Simple Practical Estimation of Sub-Portfolio Catastrophe Loss Exceedance Curves with Limited Information

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Abstract

A very simple method is shown for the estimation of the catastrophe loss exceedance curve of a sub-portfolio, when information available is limited to a total portfolio catastrophe loss exceedance curve, and just enough information about the sub-portfolio to make reasonable selections for two parameters: relative frequency and relative severity. Practical examples are shown in the contexts of exposure rating catastrophe excess reinsurance, catastrophe risk based capital requirements, and catastrophe deductible credits. The relationship of relative frequency and relative severity to the concentration diversification structure of sub-portfolios is described.

Keywords: reinsurance, catastrophe, modeling, solvency

1. INTRODUCTION AND BACKGROUND

1.1 Research Context

This paper falls primarily into the CAS research taxonomy categories I.G.9, II.Q.2, and III.I. The CAS literature contains a number of papers on the general theory of catastrophe modeling (Woo [15], Boisonnade et al[2], Friedman [6], Clark[5], Kozlowski[8]), and the estimation of risk loads (for example Meyers et al [10]), particularly by per occurrence layer for catastrophe reinsurance, using loss exceedance curves. Little or nothing has been written addressing the decomposition of catastrophe loss exceedance curves.

1.2 Objective

A very simple method is shown for the estimation of the catastrophe loss exceedance curve of a sub-portfolio, when information available is limited to a total portfolio catastrophe loss exceedance curve, and just enough information about the sub-portfolio to make reasonable selections for two parameters: relative frequency and relative severity. This method can greatly facilitate analysis of real world catastrophe sub-portfolios, particularly in reinsurance context (Carter [4], Kiln [7], Strain [14]). Even where the method may not provide meaningful absolute numbers, it may be valuable for determining relative differences between different reinsurance transactions (Stanard et al [12]).

1.3 Outline

Section 1.4 provides general background and caveats are listed in Section 1.5. Section 1.6 provides an introduction to loss exceedance curve representations of catastrophe model output. Section 2 presents several example applications and a description of the model underlying the method. Section 3 is a discussion of generalization to variable relative frequency and severity. Conclusions are presented in Section 4. Appendix A provides details of an interesting measure of loss correlation between sub-portfolios. Appendix B provides details of a key consistency constraint for choosing relative frequencies and severities. Readers only interested in a quickly understanding the method at a functional level need only read Section 1.4 and 1.6 and Section 2 through 2.6, and Appendix B.

1.4 Background

Modern catastrophe modeling software programs accept exposure profiles for a single risk or a portfolio of risks. For example, a portfolio might be all the homes in Florida insured by a large national insurer, a single risk might be just one of these homes in Miami, and a more interesting sub-portfolio might be all of these homes which are in the Florida Keys. Exposure profiles are sometimes very detailed, including information such as geocodes (latitude, longitude), construction type, insured value, etc. for every single risk included. If the modeling software, detailed exposure profiles, adequate computers, enough time, and staff trained to use the software are all simultaneously available, it is straightforward to produce loss exceedance curves, either for entire the portfolio combined or any sub-portfolio. Loss exceedance curves show the annual probability that one or more events of a given size will occur, and can be easily translated into frequencies or return periods for losses of a given size (Section 1.6).

However, in many real world situations an actuary may need to make some sort of catastrophe sub-portfolio estimates, having access to neither the modeling software nor the detailed exposure profiles, but only a total portfolio loss exceedance curve and a little bit of sub-portfolio descriptive information (i.e., percentage of total portfolio premium, percentage of total portfolio policy limits, general location, etc.). For example, the actuary might have the loss exceedance curve for the portfolio of every home in the state of Florida, the Florida homeowners market share for two different insurers, and a little bit of descriptive information about relative geographical spread of the two insurers. Detailed exposure information is often considered of high proprietary value by insurers and kept as confidential as possible. Sometimes detailed exposure information is available, but even with the dramatic increase in computer speed and capacity in recent years, catastrophe models can still take a very long time to run (Major[9]). This paper demonstrates a simple method for estimating the loss exceedance curve of the sub-portfolio in such situations.

1.5 Caveats

1. Examples in this paper are entirely hypothetical illustrations of methodology, using artificial numbers together with situations which are not based on any actual insurers, reinsurers, brokers, catastrophe modeling firms, or catastrophe modeling results.
2. The term "risk based capital" is used in this paper in the generic sense of how much capital an insurer must provide to cover the risk of higher than expected losses, not the specific context of the National Association of Insurance Commissioners' RBC requirements.
3. Most statements in this paper are the generalizations of a simple practical model which is not always guaranteed to be mathematically consistent. Counterexamples and inconsistencies can often be generated. (See Appendix B)

1.6 CATASTROPHE LOSS EXCEEDANCE CURVES

The output of catastrophe models is usually stated in terms of a finite number of points (L_i, $T(L_i)$), where $L_{i+1} > L_i$, called a loss exceedance curve. This curve shows the probability that one or more loss events will occur in a year that are at least as great as a given loss amount. For example, if T(50 million) = 1% there is a 99% chance that the largest event that occurs in a single year will be less than $50 million. Usually the exceedance curve is consistent with a collective risk model of event frequency and severity (Bowers et al [3]), with the number of events following a Poisson distribution. The sum of independent

Poisson distributed random variables is a Poisson distributed random variable. Similarly, Poisson distributed random variables can be decomposed into sums of independent Poisson distributed random variables. Each point on the exceedance curve can be thought of as representing a single event, independent of all other events. For a Poisson distribution with mean λ, the probability of at least one occurrence is $1 - e^{-\lambda}$. So the mean annual frequency of events at least as great as L_i, is $\Lambda(L_i) = -Ln(1 - T(L_i))$ and the annual frequency of L_i specifically is $\lambda(L_i) = \Lambda(L_i) - \Lambda(L_{i+1})$. If the total frequency of all events is $\Lambda(0)$ then the size of loss distribution for individual events is $F(L_i) = 1 - \Lambda(L_i) / \Lambda(0)$. The return period of a loss amount is simply define as the inverse of the mean annual frequency of losses at least as great, $R(L_i) = 1/\Lambda(L_i)$. Sometimes the return period is defined as the inverse of the probability of exceedance, $R(L_i) = 1/T(L_i)$. This definition is numerically very close to the inverse of annual frequency for low frequency events but can become fairly meaningless for very high frequency events. Table 1.6.1 shows an example of the various components common in the representation of a loss exceedance curve.

Table 1.6.1

Example of Various Components of Common Representations of Loss Exceedance Curves

Event Loss	Probability of Exceedance	Frequency of Exceedance	Return Period
[1]	[2]	[3]	[4]
= Model Output	= Model Output	= -ln(1-[2])	= 1 / [3]
1,000,000,000,000	0.1998%	0.00200	500
100,000,000,000	0.9950%	0.01000	100
10,000,000,000	9.5163%	0.10000	10
1,000,000,000	18.1269%	0.20000	5
Total	18.1269%		

Event Loss	Incremental Frequency	Severity Distribution	Severity Density
[1] = Model Output	[5] = Difference ([4])	[6] = [3] / Total([5])	[7] = Difference ([6])
1,000,000,000,000	0.00200	100%	1.0%
100,000,000,000	0.00800	99%	4.0%
10,000,000,000	0.09000	95%	45.0%
1,000,000,000	0.10000	50%	50.0%
Total	0.20000		100.0%

Catastrophe excess reinsurance treaties often limit the number of separate events covered to a finite number of "reinstatements" m. Summing the product of loss amounts multiplied by incremental frequencies corresponds to unlimited reinstatements. However, Formula 1.6.1 shows an example adjustment factor, where reinstatements are limited by the number of occurrence and L_1 is the attachment point, that the unlimited reinstatement expected losses can be multiplied by to account for the finite limit on reinstatements. Further details on reinstatement adjustment factors can be found in Anderson[1] and Simon[11].

$$A(\Lambda(L_1), m) = 1 - \frac{e^{-\Lambda(L_1)}}{\Lambda(L_1)} \sum_{n=m+1}^{\infty} (n-m) \frac{(\Lambda(L_1))^n}{n!} \tag{1.6.1}$$

2. EXAMPLES AND THE UNDERLYING MODEL

The examples in this section use loss exceedance curves with only a handful of points for illustration. Real world situations typically involve curves with hundreds of points. The curves are presented in the very useful representation of loss amount, return period, and incremental frequency. Loss exceedance curves in other representations may be easily converted using the methods in Section 1.6. Calculations are made only for expected losses by layer, but loss exceedance curves contain much more probabilistic information that can be useful for calculating many other quantities such as variances. Expected loss is defined to be the sum of loss amounts multiplied by corresponding incremental frequencies without adjustment for reinstatement limitations, but a formula for an adjustment factor is also shown in Section 1.6.

2.1 A Motivational Example: Exposure Rating Property Catastrophe Treaties

REINSURER, a multi-line reinsurance company, assumes large accounts of reinsurance for two companies, MUTUAL and COMMERCIAL. Both accounts involve almost comprehensive programs of quota share, liability per occurrence, and property per risk cessions, but no catastrophe per occurrence excess. One day before renewal each cedant company expresses interest in adding $200 million xs $100 million catastrophe coverage on its Florida homeowners portfolio. There is no readily available detailed exposure profile ready for either company's Florida homeowners portfolio. Neither is there enough time to run models and review the output for a cession rate quote, even if the profile data was available.

However, the catastrophe modeling firm CATMOD has published an estimate that the total industry hurricane loss exceedance curve for homeowners insurance in Florida in a trade publication as shown in Table 2.1.1

Table 2.1.1

Hypothetical Loss Exceedance Curve for Florida Hurricanes

Event Loss	Return Period in Years	Incremental Frequencies
1,000,000,000,000	500	0.00200
100,000,000,000	100	0.00800
10,000,000,000	10	0.09000
1,000,000,000	5	0.10000

MUTUAL and COMMERCIAL each report $100 million of Florida homeowners premium, and total industry Florida homeowners premium underlying the CATMOD curve is estimated to be $5 billion. The ACTUARY at REINSURER reviews the article with the

CATMOD loss exceedance curve and begins to consider what relative frequency and severity assumptions should made for MUTUAL and COMMERCIAL, respectively.

MUTUAL's exposures tend to be spread throughout the state of Florida. Some of its business is urban, but most is suburban or rural. ACTUARY decides that 100% of Florida hurricane events will noticeably affect MUTUAL. All other things being equal this assumption, together with MUTUAL's 2% of premium, would imply about 2% of losses from every event go to MUTUAL, but ACTUARY thinks that the share of event losses should be reduced to 1%, since MUTUAL probably has a higher proportion of the market that is far inland and in the northern part of Florida. ACTUARY decides that the loss exceedance curve for MUTUAL should look something like Table 2.1.2.

Table 2.1.2

Estimated MUTUAL Florida Hurricane Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
10,000,000,000	500	0.00200
1,000,000,000	100	0.00800
100,000,000	10	0.09000
10,000,000	5	0.10000

COMMERCIAL's exposures tend to be heavily concentrated in Miami and Tampa. ACTUARY thinks that only about 20% of Florida hurricane events will noticeably affect COMMERCIAL. All other things being equal COMMERCIAL's 2% of premium would imply about 2% of overall expected losses and hence 10% of event losses from those events for which it is noticeably affected. ACTUARY thinks that the share of event losses should be increased to 20%, since COMMERCIAL has a higher proportion of the costal and near costal market. ACTUARY decides that the loss exceedance curve for MUTUAL should look something like Table 2.1.3.

Table 2.1.3

Estimated COMMERCIAL Florida Hurricane Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
200,000,000,000	2500	0.00040
20,000,000,000	500	0.00160
2,000,000,000	50	0.01800
200,000,000	25	0.02000

ACTUARY multiplies layered loss amounts by incremental frequencies and sums them up to estimate expected losses for the two treaties in Table 2.1.4.

Table 2.1.4

Exposure Rates For MUTUAL and COMMERCIAL

MUTUAL

Layered Loss	Incremental Frequencies	Expected Layered Loss
200,000,000	0.00200	400,000
200,000,000	0.00800	1,600,000
0	0.09000	0
0	0.10000	0
	Total Expected Layered Loss	2,000,000

COMMERCIAL

Layered Loss	Incremental Frequencies	Expected Layered Loss
200,000,000	0.00040	80,000
200,000,000	0.00160	320,000
200,000,000	0.01800	3,600,000
100,000,000	0.02000	2,000,000
	Total Expected Layered Loss	6,000,000

Overall REINSURER usually targets a loss ratio of about 60% corresponding to its target ROE of 15%. ACTUARY considers how the risk/return tradeoffs of the two new cat treaties, respectively, will marginally REINSURER's overall risk/return profile (Stanard et al [13]). ACTUARY believes that the lower frequency, higher severity COMMERCIAL treaty will place a higher than average burden on REINSURER's marginal risk based capital and retrocessional costs, and MUTUAL will cause a correspondingly lower than average burden. So, ACTUARY selects 55% and 65% as the target loss ratios for COMMERCIAL and MUTUAL, respectively, resulting in the quotes in Table 2.1.5.

Table 2.1.5

Quotes for MUTUAL and COMMERCIAL

Ceding Company	MUTUAL	COMMERCIAL
Expected Layered Loss	2,000,000	6,000,000
Target Loss Ratio	65%	55%
Ceded Premium Quote	3,076,923	10,909,091
Subject Premium	100,000,000	100,000,000
Cession Rate Quote	3.1%	10.9%

2.2 The Underlying Model

Suppose a sub-portfolio is described by three parameters:

r = relative frequency or the fraction of portfolio catastrophe loss events, at each size of loss level, in which the sub-portfolio experiences losses.

s = relative severity or the fraction of portfolio losses which are incurred by the sub-portfolio for each catastrophe loss event in which the sub-portfolio experience losses.

p = relative exposure or the ratio of sub-portfolio expected catastrophe losses to portfolio expected catastrophe losses.

These definitions immediately imply the following constraints in Formulae 2.2.1.

$$0 < r, s, p < 1$$
$$p = rs \tag{2.2.1}$$

Notice that two of the three quantities r, s, and p, uniquely determine the third.

Suppose that (L_i , λ (L_i)) one of a finite number of points on a discrete catastrophe loss exceedance curve for the entire portfolio, where $L_{i+1} > L_i$, and λ (L_i) is the incremental frequency of events of size L. Since F strictly non-decreasing as L increases, (r L_i, s λ (L_i)) is a point on the loss exceedance curve of the sub-portfolio.

The key implicit assumptions of this model are:

1. The severity distribution for the sub-portfolio is equal to the severity distribution of the total portfolio for losses, with losses rescaled by a positive number.
2. The sub-portfolio is only affected a certain fraction of the events affecting the total portfolio.

In reality the r and s would vary by Li, but for practical purposes in situations with limited information constant values often the best estimate that can be made. Variability of r and s by size of loss event will be dealt with in Section 3.

It is useful to conceptually relate relative frequency and severity to the diversification and concentration structure both for risks inside the sub-portfolio and for the sub-portfolio relative to the rest of the total portfolio. Sections 2.3 and 2.4 describe the correlation implications of r = 1 and s = 1, respectively and Appendix A presents a more detailed relationship between r, s, and correlation of sub-portfolios.

2.3 Extreme Situation: 100% Relative Frequency, All Events Affect All Sub-Portfolios, Highly Diversified Sub-Portfolios

A portfolio could be partitioned into non-overlapping sub-portfolios, each of which is assumed to share losses from any event which are in exactly the same proportion as its proportion of overall expected losses. In this case the losses of the sub-portfolio are 100% correlated, the relative frequencies are all 100% and the relative severities are equal to the relative exposures. Any sub-portfolio experiences the same frequency and a lower severity of losses. Generally speaking, this would be the case where risks in each sub-portfolio were randomly from the total portfolio.

2.4 Extreme Situation: Individual Events Affect Only One Sub-Portfolio, Highly Concentrated Sub-Portfolios

Alternatively, each sub-portfolio in a partition of the total portfolio could be assumed to experience all of the losses for only a certain fraction total portfolio events. In this case the losses of the sub-portfolio are 0% correlated, and the sum of relative frequencies of sub-portfolios is 100%. Generally speaking, this would be the case where risks in each sub-portfolio were selected to be all the risks with the same key risk characteristics, such as geographical location, which determine whether a particular risk is affected by a certain catastrophe event.

2.5 Characteristic Relationships Between Risk Correlations, Relative Frequency, and Relative Severity

For a fixed relative exposure p as relative frequency r goes up and relative severity s goes down, the sub-portfolio is more correlated with the entire portfolio. Correspondingly, one can expect that as the correlation between a sub-portfolio and the rest of the total portfolio goes up, the relative frequency tends to be higher (sub-portfolio affected by more events} and relative severities tends to be lower (affected less severely by events).

The higher correlation between a sub-portfolio and the rest of the total portfolio tends to accompany lower correlation of risks within the sub-portfolio. So diversification within a sub-portfolio often corresponds to a high correlation with the rest of the portfolio. Similarly, concentration inside the sub-portfolio often corresponds to low correlation with the rest of the total portfolio.

The general relationships described above are important considerations for making practical selections of relative frequencies, relative severities, and/or relative exposures to estimate sub-portfolio loss exceedance curves. Formula 2.5.1 shows two particularly useful heuristic formulae (derived from a model in Appendix A) for the correlation coefficient between the sub-portfolio and the rest on the total portfolio when constant relative frequency and severity are assumed.

$$\begin{aligned} \rho &= (r-p)/(1-p) \\ \rho &= r(1-s)/(1-rs) \end{aligned} \qquad (2.5.1)$$

For example, if r =0.5 and s = 0.5 then the sub-portfolio losses will tend to have a 33% correlation with losses in the rest of the total portfolio.

2.6 Example – Risk Based Capital and Solvency

Since September 11, 2001 the insurance industry has recognized terrorism as a potential catastrophic peril. The insurance company COMMERCIAL is doing a general review of its capitalization and risk based capital adequacy with consideration to terrorism exposure. The primary catastrophe solvency criterion used by COMMERCIAL is the rule of thumb that its 100 year return period catastrophic loss, net of reinsurance and alternative risk transfers (ARTs), should be less than its surplus.

COMMERCIAL has modeled all of its property and casualty business for natural perils such as earthquakes and hurricanes, but not for terrorism, to produce the overall loss exceedance curve in Table 2.6.1.

Table 2.6.1

COMMERCIAL Natural Peril Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
220,000,000,000	350	0.00286
10,000,000,000	100	0.00714
5,000,000,000	50	0.01000
2,000,000,000	10	0.08000
900,000,000	5	0.10000

Since COMMERCIAL's surplus is $15 billion and its 100 year return period loss is $10 billion it easily satisfies its rule of thumb criterion, except possibly for terrorism exposure.

The cat modeler CATMOD has modeled terrorism for the entire United States property and casualty market to produce the loss exceedance curve in Table 2.6.2.

Table 2.6.2

Hypothetical Industry Terrorism Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
600,000,000,000	1000	0.00100
150,000,000,000	300	0.00233
60,000,000,000	50	0.01667
1,000,000,000	25	0.02000
500,000,000	10	0.06000

In the absence of model results specifically for COMMERCIAL which include terrorism, COMMERCIAL performs a simple estimate based on selections for relative frequency and relative severity applied to the CATMOD industrywide terrorism loss exceedance curve. The CATMOD model assumes that 50% of all future terrorism events will happen in just four of the largest U.S. cities: New York, Los Angeles, Chicago, and Washington, D.C. As COMMERCIAL's market share is around 35% in these cities, COMMERCIAL selects a terrorism relative frequency of 55% and relative severity of 30% for itself. These parameters imply that since the industrywide 50 year return period terrorism loss is $60 billion, COMMERCIAL's 91 year return period loss, just for terrorism, is $18 billion, as shown in Table 2.6.3.

Table 2.6.3

Estimated COMMERCIAL Terrorism Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
180,000,000,000	1818	0.00055
45,000,000,000	545	0.00128
18,000,000,000	91	0.00917
300,000,000	45	0.01100
150,000,000	18	0.03300

COMMERCIAL decides to perform more detailed terrorism modeling and analysis to determine whether its terrorism exposure now requires additional surplus, reinsurance, alternative risk transfers, and/or other risk management adjustments to bolster its solvency position.

2.7 Example – Individual Risk Catastrophe Deductible Credits

COMMERCIAL offers a special comprehensive commercial package policy, which excludes catastrophic perils, but offers the option of a large dollar deductible. A policyholder PHOLDER with 500 employees and $150 million of annual revenue is charged a no deductible premium of $2 million. Based on a frequency severity simulation model of non-catastrophic losses COMMERCIAL offers a deductible credit of 60% for an aggregate annual deductible of $2 million. So with the $2 million deductible the premium is $0.8 million for the non-catastrophe policy.

Because of increasing concern about natural catastrophe perils, PHOLDER is requesting extra coverage for natural catastrophes. COMMERCIAL does not use a catastrophe model for individual risks, but believes that extra premium for such extra coverage with no deductible should be about $100,000 for PHOLDER based on consideration of certain key risk characteristics. Since all 500 work in a single building in downtown Miami, which is the PHOLDER's only property, COMMERCIAL believes that catastrophe losses for the PHOLDER will be low frequency and high severity, and that the 60% deductible is far too high for the catastrophe coverage.

Since COMMERCIAL's total annual premium for natural catastrophe losses is $2 billion, the relative exposure of this policyholder is 0.005%. Due to PHOLDER's highly concentrated risk characteristics COMMERCIAL selects a relative frequency of 1%, making relative severity 0.5%. So, the estimated natural catastrophe loss exceedance curve for the PHOLDER derived from Table 2.6.1 is shown in Table 2.7.1.

Table 2.7.1

Estimated PHOLDER Natural Catastrophe Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
1,100,000,000	35,000	0.000029
50,000,000	10,000	0.000071
25,000,000	5,000	0.000100
10,000,000	1,000	0.000800
4,500,000	500	0.001000

This leads to an estimated deductible credit based on expected pure losses eliminated of 4,000/50,000 = 8% from the calculations in Table 2.7.2.

Table 2.7.1

Estimated PHOLDER Deductible Loss Elimination

Gross Loss	Deductible Loss	Incremental Frequency	Expected Gross Loss	Expected Deductible Loss
1,100,000,000	2,000,000	0.000029	31,429	57
50,000,000	2,000,000	0.000071	3,571	143
25,000,000	2,000,000	0.000100	2,500	200
10,000,000	2,000,000	0.000800	8,000	1,600
4,500,000	2,000,000	0.001000	4,500	2,000
			50,000	4,000

3. VARIABLE RELATIVE FREQUENCY AND SEVERITY

The assumption of constant relative frequency and severity across all loss sizes is grossly unrealistic in some circumstances. Consider a single $4 million property risk PROPERTY located in Brooklyn, New York with annual premium of $20,000. Suppose a terrorism loss exceedance curve for all property in New York City with total annual premium of $2 billion is given by Table 3.1.

Table 3.1

Hypothetical NYC Terrorism Property Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
1,000,000,000,000	10000	0.00010
100,000,000,000	1000	0.00090
10,000,000,000	100	0.00900
1,000,000,000	10	0.09000
1,000,000	1	0.90000

For the 1 year return period event a reasonable selection might be r = 0.001% and s = 100%, as such a small event would probably only affect about a single property policy and if PROPERTY were the policy hit it would probably sustain about all of the $1 million in damages. However, the 10,000 year event would tend to saturate the entire New York City area, representing an event comparable to a hydrogen bomb detonation in Manhattan.

PROPERTY would stand much greater than a 1 in 100,000 chance of being affected and obviously could only experience a tiny fraction of the $1 trillion in damages. A better selection for this level of loss would be r =100% and s =0.0004%.

In this sort of a situation, where sufficient information about the nature of the sub-portfolio and the type of events underlying the total loss exceedance curve is available, there is no reason why the constant relative frequency and severity model cannot be generalized to functions $r(L_i)$ and $s(L_i)$ as shown in Table 3.2 and used to derive an estimated sub-portfolio loss exceedance curve such as shown in Table 3.3.

Table 3.2

Terrorism Variable Relative Frequency and Severity for PROPERTY

Return Period in Years	$r(L_i)$	$s(L_i)$.
10000	1.00000	0.0000040
1000	0.10000	0.0000313
100	0.01000	0.0002500
10	0.00100	0.0020000
1	0.00001	1.0000000

Table 3.3

Estimated Terrorism Loss Exceedance for PROPERTY

Event Loss	Return Period in Years	Incremental Frequencies
4,000,000	10000	0.00010
3,125,000	1000	0.00090
2,500,000	100	0.00900
2,000,000	10	0.09000
1,000,000	1	0.90000

4. CONCLUSIONS

Modern catastrophe modeling computer programs can be tremendously valuable. These software models quantify the exposure of insurance portfolios to catastrophic losses based on what physical scientists, engineers, and social scientists know about perils which may cause catastrophic losses, such as earthquakes, hurricanes, floods, tornados, and terrorism. In many practical situations the resources, time, and detailed portfolio exposure information needed to fully utilize such models for analysis of sub-portfolios – or even the model software itself – may not be available. In such situations it is possible to use total portfolio loss exceedance curves and other information about a sub-portfolio's risk concentration diversification characteristics to select a relative frequency and a relative severity. Then the calculation of a loss exceedance curve for the sub-portfolio and a variety of analyses based on it can easily be performed in very little time with a simple spreadsheet. The resulting loss estimated exceedance curve is by no means equivalent to that which would be produced by the actual catastrophe model. However, in some situations with limited information, time, or resources it may be a practical and reasonable substitute.

Appendix A - The Correlation Coefficient Interpretation

If the total portfolio loss random variable is Z with frequency given by the Poisson random variable N and severity given by the random variable X the variance of losses is given by the Formula A.1.

$$\mathrm{Var}[Z]=E[N]\,E[X^2] \tag{A.1}$$

We can think of Z as a sum of loss random variables for disjoint sub-portfolios. If these sub-portfolios were 100% pair wise correlated then the total portfolio standard deviation would be equal to the sum of the standard deviations of the sub-portfolios. Alternatively if the sub-porfolios were 0% pair-wise correlated then the total portfolio standard deviation would be equal to the square root of the sum of the variances (standard deviations squared) of the sub-portfolios.

For a sub-portfolio with r and s relative frequency and severity, respectively, and loss random variable Y the variance of losses is given by the Formula A.2.

$$\mathrm{Var}[Y] = E[rN]\,E[(sX)^2]=rs^2\,E[N]\,E[X^2] = rs^2\,\mathrm{Var}[Z] \tag{A.2}$$

Now if Z is equal to the sum of k sub-portfolios identical to Y and the pair-wise correlation coefficient between different sub-portfolios is equal to the constant ρ the total portfolio variance of losses is given by the Formula A.3.

$$\begin{aligned}\mathrm{Var}[Z] &= k\,\mathrm{Var}[Y] + k\,(k-1)\,\rho\,(\mathrm{Var}[Y])^{1/2}\,(\mathrm{Var}[Y])^{1/2}\\ &= (k + k\,(k-1)\,\rho)\,\mathrm{Var}[Y]\\ &= (k + k\,(k-1)\,\rho)\,rs^2\,\mathrm{Var}[Z]\end{aligned} \tag{A.3}$$

Algebraically this implies Formula A.4.

$$(k + k\,(k-1)\,\rho)\,rs^2 = 1 \tag{A.4}$$

Since $1/k = p = rs$ Formula A.4 can be rewritten as Formula A.5.

$$(p + (1-p)\,\rho)\,/\,r = 1 \tag{A.5}$$

Formula A.5 can be restated into the particularly interesting forms in Formulae A.6.

$$\rho = (r-p)/(1-p)$$
$$\rho = r(1-s)/(1-rs) \tag{A.6}$$

So, if r = 1 then s = p and the correlation between the sub-portfolios is 100%. Similarly, if r = p then s = 1 and the correlation is 0%.

Appendix B - Consistency of Estimates of Relative Frequency and Relative Severity

Since expected losses for the total portfolio must equal the sum of the individual expected losses for a partition into disjoint sub-portfolios we have the constraint on constant relative frequencies and severities given by Formulae B.1.

$$E[N]\,E[X] = \sum_j E[r_j\,N]\,E[s_j\,X] = E[N]\,E[X]\sum_j r_j\,s_j$$

$$\sum_j r_j\,s_j = 1 \tag{B.1}$$

However, this constraint is not sufficient to prevent inconsistencies. For example Table B.1 satisfies Formulae B.1 but still leads to an inconsistent interpretation.

Table B.1

Inconsistent Partition into Sub-portfolios

Sub-Portfolio	r_j	s_j
A	0.50	0.50
B	1.00	0.75

These selections satisfy the previous constraint but imply that 50% loss events affect both A and B which together experience 125% of the losses from such events. To avoid this type of inconsistency in selecting constant relative frequencies and severities for a partition of the total portfolio into a finite set of sub-portfolios, the selections of r and s must also meet following tiling condition:

- The set of rectangles with length r_j and width s_j, respectively, must completely cover the unit

The set of rectangles with length r_j and width s_j, respectively, must completely cover the unit square (length and width both equal to 1) without any overlap and without rotation. For example the relative frequencies and severities for the partition into 6 sub-portfolios given in Table B.2 is shown to be consistent by the tiling in Figure B.1.

Table B.2

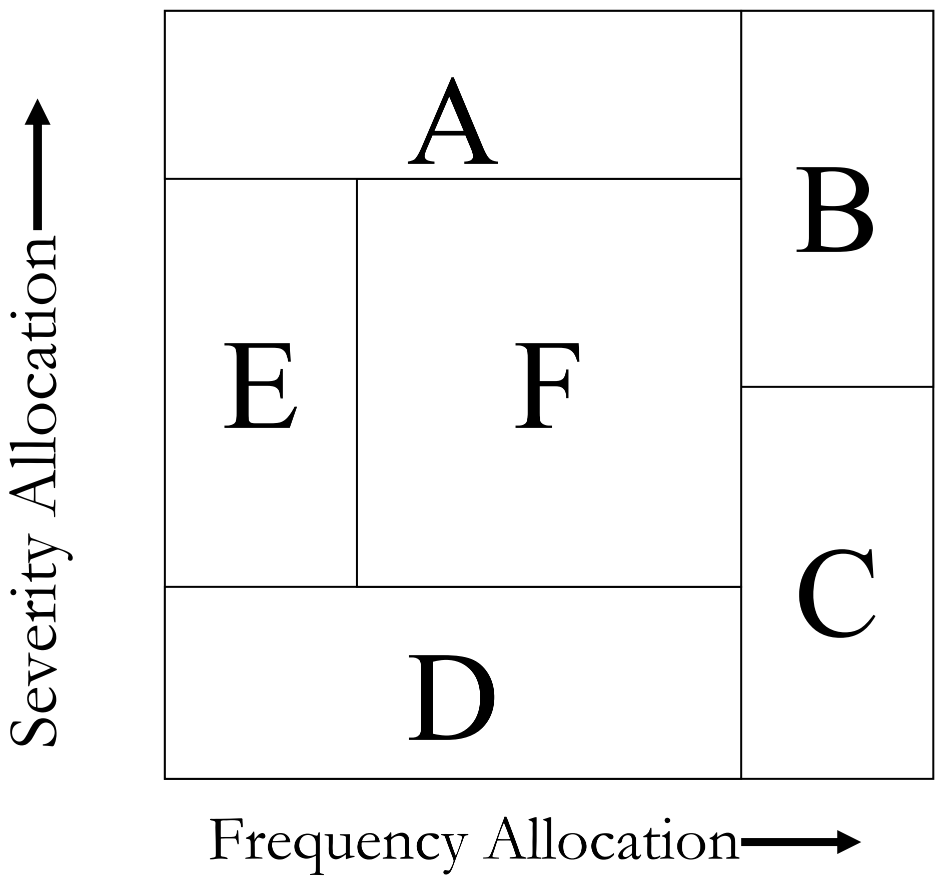
Consistent Partition Into Sub-portfolios

Sub-Portfolio	r_j	s_j
A	0.75	0.25
B	0.25	0.50
C	0.25	0.50
D	0.75	0.25
E	0.50	0.25
F	0.50	0.50

In the general case where r_j and s_j are allowed to vary as functions of the size of loss L_i, a tiling must exist for each L_i for consistency. As a practical matter, using judgment to divide up a unit square into rectangles such as in Figure X may be a good way to select r_j and s_j. The unit square offers a graphical tool where more horizontal coordinate overlap corresponds between rectangles tends to indicate higher correlation and the area of a rectangle is proportional to the share of overall expected losses.

Figure B.1

Tiling Demonstration of Relative Frequency and Severity Consistency



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Biography of Author

Jonathan Evans, FCAS, MAAA has 10 years of actuarial experience including ratemaking and reserving, but has specialized in the areas of reinsurance, catastrophe modeling, and Puzzlements in "The Actuarial Review".