Exposure Rating Casualty Reinsurance Excess Layers With Closed Form Annuity Models

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Abstract

Casualty excess reinsurance terms are typically stated in fixed attachment and limit amounts. Unless a lump sum settlement or commutation is made ultimate recoveries are settled years later as total payments penetrate the excess layer. This paper demonstrates that annuity models incorporating claim life, late reporting, benefit inflation, and discounting can be formulated with simple functional form components that lead to closed form solutions, or at least efficient numerical integration solutions, for exposure rating quantities. Applications are shown to problems such as commutation of existing claims, prospective reinsurance rating, excess loss development factors, and volatility of excess layers.

Keywords. Annuity, reinsurance, exposure rating, casualty

1. INTRODUCTION

Consider a very motivating practical example along the lines of (Ferguson [6]) or (Bluhmson [1]). Suppose a permanent disability claim is reported to an insurer. The disabled beneficiary is 40 years old and expected to live 40 more years, annual indemnity and medical benefits are currently $100,000 but subject to future inflation throughout the life of the beneficiary. The claim is covered by one reinsurance treaty for $15 million excess of $10 million reinsurance treaty and by a second treaty $25 million excess of $25 million. What is the gross reserve for the claim, nominal and discounted, respectively? Traditional tools of life contingencies do a respectable job of answering this type of question (Bowers [2]).

What are the reserves, ceded to each reinsurance treaty and net, nominal and discounted, respectively? How much should the reinsurers pay to commute the claim? The reinsurers say the ceded amount will be about 0 since even if average benefit inflation is 4% the beneficiary would have to live to 82 years old to even begin to penetrate the first treaty layer. The insurer believes that average benefit inflation will be 11%, mostly driven by exploding medical expenses, and that both treaty layers will be totally exhausted if the beneficiary lives to 80. How are the answers affected by uncertainty in mortality and interest rates? For questions involving IBNR, prospective rating of treaties, and risk adjusted discounting the confusion and disagreements are even worse. Actuarial tools have not been well developed to answer these types of questions.

Large casualty claims, absent of a lump sum settlement, involve long streams of indemnity and medical payments that can last many decades. The ultimate value of these claims is highly uncertain due to many risk factors, including:

1. claim life or mortality of the disabled individual
2. inflation of future benefit amounts
3. interest rate (or appropriate discount rate) where present values are calculated

Excess reinsurance layers are generally stated in fixed nominal dollar amounts for attachment points and limits. Consequently, losses in these layers can be even more uncertain. This phenomenon has been dealt with through the use of discrete spreadsheet simulation models in (Bluhmsohn [1]).

However, closed form solutions for exposure rate estimates for excess reinsurance layers can be obtained if simple analytic functional forms are assumed for components such as claim life, reporting delay, inflationary trends, etc. of a life contingent annuity model.

1.1 Research Context

The net and ceded problem for annuity reserves was considered in (Ferguson[6]). The sensitivity of layers of a casualty claim to parameter assumptions has been considered in (Bluhmsohn [1]), and annuity models for paid tail factor development have been considered in (Corro [5]). This paper falls primarily into CAS research categories I.G.3, I.G.9, I.G.12, I.I.1.a, and I.I.1.b.

1.2 Objective

The techniques demonstrated in this paper connect the methods of life contingencies (Bowers [2]), as used for annuity calculations, with the methods of loss distributions (Klugman [8]), as used for per claim layered exposure rating calculations, to solve practical casualty reinsurance problems (Carter [4], Kiln [7], Strain [12]). The techniques are also somewhat applicable to low frequency per occurrence layers, but effectively inapplicable to high frequency catastrophic clash layers. Applications are demonstrated for commutation of claims, exposure rating of reinsurance treaties, loss development by layer, and volatility of excess layers.

1.3 Outline

This paper will explore annuity models with closed form solutions with the help of the symbolic manipulation and numerical calculation program MATHEMATICA. Attention will also be paid to parameter uncertainty – in addition to process variance.

Section 2 contains general setup for annuity models and several specific models with closed form solutions. Then a series of practical applications is presented in Section 3. Section 4 compares the advantages and disadvantages of closed form models versus simulation models. Section 5 discussions possibilities for further development of closed form models.

2. BACKGROUND AND METHODS

2.1 General Framework

The general notation and framework assumptions of this paper are presented in this section. Some readers, anxious to see the results in action, may benefit by skipping ahead to the examples in Section 3 and referring to Section 2.2 and Appendix C for model and formulae details.

The general framework and notation is:

1. A claim is reported at the time R and closed at time $R + T$, where $R \geq 0$ and $T \geq 0$, and the units of time are years.
2. Payments on the claim begin at time R and end at time $R + T$.
3. The survival function for R is $S_R(t) = 1 - F_R(t)$, where $F_R(t)$, is the probability distribution of R, and similarly the survival function of T is $S_T(t) = 1 - F_T(t)$.

4. The instantaneous rate of payment on a claim at time t is $P(t)$ dollars per year, where $P(t) \geq 0$.
5. The cumulative payment through time t, where $R+T \geq t \geq R$, is given by Formula 2.1.1.

$$C(t, R) = \int_R^t P(t)\, dt \tag{2.1.1}$$

6. The risk adjusted present value at time 0 of one dollar paid at time t is $D(t) \geq 0$.
7. The earliest time for the claim to close, or equivalently the minimum value for $R + T$, so that $C(t, R) \geq m$, where $m \geq 0$, is denoted $C^{-t}(l, R)$.
8. The ground up ultimate nominal claim payment is denoted Y and the layered payment is Y – A, limited to a minimum of 0 and a maximum of L.

This framework allows for process variance in the report time and claim life, which are stochastic, but leaves the benefit amount, benefit inflation, and discount rate deterministic. However, parameter uncertainty can be incorporated for all of these quantities. The examples in the next section will rely primarily on examination of a few parameter scenarios. Models including continuous distributions for parameters are also of interest, but are not demonstrated in this paper.

There are two key exposure rating formulae that form the focus of the rest of this paper. These formulae transform integration through the layer into an integral through an interval of time. This is possible since for a given report time the total value of payments, with or without discount, is a strictly increasing function of time as long as the claim is open and constant after the claim is closed. First, the expected nominal value of losses ceded to an excess layer with attachment $A \geq 0$ and limit $L \geq 0$ is defined by Formula 2.1.2.

$$NX(A, L) = \int_0^\infty d\, F_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} P(t)\, S_T(t-R)\, dt \tag{2.1.2}$$

Second, the expected present value of losses ceded to an excess layer with attachment A

≥ *0* and limit *L* ≥ *0* is defined by Formula 2.1.3.

$$PX(A, L) = \int_0^\infty dF_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} D(t)\, P(t)\, S_T(t-R)\, dt \tag{2.1.3}$$

Formulae 2.1.2 and 2.1.3 have convenient closed form solutions for some basic but reasonable models for $S_R(t)$, $S_T(t)$, $P(t)$, and $D(t)$.

2.2 Some Basic Models with Closed Form Solutions

Formulae 2.1.2 and 2.1.3 have convenient closed form solutions for some basic models for $S_R(t)$, $S_T(t)$, $P(t)$, and $D(t)$, such as the following extremely simple model (MOD1):

- $R = 0$, claims reported immediately
- $S_T(t) = e^{-t/l}$, exponential distribution for time the claim is open with mean l
- $P(t) = B\ e^{at}$, constant payment rate B with constant force of inflation a
- $D(t) = e^{-dt}$, constant force of discount d

Formula 2.1.1 becomes:

$$C(t, R) = \frac{B(e^{aT} - 1)}{a} \tag{2.2.1}$$

The closure time inversion formula becomes:

$$C^{-t}(m, R) = \frac{\log\left(\frac{a\,m}{B} + 1\right)}{a} \tag{2.2.2}$$

Formula 2.1.2 becomes:

$$\mathrm{NX}(A, L) = \frac{l\left(B\left(\frac{B+a\,(A+L)}{B}\right)^{1-\frac{1}{a\,l}} - \left(\frac{a\,A}{B} + 1\right)^{-\frac{1}{a\,l}} (a\,A + B)\right)}{al - 1} \tag{2.2.3}$$

Formula 2.1.3 becomes:

$$\mathbf{PX}(A, L) = \frac{l\left(\left(1+\frac{aA}{B}\right)^{-\frac{d+\frac{1}{l}}{a}}(-aA-B)+B\left(\frac{B+a\,(A+L)}{B}\right)^{1-\frac{d+\frac{1}{l}}{a}}\right)}{al-dl-1} \tag{2.2.4}$$

Notice that formulae 2.2.3 and 2.2.4 are pretty simple, involving only basic arithmetic operations and exponentiation. These formulae are very easy to program in a spreadsheet.

Next consider a slight generalization to a constant report lag (MOD2):

- $R = s$, claims reported with a constant lag

Now the formulae become:

$$C(t, R) = \frac{B e^{a s}(e^{a T}-1)}{a} \tag{2.2.5}$$

$$C^{-t}(m, R) = \frac{\log\left(\frac{a m}{B}+e^{a s}\right)}{a} - s \tag{2.2.6}$$

$$\mathbf{NX}(A, L) = \frac{l\left(-(1+aA\,\mathrm{B}e^{-\mathrm{as}})^{-\frac{1}{a l}}(aA+\mathrm{B}e^{\mathrm{as}})+\mathrm{B}e^{\mathrm{as}}(1+a\,\mathrm{B}e^{-\mathrm{as}}(A+L))^{1-\frac{1}{a l}}\right)}{al-1} \tag{2.2.7}$$

Formula 2.1.3 becomes:

$$\mathbf{PX}(A, L) = \frac{e^{-\mathrm{ds}}\, l\left((1+aA\,\mathrm{B}e^{-\mathrm{as}})^{-\frac{d+\frac{1}{l}}{a}}(-aA-\mathrm{B}e^{\mathrm{as}})+\mathrm{B}e^{\mathrm{as}}(1+a\,\mathrm{B}e^{-\mathrm{as}}(A+L))^{1-\frac{d+\frac{1}{l}}{a}}\right)}{al-dl-1} \tag{2.2.8}$$

So far the solution expressions are pretty simple. For calculation these formulae can be programmed into spreadsheets or supporting macro programs easily, and use trivial computing capacity when running.

Another natural extension of the report time assumption leads to MOD3:

- $S_R(t) = e^{-t/s}$, exponential distribution for report time with mean s

MOD3 still produces a closed form solution, but the expressions are much longer and use hypergeometric functions (Appendix A). The reader is cautioned that although these formulae may superficially seem intimidating, they are still relatively easy to program for calculation and consume trivial computer resources, typically only a few hundred floating point calculations, and offer many other advantages over simulation that will be discussed in Section 4.

One concern about the exponential distribution as used for reporting time and claim life is that it has constant mean residual life (i.e. a claim that begins with expected time to closure of 20 years but happens to remain open after 60 years, as 5% of such claims do, still is expected to remain open 20 more years.). Generally beyond some time limit, for example 100 years, it is reasonable to expect that all claims have been reported and closed. The functional forms chosen for report time and claim life should have very low survival probability beyond this time limit. So the exponential distribution model would tend to be appropriate for situations where average claim life (or report lag) is relatively short, perhaps 5-10 years or less, but a significant fraction of claims do remain open for (or are reported at) periods several times longer, perhaps 30-60 years. A better model for claim life in most situations would be:

- $S_T(t) = e^{-\frac{\pi x^2}{4 l^2}}$, for time the claim is open with mean l.

However, the author was unable to find a corresponding continuous probability distribution for report time that led to a final closed form solution. The author would be very appreciative to any reader who can find such a model – or any other interesting model with a closed form solution – and forward it to him. Nevertheless, a closed form solution can still be obtained by using a less elegant finite discrete model for R. As a simple example:

- R = *0, s,* or *2s* each with probability *1/3* respectively

The last two assumptions for T and R lead to MOD6, whose solution formulae are detailed in Appendix C, along with MOD4 and MOD5, which have the same report lag assumptions as MOD1 and MOD2, respectively. This illustrates an important point. In cases where continuous distributions form expressions so complicated that closed form solutions cannot be found, a finite distribution can be substituted. Unfortunately these finite distributions must be limited to a few allowed values to avoid long and unwieldy expressions with many terms to sum.

Another modeling concern is that report lag and time between report and closure are independent. In some real world situations claims reported at long lags after an exposure period would likely involve much older beneficiaries and would tend to remain open for shorter periods of time than claims reported earlier. A model that allowed for anti-correlation between T and R might be desirable but will not be considered in this paper.

3. EXAMPLE APPLICATIONS

This section presents applications to hypothetical situations that are representative of real world situations. The solution applications are somewhat simplified for expository purposes. In practice closed form annuity models should not be applied with overconfidence. Much attention should be paid to parameter and model uncertainties. Prominent reinsurance industry chief executives, with actuarial backgrounds, have pointed out historical pitfalls due overconfidence in actuarial models (Stanard and Wacek [11]). Another important consideration is that the models allow for the return on capital and income taxes through the discount rate (Butsic [3]), with a lower discount rate corresponding to a higher rate of return or more required capital.

3.1 Individual Claim Commutation Net vs Ceded Exposure Rating

MOD4 is applicable to the example in the introduction. Tables 1 and 2 show the expected values of the layered losses (See formula details in Appendix A), at benefit inflation assumptions that vary from 4% to 11%, on a undiscounted and a 3% discounted basis, respectively. The annuity model itself reduces the spread between expected losses in the reinsurance layers the 4% and 11% inflation assumptions. For a discounted reserve the spread is a bit wider since the higher inflation rate simultaneously increases the expected losses to the layers and speeds up the timing, decreasing the discount.

Table 1

Example Commutation Loss Nominal Expected Values, MOD 4

Benefit Inflation	10m xs 0	15m xs 10m	25m xs 25m
4%	7,098,059	4,204,456	2,526,378
5%	7,512,175	5,599,403	4,519,238
6%	7,839,130	6,798,060	6,573,467
7%	8,101,604	7,806,251	8,503,398
8%	8,315,646	8,649,693	10,236,941
9%	8,492,693	9,356,646	11,760,854
10%	8,641,010	9,952,287	13,087,796
11%	8,766,671	10,457,463	14,239,711

Table 2

Example Commutation Loss Present Values at 3% Discount, MOD4

Benefit Inflation	10m xs 0	15m xs 10m	25m xs 25m
4%	3,804,800	957,399	337,133
5%	4,161,881	1,506,256	786,403
6%	4,481,880	2,083,394	1,395,349
7%	4,768,343	2,656,426	2,106,887
8%	5,025,315	3,207,666	2,871,281
9%	5,256,633	3,728,593	3,652,392
10%	5,465,706	4,215,998	4,426,331
11%	5,655,473	4,669,605	5,178,355

Tables 1 and 2 tend to suggest compromise a nominal reserve value and commutation value of about \$8m and \$3m, respectively, for the 15m xs 10m treaty and about \$9.5m and \$2.5m, respectively, for the 25m xs 25m treaty.

3.2 Prospective Per Claim Excess Exposure Rating

Consider prospectively rating the \$15m xs \$10m layer from the example in the introduction. MOD6 is applicable to the example in the introduction. Since the layer is per claim frequency will not affect the exposure rates. Various parameter assumption scenarios are shown in Table 3 and corresponding exposure rates are shown in Table 4.

Table 3

Example Scenarios

Scenario	Average Claim Life	Benefit Inflation	Discount	Average Report Lag	Annual Benefits
1	30	5%	0%	1	75,000
2	30	5%	0%	1	150,000
3	30	5%	0%	3	75,000
4	30	5%	0%	3	150,000
5	30	5%	3%	1	75,000
6	30	5%	3%	1	150,000
7	30	5%	3%	3	75,000
8	30	5%	3%	3	150,000
9	30	9%	0%	1	75,000
10	30	9%	0%	1	150,000
11	30	9%	0%	3	75,000
12	30	9%	0%	3	150,000
13	30	9%	3%	1	75,000
14	30	9%	3%	1	150,000
15	30	9%	3%	3	75,000
16	30	9%	3%	3	150,000
17	50	5%	0%	1	75,000
18	50	5%	0%	1	150,000
19	50	5%	0%	3	75,000
20	50	5%	0%	3	150,000
21	50	5%	3%	1	75,000
22	50	5%	3%	1	150,000
23	50	5%	3%	3	75,000
24	50	5%	3%	3	150,000
25	50	9%	0%	1	75,000
26	50	9%	0%	1	150,000
27	50	9%	0%	3	75,000
28	50	9%	0%	3	150,000
29	50	9%	3%	1	75,000
30	50	9%	3%	1	150,000
31	50	9%	3%	3	75,000
32	50	9%	3%	3	150,000

Table 4

Scenario Estimates, MOD6

Scenario	Expected Loss 15m xs 10m	Expected Loss xs 0	Exposure Rate
1	1,909,437	8,561,971	22.3%
2	4,605,080	17,123,942	26.9%
3	2,210,921	9,462,441	23.4%
4	5,097,247	18,924,882	26.9%
5	451,287	3,521,110	12.8%
6	1,534,216	7,042,221	21.8%
7	517,975	3,664,810	14.1%
8	1,676,107	7,329,619	22.9%
9	5,809,299	49,344,415	11.8%
10	8,347,250	98,688,829	8.5%
11	6,447,687	59,075,990	10.9%
12	9,016,099	118,151,980	7.6%
13	2,124,825	12,390,469	17.1%
14	3,765,838	24,780,937	15.2%
15	2,347,693	13,970,214	16.8%
16	4,036,931	27,940,428	14.4%
17	7,010,277	56,319,066	12.4%
18	9,718,957	112,638,132	8.6%
19	7,402,612	62,242,194	11.9%
20	10,091,418	124,484,387	8.1%
21	1,597,912	8,284,288	19.3%
22	3,164,089	16,568,576	19.1%
23	1,675,854	8,622,376	19.4%
24	3,248,388	17,244,752	18.8%
25	10,630,733	5,168,591,731	0.2%
26	12,127,826	10,335,447,376	0.1%
27	11,041,391	6,187,721,601	0.2%
28	12,472,307	12,372,969,781	0.1%
29	3,856,218	138,561,666	2.8%
30	5,439,707	277,122,982	2.0%
31	3,989,718	156,227,807	2.6%
32	5,555,503	312,455,115	1.8%

Aside from the extreme Scenarios 25-32, which have both high benefit inflation and high average claim life, possibly raising doubts about the implicit assumption of primary rate adequacy, the exposure rates in Table 4 seem to cluster around the 15% to 25% range. In practice these exposure rates would need to be reduced by primary insurer underwriting expenses not passed to the reinsurer and the fraction of losses resulting from severe disability claims that are capable of penetrating the layer. For example if primary underwriting expense credit is 20% and only about 30% of losses in the underlying casualty exposures are severe disability then the rate, then range of final cession rates corresponds to about (100% - 20%) x 30% = 24% of the exposure rate, for a cession rate range of about 4% to 6%. Another possible refinement would be to use a lower discount rate for the excess layer to account for a higher cost of capital.

3.3 Prospective Non-Catastrophic Per Occurrence Exposure Rating

Although the technique in this paper does not readily lend itself to the per occurrence context, allowing for several dramatically different values in the annual benefit provides a crude adaptation to low frequency multiple claim occurrence situations. The simplifying assumption can be made that each claim clashing into the layer from the same occurrence is identical with the same benefit amount, lifetime, and discount. Then for a fixed number of claims the single claim model can be used with the total annual benefit for all of the claims substituted for the annual benefit. Although this does not let the claim characteristics vary for a given occurrence, it is easy to let them vary by the number of claims from a given occurrence. This is highly desirable as in many situations multiple claim occurrences tend to result in more severe and longer term disabilities. Tables 5 and 6 show this technique using MOD6 with the fixed assumptions that benefit inflation is 6% and the discount rate is 3%.

Table 5

Example Per Occurrence Undiscounted, MOD6

Number of Claims From Occurrence	Probability	Annual Benefit Per Claim	Average Claim Life
1	50.00000%	50,000	10
2	25.00000%	75,000	14
3	12.50000%	100,000	18
4	6.25000%	125,000	22
5	3.12500%	150,000	26
6	1.56250%	175,000	30
7	0.78125%	200,000	34
8	0.78125%	225,000	38

Table 6

Example Per Occurrence 3% Discounted, MOD6

Number of Claims	Expeced Loss 15m xs 10m	Expected Loss xs 0	Exposure Rate
1	1	650,485	0.0%
2	6,799	1,490,633	0.5%
3	170,650	2,799,830	6.1%
4	752,307	4,706,863	16.0%
5	1,653,234	7,378,498	22.4%
6	2,625,240	11,031,574	23.8%
7	3,522,071	15,949,257	22.1%
8	4,297,517	22,502,976	19.1%
Overall	223,824	2,045,413	10.9%

3.4 Excess Layer Paid Loss Development

Empirical triangles of loss development for excess layers are extremely sparse (Pinto [9]). Worse still they are more vulnerable than ground up triangles to changes over time in average claim severities. It has shown how a loss distribution can be interpreted to be an annuity density assuming a constant payment rate and this can then be used to produce tail factors for loss development (Corro [5]). The framework of this paper can be used to estimate loss development factors for excess layers. The expected amount paid in the layer at time T can be determined by limiting the claim life time limits of integration to T in Formula 2.1.2 as shown in Formula 3.4.1.

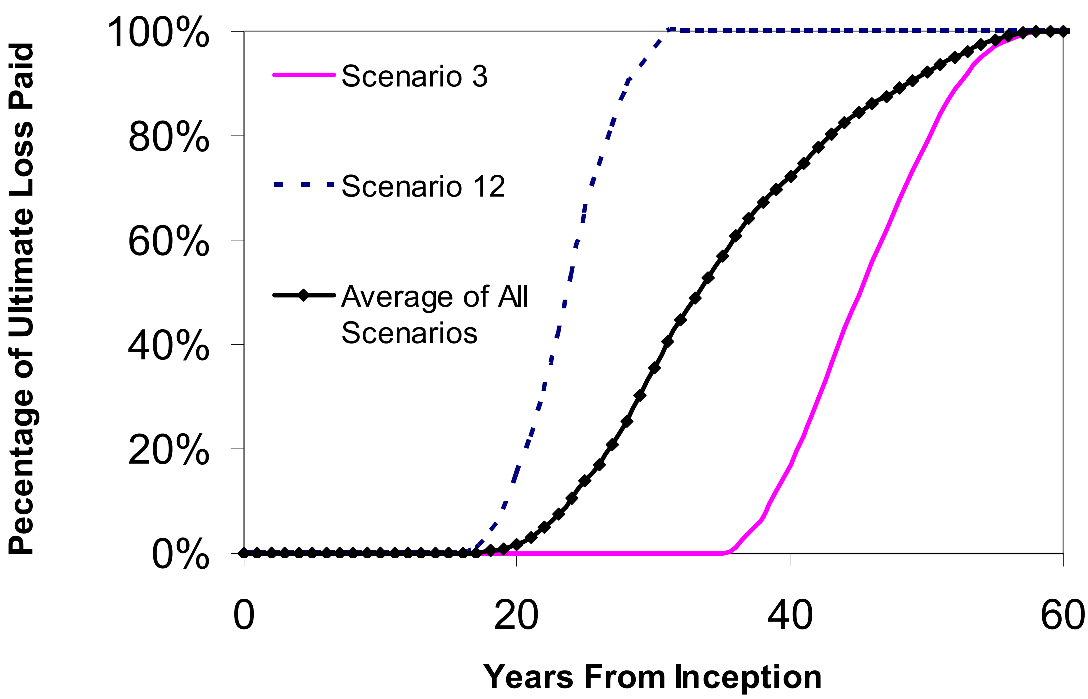
$$NXT(A, L, T) = \int_0^T d\,F_R(r) \int_{\mathrm{Min}(T, C^{-t}(A,R))}^{\mathrm{Min}(T, C^{-t}(A+L,R))} P(t)\, S_T(t-R)\, dt \qquad (3.4.1)$$

NXT(A, L, T) / NXT(A, L, Infinity) is a reasonable proxy for the expected percentage of ultimate losses paid by time *T*.

Figure 1 shows the paid development patterns for MOD6 using Scenarios 3 and 12, and the overall average for all of the scenarios in Table 3. Scenario 3 does not begin to penetrate the layer until 36th year after inception, but totally exhausts the layer by the 58th year. Scenario 12 begins to penetrate the layer in the 17th year and exhausts the layer by the 31st year. Any of the Scenarios will produce 0 losses in the layer for many years and then exhaust the layer in a relatively quick period of time thereafter, even accounting for the stochastic pattern of reporting time and claim life. The timing of the layer payment is very sensitive to the scenario assumptions. For overall paid development percentages it would make sense to average over all the scenarios. These percentages can then be used in loss development factor, Bornhuetter-Ferguson, or other aggregate loss reserving methods. Generally, the Bornhuetter-Ferguson would need to be used as actual paid loss experience for the layer is very sparse and generally 0 for a long period of time after inception.

Figure 1

Example Paid Development Patterns for $15m xs $10m, MOD6



3.5 Excess Layer Case Incurred Loss Development

Whereas estimating paid loss development with an annuity model is straightforward in principal, the incurred loss development problem is very ambiguous. Case reserves should generally always reflect expect ultimate loss, and the only systematic development over time should be due to late reporting of claims and possibly the unraveling of tabular discount. However, it is often the case that case reserves at any point in time reflect an implicit discounting beyond that sometimes allowed for tabular discount. The expected case discounted amount paid in the layer up to time T can be determined by limiting the report time and payment time limits of integration to T in Formula 2.1.3 as shown in Formula 3.5.1.

$$PXT(A, L, T) = \int_0^T dF_R(r) \int_{\mathrm{Min}\left(T, C^{-t}(A,R)\right)}^{\mathrm{Min}\left(T, C^{-t}(A+L,R)\right)} D(t)\, P(t)\, S_T(t-R)\, dt \qquad (3.5.1)$$

The expected discounted case incurred amount in the layer based on claims reported through time T can be determined by limiting the report time limits of integration to T in Formula 2.1.3 as shown in Formula 3.5.2.

$$PXTR(A, L, T) = \int_0^T dF_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} D(t)\, P(t)\, S_T(t-R)\, dt \qquad (3.5.2)$$

Assuming case adjusters have a pretty good idea of ultimate nominal amount on cases reported, but effectively apply an implicit discount to the layered reserve $(NXT(A, L, T) + e^{dT}(PXTR(A, L,T) - PXT(A, L, T))) / NX(A, L)$ is a reasonable proxy for the ratio of case incurred losses to ultimate losses incurred at time *T*. In this expression case reserves at time T correspond to the expected nominal total payments to date in the layer and the discounted expected future payments in the layer. This is not the same as combining the expected ground-up payments to date at time T with the discounted reserve and then layering. There is an argument for the latter but it requires a much more complicated expression and tends to greatly delay the recognition of any reserves in the layer.

Figure 2 shows all scenario averages for paid and case incurred development patterns. Only the scenarios with the 3% discount, Scenarios 5-8, 13-16, 21-24, 29-32, are used for the case incurred patterns. The incurred line shows a jagged pattern early on. This is due to the three discrete values allowed for report time. The report times are very early relative to the actual payments in the layers. Of course, MOD6 could be easily modified to include more report times and later report times.

Figure 2

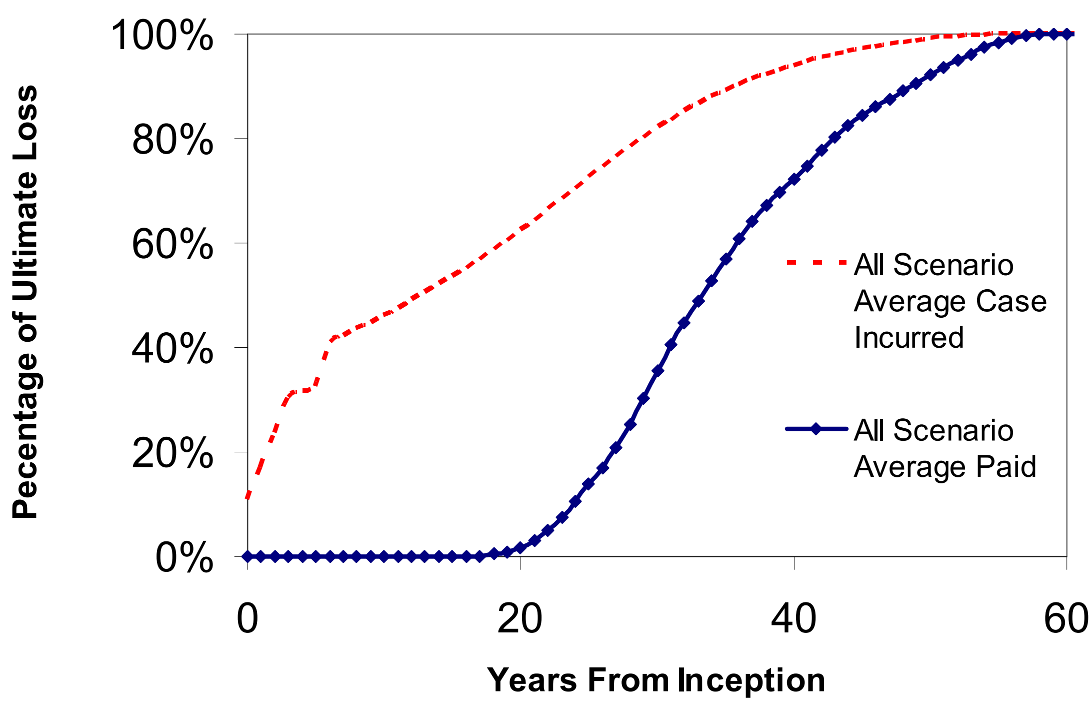


Table 7
Example Loss Development Patterns, MOD6

	% of Ultimate Paid			% of Ultimate Case Incurred
Years From Inception	Scenario 12	Scenario 3	All Scenario Average	All Discounted Scenario Average
1	0.0%	0.0%	0.0%	17.0%
2	0.0%	0.0%	0.0%	23.7%
3	0.0%	0.0%	0.0%	30.7%
4	0.0%	0.0%	0.0%	31.6%
5	0.0%	0.0%	0.0%	32.6%
6	0.0%	0.0%	0.0%	41.0%
7	0.0%	0.0%	0.0%	42.2%
8	0.0%	0.0%	0.0%	43.5%
9	0.0%	0.0%	0.0%	44.8%
10	0.0%	0.0%	0.0%	46.2%
11	0.0%	0.0%	0.0%	47.6%
12	0.0%	0.0%	0.0%	49.1%
13	0.0%	0.0%	0.0%	50.6%
14	0.0%	0.0%	0.0%	52.1%
15	0.0%	0.0%	0.0%	53.7%
16	0.0%	0.0%	0.0%	55.3%
17	0.0%	1.0%	0.1%	57.0%
18	0.0%	4.5%	0.5%	58.8%
19	0.0%	8.3%	1.0%	60.7%
20	0.0%	14.9%	1.8%	62.6%
21	0.0%	22.3%	3.1%	64.5%
22	0.0%	31.1%	5.0%	66.6%
23	0.0%	41.9%	7.5%	68.6%
24	0.0%	53.4%	10.5%	70.6%
25	0.0%	65.4%	13.8%	72.7%
26	0.0%	74.2%	17.1%	74.6%
27	0.0%	81.7%	20.7%	76.6%
28	0.0%	89.6%	25.2%	78.6%
29	0.0%	93.2%	30.2%	80.4%
30	0.0%	96.9%	35.6%	82.1%
31	0.0%	100.0%	40.5%	83.7%
32	0.0%	100.0%	44.8%	85.3%
33	0.0%	100.0%	48.9%	86.7%
34	0.0%	100.0%	52.8%	87.9%
35	0.0%	100.0%	56.8%	89.2%
36	1.1%	100.0%	60.7%	90.3%
37	4.0%	100.0%	64.3%	91.3%
38	6.9%	100.0%	67.4%	92.2%
39	11.8%	100.0%	69.7%	93.1%
40	17.0%	100.0%	72.1%	94.0%
41	22.6%	100.0%	74.7%	94.7%
42	29.6%	100.0%	77.7%	95.4%
43	36.4%	100.0%	80.3%	96.1%
44	43.0%	100.0%	82.6%	96.6%
45	49.5%	100.0%	84.5%	97.1%

46	55.8%	100.0%	86.1%	97.6%
47	61.9%	100.0%	87.6%	98.0%
48	67.8%	100.0%	89.1%	98.4%
49	73.5%	100.0%	90.6%	98.7%
50	79.0%	100.0%	92.1%	99.0%
51	84.3%	100.0%	93.6%	99.3%
52	89.0%	100.0%	95.0%	99.5%
53	92.0%	100.0%	96.2%	99.7%
54	94.9%	100.0%	97.4%	99.8%
55	97.1%	100.0%	98.5%	99.9%
56	98.3%	100.0%	99.3%	100.0%
57	99.5%	100.0%	99.8%	100.0%
58	100.0%	100.0%	100.0%	100.0%
59	100.0%	100.0%	100.0%	100.0%
60	100.0%	100.0%	100.0%	100.0%

3.6 Comparison of Different Reinsurance Treaties

A table like Table 4 can be useful in comparing reinsurance treaties with different underlying exposures or even different treaty terms. Even when actuarial models for reinsurance may be unreliable for absolute numerical estimates, these may be very valuable at determining relative differences in expected losses due to differences in underlying exposures or contract terms (Stanard [10]). Suppose the details known about Treaties A and B are:

1. Both treaties cover the layer $15m xs $10m on an effectively per claim basis, as clash effects are expected to be trivial.
2. Treaty A covers very much younger beneficiaries and claims will tend to remain open about 50 years versus 30 years for B.
3. Treaty A's younger beneficiaries tend to require a much lower annual medical benefit, bringing the total annual benefit closer to 75,000, versus 150,000 for Treaty B.
4. Treaty B benefits are much more dominated by higher inflation medical expenses leading to an overall benefit inflation of about 9% versus about 5% for Treaty A.
5. Both treaties involve $100m subject premium, 20% underwriting expense credits, and 30% of losses coming from serious disabilities, and a market cession rate of 2.8%.

Given a choice which treaty should the reinsurer participate in? As the details say nothing about reporting lag or discount the relevant scenarios are 17, 19, 21, 23 for Treaty A and 10, 12, 14, 16 for Treaty B. The exposure rate ranges for these groups of scenarios are 11.9% to 19.4% and 7.6% to 15.2%, respectively. The corresponding cession rate ranges are 2.9% to 4.7% and 1.8% to 3.6%, respectively. Since the market cession rate is just below the modeled range for Treaty A but above the modeled range midpoint for Treaty B, Treaty B appears to be the more attractive offer. The layer exposure to longer average claim life for Treaty A tends to dominate the higher benefit and higher benefit inflation of Treaty B.

3.7 Excess Layer Frequency, Severity, and Variance

Excess layer frequencies, severities, and variances can be calculated from the distribution of ultimate layer amount, and it is easy to determine this distribution from derivative of Formula 2.1.2 (Appendix B). The excess loss functions NX and PX only accounted for variability (MOD1-6) in the claim life and the report lag (MOD3, 6). Variability in the other parameters, such as benefit inflation, has been dealt with through scenario testing in the previous examples. The ultimate layer amount distribution underlying MOD6 is given by Equation 3.7.1.

$$F_{\text{layer loss}}(L) = \frac{1}{3}\left(-e^{-\frac{\pi\left(4a^2s^2+\log^2\left(\frac{aL}{B}+e^{2as}\right)\right)}{4a^2l^2}}\left(\frac{aL}{B}+e^{2as}\right)^{\frac{\pi s}{a l^2}} - e^{-\frac{\pi\left(a^2s^2+\log^2\left(\frac{aL}{B}+e^{as}\right)\right)}{4a^2l^2}}\left(\frac{aL}{B}+e^{as}\right)^{\frac{\pi s}{2 a l^2}} - e^{-\frac{\pi\log^2\left(\frac{aL}{B}+1\right)}{4a^2l^2}}+3\right) \tag{3.7.1}$$

For Scenario 1 from Section 3.2, 25% of all claims penetrate the layer and 6.2% of claims exhaust the layer. The average severity of claims penetrating the layer is 1,909,437 / 25% = 7,637,748.

The cumulative ultimate amount density underlying MOD6 is given by Equation 3.7.2.

$$f_{\text{layer loss}}(L) = \frac{\pi}{6 a l^2} \left(\frac{e^{-\frac{\pi\left(4a^2 s^2+\log^2\left(\frac{aL}{B}+e^{2as}\right)\right)}{4a^2 l^2}} \left(\log\left(\frac{aL}{B}+e^{2as}\right)-2as\right)\left(\frac{aL}{B}+e^{2as}\right)^{\frac{\pi s}{a l^2}}}{e^{2as} B + aL} + \frac{e^{-\frac{\pi \log^2\left(\frac{aL}{B}+1\right)}{4a^2 l^2}} \log\left(\frac{aL}{B}+1\right)}{B+aL} + \frac{\left(\frac{aL}{B}+e^{as}\right)^{\frac{\pi s}{2 a l^2}} e^{-\frac{\pi\left(a^2 s^2+\log^2\left(\frac{aL}{B}+e^{as}\right)\right)}{4a^2 l^2}} \left(\log\left(\frac{aL}{B}+e^{as}\right)-as\right)}{e^{as} B + aL} \right) \tag{3.7.2}$$

The conditional second moment of the layer loss is given by Equation 3.7.3.

$$\frac{(1-F(25000000))\,(25000000-10000000)^2 + \int_{10000000}^{25000000} (L-10000000)^2\, f(L)\, dL}{1-F(10000000)} \tag{3.7.3}$$

Numerically this is 8.85 x 10^{13}, making the severity variance 3.02 x 10^{13} and the severity standard deviation 5,490,000. If the number of ground-up claims is Poisson with mean 10, then the number of claims penetrating the layer is Poisson with mean 2.5 and variance 2.5. The total layer variance is then easily determined by the standard formula:

$E[N]Var[L]+Var[N]E[L]^2 = 2.5\ (3.02 \times 10^{13}) + 2.5\ (7{,}637{,}748)^2 = 2.21 \times 10^{14}$,

and total layer standard deviation is 1.48 x 10^7, for a coefficient of variation of 1.48 x 10^7 / 7,637,748 = 195%.

The analysis above omitted the very important considerations of parameter uncertainty in the benefit amount, claim life, report lag, and benefit inflation. These can be incorporated by running the calculation for all of Scenarios 1-32 in Table 3. Each scenario can be given equal weight and the total variance for the layer is then the average variance for each scenario plus the variance of the scenario layer means.

The analysis above can also be performed fairly easily on a discounted basis for MOD1, 2, 4, 5 where the report lag is fixed. However, when the report time is stochastic the discounted basis analysis becomes much harder because the discounted layered amount of the claim is no longer uniquely determined by the total discounted amount of the claim. Two claims with the same total discounted amount, but different report times, can have different layered amounts, nominal or discounted.

4. CLOSED FORM MODELS VERSUS SIMULATION

Closed form solutions, or even numerical integration solutions, are often much more computationally efficient than simulation. This higher efficiency makes testing many different parameter assumptions much easier. Such solutions also avoid concerns about structural patterns or biases in pseudorandom numbers. Simulation also may require tremendous amounts of memory or disk storage space if simulated data is not tabulated in bins during the simulation. The primary limitation of models with closed form, or numerical integration, solutions is that they do not allow for the kind of opened ended structural formulations that may include all sorts of special limitations or dependencies and can always be programmed into a simulation.

5. POSSIBLE FURTHER DEVELOPMENTS

5.1 Other Models and Standardized Software Modules

There are several obvious areas for improvement on the models presented in this paper.

- The most important area for improvement is probably incorporating good functional forms for stochastic uncertainty in the parameters.
- Also, of interest probability models for report lag R and claim life T. It

would be desirable to have simple functional forms that fit actual mortality or decrement experience tables well and possibly even involve a dependence relationship between R and T.

- Another concern is that in many situations long term disability claims have large medical expense early on. The models in this paper assume that the initial annual benefit simply increases with inflation. An extra benefit amount paid only at the time of report could be added.

The most burdensome part of using these models is actually finding models with closed form or efficient numerical integration solutions and programming software to calculate the solution values. Once found, the solutions can be programmed into standardized software modules, or macros in spreadsheets, and efficiently used. A major concern is debugging such modules - and even verifying the solution itself – as the complexity of the solution expressions can easily hide errors. A key tool that greatly facilitates debugging is testing the software calculation for several key properties that must hold (Appendix C).

5.2 Simplified Benchmarks for Non-actuaries

The mathematical sophistication required to define models, find solutions, and program them is on par with that of a credentialed actuary or a Ph.D. in a mathematical science. For practical use by managerial decision makers the model solutions must be turned into numerical exhibits and/or graphs, fixed in print or interactively generated through user friendly software. Since the initially there is a significant cost in high skilled labor to produce the software, care should be taken in determining what practical situations typically arise and what kind of condensed presentation will be most useful. Tables 1-6 and Figure 1 are very modest examples of such formats and much more can be done.

6. CONCLUSIONS

Annuity models offer a practical solution to problems of analyzing casualty excess reinsurance. Finding models with closed form solutions or easy numerical integration has been greatly facilitated by the advent of software, such as MATHEMATICA, capable of performing mathematical symbolic manipulation and efficient numerical analysis. These models are not subject to several problems of simulation analysis such as: biases of random number generators, large overly complex spreadsheets, extremely long times to run simulations and test parameters, and huge sizes of output. The primary challenge in implementation is to define a reasonable model that has closed form and then program a calculation module in a spreadsheet. Once the module is checked for reliability, spreadsheet analyses, including parameter sensitivity and graphical inspection, may be done in small simple spreadsheets that use very little computational capacity. Annuity models, like most actuarial models, will never be "right" and should not be blindly relied on for point estimates. However, if used properly these models do facilitate the understanding of qualitative effects, and the quantification of relative differences and magnitudes of uncertainty.

Appendix A – Other Model Solutions

MOD3

$$
\begin{aligned}
\mathbf{NX}(A, L) = -\,&\frac{l}{(a l-1)(a s-1)} \\
&-B\,{}_2F_1\!\left(\frac{1}{al},\ \frac{1}{\mathrm{as}}-1;\ \frac{1}{a s};-\frac{a A}{B}\right) \\
&+B\,{}_2F_1\!\left(\frac{1}{al},\ \frac{1}{\mathrm{as}}-1;\ \frac{1}{a s};-\frac{a (A+l)}{B}\right) \\
&+(a s-1)\,a\left(A\,{}_2F_1\!\left(\frac{1}{al},\ \frac{1}{a s};\ \frac{\mathrm{as}+1}{a s};-\frac{a A}{B}\right)\right. \\
&\qquad\left.-(A+l)\,{}_2F_1\!\left(\frac{1}{al},\ \frac{1}{a s};\ \frac{\mathrm{as}+1}{a s};-\frac{a (A+l)}{B}\right)\right)
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\mathbf{PX}(A, L) = -\,&\frac{l}{(a l-d l-1)(a s-d s-1)(d s+1)} \\
&-(d s B+B)\,{}_2F_1\!\left(\frac{d l+1}{al},\ \frac{\mathrm{sd}+1}{\mathrm{sa}}-1;\ \frac{d s+1}{a s};-\frac{a A}{B}\right) \\
&+(d s B+B)\,{}_2F_1\!\left(\frac{d l+1}{al},\ \frac{\mathrm{sd}+1}{\mathrm{sa}}-1;\ \frac{d s+1}{a s};-\frac{a (A+l)}{B}\right) \\
&+(a s-d s-1)\,a\left(A\,{}_2F_1\!\left(\frac{d l+1}{al},\ \frac{d s+1}{a s};\ \frac{(a+d) s+1}{a s};-\frac{a A}{B}\right)\right. \\
&\qquad\left.-(A+l)\,{}_2F_1\!\left(\frac{d l+1}{al},\ \frac{d s+1}{a s};\ \frac{(a+d) s+1}{a s};-\frac{a (A+l)}{B}\right)\right)
\end{aligned}
\tag{A.2}
$$

$_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} (a)_k\, (b)_k / (c)_k\; z^k / k\,!$ is the hypergeometric function where $(a)_k = a\,(a + 1)\ldots(a + k - 1)$.

MOD4

$$\mathbf{NX}(A, L) = B\,e^{\frac{a^2 l^2}{\pi}}\, l \left(\operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,(A+L)}{B} + 1\right) - 2\,a^2\,l^2}{2\,a\sqrt{\pi}\;l}\right) - \operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,A}{B} + 1\right) - 2\,a^2\,l^2}{2\,a\sqrt{\pi}\;l}\right)\right) \tag{A.3}$$

$$\mathbf{PX}(A, L) = B\,e^{\frac{(a-d)^2 l^2}{\pi}}\, l \left(\operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,(A+L)}{B} + 1\right) - 2\,a\,(a-d)\,l^2}{2\,a\sqrt{\pi}\;l}\right) - \operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,A}{B} + 1\right) - 2\,a\,(a-d)\,l^2}{2\,a\sqrt{\pi}\;l}\right)\right) \tag{A.4}$$

$\operatorname{erf}(z) = 2 \big/ \sqrt{\pi} \int_0^z e^{-t^2}\, d\,t$ is the error function.

MOD5

$$\mathbf{NX}(A, L) = B\, e^{\frac{a\left(a\, l^2+\pi\, s\right)}{\pi}}$$

$$l\left(\mathrm{erf}\left(\frac{-2a^2 l^2 - a\pi\, s + \pi \log\left(\frac{a\,(A+L)}{B} + e^{as}\right)}{2a\sqrt{\pi}\, l}\right) - \mathrm{erf}\left(\frac{-2a^2 l^2 - a\pi\, s + \pi \log\left(\frac{a\,A}{B} + e^{as}\right)}{2a\sqrt{\pi}\, l}\right)\right) \quad \text{(A.5)}$$

$$\mathbf{PX}(A, L) = B\, e^{\frac{(a-d)\left((a-d)\, l^2+\pi\, s\right)}{\pi}}$$

$$l\left(\mathrm{erf}\left(\frac{-2a(a-d)\, l^2 - a\pi s + \pi \log\left(\frac{a\,(A+L)}{B} + e^{a\,s}\right)}{2a\sqrt{\pi}\, l}\right) - \mathrm{erf}\left(\frac{-2a(a-d)\, l^2 - a\pi\, s + \pi \log\left(\frac{a\,A}{B} + e^{a\,s}\right)}{2a\sqrt{\pi}\, l}\right)\right) \quad \text{(A.6)}$$

MOD6

NX(A,L) and PX(A,L) are simply the formulas for MOD5 averages over the values 0, s, 2s substituted for s.

Appendix B – Converting Limited Pure Premiums into Loss Distributions

The nominal expected losses in a layer are equal to the integral of the survival function in the layer. Therefore the distribution can be derived from the derivative with respect to L of the expected losses in the layer [0, L].

$$NX(0, L) = \int_0^L (1 - F_{\text{total loss}}(l))\, dl \tag{B.1}$$

$$F_{\text{total loss}}(L) = 1 - \frac{d\,NX(0, L)}{dL} \tag{B.2}$$

Formula B.2 is particularly useful, as it allows easy conversion of the limited pure premium into the loss distribution. Assuming a constant report time s, the total discounted losses in a layer is an increasing function of the total losses in the layer, and hence uniquely determined by the total losses in the layer.

$$\text{PV}(L) = \int_s^{C^{-t}(L,s)} D(t)\, P(t)\, dt \tag{B.3}$$

$$F_{\text{discounted total loss}}(L) = F_{\text{total loss}}\left(\text{PV}^{-1}(L)\right) \tag{B.4}$$

When the report time is stochastic determining the loss distribution is more difficult.

Appendix C –Test Checklist for Solution Software

The following questions should all be answered affirmatively to check a solution calculation program. Affirmative answers do not guarantee correctness, but negative answers indicate an error.

1. Do PX(A, L) and NX(A, L) both increase as L increases and decrease as A increases?
2. Does PX(A, L) = NX(A, L) when D(t) = 1 for all t?
3. Is PX(A, L) < NX(A, L) when D(t) < 1 for all t?
4. Is PX(A, L) > NX(A, L) when D(t) > 1 for all t?
5. Does PX(A, L) increase when D(t) increases for all t?
6. Does NX(A, L) increase as P(t) increases for all t?
7. Does NX(A, L) increase as $S_T(t)$ increases for all t?

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