< 1 \Rightarrow \text{there exists } b > 0 \text{ such that} \\
E_Y(r) - E_X(r) &= \delta(r) < 0 \Rightarrow E_Y(r) < E_X(r) \text{ for all } r \geq b
\end{aligned}$$

completing the proof. ■

An immediate consequence is that distributions with an infinite but comparatively thin tail act like distributions with finite support:

Corollary 21 *In the case of the differential frequency trend model $g(x) = \widehat{h}(x)f(x)$, as defined above, assume that $M_X = \infty$, that h is differentiable on $(0,\infty)$ (except at perhaps finitely many points), that there exists $c > 0$ with $\frac{dh}{dx} = 0$ for $x \geq c$, and further that $\lim_{x\to\infty}\frac{S((\frac{\nu}{\mu})x)}{S(x)} = 0$. Then there exists $b > 0$ such that $E_Y(r) < E_X(r)$ for all $r \geq b$.*

Example 22 *As a general example of a differential freqency trend model we may take $h = F$, then $g(x) = aF(x)f(x)$ for a uniquely determined constant a. But clearly F^2 is itself a distribution function and setting:*

$$\begin{aligned}
G &= F^2 \\
&\Rightarrow \frac{dG}{dx} = 2F\frac{dF}{dx} = 2Ff \textit{ is a PDF} \Rightarrow a = 2
\end{aligned}$$

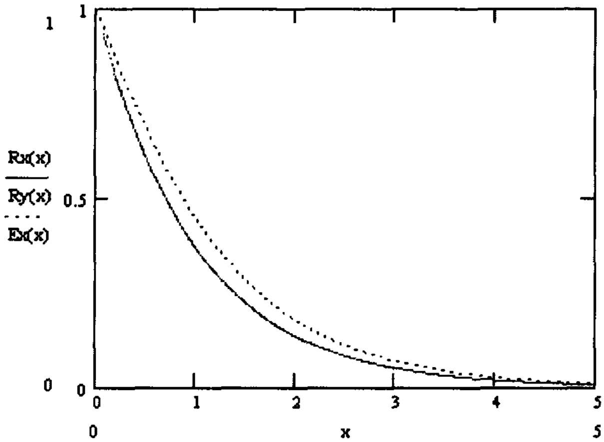
and the increase in the mean is:

$$\begin{aligned}
\nu - \mu &= \int_0^\infty (1 - G(y))dy - \int_0^\infty S(y)dy \\
&= \int_0^\infty (1 - F(y)^2)dy - \int_0^\infty S(y)dy \\
&= \int_0^\infty (1 - F(y))(1 + F(y))dy - \int_0^\infty S(y)dy \\
&= \int_0^\infty S(y)(1 + F(y))dy - \int_0^\infty S(y)dy \\
&= \int_0^\infty S(y)(1 + F(y) - 1)dy \\
&= \int_0^\infty S(y)F(y)dy.
\end{aligned}$$

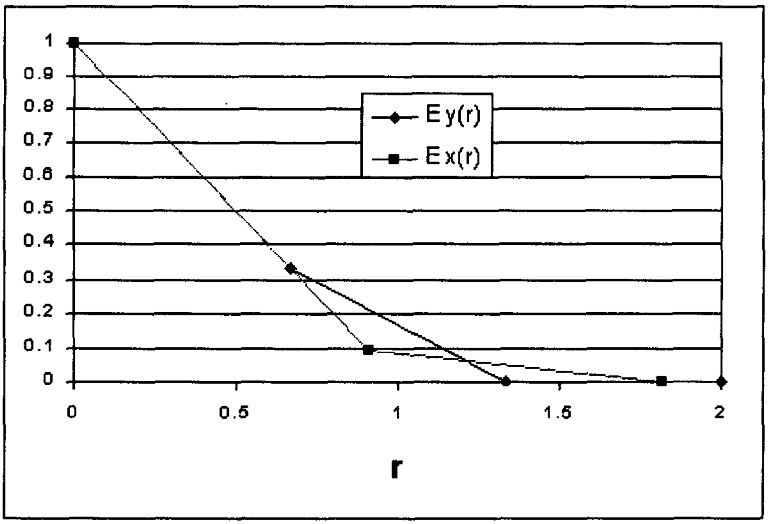
Example 23 *Let X be an exponential denisty with $f(x) = e^{-x}$ and set $h(x) = \frac{x}{x+1}$. Then from numerical integration applied directly to the definitions:*

$$\begin{aligned}
E[h(X)] &= 0.404 \\
\mu &= 1 \\
\nu &= 1.477
\end{aligned}$$

The following graphs the excess ratio functions $R_X(x) = E_X(x), R_Y(x)$, and $E_Y(x)$; from the graph we see that: $E_Y(x) < E_X(x) = R_X(x) < R_Y(x)$.



Example 24 *Consider the case of* 10 *losses per year:* 9 *of amount* 1 *and* 1 *of amount* 2 *and let* X *denote the corresponding random variable. Suppose there is a decline in frequency to a rate of just* 2 *losses per year:* 1 *of amount* 1 *and* 1 *of amount* 2 *with random variable* Y. *The following graphs the excess ratio functions* $E_X(r)$, *and* $E_Y(r)$. *In this case we see that* $E_Y(1) > E_X(1)$ *and* $E_Y(1.5) < E_X(1.5) > 0$.



Example 25 *Consider a Pareto density with survival function* $S(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$ *and a linear frequency decline of the form* $h(x) = Min\left(\frac{x+d}{c+d}, 1\right)$. *We provide the results of a direct evaluation via numerical methods for two cases:*

$$\begin{aligned} \theta &= 2, \alpha = 5, c = 2, d = 1 \\ \mu &= 0.5, \nu \approx 0.695, \rho = \frac{\nu}{\mu} \approx 1.39 \\ \frac{1}{\lambda} &= \lim_{x\to\infty} \frac{S(x)}{S(\rho x)} \approx 5.16 > 2.04 \approx \widehat{h}(c) \\ E_Y(x) &< E_X(x) \end{aligned}$$

and:

$$\begin{aligned} \theta &= 2, \alpha = 5, c = 10, d = 5 \\ \mu &= 0.5, \nu \approx 0.575, \rho = \frac{\nu}{\mu} \approx 1.149 \\ \frac{1}{\lambda} &= \lim_{x\to\infty} \frac{S(x)}{S(\rho x)} \approx 1.978 < 2.728 \approx \widehat{h}(c) \\ E_Y(x) &> E_X(x). \end{aligned}$$

In both cases, Proposition 17 holds for any $b > 0$. *This gives an instance for which the same untrended loss variable and two functions for h, both of linear frequency decline proportions with the same range of* $\left[\frac{1}{3}, 1\right]$, *can produce opposite sign impacts on the normalized excess ratio function.*

As to the rAB table for this differential frequency trend model, as before let

$$r_0 = 0 < r_1 < r_2 < \cdots < r_M$$

be a sequence of entry ratios and set

$$A_i = A_X(r_i), B_i = B_X(r_i), 0 \leq i \leq M.$$

Suppose that $A_i = A_X(r_i) > A_X(r_{i-1}), 1 \leq i \leq M$ and $A_M = 1$. Set $\Delta A_i = A_i - A_{i-1}, \Delta B_i = B_i - B_{i-1}$. Again note that $\mu\frac{\Delta B_i}{\Delta A_i}, 1 \leq i \leq M$, is the mean value of the untrended loss over the interval $[\mu r_{i-1}, \mu r_i]$, which we assume can be taken as an estimate for the mean of the trended loss. This would hold provided that, within sufficiently narrow entry ratio layers, the removed claims (and whence the retained) are representative of all claims in that layer. This would hold exactly, for example, in case the function h is a step function that is constant on the intervals $[r_{i-1}, r_i)$. For $1 \leq i \leq M$, set

$$\begin{aligned}
\widetilde{A}_i &= \widehat{h}(\mu r_i)A_i,\ \Delta\widetilde{A}_i = \widetilde{A}_i - \widetilde{A}_{i-1} \\
\Delta\widetilde{B}_i &= \Delta\widetilde{A}_i\left(\mu\frac{\Delta B_i}{\Delta A_i}\right) \\
\widetilde{B}_i &= \sum_{k=1}^{i}\Delta\widetilde{B}_k.
\end{aligned}$$

Assuming then that $\mu\frac{\Delta B_i}{\Delta A_i}$ is a good estimate of the mean value of the trended losses on the interval $[\mu r_{i-1}, \mu r_i]$, we have:

$$\begin{aligned}
\frac{\widetilde{B}_M}{\widetilde{A}_M} &= \sum_{k=1}^{M}\frac{\Delta\widetilde{B}_k}{\widetilde{A}_M} = \sum_{k=1}^{M}\frac{\Delta\widetilde{A}_k}{\widetilde{A}_M}\left(\mu\frac{\Delta B_k}{\Delta A_k}\right) \\
&\approx \sum_{k=1}^{M}\Pr\left(\mu r_{k-1} < Y \leq \mu r_k\right)\left(\mu\frac{\Delta B_k}{\Delta A_k}\right) \\
&\approx \sum_{k=1}^{M}\Pr\left(\mu r_{k-1} < Y \leq \mu r_k\right)E[Y\ |\mu r_{k-1} < Y \leq \mu r_k] \\
&= E[Y] = \nu
\end{aligned}$$

and we infer, as before, that $\nu \approx \frac{\widetilde{B}_M}{\widetilde{A}_M}$ and that the two sequences $\left\{\widetilde{A}_i\right\}$ and $\left\{\widetilde{B}_i\right\}$ are nearly equal to the cumulative cases and losses of not necessarily normalized trended losses. So they only need to be rescaled to give the A and B columns of the trended losses. Whence they are very nearly proportional to the A and B columns of the entry ratio for the trended losses (and albeit with different proportionality constants). So setting:

$$\widehat{r}_i = \frac{\mu r_i}{\frac{\widetilde{B}_M}{\widetilde{A}_M}} = \frac{\mu r_i\widetilde{A}_M}{\widetilde{B}_M},\ \widehat{A}_i = \frac{\widetilde{A}_i}{\widetilde{A}_M},\ \widehat{B}_i = \frac{\widetilde{B}_i}{\widetilde{B}_M},\ 0 \leq i \leq M$$

we have approximated the rAB table for the trended losses: $rAB_Y \approx \widehat{rAB}$. Finally, note that this simple three-step differential frequency trend adjustment to the rAB table (adjust A, estimate B, renormalize r, A, and B) can be done quite generally to account for a change in frequency by size of loss and does not formally demand that $\frac{dh}{dx} > 0$ on $(0, \infty)$, although order preserving is needed to justify the calculation.

5.1 WC Case Study of Differential Frequency

The tables for excess ratios in WC are produced by five types of WC injury: Fatal, Permanent Total Disability [PTD], Permanent Partial Disability [PPD], Temporary Total Disaility [TTD], and medical only [MO]. The WC system in the US has seen a persistent decline in claim frequency over the past 10-15 years. The decline is observed within each of the injury types and over the spectrum of US industries. There is no consensus on how long this pattern can persist, or even on its underlying causes. One pattern that has emerged, both in NCCI investigations as well as from studies by the Department of Labor, is that this decline has not been uniform by size of loss. Small WC claims have declined proportionally more than have large WC claims. That is the motivation for this look at how differential frequency trend impacts entry ratio tables.

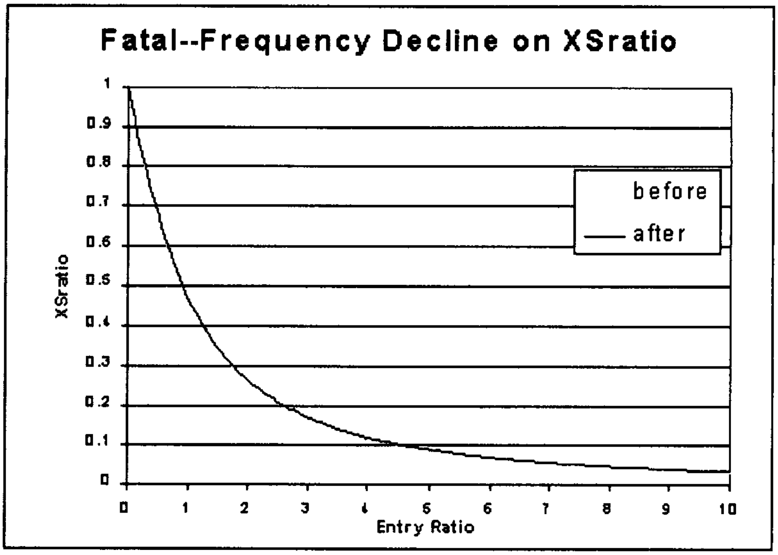
A recent NCCI study produced the following table of percentage changes in claim frequency (per unit of wage-adjusted payroll exposure):

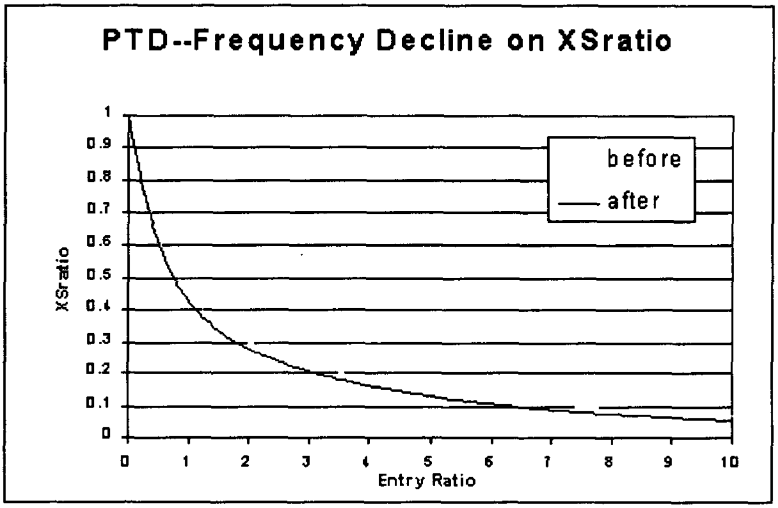
	Fatal	PTD	PPD	TTD	MO
Smallest third of claims	−6.2%	−52.4%	−23.7%	−32.8%	−26.7%
Middle third of claims	−7.9%	−18.5%	−12.8%	−20.4%	−29.9%
Largest third of claims	−10.3%	4.3%	−8.7%	−8.5%	−13.8%

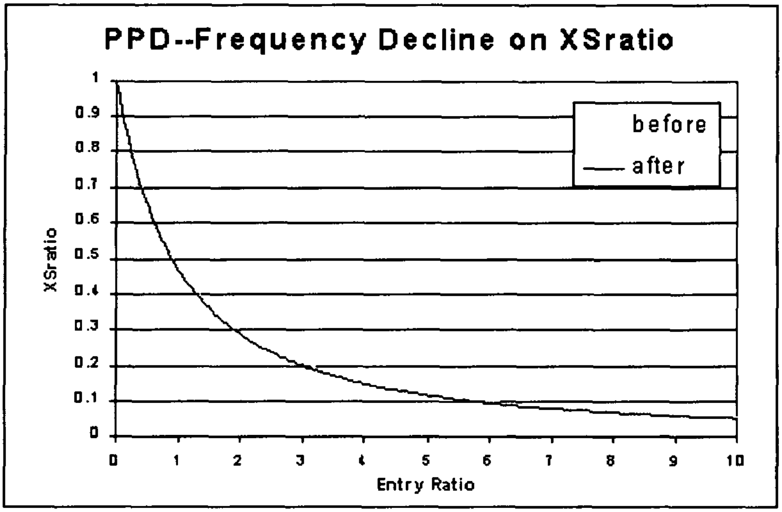
With the exceptions of the fatal and medical only injury types, the table conforms to the by now familiar pattern of a smaller decline in frequency with increasing claim size. These percent changes were used to define a proportional change in frequency function $h_i(r)$ as a step function of entry ratio r for each injury type i. Even a smoothed version of $h_i(r)$ would not likely conform to the differential frequency trend model assumptions for injury types Fatal $[i = 1]$ and Medical Only$[i = 5]$:

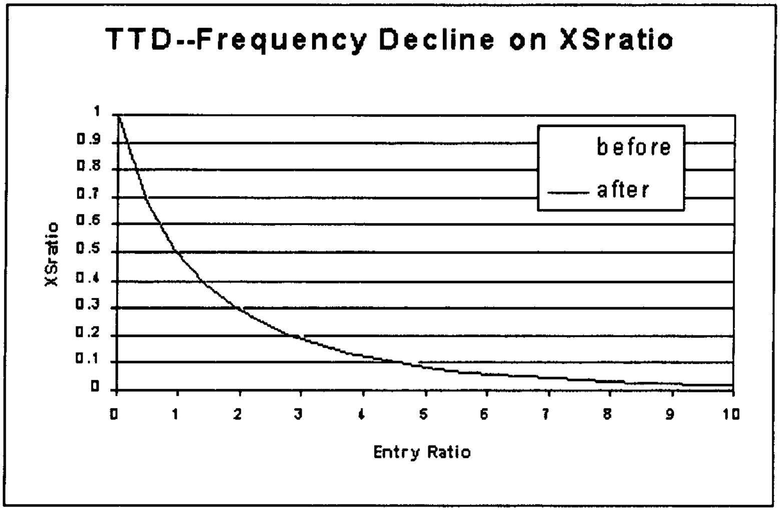
Range of r	$h_1(r)$	$h_2(r)$	$h_3(r)$	$h_4(r)$	$h_5(r)$
$0 \leq A(r) < \frac{1}{3}$	0.9382	0.476	0.7628	0.6718	0.7329
$\frac{1}{3} \leq A(r) < \frac{2}{3}$	0.9211	0.8151	0.8723	0.7957	0.7014
$\frac{2}{3} \leq A(r) \leq 1$	0.8967	1.043	0.9134	0.9151	0.8624

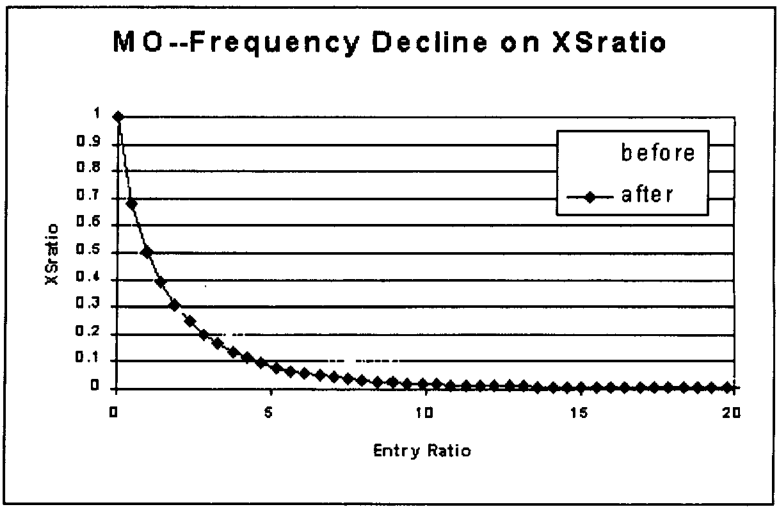
Even though the assumptions of the differential frequency trend model are technically not met in this case study, the discussion still makes it clear how to determine, for each injury type, a trended entry ratio table from the untrended table. The graphs below show the excess ratio functions $E_{X_i}(r)$, and $E_{Y_i}(r)$ by injury type i before (X_i) and after (Y_i) trend. With the exceptions of the fatal and medical only injury types, we again see that $E_Y(r) - E_X(r) \leq 0$. For each injury type except perhaps medical only, the two curves are very close, which indicates that little or no frequency trend adjustment to the rAB table is indicated.




PTD--Frequency Decline on XSratio
XSratio
Entry Ratio
before
after
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10


PPD--Frequency Decline on XSratio
XSratio
Entry Ratio
before
after
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10





As in the earlier case study, it is straightforward to combine differential frequency trend impacts by injury type into a combined impact on the normalized excess ratio.

6 Matching the Mean and Median Loss

Suppose we are presented with an entry ratio table rAB_X together with some constant $\varepsilon \neq 0$, we next discuss how to build the entry ratio table rAB_{X^ε}. Here we consider the trended random variable to be $Y = \psi(X) = X^\varepsilon$ where the transformation $\psi(x) = x^\varepsilon$ has $\frac{d\psi}{dx} = \varepsilon x^{\varepsilon-1}$ and is order preserving for $\varepsilon > 0$ and order reversing for $\varepsilon < 0$. Thus, as we did for differential severity trend, we have:

$$\begin{aligned} G(\psi(x)) &= \Pr(Y \leq \psi(x)) \\ &= \Pr(\psi(X) \leq \psi(x)) = \left\{ \begin{array}{l} \Pr(X \leq x) = F(x) \Leftrightarrow \varepsilon > 0 \\ \Pr(X \geq x) = S(x) \Leftrightarrow \varepsilon < 0 \end{array} \right\}. \end{aligned}$$

Let $r_0 = 0 < r_1 < r_2 < \cdots < r_M$ be a sequence of entry ratios and set

$$A_i = A_X(r_i), B_i = B_X(r_i), 0 \leq i \leq M.$$

As before, suppose that $A_i = A_X(r_i) > A_X(r_{i-1}), 1 \leq i \leq M$ and $A_M = 1$. Set $\Delta A_i = A_i - A_{i-1}, \Delta B_i = B_i - B_{i-1}$. Note that $\mu\frac{\Delta B_i}{\Delta A_i}, 1 \leq i \leq M$, is the mean value of the untrended loss over the interval $[\mu r_{i-1}, \mu r_i]$. For $1 \leq i \leq M$, set

$$\begin{aligned} \Delta\widetilde{B}_i &= \Delta A_i \left(\psi\left(\mu\frac{\Delta B_i}{\Delta A_i}\right)\right) = \Delta A_i \left(\mu\frac{\Delta B_i}{\Delta A_i}\right)^\varepsilon \\ \widetilde{B}_i &= \sum_{k=1}^{i} \Delta\widetilde{B}_k. \end{aligned}$$

Assuming, as usual, that $\left(\mu\frac{\Delta B_i}{\Delta A_i}\right)^\varepsilon$ is a good estimate of the mean value of the trended losses within the interval $\left[\mu^\varepsilon r_{i-1}^\varepsilon, \mu^\varepsilon r_i^\varepsilon\right]$ leads to the familiar estimate $\nu \approx \widetilde{B}_M$ and, as before, the two sequences $\{A_i\}$ and $\left\{\widetilde{B}_i\right\}$ approximate the cumulative claim and loss percentages of the trended losses. A change of scale to normalize the trended losses corresponds to adjusting the two sequences $\{A_i\}$ and $\left\{\widetilde{B}_i\right\}$ by constant factors. So the sequences are very nearly proportional to the A and B columns of the entry ratio for the trended losses. Setting:

$$\widehat{r}_i = \frac{\mu^\varepsilon r_i^\varepsilon}{\widetilde{B}_M},\ \widehat{A}_i = A_i, \text{ and } \widehat{B}_i = \frac{\widetilde{B}_i}{\widetilde{B}_M},\ 0 \leq i \leq M$$

we have approximated the rAB table for the trended losses: $rAB_Y = rAB_{X^\varepsilon} \approx \widehat{rAB}$.

Now abstract from this and suppose only that you are provided an entry ratio table Θ in the form of three finite increasing sequences of M numbers:

$$\begin{aligned} r_0 &= 0 < r_1 < r_2 < \cdots < r_M \\ A_0 &= 0 < A_1 < A_2 < \cdots < A_M = 1 \\ B_0 &= 0 < B_1 < B_2 < \cdots < B_M = 1 \end{aligned}$$

We will assume that these table values were constructed using some loss variable X and so Θ at least conforms to the properties of an entry ratio table. Given $\varepsilon > 0$ we can formally construct a new entry ratio by mimicking the above and assuming, with no loss of generality, that $\mu_X = 1$. For $1 \le i \le M$, set $\Delta A_i = A_i - A_{i-1}$ and $\Delta B_i = B_i - B_{i-1}$ and define

$$\begin{aligned} \Delta\widetilde{B}_i &= \Delta A_i \left(\frac{\Delta B_i}{\Delta A_i}\right)^{\varepsilon} \\ \widetilde{B}_i &= \sum_{k=1}^{i} \Delta\widetilde{B}_k. \end{aligned}$$

And construct a new table $\widehat{\Theta}$ from the increasing sequences:

$$\widehat{r}_i = \frac{r_i^\varepsilon}{\widetilde{B}_M}, \ \widehat{A}_i = A_i, \ \text{and } \widehat{B}_i = \frac{\widetilde{B}_i}{\widetilde{B}_M}, \ 0 \le i \le M.$$

The significance of this construction for adapting entry ratio tables to changing conditions will become clear from the following:

Proposition 26 *Let $1 \le x_1 < x_2 < \cdots < x_M$ be an increasing sequence of $M > 1$ numbers. Then for any fixed number w with $0 < w < 1$ and integer k, $1 \le k < M$, there exist $\alpha, \beta > 0$ such that setting $y_i = \alpha {x_i}^\beta$ we have:*

$$\frac{1}{M}\sum_{i=1}^{M} y_i = 1 \quad \textit{and} \quad y_k = w$$

Proof. Let $z_i = \frac{x_i}{x_k}, 1 \le i \le M$ and define $\varphi(v) = \frac{1}{M}\sum_{i=1}^{M} z_i^v$ then φ is a continuous function of v and invoking the Intermediate Value Theorem[IVT]:

$$\begin{aligned}
\varphi(0) &= 1 \\
z_M &> 1 \Rightarrow \lim_{v \to \infty} \varphi(v) = \infty \\
1 &< \frac{1}{w} < \infty, \text{IVT} \Rightarrow \text{there exists } \beta > 0 \text{ such that } \varphi(\beta) = \frac{1}{w}
\end{aligned}$$

Now set $\alpha = \frac{w}{x_k^\beta}$, then we have:

$$\begin{aligned}
y_i &= \alpha {x_i}^\beta = \frac{w}{x_k^\beta} {x_i}^\beta = w z_i^\beta, 1 \le i \le M \\
&\Rightarrow \frac{1}{M} \sum_{i=1}^{M} y_i = \frac{w}{M} \sum_{i=1}^{M} z_i^\beta \\
&= w\varphi(\beta) = \frac{w}{w} = 1 \quad \text{and} \quad y_k = w z_k^\beta = w 1^\beta = w
\end{aligned}$$

completing the proof. ∎

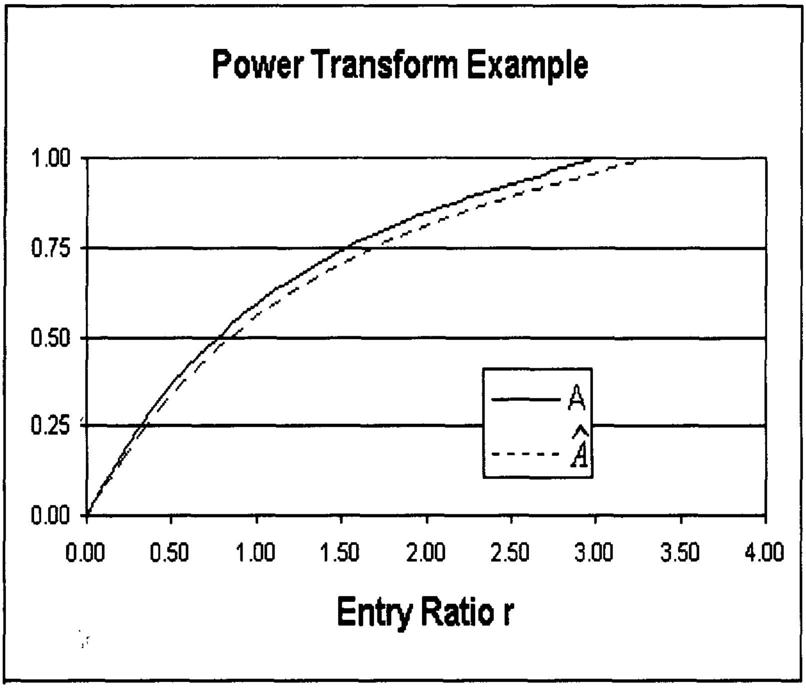
This means that, quite generally, for discrete loss data the power transform $Y = \alpha X^\beta$ enables us not only to normalize to mean 1 but also to simultaneously specify the entry ratio $w(= r)$ of any selected percentile $\frac{k}{M}(= A(r))$. As a very general example, suppose you are provided an rAB table and some loss data with random variable X. Suppose further that you observe a median $= m$ and mean $= \mu$, so the observed entry ratio of the median $= \frac{m}{\mu}$. Now suppose further that in the given rAB table you observe that $A(\frac{m}{\mu})$ is well removed from $\frac{1}{2}$. This suggests to you that the given rAB table may not be suited to the task, say, of looking up excess ratios $R_X(x)$ for the given loss data. Now assume that the given entry ratio table rAB has $A(w) = \frac{1}{2}$ for some $w < 1$—this is not unreasonable since loss distributions typically have median less than the mean. From Proposition 23, there is a power transform $Y = \alpha X^\beta$ whose median has entry ratio equal to w. But this, in turn, suggests that the given entry ratio table rAB may be suitable as an entry ratio table for the transformed losses $Y = \alpha X^\beta$, i.e. $rAB \approx rAB_Y$, inasmuch as the transformed losses have the ratio of mean to median implicit in the table. While a power transform may not be the exact relationship for how losses trend, it is reasonable to assume some structural relationship between the given rAB table and the given losses. By Proposition 5, $R_X(x) = R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}} x)$ and we find that all we require to customize the table lookup of excess ratios is an entry ratio table for $Y^{\frac{1}{\beta}}$. But the above discussion provides an algorithm for determining the entry ratio table of a power transform. So let $\widehat{rAB}$ be determined, as above, from the original rAB table under the power transformation $\varepsilon = \frac{1}{\beta}$, then $\widehat{rAB} \approx rAB_{Y^{\epsilon}} = rAB_{Y^{\frac{1}{\beta}}}$. This enables us to look up the excess ratio $R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}}x)$. Finally, note that all this simplifies to the usual process of looking up the entry ratio of the loss limit, but in the adjusted entry ratio table:

$$\begin{aligned}
Y &= \alpha X^{\beta} \Rightarrow Y^{\frac{1}{\beta}} = \alpha^{\frac{1}{\beta}} X \Rightarrow \mu_{Y^{\frac{1}{\beta}}} = \alpha^{\frac{1}{\beta}} \mu_X \\
&\Rightarrow R_X(x) = R_{Y^{\frac{1}{\beta}}}(\alpha^{\frac{1}{\beta}}x) = E_{Y^{\frac{1}{\beta}}}\left(\frac{\alpha^{\frac{1}{\beta}}x}{\mu_{Y^{\frac{1}{\beta}}}}\right) \\
&= E_{Y^{\frac{1}{\beta}}}\left(\frac{\alpha^{\frac{1}{\beta}}x}{\alpha^{\frac{1}{\beta}}\mu_X}\right) = E_{Y^{\frac{1}{\beta}}}\left(\frac{x}{\mu_X}\right) \\
&\Rightarrow R_X(x) \approx \widehat{E}\left(\frac{x}{\mu_X}\right)
\end{aligned}$$

So to summarize, this example illustrates a general technique to deal with the case in which "trend" has impacted the shape of the severity distribution as evidenced by a change in the relationship between the mean and the median loss. In fact, the discussion details how to "trend" the old entry ratio table, rAB, to a new table $\widehat{rAB}$.

The challenge with this approach comes in finding α and β. At first, it would seem to require a calculation involving the complete loss variable X, or at least a very robust and representative claim subsample. And such calculation (the proof of Propostion 23 coupled with a binary search algorithm might prove useful), if doable at all, would suggest that direct calculation of the excess ratio, or even an entirely new rAB table, may be more practical. However, notice that only β is required to construct $\widehat{rAB}$ from rAB and it is a straightforward spreadsheet application to try different values for β until the resulting $\widehat{rAB}$ satisfies $\widehat{A}(w) = \frac{1}{2}$. This approach may well provide a β that works even when $w \geq 1$ and the technique can be applied equally well to other percentiles than the median. Consequently, the technique is both general and constructive.

Example 27 *This example considers an entry ratio table rAB (columns r,A,B) that reflects a loss distribution for which the median is about $\frac{4}{5}^{th}$ of* *the mean. Assume that later data revealed that the entry ratio of the median loss had grown from* 0.8 *to* 0.85. *A power transform with* $\beta = 2$ *is illustrated. Appendix B includes the table and displays a trended entry ratio table* $\widehat{rAB}$ *(columns* $\widehat{r}$, $\widehat{A}$, $\widehat{B}$*) which may better fit the newer data. The following chart shows the corresponding change in the normalized cumulative distribution function, from* $A \rightarrow \widehat{A}$*:*



We just saw how a calculation similar to that of the differential severity trend approach can adapt the rAB table to a power transform $Y = \alpha X^{\beta}$. We conclude this section by describing how the set up of the differential frequency trend calculation can adapt the rAB table to a proportional hazard transform $S_Y = (S_X)^{\alpha}$. In the notation used for differential frequency trend, we have:

$$g(x) = -\frac{dS_Y}{dx} = \alpha S_X(x)^{\alpha-1} f(x) \Rightarrow \widehat{h}(x) = \alpha S_X(x)^{\alpha-1}.$$

Now abstract from this as above and suppose again that you are provided an entry ratio table Θ in the form of three finite increasing sequences of M

numbers:

$$\begin{aligned} r_0 &= 0 < r_1 < r_2 < \cdots < r_M \\ A_0 &= 0 < A_1 < A_2 < \cdots < A_M = 1 \\ B_0 &= 0 < B_1 < B_2 < \cdots < B_M = 1. \end{aligned}$$

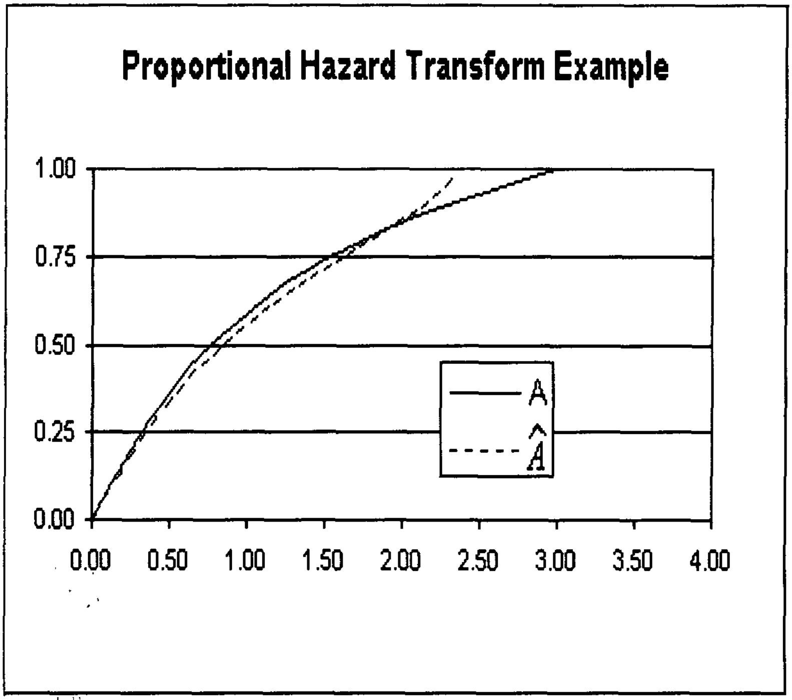
Given $\alpha > 0$ we can formally construct a new entry ratio table by employing the three-step process for the frequency differential trend, again assuming for convenience and with no loss of generality that the mean of the loss variable of the given table is 1. Set $\Delta A_i = A_i - A_{i-1}, \Delta B_i = B_i - B_{i-1}$ and define

$$\begin{aligned} \widetilde{A}_i &= 1 - (1 - A_i)^{\alpha}, \Delta\widetilde{A}_i = \widetilde{A}_i - \widetilde{A}_{i-1},\ 0 \le i \le M \\ \Delta\widetilde{B}_i &= \Delta\widetilde{A}_i \left(\frac{\Delta B_i}{\Delta A_i}\right) \\ \widetilde{B}_i &= \sum_{k=1}^{i} \Delta\widetilde{B}_k, 1 \le i \le M \end{aligned}$$

From which we construct a new table $\widehat{\Theta}$ from the increasing sequences:

$$\widehat{r}_i = \frac{r_i \widetilde{A}_M}{\widetilde{B}_M},\ \widehat{A}_i = \frac{\widetilde{A}_i}{\widetilde{A}_M},\ \widehat{B}_i = \frac{\widetilde{B}_i}{\widetilde{B}_M},\ 0 \le i \le M$$

Example 28 *This example begins with the same entry ratio table rAB as the previous example. A proportional hazard transform $\alpha = \frac{5}{7}$, selected to again adjust the median to an entry ratio of about 0.85–the table is included in Appendix.B. The following chart shows the corresponding change in the normalized cumulative distribution function, from $A \rightarrow \widehat{A}$:*



The two examples illustrate the rather different ways in which the power transform (which bears a formal similarity with the differential severity trend set up) and the proportional hazard transform (which bears a formal similarity with the differential frequency trend set up) achieve raising the relativity of the median to the mean loss. The power transform disproportionately increases the larger losses, including increasing the maximum loss amount from 3 to around 3.3, so that proportionally fewer losses above 0.8 are needed for an overall mean = 1. By contrast, the proportional hazard adjustment removes the largest losses, including dropping the maximum loss amount from 3 to about 2.3, forcing the smaller losses to increase in order to maintain an overall mean = 1. Accordingly, it is advisable to consider the impact of trend on the largest losses when selecting a trend adjustment to update an entry ratio table.

It is also worth comparing what the WC case studies suggest in regard to the justification for trending an entry ratio table. Medical inflation has outstripped overall wage growth very consistently and the reasons why are well understood. Also, WC medical coverage is not subject to the statutory limitations imposed on wage-replacement benefits. Finally, in the case of excess ratios, the direction of the change in the tablular values is consistent and readily explained. So in the case of differential severity trend, there is a strong argument to be made that the underlying dynamics are persistent.

The case of differential frequency trend provides a contrast. The decline in WC claim frequency, while persistent over the past decade, is neither readily explained nor well understood. Experts disagree on whether the decline will, or even can, continue. While no one is surprised that medical inflation outstrips wage growth, the observation that the WC frequency decline is greater for smaller claims is a fairly new and a largely unforseen observation. In the case of excess ratios and differential frequency trend, the direction of the change in the tablular values is neither consistent nor straightforward. While the dymanics of differential severity trend are extemely unlikely to reverse, that cannot be said for differential frequency trend.

As with any trend adjustment, there is the concern that missing turning points will result in trend adjustments leading to worse estimates rather than better estimates. This is especially so when the direction of the numerical change is itself problematic. In the case of entry ratio tables, there is a built in correction for short term changes in severity that works very well. And so any "trend" adjustment must be justified over a long time window as improving the estimate. This study suggests that while a fairly strong argument can be made for incorporating the differential severity trend adjustment to WC entry ratio tables, the case is much weaker for differential frequency trend.

7 Conclusion

In the case of a differential severity trend in which large losses trend upward faster (slower) than do smaller losses, the use of an entry ratio table assumes an average trend which corresponds with a severity distribution whose tail is not thickening (thinning) in response to the non-uniform trend. Ideally, the normalized excess ratios from the rAB table should be increased (decreased) to offset this.

In the case of a differential frequency trend in which the frequency of small losses declines faster (slower) than for large losses, the impact of the frequency decline on the mean severity is leveraged. Over the range of attachment points, the use of an untrended entry ratio table may sometimes overstate or sometimes understate the change in the excess ratio.

The two models described here, the differential severity trend and differential frequency trend scenarios, are meant to act independently of one another. Differential severity trend assumes that all trend is due to inflationary movement and none is due to a change in claim emergence. Differential frequency trend holds loss amounts fixed while applying a proportional change in the density. Therefore, it is perhaps not too surprising that while both act to increase the mean severity, they can impact the normalized excess ratio in opposite directions and may offset one another when updating an entry ratio table.

Another very general technique that can be used to accomodate a non-uniform trend is to use a power transformation or a proportional hazard transformation, in lieu of just dividing by the mean loss when performing the lookup into the entry ratio table. The technique provides another way to trend an entry ratio table. More precisely, the ratio between the mean loss and a fixed percentile loss may be observed to change over time. And this calculation gives a way to periodically modify the entry ratio table to accomodate that movement.

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APPENDIX A

In this appendix we invoke the notation and assumptions of the Differential Frequency Trend (section 4) of the main paper and let X be a loss variable with survival function $S(x)$ for which $M_X = \infty$. We consider the implications of the assumption that the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$. Proposition 14 of the paper gives:

Proposition 29 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$, then for any $\rho > 1$ for which the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists:*

$$\lambda(\rho) \le \frac{1}{\rho} < 1.$$

Note that when the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists:

$$\begin{aligned}
\lim_{x\to\infty} \frac{S(\rho^2 x)}{S(x)} &= \lim_{x\to\infty} \frac{S(\rho^2 x)}{S(\rho x)} \frac{S(\rho x)}{S(x)} \\
&= \lim_{x\to\infty} \frac{S(\rho\,(\rho x))}{S(\rho x)} \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} \\
&= \left(\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}\right)^2 \\
&\Rightarrow \lambda(\rho^2) = \lambda(\rho)^2.
\end{aligned}$$

More generally, we have:

Proposition 30 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$, then for any $m \in \mathbb{N}$, if the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists then*

$$\lambda(\rho^m) = \lim_{x\to\infty} \frac{S(\rho^m x)}{S(x)} = \lambda(\rho)^m.$$

Proof. The verification is a straightforward induction, the result has been observed to hold for $m = 1, 2$, we have:

$$\begin{aligned}
\lim_{x\to\infty} \frac{S(\rho^{m+1} x)}{S(x)} &= \lim_{x\to\infty} \frac{S(\rho^{m+1} x)}{S(\rho^m x)} \frac{S(\rho^m x)}{S(x)} \\
&= \lim_{x\to\infty} \frac{S(\rho\,(\rho^m x))}{S(\rho^m x)} \lim_{x\to\infty} \frac{S(\rho^m x)}{S(x)} \\
&= \lambda(\rho)\lambda(\rho^m) \\
&= \lambda(\rho)\lambda(\rho)^m = \lambda(\rho)^{m+1}
\end{aligned}$$

completing the induction and the proof. ∎

When such limits all exist, this generalizes to:

Proposition 31 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$, and assume that the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$. Then*

$$\lambda(\rho)^\omega = \lambda(\rho^\omega) \text{ for any positive real number } \omega.$$

Proof. Observe that since the limit $\lambda(\rho^{\frac{1}{n}}) = \lim_{x\to\infty} \frac{S(\rho^{\frac{1}{n}} x)}{S(x)}$ is assumed to exist, we must have:

$$\begin{aligned}
\frac{S(\rho x)}{S(x)} &= \frac{S\left(\rho^{\frac{1}{n}}\left(\rho^{\frac{n-1}{n}}x\right)\right)}{S\left(\rho^{\frac{n-1}{n}}x\right)} \frac{S\left(\rho^{\frac{1}{n}}\left(\rho^{\frac{n-2}{n}}x\right)\right)}{S\left(\rho^{\frac{n-2}{n}}x\right)} \cdots \frac{S\left(\rho^{\frac{1}{n}}\left(\rho^{\frac{n-n}{n}}x\right)\right)}{S\left(\rho^{\frac{n-n}{n}}x\right)} \\
&\Rightarrow \lim_{x\to\infty} \left(\frac{S\left(\rho^{\frac{1}{n}}x\right)}{S(x)}\right)^n = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = \lambda(\rho) \\
&\Rightarrow \lim_{x\to\infty} \frac{S\left(\rho^{\frac{1}{n}}x\right)}{S(x)} = \lambda(\rho)^{\frac{1}{n}}.
\end{aligned}$$

But then for any positive integers m, n we have:

$$\lambda(\rho^{\frac{m}{n}}) = \lim_{x\to\infty} \frac{S(\rho^{\frac{m}{n}}x)}{S(x)} = \lim_{x\to\infty} \frac{S(\left(\rho^{\frac{1}{n}}\right)^m x)}{S(x)} = \left(\lambda(\rho)^{\frac{1}{n}}\right)^m = \lambda(\rho)^{\frac{m}{n}}.$$

Whence $\lambda(\rho^a) = \lambda(\rho)^a$ for any positive rational a. Now let ω be a positive real, then there are sequences of positive rationals:

$$\begin{aligned}
a_k, b_k &\in \mathbb{Q}, k \in \mathbb{N} \\
\text{such that } 0 &< a_1, a_k \leq a_{k+1}, b_k \geq b_{k+1} \\
\text{and with } \lim_{k\to\infty} a_k &= \lim_{k\to\infty} b_k = \omega.
\end{aligned}$$

This clearly forces $a_k \leq \omega \leq b_k$ and since S is a continuous, non-increasing function, we have:

$$\begin{aligned}
a_k &\leq \omega \leq b_k \Rightarrow \rho^{a_k} \leq \rho^{\omega} \leq \rho^{b_k} \\
&\Rightarrow \rho^{a_k} x \leq \rho^{\omega} x \leq \rho^{b_k} x \text{ for all } x > 0 \\
&\Rightarrow S\left(\rho^{a_k} x\right) \geq S\left(\rho^{\omega} x\right) \geq S\left(\rho^{b_k} x\right) \text{ for all } x > 0 \\
&\Rightarrow \frac{S\left(\rho^{a_k} x\right)}{S(x)} \geq \frac{S\left(\rho^{\omega} x\right)}{S(x)} \geq \frac{S\left(\rho^{b_k} x\right)}{S(x)} \text{ for all } x > 0 \\
&\Rightarrow \lim_{x\to\infty} \frac{S\left(\rho^{a_k} x\right)}{S(x)} \geq \lim_{x\to\infty} \frac{S\left(\rho^{\omega} x\right)}{S(x)} \geq \lim_{x\to\infty} \frac{S\left(\rho^{b_k} x\right)}{S(x)} \\
&\Rightarrow \lambda(\rho)^{a_k} = \lambda(\rho^{a_k}) \geq \lambda(\rho^{\omega}) \geq \lambda(\rho^{b_k}) = \lambda(\rho)^{b_k} \\
\lambda(\rho)^{\omega} &= \lambda(\rho)^{\lim_{k\to\infty} a_k} \geq \lambda(\rho^{\omega}) \geq \lambda(\rho)^{\lim_{k\to\infty} b_k} = \lambda(\rho)^{\omega} \\
&\Rightarrow \lambda(\rho)^{\omega} = \lambda(\rho^{\omega})
\end{aligned}$$

and we see that $\lambda(\rho)^{\omega} = \lambda(\rho^{\omega})$ for any positive real number ω, completing the proof. ■

An immediate consequence is:

Corollary 32 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$, and assume that the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$ Then*

1. there exists $\rho > 1$ such that $\lambda(\rho) = 0 \Leftrightarrow \lambda(\rho) = 0$ for every $\rho > 1$
2. there exists $\rho > 1$ such that $\lambda(\rho) \neq 0 \Leftrightarrow \lambda(\rho) \neq 0$ for every $\rho > 1$.

Consider the one-parameter survival function:

$$\begin{aligned}
T(x) &= T(\beta; x) = \left\{ \begin{array}{cc} 1 & x \leq 1 \\ x^{-\beta} & x > 1 \end{array} \right\} \\
\rho, \beta, x &\geq 1 \Rightarrow \frac{T(\rho x)}{T(x)} = \frac{(\rho x)^{-\beta}}{x^{-\beta}} = \rho^{-\beta} = \lim_{y\to\infty} \frac{T(\rho y)}{T(y)}
\end{aligned}$$

Note that $T(\beta; x)$ has a finite mean if and only if $\beta > 1$. By convention, we include the (discontinuous) possibility that $\beta = \infty$ by setting $T(\infty; x) = x^{-\infty} = 0$ for $x > 1$.

Proposition 33 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$ and assume that the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$. Then*

$$\begin{aligned}
\lambda(\rho) &= \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = \rho^{-\beta} \\
\textit{for all } \rho &> 1, \textit{ where } \beta = -\ln(\lambda(e)) \geq 1.
\end{aligned}$$

Proof. Consider first the case when there is some $\rho_0 > 1$ such that $\lambda(\rho_0) \neq 0$. Then from Proposition 29 and Corollary 32 we find that $\lambda(e) \in (0,1)$. Then for any real $\rho > 1$ we have:

$$\begin{aligned} \lambda(\rho) &= \lambda(e^{\ln \rho}) = \lambda(e)^{\ln \rho} = \left(e^{-\beta}\right)^{\ln \rho} = \left(e^{\ln \rho}\right)^{-\beta} = \rho^{-\beta} \\ \text{where } \lambda(e) &= e^{-\beta} \Leftrightarrow \beta = -\ln(\lambda(e)) \end{aligned}$$

and since by Proposition 29:

$$\lambda(e) \leq \frac{1}{e} \Rightarrow e \leq \frac{1}{\lambda(e)} \Rightarrow 1 = \ln(e) \leq \ln\left(\frac{1}{\lambda(e)}\right) = -\ln(\lambda(e)) = \beta$$

the result follows in this case. For the remaining case $\lambda(\rho) = 0$ for all $\rho > 1$ we have from Corollary 32, with minimally abusive notation and our conventions:

$$\begin{aligned} -\ln(\lambda(e)) &= -\ln(0) = \infty \\ &\Rightarrow \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = 0 = \rho^{-\infty} \text{ for all } \rho > 1 \end{aligned}$$

and the result holds in this case as well. The proof is complete. ■

Corollary 34 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$ and assume that the limit $\lambda(\rho) = \lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho > 1$ and further that there is some $\rho_0 > 1$ such that $\lambda(\rho_0) \neq 0$. Then*

$$\begin{aligned} \lambda(\rho) &= \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = \lim_{x\to\infty} \frac{T(\beta;\rho x)}{T(\beta;x)} = \rho^{-\beta} \\ \textit{for all } \rho &> 1, \textit{ where } 1 \leq \beta = -\ln(\lambda(e)) < \infty. \end{aligned}$$

Proposition 35 *Let X be a loss variable with $M_X = \infty$ and $S = S_X$, then the following are equivalent:*

1. $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ *exists for all* $\rho > 1$
2. *there exist* $\rho_k > 1, k \in \mathbb{N}$ *such that* $\lim_{k\to\infty} \rho_k = 1$ *and* $\lim_{x\to\infty} \frac{S(\rho_k x)}{S(x)}$ *exists for every* $k \in \mathbb{N}$.

Proof. It is apparent that $1 \Rightarrow 2$. To establish the meaningful direction $2 \Rightarrow 1$, we begin with the claim that:

$\alpha_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \alpha_k = 0 \Rightarrow \{m\alpha_k | k, m \in \mathbb{N}\}$ is dense in $[0, \infty)$.

Indeed, given $\epsilon > 0, b \in (0, \infty)$:

$$\lim_{k\to\infty} \alpha_k = 0 \Rightarrow \exists k \in \mathbb{N} \ni 0 < \alpha_k < \frac{\epsilon}{2}$$

and setting

$$\begin{aligned} b_m &= m\alpha_k \Rightarrow b_{m+1} - b_m = \alpha_k < \frac{\epsilon}{2} \\ &\Rightarrow \text{ there exists } m \in \mathbb{N} \text{ such that } b_m \in (b - \epsilon, b + \epsilon) \\ &\Rightarrow m\alpha_k \in (b - \epsilon, b + \epsilon), k, m \in \mathbb{N} \end{aligned}$$

Since this holds for any $\epsilon > 0$, it follows that $\{m\alpha_k | k, m \in \mathbb{N}\}$ is dense in $[0, \infty)$ as claimed. And since the log function $\ln : [1, \infty) \to [0, \infty)$ is bicontinuous and bijective, we see that

$\rho_k > 1$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \rho_k = 1 \Rightarrow \{\rho_k^m | k, m \in \mathbb{N}\}$ is dense in $[1, \infty)$.

Now we have our assumption:

$$\begin{aligned} \text{there exist } \rho_k &> 1, k \in \mathbb{N} \text{ such that} \\ \lim_{k\to\infty} \rho_k &= 1 \text{ and } \lim_{x\to\infty} \frac{S(\rho_k x)}{S(x)} \text{ exists for every } k \in \mathbb{N} \end{aligned}$$

and we select any $\rho > 1$ and seek to prove that this assumption is sufficient to imply that the limit $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exits. So assume, by way of contradiction, that $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ does not exist. We have, by density:

$$\begin{aligned} \text{there exist } a_k, b_k &\in \{\rho_l^m | l, m \in \mathbb{N}\}, k \in \mathbb{N} \\ \text{such that } 1 &< a_1, a_k \leq a_{k+1}, a_k \leq \rho \text{ and with } \lim_{k\to\infty} a_k = \rho \\ \text{and such that } b_k &\geq b_{k+1}, b_k \geq \rho \text{ and with } \lim_{k\to\infty} b_k = \rho. \end{aligned}$$

Now S is a continuous, non-increasing function on $[0, \infty)$ and so we have:

$$\begin{aligned} a_k &\leq \rho \leq b_j \text{ for all } j, k \in \mathbb{N} \\ &\Rightarrow a_k x \leq \rho x \leq b_j x \text{ for all } x > 0, \text{ and for all } j, k \in \mathbb{N} \\ &\Rightarrow S(a_k x) \geq S(\rho x) \geq S(b_j x) \text{ for all } x > 0, \text{ and for all } j, k \in \mathbb{N} \\ &\Rightarrow \frac{S(a_k x)}{S(x)} \geq \frac{S(\rho x)}{S(x)} \geq \frac{S(b_j x)}{S(x)} \text{ for all } x > 0, \text{ and for all } j, k \in \mathbb{N}. \end{aligned}$$

Consider the two sets:

$$\begin{aligned} A &= \left\{ \lim_{x\to\infty} \frac{S(a_k x)}{S(x)}, k \in \mathbb{N} \right\} \\ B &= \left\{ \lim_{x\to\infty} \frac{S(b_k x)}{S(x)}, k \in \mathbb{N} \right\} \end{aligned}$$

The above inequalities clearly force:

$$\beta \leq \alpha \leq 1 \quad \text{for all } \alpha \in A \text{ and for all } \beta \in B.$$

Observe that by Proposition 29:

$$\begin{aligned} \lim_{x\to\infty} \frac{S(a_k x)}{S(x)} &\leq \frac{1}{a_k} < 1 \\ &\Rightarrow \alpha < 1 \text{ for all } \alpha \in A. \end{aligned}$$

We also have, for any $k \in \mathbb{N}$, that:

$$\begin{aligned} a_k &\leq \rho \\ &\Rightarrow a_k x \leq \rho x \text{ for all } x > 0 \\ &\Rightarrow S(a_k x) \geq S(\rho x) \text{ for all } x > 0 \\ &\Rightarrow \frac{S(a_k x)}{S(x)} \geq \frac{S(\rho x)}{S(x)} \text{ for all } x > 0 \\ &\Rightarrow \lim_{x\to\infty} \frac{S(a_k x)}{S(x)} \geq \frac{S(\rho x)}{S(x)} \geq 0 \text{ for all } x > 0. \end{aligned}$$

We claim that $\lim_{x\to\infty} \frac{S(a_k x)}{S(x)} > 0$ for every $k \in \mathbb{N}$, since otherwise:

$$\lim_{x\to\infty} \frac{S(a_k x)}{S(x)} = 0 \Rightarrow \text{existence of the limit } \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = 0 \Rightarrow\Leftarrow .$$

And we established:

$$\begin{aligned} 0 &< \alpha < 1 \text{ for all } \alpha \in A \\ &\Rightarrow A \subset (0,1). \end{aligned}$$

Now set:

$$\alpha = \inf A, \beta = \sup B$$

then clearly $0 \leq \beta \leq \alpha \leq 1$. We claim that:

$$\alpha = \beta.$$

Indeed, suppose, again by way of contradiction, that $\alpha \neq \beta$. Then we would have:

$$\beta < \alpha.$$

Now

$$A \subset (0,1) \Rightarrow \text{ there exists } c \in \{a_k | k \in \mathbb{N}\} \text{ with } 1 > \gamma = \lim_{x\to\infty} \frac{S(cx)}{S(x)} > 0$$

and we have, for any given $\epsilon > 0$:

$$\begin{aligned}
1 &> \gamma > 0 \Rightarrow \exists n \in \mathbb{N} \text{ such that } \gamma^{\frac{1}{n}} > 1 - \epsilon \\
\lim_{k\to\infty} \rho_k &= 1 \Rightarrow \exists m \in \mathbb{N} \text{ such that } \rho_m < c^{\frac{1}{n}} \\
&\Rightarrow \rho_m^n < c \\
&\Rightarrow \rho_m^n x < cx \text{ for all } x > 0 \\
&\Rightarrow S\left(\rho_m^n x\right) \geq S\left(cx\right) \text{ for all } x > 0 \\
&\Rightarrow \frac{S\left(\rho_m^n x\right)}{S\left(x\right)} \geq \frac{S\left(cx\right)}{S\left(x\right)} \text{ for all } x > 0 \\
&\Rightarrow \lim_{x\to\infty} \frac{S(\rho_m^n x)}{S(x)} \geq \lim_{x\to\infty} \frac{S\left(cx\right)}{S(x)} = \gamma \\
&\Rightarrow \left(\lim_{x\to\infty} \frac{S(\rho_m x)}{S(x)}\right)^n = \lim_{x\to\infty} \frac{S(\rho_m^n x)}{S(x)} \geq \gamma \\
\text{Proposition 29} &\Rightarrow 1 > \frac{1}{\rho_m} \geq \lim_{x\to\infty} \frac{S(\rho_m x)}{S(x)} \geq \gamma^{\frac{1}{n}} > 1 - \epsilon
\end{aligned}$$

And we have established:

$$\begin{aligned}
\text{For any given } \epsilon &> 0 \text{ there exists } \varphi_1 > 1 \\
&\text{such that the limit } \lim_{x\to\infty} \frac{S(\varphi_1 x)}{S(x)} \text{ exists and is in } (1 - \epsilon, 1).
\end{aligned}$$

We next claim that:

$$\text{There exists } \varphi_2 > 1 \text{ such that } \lim_{x\to\infty} \frac{S(\varphi_2 x)}{S(x)} \in (\beta, \alpha).$$

Let $\beta_1 = \frac{\alpha+\beta}{2}$, then clearly $0 < \beta_1 < \alpha$ and $(\beta_1, \alpha) \subseteq (\beta, \alpha)$. Now let $\delta = \ln\alpha - \ln\beta_1 > 0$. Then letting $\epsilon = 1 - e^{-\delta}$ we have $\epsilon > 0$ and so by an earlier claim there exists $\varphi_1 > 1$ such that

$$\begin{aligned}\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)} &\in (1-\epsilon, 1)\\ &\Rightarrow \ln\left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right) \in (-\delta, 0).\end{aligned}$$

Set $\eta = -\ln\left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right)$, then:

$$\begin{aligned}-\eta &= \ln\left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right) \in (-\delta, 0) = (\ln\beta_1 - \ln\alpha, 0), 0 < \eta < \delta\\ \left|\frac{\ln\beta}{-\eta} - \frac{\ln\alpha}{-\eta}\right| &= \frac{\ln\alpha}{\eta} - \frac{\ln\beta}{\eta} = \frac{\delta}{\eta}\left(\frac{\ln\alpha - \ln\beta}{\delta}\right) > \frac{\ln\alpha - \ln\beta}{\delta} = 1\\ &\Rightarrow \text{there exists } l \in \mathbb{N} \text{ such that } l \in \left(\frac{\ln\alpha}{-\eta}, \frac{\ln\beta_1}{-\eta}\right)\\ &\Rightarrow -\eta l \in (\ln\beta_1, \ln\alpha) \Rightarrow e^{-\eta l} \in (\beta_1, \alpha) \subseteq (\beta, \alpha)\end{aligned}$$

and it follows that, setting $\varphi_2 = \varphi_1^l$ we have:

$$\begin{aligned}e^{-\eta l} &= e^{l\ln\left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right)} = \left(e^{\ln\left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right)}\right)^l\\ &= \left(\lim_{x\to\infty}\frac{S(\varphi_1 x)}{S(x)}\right)^l = \lim_{x\to\infty}\frac{S(\varphi_1^l x)}{S(x)} = \lim_{x\to\infty}\frac{S(\varphi_2 x)}{S(x)}\\ &\Rightarrow \lim_{x\to\infty}\frac{S(\varphi_2 x)}{S(x)} = e^{-\eta l} \in (\beta, \alpha)\end{aligned}$$

and the claim is established. Recalling how α and β were defined, we have:

$$\begin{aligned}\lim_{x\to\infty}\frac{S(\varphi_2 x)}{S(x)} &\in (\beta, \alpha) = (\sup B, \inf A)\\ &\Rightarrow \lim_{x\to\infty}\frac{S(b_j x)}{S(x)} < \lim_{x\to\infty}\frac{S(\varphi_2 x)}{S(x)} < \lim_{x\to\infty}\frac{S(a_k x)}{S(x)}, \forall j, k \in \mathbb{N}\end{aligned}$$

and we also have that:

$$\begin{aligned}
a_k &> \varphi_2 \Rightarrow \frac{S\left(a_k x\right)}{S(x)} \leq \frac{S\left(\varphi_2 x\right)}{S(x)} \\
&\Rightarrow \lim_{x\to\infty} \frac{S\left(a_k x\right)}{S(x)} \leq \lim_{x\to\infty} \frac{S\left(\varphi_2 x\right)}{S(x)} < \alpha = \inf A \Rightarrow\Leftarrow \\
b_j &< \varphi_2 \Rightarrow \frac{S\left(b_j x\right)}{S(x)} \geq \frac{S\left(\varphi_2 x\right)}{S(x)} \\
&\Rightarrow \lim_{x\to\infty} \frac{S\left(b_j x\right)}{S(x)} \geq \lim_{x\to\infty} \frac{S\left(\varphi_2 x\right)}{S(x)} > \beta = \sup B \Rightarrow\Leftarrow
\end{aligned}$$

and we are lead to:

$$a_k \leq \varphi_2 \leq b_j \text{ for all } k, j \in \mathbb{N}.$$

But this, in turn, leads to

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \rho = \lim_{k\to\infty} b_k \Rightarrow \varphi_2 = \rho \\
&\Rightarrow \text{ existence of the limit } \lim_{x\to\infty} \frac{S\left(\varphi_2 x\right)}{S(x)} = \lim_{x\to\infty} \frac{S\left(\rho x\right)}{S(x)} \Rightarrow\Leftarrow
\end{aligned}$$

and with this contradiction we have established our claim that that $\alpha = \beta$. Now by the definition of the set A and $\alpha = \inf A$ we find that for any given $\epsilon > 0$:

$$\begin{aligned}
\text{there exists } k_1 &\in \mathbb{N} \text{ such that } \alpha + \frac{\epsilon}{2} > \lim_{x\to\infty} \frac{S(a_{k_1} x)}{S(x)} \\
&\Rightarrow \text{ there exists } x_1 > 0 \text{ such that} \\
\alpha + \epsilon &> \frac{S\left(a_{k_1} x\right)}{S(x)} \geq \frac{S\left(\rho x\right)}{S(x)}, \text{ for every } x \geq x_1 \\
&\Rightarrow \text{ there exists } x_1 > 0 \text{ such that} \\
\alpha + \epsilon &> \frac{S\left(\rho x\right)}{S(x)}, \text{ for every } x \geq x_1.
\end{aligned}$$

And similarly, by the definition of the set B with $\alpha = \beta = \sup B$, we find that for any given $\epsilon > 0$

$$\begin{aligned}
\text{there exists } k_2 &\in \mathbb{N} \text{ such that } \beta - \frac{\epsilon}{2} < \lim_{x\to\infty} \frac{S(b_{k_2}x)}{S(x)} \\
&\Rightarrow \text{ there exists } x_2 > 0 \text{ such that} \\
\beta - \epsilon &< \frac{S\left(b_{k_2}x\right)}{S(x)} \le \frac{S\left(\rho x\right)}{S(x)}, \text{ for every } x \ge x_2 \\
&\Rightarrow \text{ there exists } x_2 > 0 \text{ such that} \\
\alpha - \epsilon &= \beta - \epsilon < \frac{S\left(\rho x\right)}{S(x)}, \text{ for every } x \ge x_2.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\text{given any } \epsilon &> 0, \text{there exists } x_3 > 0 \text{ such that} \\
\left\{\frac{S\left(\rho x\right)}{S(x)}, x \ge x_3\right\} &\subseteq (\alpha - \epsilon, \alpha + \epsilon) \\
&\Rightarrow \text{ the limit } \lim_{x\to\infty} \frac{S(\rho x)}{S(x)} = \alpha \text{ exists } \Rightarrow\Leftarrow
\end{aligned}$$

and this final contradiction establishes that the limit $\lim_{x\to\infty} \frac{S(\rho x)}{S(x)}$ exists for all $\rho \ge 1$ and completes the proof. ■

APPENDIX B

Power Transform Example ($\beta = 2$)

r	A	B	ΔA	ΔB	$\tilde{r}$	$\Delta\tilde{B}$	$\tilde{B}$	$\hat{r}$	$\hat{A}$	$\hat{B}$
0	0	0	0	0	0	0	0	0	0	0
0.1	0.082907	0.004145	0.082907	0.004145	0.316228	0.01853862	0.018539	0.110346	0.082907	0.020457
0.2	0.163781	0.016276	0.080874	0.012131	0.447214	0.03132221	0.049861	0.220692	0.163781	0.05502
0.3	0.236026	0.034338	0.072245	0.018061	0.547723	0.03612254	0.085983	0.331039	0.236026	0.094879
0.4	0.303221	0.057856	0.067195	0.023518	0.632456	0.03975295	0.125736	0.441385	0.303221	0.138745
0.5	0.363143	0.084821	0.059922	0.026965	0.707107	0.04019696	0.165933	0.551731	0.363143	0.183101
0.6	0.417715	0.114835	0.054572	0.030015	0.774597	0.04047166	0.206405	0.662077	0.417715	0.22776
0.7	0.467217	0.147012	0.049502	0.032177	0.83666	0.03991004	0.246315	0.772424	0.467217	0.271799
0.8	0.512363	0.180871	0.045146	0.033859	0.894427	0.03909732	0.285412	0.88277	0.512363	0.314942
0.9	0.554109	0.216355	0.041746	0.035484	0.948683	0.03848772	0.3239	0.993116	0.554109	0.357411
1	0.591878	0.252236	0.03777	0.035881	1	0.03681339	0.360713	1.103462	0.591878	0.398034
1.1	0.626693	0.288792	0.034815	0.036555	1.048809	0.03567449	0.396388	1.213809	0.626693	0.437399
1.2	0.658777	0.325688	0.032084	0.036897	1.095445	0.03440628	0.430794	1.324155	0.658777	0.475365
1.3	0.688675	0.363061	0.029898	0.037373	1.140175	0.03342733	0.464222	1.434501	0.688675	0.512251
1.4	0.716234	0.400266	0.027559	0.037204	1.183216	0.03202034	0.496242	1.544847	0.716234	0.547584
1.5	0.741513	0.43692	0.025279	0.036655	1.224745	0.03044021	0.526682	1.655193	0.741513	0.581174
1.6	0.765774	0.474524	0.024261	0.037604	1.264911	0.03020417	0.556886	1.76554	0.765774	0.614503
1.7	0.788177	0.51149	0.022404	0.036966	1.30384	0.02877787	0.585664	1.875886	0.788177	0.646258
1.8	0.809417	0.54866	0.02124	0.037169	1.341641	0.02809741	0.613762	1.986232	0.809417	0.677263
1.9	0.828932	0.584763	0.019515	0.036103	1.378405	0.02654347	0.640305	2.096578	0.828932	0.706552
2	0.847016	0.620025	0.018084	0.035263	1.414214	0.02525227	0.665557	2.206925	0.847016	0.734417
2.1	0.864609	0.656092	0.017594	0.036067	1.449138	0.02519015	0.690747	2.317271	0.864609	0.762214
2.2	0.88129	0.691955	0.01668	0.035863	1.48324	0.02445823	0.715206	2.427617	0.88129	0.789202
2.3	0.897579	0.728605	0.016289	0.03665	1.516575	0.02443335	0.739639	2.537963	0.897579	0.816164
2.4	0.912953	0.764736	0.015375	0.036131	1.549193	0.02356912	0.763208	2.64831	0.912953	0.842171
2.5	0.927842	0.801212	0.014888	0.036476	1.581139	0.02330379	0.786512	2.758656	0.927842	0.867886
2.6	0.942066	0.837484	0.014224	0.036272	1.612452	0.02271451	0.809226	2.869002	0.942066	0.892951
2.7	0.955979	0.874354	0.013913	0.03687	1.643168	0.02264905	0.831875	2.979348	0.955979	0.917943
2.8	0.970255	0.913613	0.014276	0.039259	1.67332	0.0236738	0.855549	3.089694	0.970255	0.944066
2.9	0.98386	0.952387	0.013605	0.038774	1.702939	0.02296765	0.878517	3.200041	0.98386	0.96941
3	1	1	0.01614	0.047613	1.732051	0.02772158	0.906238	3.310387	1	1

Proportional Hazard Transform Example $(\alpha = 5/7)$

r	A	B	ΔA	ΔB	$\tilde{A}$	$\Delta\tilde{A}$	$\Delta\tilde{B}$	$\tilde{B}$	$\hat{r}$	$\hat{A}$	$\hat{B}$
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0.08290724	0.004145	0.082907	0 004145	0.059947	0.05994702	0.00299735	0.002997	0.078635	0.059947	0.002357
0.2	0.16378083	0.016276	0.080874	0.012131	0.119936	0 05998875	0.00899831	0.011996	0.15727	0.119936	0.009433
0 3	0.23602591	0.034338	0.072245	0.018061	0.174942	0.055005747	0.01375144	0.025747	0.235905	0.174942	0.020246
0 4	0.30322066	0 057856	0.067195	0 023518	0.227453	0.052510987	0.01837885	0.044126	0.31454	0.227453	0.034698
0.5	0.36314275	0 084821	0.059922	0.026965	0 275514	0.048061374	0.02162762	0.065754	0.393175	0.275514	0.051705
0.6	0.41771473	0.114835	0.054572	0.030015	0.320421	0.044907259	0.02469899	0.090453	0.47181	0.320421	0 071127
0 7	0.46721704	0.147012	0.049502	0 032177	0.362208	0.041787246	0.02716171	0.117614	0.550445	0.362208	0.092486
0 8	0.51236273	0.180871	0.045146	0 033859	0.401296	0.03908775	0 02931581	0.14693	0.629081	0.401296	0.115539
0.9	0 55410853	0.216355	0.041746	0.035484	0.438371	0 037075025	0.03151377	0.178444	0.707716	0.438371	0.140319
1	0.59187827	0.252236	0 03777	0.035881	0.472779	0.034407898	0.0326875	0.211131	0.786351	0 472779	0.166023
1.1	0.62669301	0.288792	0.034815	0 036555	0.50531	0.032531155	0.03415771	0.245289	0.864986	0.50531	0.192883
1.2	0.65877703	0 325688	0.032084	0 036897	0.536066	0.030756108	0.03536952	0.280659	0.943621	0.536066	0 220696
1 3	0.68867534	0.363061	0.029898	0.037373	0.56548	0.02941379	0 03676724	0.317426	1.022256	0.56548	0.249608
1.4	0.71623406	0.400266	0.027559	0.037204	0.593316	0 027835655	0 03757813	0.355004	1.100891	0.593316	0.279158
1.5	0.74151327	0.43692	0.025279	0.036655	0.619536	0.026220703	0.03802002	0.393024	1.179526	0.619536	0.309055
1.6	0.76577385	0.474524	0.024261	0 037604	0.645399	0 025862853	0.04008742	0.433111	1.258161	0.645399	0.340577
1.7	0.78817739	0.51149	0.022404	0.036966	0 669971	0.024572019	0 04054383	0 473655	1.336796	0.669971	0.372459
1.8	0.80941703	0.54866	0.02124	0 037169	0.693963	0 023991374	0.0419849	0.51564	1.415431	0 693963	0.405474
1.9	0.82893218	0 584763	0.019515	0.036103	0.716689	0.022726515	0.04204405	0.557684	1.494066	0.716689	0 438535
2	0.84701571	0.620025	0 018084	0.035263	0.73842	0.02173056	0.04237459	0.600059	1.572701	0.73842	0.471857
2.1	0.86460927	0 656092	0.017594	0 036067	0.760279	0.02185905	0 04481105	0.64487	1.651336	0.760279	0.507094
2.2	0.88128965	0.691955	0.01668	0.035863	0.781767	0.021488162	0.04619955	0.691069	1.729971	0.781767	0.543423
2.3	0.89757855	0 728605	0.016289	0.03665	0.803602	0.021835308	0.04912944	0.740199	1 808607	0 803602	0.582056
2.4	0.91295335	0 764736	0.015375	0 036131	0.825144	0.021541886	0 05062343	0.790822	1.887242	0.825144	0.621864
2.5	0.92784159	0 801212	0.014888	0 036476	0.847071	0.021926519	0 05371997	0.844542	1.965877	0.847071	0.664106
2.6	0.94206597	0.837484	0.014224	0.036272	0.869268	0.022197493	0.05660361	0.901146	2.044512	0.869268	0 708617
2.7	0.95597917	0.874354	0.013913	0.03687	0.892555	0.023287235	0 06171117	0.962857	2.123147	0.892555	0.757143
2.8	0.97025501	0.913613	0.014276	0.039259	0.918795	0.026239833	0.07215954	1.035017	2.201782	0.918795	0.813886
2.9	0.98385987	0.952387	0.013605	0.038774	0.947527	0.028732009	0.08188622	1 116903	2.280417	0.947527	0.878277
3	1	1	0 01614	0 047613	1	0.052472719	0 15479452	1.271697	2.359052	1	1