Estimation of Trends Using Ordinary Differential Equations: An Application to Occupational Injuries

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Abstract

In this paper, we present and discuss the dynamic time varying version of trend estimation. These models often underlie the analytical functions that are used in practice by actuaries and economists. We also show how one of the most frequently used softwares (the SAS systems) by practitioners and researchers can be used to fit the dynamics to data. An alternate formulation of the Ordinary Least Squares (OLS) is also given. Using this technique, we analyze occupational injury and illness data from 12 countries. The results for most countries have shown an average decline.

I. Introduction

In this paper, we estimate trends in occupational injuries and illnesses for 12 countries using the dynamics as represented by the ordinary differential equations (ODEs)[1]. We show that the most commonly employed models in practice are analytical solutions of the basic differential equations. Differential equations are used in many applications in real life such as engineering. For the cases of the linear and exponential trend models, we demonstrate that these models yield the same results.

To provide further insight into the modeling and estimation of trends, we present and discuss in some detail the relationship between the continuous time dynamics of the time series variable (injuries) and their analytical solutions. In other words, we want to highlight the link between the continuous time dynamics and their solutions, which are often used in regression analysis. Dynamic estimation, i.e., fitting models that are represented by equations that describe the time evolution of the economic/actuarial variables, is also performed (Ussif, Sandal and Steinshamn, 2002a, b). In areas such as oceanography and meteorology such dynamic parameter estimation technique is called data assimilation (Evensen, Dee and Schroeter, 1998, Matear, 1995). We use the SAS dynamic estimation capability (see the SAS Institute's online documentation) which is not provided by most software packages (see also Ussif et al., 2002 a, b). The goal is to point out that such a capability exists and can be used to perform more advanced dynamic systems estimation. This can be used to fit nonlinear dynamic systems that arise from the relaxation of the linearity assumptions often made in economics. Pesaran and Potter

[1] Please note the differences in the number of observations available for each country. This should however note be a problem in this application.

(1992) in their introductory notes on nonlinear dynamics and econometrics argue that the rich dynamics in nonlinear models in economics be explored.

The structure of this paper is as follows. In the next section, we discuss the dynamic representation of the models. This is followed by the estimation of the trend coefficients using SAS software and a discussion of the results. A technical note on the adjoint method is also presented. We then summarize and conclude the paper.

Dynamic representation of the trend models

In this section, we show how the purely linear and exponential functions of time that are being used for trend estimation can be derived as solutions of their corresponding continuous time dynamic equations, i.e., equations used to describe how systems change or evolve over time. This is important because understanding the relationships can be very useful to economists and researchers. It is often the case that reality necessitates the relaxation of the linearity assumptions in economics giving rise to nonlinear dynamic systems. Analytical solutions of these systems are in general unattainable for some relatively more complicated dynamics and the only method of estimation may be the dynamic approach. The dynamic capability is quite rare in most software packages in part because conventional economic analysis has in the past focused on simpler models.

Linear trend function: In the dynamic continuous time formulation, it is assumed that the absolute change with respect to time of the series is equal to a constant. That is, the average growth is constant during the period. Hence, the dynamics are given by

$$\frac{dy}{dt}=\beta,\ y(0)=\alpha \qquad (1)$$

where α is the initial value of the series. This is equivalent to assuming that $\frac{d^2y}{dt^2}=0$, where $\frac{dy(0)}{dt}=\beta$, $y(0)=\alpha$ are the initial conditions. It is quite easy to see that this equation has the solution

$$y_t=\alpha+\beta t \qquad (2)$$

which is the linear trend function in time. Thus, we can view the estimation of the parameters in (1) as fitting the solution (2) to a discrete data set. Note that $y(t)$ and y_t are used interchangeably in this case.

Exponential trend function: The dynamics in this case can be described by

$$\frac{dy}{dt}=\beta y,\ y(0)=y_0=e^{\alpha} \qquad (3)$$

that is, the percent growth rate is equal to a constant or that the absolute change is proportional to the current value of the series. We denote by y_0 the initial condition for the problem. It can be seen by inspection that this equation has the solution

$$y_t=\exp(\alpha+\beta t). \qquad (4)$$

This is the familiar exponential trend function of time used in estimating trends and growth rates. Its advantage is that the estimated coefficient is the average growth rate. The linear and exponential functions of time are often used in economics, business, and finance to forecast trends. Recent advances and progress in most statistical software packages allow us to fit nonlinear regression equations using nonlinear least squares techniques without having to use, for example, logarithmic transformation.

The SAS software has the additional capability of fitting dynamic systems to data without requiring that analytical or closed form solutions be available. This is important and very useful because for some relatively complicated dynamics, closed form solutions are often not attainable. Hence, dynamic estimation becomes the only option available. In dynamic estimation, the parameters of interest are estimated by fitting the dynamic equations rather than their solutions.

To illustrate the use of the dynamic estimation capability of SAS, we fit (see sample program) the dynamics represented by equation (3) and the results are compared with the results of the purely exponential function in Table 1. It can be observed that the results are consistent with those obtained using the log transformed function (shown in the log-linear column of Table 1). The agreement is quite impressive both qualitatively and quantitatively. The dynamic estimation was performed using the procedure "proc model" in SAS. Included is a sample SAS program for interested readers. In static dynamic option, the initial data point is used as the initial condition of the differential equation, while in the dynamic option; the initial condition(s) is estimated as an additional parameter. The nice thing about this procedure is that the dynamics are written as they are seen in the model equations. It is very important to understand the difference between the static and dynamic options when fitting dynamic models to data. For further details, readers are referred to the SAS manual (www.sasonline.com).

Technical note

It is the goal in this note to describe an alternative technique for estimating dynamic systems. The method in this section is much used in areas such as meteorology, oceanography, etc. (Evensen et al., 1998; Thacker, 1989). It has recently been applied to resource economics (see Ussif et al., 2002a-b). The approach is a data assimilation technique called the Adjoint Method (AM). In the AM, a loss or penalty function measuring the distance between the model solution and the observations is minimized. This is formulated in the following sections.

The Adjoint Method

The formulation of the adjoint method is as follows. We minimize the penalty function $J = \sum_{t=0}^{T-1} (y_t - y_t^{obs})^2$ subject to the dynamics in (3). Thus the statement of the problem is

$$\min_{\alpha,\beta} \sum_{t=0}^{T-1} (y_t - y_t^{obs})^2 \qquad (5)$$

subject to

$$\frac{dy}{dt} = \beta\, y, y(0) = y_0 = e^{\alpha}$$

where y_t, y_t^{obs} are the model solution and the observed value respectively and y_0 is the initial condition. The model solution is the numerical approximation often obtained by finite difference methods (Ussif et al., 2002b, Gerald and Wheatley, 1992). Note that, in determining the best fit to the data we estimate the initial condition as a parameter. It is important to note that the static option in SAS uses the first data point as the initial condition and thus does not estimate it. The problem in this example is trivial because the dynamics are linear. It becomes more complicated if the dynamics are nonlinear and coupled, that is, a simultaneous system of differential equations that are linked together through the variables (see Ussif et al., 2002b).

The constrained optimization problem can be solved by using the calculus of variations or optimal control theory. By constructing the continuous form of the Lagrangian L using μ and λ as the Lagrange multipliers, we have,

$$L[y, y_0, \beta] = J + \mu(y(0) - y_0) + \int_0^T \lambda(\frac{dy}{dt} - \beta\, y)dt \ . \qquad \text{(A6)}$$

The constrained problem (5) is now transformed into the unconstrained optimization problem of finding the extreme values of $L[y, y_0, \beta]$ in (6).

Using the calculus of variations, we can derive the adjoint equation. Detailed derivation of the adjoint equation is not given in this paper since the goal here is to present and formulate the problem. However, interested readers are referred to (Ussif et al., 2002 a-b or Evensen et al., 1998). The first order conditions are

$$\frac{\partial L}{\partial \mu} = 0, \quad \frac{\partial L}{\partial \lambda} = 0 \qquad (7)$$

$$\frac{\partial L}{\partial y}=0\,, \qquad (8)$$

Note that differentiating L with respect to μ and λ give back the initial condition and the model dynamics respectively, while differentiating with respect to y results in the so-called adjoint equation.

To make the analysis easier and for consistency with the continuous model dynamics, assume we have data continuously, i.e., the time period of observation or reporting is small (e.g. on the scale of a day or even more frequently), and also, rewriting $J=\frac{J}{2}$, then the Adjoint equation is

$$\frac{d\lambda}{dt}=-\beta\lambda+(y-y^{obs}),\quad \lambda(T)=0 \qquad (9)$$

$$\frac{\partial J}{\partial \beta}=-\int_{t=0}^{T}\lambda\, y\, dt \qquad (10)$$

$$\frac{\partial J}{\partial y_0}=-(\mu+\lambda(0)\beta)=-(1+\beta)\,\lambda(0),\ \mu=\lambda(0)\,. \qquad (11)$$

The gradients of the penalty function (10-11) are obtained by differentiating the Lagrangian with respect to the independent parameters (y_0,β) and are used together with an optimization routine (e.g. the Newton-Raphson method), to find the minimum of the penalty function. It can be shown that, one can estimate either α directly or its exponent

$y_0 = e^{\alpha}$ which is by definition the initial condition. We have chosen the latter for convenience and also for practical purposes since we may be able to guess the starting value from available data.

Implementation of the Algorithm

Implementation of the adjoint technique is quite straightforward. The algorithm is outlined below

- Choose the first guess for the disposable or free parameters i.e. the parameters that can be tuned in order to minimize the penalty function
- Integrate the forward model (3) over the time horizon[2]
- Calculate the penalty function
- Integrate the adjoint equation (9) and calculate the gradients (10-11)[3]
- Use an iterative procedure[4] to find the minimum of the penalty function

For this simple dynamic problem convergence of the iterative procedure to the absolute minimum may be possible. However, for more complex problems, i.e., highly nonlinear dynamics with many parameters to estimate, multiple extrema may exist and convergence to the absolute minimum can be difficult. Note that the problem reduces to solving a two point boundary value problem (Equations 3 and 9) and then calculating the gradients This makes it possible to calculate the gradients of several parameters simultaneously and more accurately compared to when using the finite difference methods (Huiskes, 1998).

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[2] This is usually done numerically since the models are much more complicated than the example used in this paper.
[3] See how the adjoint variable enters the gradient relationships in equations A10-11.
[4] This requires setting an appropriate convergence criterion for the minimization. Please see Ussif 2002b for example.

The Error-Covariance Matrix

While point estimates are often useful, their utility is greatly enhanced if their error bounds are also provided. Statistical tests can be performed and confidence intervals can also be constructed. When the errors in the observations are assumed to be normally distributed, the uncertainty in the optimal parameters is obtained by analyzing the Hessian matrix. The Hessian matrix is the second derivative of the penalty function with respect to the parameters. By differentiating J two times with respect to each of the parameters the Hessian matrix (H) is obtained as

$$H = \begin{bmatrix} \frac{\partial^2 J}{\partial^2 \alpha} & \frac{\partial^2 J}{\partial \alpha \partial \beta} \\ \frac{\partial^2 J}{\partial \alpha \partial \beta} & \frac{\partial^2 J}{\partial^2 \beta} \end{bmatrix}. \tag{12}$$

The Hessian matrix is symmetric and positive definite and is often called the Fisher information matrix in the econometrics literature. Inverting the Hessian matrix gives the approximate Variance-Covariance matrix (Greene, 1997; Matear, 1995). Hence, the diagonal elements of the Variance-Covariance matrix are the variances, which can be used to construct confidence intervals for the parameters.

The Adjoint Method is an efficient method for the minimization of the penalty function. It provides an efficient and reliable way of calculating the gradient(s) of the penalty function which allows for the simultaneous estimation of a large number of parameters (Huiskes, 1998; Matear, 1995). So called derivative free methods such as the simplex algorithm (Nelder and Mead, 1965) can also be used to optimize the penalty function. Other methods of minimizing the penalty function are simulated annealing (Greene, 1997; Matear, 1995) and the Markov Chain Monte Carlo technique (Harmon and Challenor, 1997). Kruger (1992) stated that, the adjoint method is about 100 times faster than simulated annealing. In general, the adjoint method is faster than the other methods.

Conclusions

This paper has demonstrated the utility of trend estimation using dynamic representation. The SAS systems have been used to fit two prototypes to data on occupational injuries for 12 countries. The conclusion is that such techniques are equally applicable. However, once the analysis becomes more complicated, this approach can be of tremendous help.

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```
* SAMPLE SAS PROGRAM ****************************************;
DATA REGDATA;
INPUT YEAR Y;
DATALINES;
   1      3.21
   2      4.01
   3      3.89
;
```

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* Dynamic Estimation Program: Model: $\dfrac{dy}{dt} = \beta\, y, y(0) = y_0$ *

**;

```
PROC MODEL DATA=REGDATA;
      PARMS A B; /* MODEL PARAMETERS*/
      DERT.Y = B *Y; /* DEFINING THE EQUATION*/
      FIT Y INITIAL=(Y=A) / TIME=YEAR DYNAMIC; /* OPTIONS*/
RUN;
***********************************************************************;
```

Table 1. Estimated Coefficients of the Dynamic Model and Log Linear Function.

	Sample Period	Log-linear	Dynamic
Country		**Value**	**Value**
Australia	1992-1999	-0.0583(0.0001)	-0.0585(0.0001)
Canada	1970-1999	-0.0158(0.0001)	-0.0145(0.0001)
Denmark	1979-1999	0.0176(0.0022)	0.0146(0.0009)
Finland	1976-1997	-0.0404(0.0001)	-0.0346(0.0001)
France	1975-1999	-0.0214(0.0001)	-0.0226(0.0001)
Japan	1990-1999	-0.0443(0.0001)	-0.0440(0.0001)
Mexico	1991-1999	-0.0058(0.1045)	-0.0078(0.0225)**
Norway	1975-1999	0.0281(0.0002)	0.0361(0.0001)
Sweden	1976-1999	-0.0633(0.0001)	-0.0522(0.0001)
Switzerland	1984-1998	-0.0293(0.0003)	-0.0325(0.0001)
United Kingdom	1987-1999	-0.0674(0.0001)	-0.0908(0.1048)
United States	1978-1999	0.0098 (0.0019)	0.0093(0.0026)

Table A1: Annual average trend estimates (p-values in parentheses) for 12 countries. ** the estimate for Mexico is suspect because, there are only 4 observations with missing values in between them. Results not corrected for serial correlation.