Discussion of Generalized Minimum Bias Models

by Luyang Fu, Ph. D. and Cheng-sheng Peter Wu, FCAS, ASA, MAAA

Discussion by Stephen J. Mildenhall, FCAS, ASA, MAAA

Abtract: Fu and Wu have presented three generalizations of the minimum bias model iterations and demonstrated the impact these generalizations have on fitted parameters. This discussion explains how their generalized minimum bias models correspond to generaliezd linear models.

Fu and Wu's paper introduces 2- and 3-parameter Generalized Minimum Bias Models (GMBMs) which the authors claim extend Generalized Linear Models (GLMs). The GMBM depends on three parameters p, q and k, and is specified by the iterative scheme

$$x_i^k = \frac{\sum_j w_{ij}^p r_{ij}^k y_j^{q-k}}{\sum_j w_{ij}^p y_j^q} \tag{1}$$

where the w are prior weights, r the observations and x, y the parameters. The model is multiplicative: the fitted expected value of r_{ij} is given by $x_i y_j$. We will call this model a GMBM(p, q, k). The authors provide numerous examples of fits with different p, q, k. Unfortunately, by examining the effect of the three parameters p, q and k we can show that every GMBM corresponds to a GLM, so the new models do not extend the existing statistical models. Statistical models and approaches should always be preferred to non-statistical minimum bias models.

The parameter p is used to adjust the weights used from w_{ij} to w_{ij}^p. This adjustment can also be made in a GLM; the weights can be chosen however the modeler likes so long as they are specified ahead of time.

The parameter k replaces the responses r_{ij} with r_{ij}^k. A model is then fitted to the new responses to get parameters x_i and y_j. Finally, these parameters are converted back to the scale of the original responses by taking kth roots. Again, this procedure carries over to GLMs. Prior to modeling, replace each r_{ij} with r_{ij}^k, fit the model, and then replace the resulting fit parameters x_i, y_j with $x_i^{1/k}$ and $y_j^{1/k}$ respectively.

The parameter q is the most interesting. Comparing Equation 12 in the paper with Mildenhall [1999, Equation 7.13] shows that a value of q corresponds to using a variance function $V(\mu) = \mu^{2-q}$ in the GLM. As discussed in Mildenhall [1999, Section 8] there is a whole family of exponential distributions with variance $V(\mu) = \mu^\zeta$ where μ is the mean. The correspondence between ζ and distributions is shown in the table below. The common special cases are the normal $\zeta = 0$,

Table 1: Variance Functions

ζ	Distribution
$\zeta < 0$	Extreme Stable
0	Normal
$0 < \zeta < 1$	Not Exponential Family
1	Poisson
$1 < \zeta < 2$	Tweedie
2	Gamma
$2 < \zeta < \infty, \zeta \neq 3$	Positive Stable
3	Inverse Gaussian

Poisson $\zeta = 1$, gamma $\zeta = 2$ and inverse Gaussian $\zeta = 3$. These families are discussed more in McCullagh and Nelder [1989], Jörgensen [1997] and Jörgensen [1987]

The table shows that when $\zeta \neq 0, 1, 2, 3$ and $\zeta \notin (0, 1)$ there is still an exponential family corresponding to the variance function $V(\mu) = \mu^\zeta$. However, these distributions do not have a closed form expression for their densities. It is still possible to fit a GLM using these densities because the basic form of the likelihood function is known from the fact the distributions are in the exponential family. As explained in Mildenhall [1999, Section 8] and McCullagh and Nelder [1989] the deviance of an individual observation r_i is

$$2w_i \int_\mu^{r_i} \frac{r_i - t}{V(t)} dt + \log(V(r_i)). \tag{2}$$

This quantity is called the extended quasi-likelihood. It can be computed given the the function V only. It does not need the whole density. When $\zeta \in (0, 1)$ Equation 2 still makes sense (r_i and μ are positive in the examples) and it can be used in the GLM algorithm. However, such a variance function does not correspond to an exponential family distribution. Thus it is possible to work with GLMs with arbitrary $\zeta = 2 - q$.

Putting all three of these adjustments together gives the following dictionary between GMBMs and GLMs. The parameters produced by a GMBM(p, q, k) correspond to the kth roots of the parameters produced by a GLM with log link and weights w^p applied to data r^k and variance function $V(\mu) = \mu^\zeta$ where $\zeta = 2-q/k$. The relativities in the appendix of Fu and Wu can be produced by GLMs in this way. When $\zeta \in (0, 1)$, for example $k = 1, q = 1.5$, there is no exponential family distribution member, but the GLM iteratively re-weighted least squares method still converges to the same values as given in the paper.

The paper also claims that using the most recent evaluation of each parameter in the iterative process greatly speeds up convergence. Subsequent to completing Mildenhall [1999] I read in Golub and Loan [1996, Section 10.1.1] that for the basic linear additive model, the minimum bias iterations were discovered by Jacobi, and are sometimes called the Jacobi iterations. Golub and Loan [1996] also contains the same idea for improving convergence that the authors suggest. For the linear additive model it is called the Gauss-Seidel iteration. In terms of overall speed of computation, the re-weighted least squares approach is like a higher-dimensional version of the Newton-Raphson method. The Newton-Raphson method converges extremely quickly. As explained in Mildenhall [1999], the basic minimum bias method converges as powers of the largest eigenvalue of a certain matrix. It can converge much more slowly than the GLM method. The improved scheme is clearly faster than the original but it may not be as quick as the re-weighted least squares algorithm.

References

Stephen J. Mildenhall. A systematic relationship between minimum bias and generalized linear models. *Proceedings of the Casualty Actuarial Society*, LXXXVI:393–487, 1999.

P. McCullagh and J. A. Nelder. *Generalized Linear Models*. Chapman and Hall, second edition, 1989.

Bent Jörgensen. *The Theory of Dispersion Models*. Chapman and Hall, 1997.

Bent Jörgensen. Exponential dispersion models. *J. R. Statist. Soc. B*, 49(2):127–162, 1987.

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, third edition, 1996.