Exposure Dependent Modeling of Percent of Ultimate Loss Development Curves

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OF

PERCENT OF ULTIMATE LOSS DEVELOPMENT CURVES

by

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Abstract

This paper presents a loss development model in which exposure period dependence is fundamental to the structure of the model. The basic idea is that an exposure period, such as an accident year or policy year, gives rise to a particular distribution of accident date lags, where the accident date lag is the time elapsed from the start of the exposure period till the accident date. The paper shows how to derive the density of the accident date lag from a familiar parallelogram diagram. A fairly general theory of development is then presented and simplified under certain conditions to arrive at a total development random variable whose cumulative distribution is related to the usual percent of ultimate development curve. After presenting the theory, the paper turns to practical applications. Simulation is used to generate consistent patterns for different exposure periods. A convenient accident period development formula is derived and then used to fit and convert factors. The average date of loss approximation is generalized. To summarize, this paper will demonstrate that modeling loss development with exposure dependent percent of ultimate curves is a theoretically sound procedure with many practical uses.

1. INTRODUCTION

A key step in the usual procedure for modeling a loss development pattern is to fit formulas to empirical age-to-age or age-to-ultimate factors. Having a fitted formula is useful because it provides an easy way to smooth the bumps found in most series of empirical factors. Also, if the fit is to age-to-ultimate factors, the formula usually provides a convenient way to interpolate the factors.

While the fitting is convenient and practical, it can hardly be said to have a substantive conceptual foundation. A formula is chosen because it is easy to compute and because it nicely fits the age-to-age factors. It is not derived from more basic assumptions in the sense that nothing is specifically built in to reflect that it is being fitted to data that represent ratios of loss for a particular exposure period as of given evaluation ages.

While a formula serves perfectly well for smoothing, it may not suffice, in and of itself, to handle other applications such as tail factor extrapolation, early age extrapolation or conversion of the factors from one exposure basis to another. Tail factor extrapolation is needed to get age-to-ultimate factors after a fit is obtained to age-to-age factors. Yet, an age-to-age factor formula may not immediately lead to the extrapolation. To obtain the desired age-to-ultimate factors the actuary may have to derive the product of an infinite series, make cut-off assumptions, or use a computerized numerical algorithm.

In early age extrapolation, the actuary is seeking factors at an evaluation age younger than the earliest evaluation age associated with the fitted factors. For example, the actuary may have accident year age-to-ultimate factors for evaluations at 12, 24, 36... months, yet may need to have factors at 6, 18, 30, ...months. The problem is that the back extrapolation of a formula fit may or may not yield plausible results at earlier ages (i.e. the factor at 6 months). Some additional techniques may be needed to get reasonable factors at these ages.

Finally with regard to conversion, the actuary may have fitted accident year factors, but may want to have policy year factors. Yet a good fit to accident year factors may not directly lead to a good fit to the corresponding policy year factors. Actuaries have usually dealt with this conversion problem by using an average date of loss adjustment. Under this adjustment, the development factor for one type of exposure period at a given evaluation age is estimated by the development factor for the original type of exposure period at an adjusted evaluation age. The adjustment is equal to the difference in the average dates of loss for the different exposure periods. While this adjustment works well at mature ages after all exposures are earned, it goes awry at immature evaluation ages.

The conclusion is that fitting with general formulas is a useful and flexible approach that must often be supplemented for extrapolation and conversion applications. The supplemental procedures may not be too difficult to implement. So, in the end, from a practical perspective, not too much should be made of the need to introduce them. However, it would be more convenient to have a model of loss development that would automatically handle extrapolation and conversion. Such a model would not start with a formula for age-to-age factors, but would instead be based on percent of ultimate or age-to-ultimate curves having an explicit dependence on the underlying exposure period.

Models such as this have been previously proposed. Yet they have not been widely adopted. Why? We speculate the reluctance stems from two essential areas of concern. First, there may be questions about the theoretical underpinnings of such models. Second, there may be doubts about whether the proposed models are practical.

In order to address these concerns, we will present a general, yet accessible, conceptual foundation for exposure dependent percent of ultimate models. We will start by relating an exposure period, such as an accident year or policy year, to an associated distribution of accident date lags. The accident date lag for a claim is defined as the length of time from the start of the exposure period to the accident date. We will show that the familiar parallelogram or rectangle diagram representation of an exposure period can be readily converted into a graph of the density of this accident date lag random variable. The cumulative distribution of the accident date lag may be identified with the percent of premium earned to date assuming the earning of premium corresponds exactly to the exposure to loss. We will argue that under certain conditions the percent of ultimate loss development curve may be expressed as the cumulative distribution of the sum of the accident date lag random variable plus another random variable that summarizes the claims process. The claims process in this context includes the delay between the accident date and report date, as well as the changes in the valuation of a claim and the time lags between these valuation changes. Perhaps the key insight underlying this construction is that exposure dependence can be isolated in the accident lag distribution.

We will then turn to applications. We will use the model to simulate patterns for different exposure periods, derive a convenient accident period development formula, fit and convert patterns, extend the average date of loss approximation, and approximate a converted pattern as the weighted sum of shifted versions of the original pattern. In the end we will hope to have shown that exposure dependent percent of ultimate models are not only pleasing to the theorist, but also useful to the practical actuary.

2. EXPOSURE MODELING

We start by establishing the key concept that an exposure period is defined by a distribution of accident date lags, where an accident date lag is the length of time from the start of an exposure period until an accident occurs.

To state this mathematically, define:

- W = Exposure Random Variable = Accident Date Lag

  = accident date - date of start of exposure period (2.1)

We identify the cumulative distribution of W with the percentage of exposure earned to date and sometimes write:

$$F_W(w) = ETD_W(w) \tag{2.2}$$

The assumption here is that the earning of premium corresponds exactly with exposure to accidents so that the percent of premium earned as of a given date equals the expected percent of accidents that have occurred by that date.

It is easy to define the accident date lag distributions for the most commonly encountered exposure periods. For an accident year under the usual uniformity assumptions, the exposure random variable is a uniform random variable.

$$f_{AY}(w) = \begin{cases} 1 & \text{for } 0 < w < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.3a}$$

$$F_{AY}(w) = \begin{cases} w & \text{for } 0 < w < 1 \\ 1 & \text{for } w \geq 1 \end{cases} \tag{2.3b}$$

The policy year exposure random variable has density that increases linearly for one year and then decreases linearly for the second year.

$$f_{PY}(w) = \begin{cases} w & \text{for } 0 < w < 1 \\ 2-w & \text{for } 1 \leq w < 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.4a}$$

$$F_{PY}(w) = \begin{cases} \frac{w^2}{2} & \text{for } 0 < w < 1 \\ \left(\frac{1}{2} + \frac{1-(2-w)^2}{2}\right) & \text{for } 1 \leq w < 2 \\ 1 & \text{for } w \geq 2 \end{cases} \tag{2.4b}$$

Though it may appear initially a bit different, this view of an exposure period as being synonymous with a distribution of accident date lags is equivalent to the standard actuarial approach involving rectangles and parallelograms. It is generally straightforward to convert these geometric objects into the density of the exposure random variable defined here. The idea is to collapse the parallelogram down towards the "x-axis" and then normalize so that the area under the curve is unity.

For example, consider how the policy year parallelogram in Figures 1 can be collapsed to yield the policy year density shown in Figure 2.

Figure 1

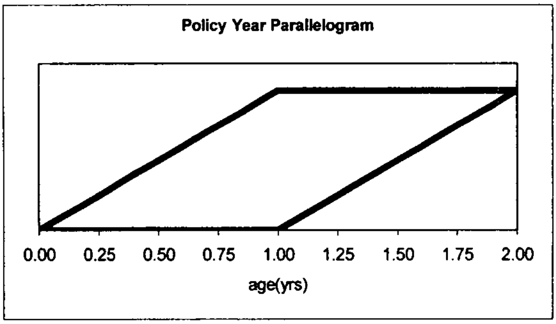
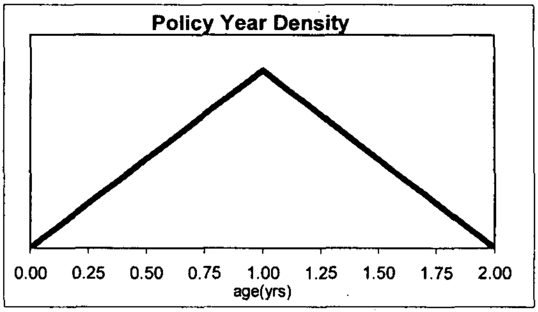


Figure 2



Similarly the policy quarter parallelogram in Figures 3 is readily converted to the policy quarter density shown in Figure 4. The policy quarter density is typical of policy periods: the density starts with an exposure growth triangle, then reaches an exposure plateau, and finally ends with an exposure decay triangle

Figure 3

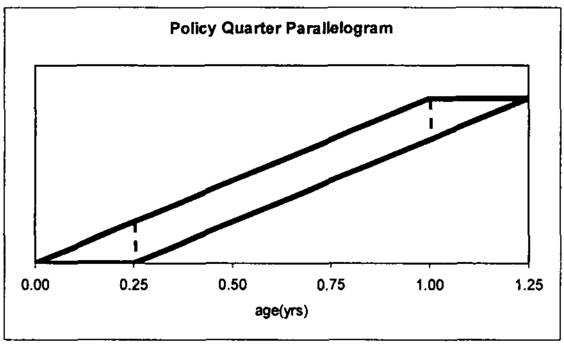
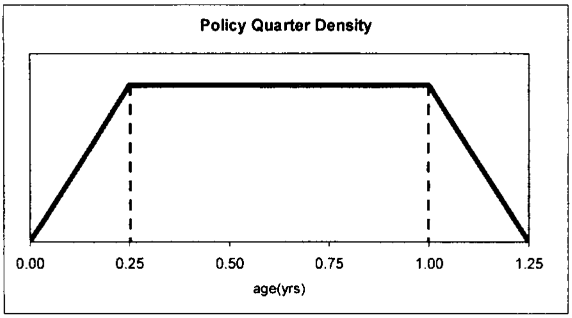


Figure 4



To summarize, the accident date lag for an exposure period is a random variable that captures differences between different types of exposure periods. The density of this random variable may be easily constructed from the parallelogram diagrams with which actuaries are familiar. To put it in other words, we start our exposure dependent development model by characterizing different exposure periods by their Earned to Date functions.

3. MODELING THE CLAIMS PROCESS

Next we model the development of a claim after the original accident has occurred. We model this development with a series of paired random variables, where each pair in the series describes a step in the claims development process. Each pair consists of:

- a time lag random variable that measures the time since the previous step and,
- an amount change random variable that equals the change in the value of the claim at that step.

After the accident has occurred, the first step in the claim process is that the claim is reported. The length of time between the accident date and report date is called the report lag. If we are interested in development of case incurred losses, the amount change variables will measure changes in the case incurred loss. If we are looking at paid development, the amount changes will equal payments made a various points in time as defined by the lags.

To describe this in general mathematical terms, we define:

- M = Number of steps (3.1)
- $\Delta V(i)$ = Process lag at the i^{th} step (3.2a)

  = the time between $(i-1)^{st}$ step and the i^{th} step

  (where the 1^{st} step is the report lag)
- $V(i)$ = Total lag since the claim occurred = $\Delta V(1)+\Delta V(2)\ldots+\Delta V(i)$ (3.2b)
- $\Delta A(i)$ = Change in the amount of a claim at the i^{th} step (3.3a)
- $A(i)$ = Claim amount after the i^{th} step = $\Delta A(1)+\Delta A(2)\ldots+\Delta A(i)$ (3.3b)

Diagrams can be helpful in understanding the definitions of these variables. Figure 5 depicts the lag variables in a claim count development model

Figure 5

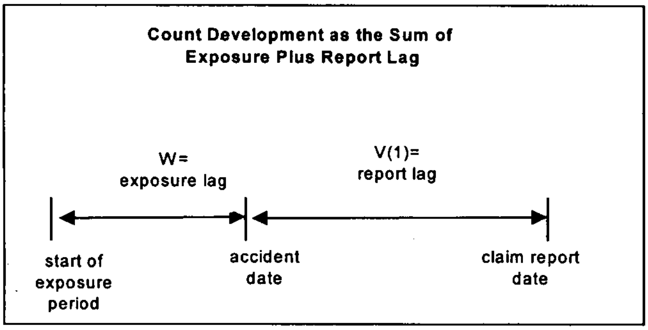
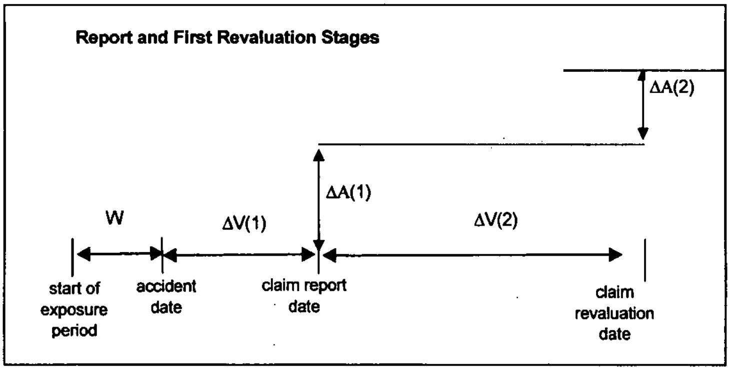


Figure 6 shows the lag and amount change variables for the claim reporting and first revaluation stages of a claim.

Figure 6



We now use the time lags to define a function, B(t), which is the claim amount expressed as a function of the time, t, that has elapsed since the accident.

$$B(t) = \begin{cases} 0 & \text{if } t < V(1) \\ A(i) & \text{if } V(i) \le t < V(i+1) \text{ for } i = 1,2,\ldots M-1 \\ A(M) & \text{if } V(M) \le t \end{cases} \tag{3.4}$$

Now we define P(t) as the ratio of the expected value of B(t) over the expected ultimate value of B.

$$P(t) = \frac{E[B(t)]}{E[B(\infty)]} \tag{3.5}$$

While the diagrams can be drawn for as many transitions as necessary, it is clear that the final evaluation of E[B(t)] could become fairly messy. One would need assumptions on the distribution of the number of revaluations a claim will undergo. One would also need assumptions about the distributions of the lags and the amount changes. Further, in general, the number of steps, the length of the lags, and the amount of the changes might not be independent of one another. Rather than try to evaluate all full model in detail, we will first attempt to simplify it.

As preparation for simplifying the model, we first note that in the general case some of the amount change variables could well be negative or even have a negative expectation. We have allowed this because we want a model that could handle negative development such as can arise from downward reserve revaluations, closing of claims without payment, salvage and subrogation, and other factors.

However, if we now restrict the model and assume that all of the amount change variables must be non-negative, it will follow that B(t) is an increasing function of t and that E[B(t)] is increasing as well. We can therefore conclude that P(t) is an increasing function between zero and unity that tends to unity as time approaches infinity. Thus P(t) is the cumulative distribution of some random variable. We call this random variable the Process Lag and denote it as S. Sometimes we may write $F_S(t)$ in place of P(t). Observe that S effectively summarizes the amount change and step lag random variables that describe the development of claims after their accident dates. It is the existence of this single Process Lag that allows us to simplify the model.

Before going further with our simplified model, we first observe that under these definitions the Report Lag (from accident date to report date) is included in the Process Lag. We also observe that the Process Lag distribution defined here is equivalent to the percent of ultimate loss development pattern for loss on an exposure of infinitesimal duration as given in Robbin and Homer [4] and similar functions defined in Brosius [1], Philbrick [3], and Wiser [6].

4. EXPOSURE DEPENDENT DEVELOPMENT

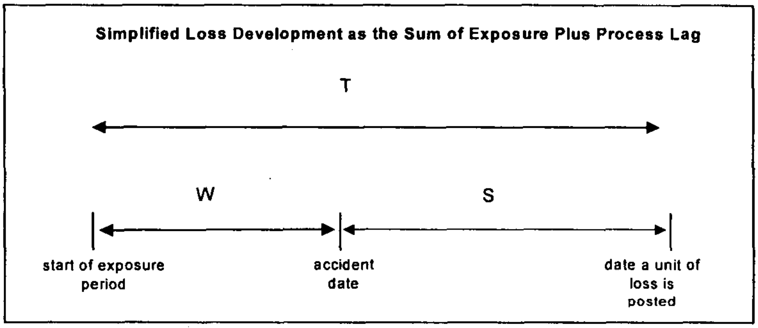
We now add the Accident Date Lag to the Process Lag to obtain the Total Lag for exposure period loss development.

Define:

- T_W = Total Lag = W+S (4.1)

We may view T as the difference between the start of the exposure period and the date a unit of loss is posted on the books. The term, "unit of loss", is here meant to be a general term that could apply to claim counts reported, loss dollars incurred, loss dollars paid or other quantities that actuaries display in triangles. The random variables are shown in the diagram in Figure 7.

Figure 7



In principle, the claims reporting and settlement process should not depend on how the claims are grouped into exposure period buckets. We formalize this by assuming that W and S are independent. As necessary, we index the total lag distribution, T, by W to indicate its dependence on the exposures.

We next make the critical observation that the cumulative distribution of T_W is the same as the percent of ultimate curve for losses arising from the exposures specified by W. Let $PCT_W(t)$ denote the expected percent of ultimate for losses arising from exposures given by accident date lag W as of time, t, since the start of the exposure period. Our observation is mathematically expressed by the equation:

$$PCT_W(t) = F_{W+S}(t) = F_{T_W}(t) \tag{4.3}$$

For example, in a claim reporting model, let N(t) be the number of claims reported as of time, t, and let N(ult) be the ultimate number of claims. The report date measured from the start of the exposure period can be regarded as a sample of the random variable, T_W. It follows that N(t) will be binomially distributed with parameters, N(ult) and $F_{T_W}(t)$. Thus E[N(t)] = N(ult) $F_{T_W}(t)$ and it follows that $PCT_W(t) = F_{T_W}(t)$. For example, if the percent of ultimate curve is at 60% as of a particular evaluation age, then if we look at the total report lags for a sufficiently large set of claims, we will find that 60% of these lags are less than or equal to the given evaluation age.

In general, the loss development factor from age t to ultimate is given as the inverse of the percent of ultimate. We can thus relate standard age-to-ultimate factors to the inverse of the cumulative distribution of the Total Lag:

$$AULDF_W(t) = \frac{1}{PCT_W(t)} = \frac{1}{F_{T_W}(t)} \tag{4.4}$$

Assuming W and S are independent, it is known that the cumulative distribution of their sum is given as a convolution integral. Thus we can write:

$$F_{T_W}(t) = \int_0^t dw\, f_W(w) \cdot F_S(t-w) = \int_0^t dw\, f_W(w) \cdot P(t-w) \tag{4.5}$$

This is equivalent to percent of ultimate loss development formulas seen in the literature (Robbin and Homer [4], Brosius[1], and Philbrick [3]). What we have done here is base the formula on well-defined random variables. The derivation is based on the assumption the underlying amount change random variables were all non-negative. Later, we will relax this assumption, but for now we see that it is critical, for it allows us to summarize all the changes a claim undergoes with a single process random variable

Next, we will use our Exposure Lag plus summarized Process Lag model to directly simulate loss development patterns.

5. SIMULATION

A big advantage in having a development model based on process and exposure random variables is that we may simulate these variables and thereby generate loss development patterns. Given any non-negative random variable as a model for S and a particular exposure period with accident lag random variable, W, we can use simulation models to quickly generate a few thousand samples of S and W. With these, we can compute the cumulative distribution of T=S+W at various evaluation ages. By retaining our original set of simulated process lags and using a different exposure random variable, we can see how the development pattern changes in response to a change in the underlying exposures.

Exhibit 1 provides a small sample demonstration of the procedure. The accident year and policy year patterns shown in the exhibit were generated from the sample of 20 random trials listed in Sheet 2 of Exhibit 1. The Process was assumed to follow a Pareto distribution with shape parameter equal to 2.0. Given the extremely small sample size, it is no surprise these simulated patterns differ significantly from the true patterns displayed in the exhibit. The small sample size was used so the reader could follow the computation of the percent of ultimate from the simulated values. Much larger samples would be required in any real application. The formulas for the true patterns are shown in Appendix A. A more realistic sample size of 2,000 was used to generate the simulated patterns displayed in Exhibit 2. These fit the true formula-generated patterns quite nicely.

When applying this simulation technique to actual problems, the required sample size ought to be large enough to guarantee that the simulated percent of ultimate values or incremental percentages are highly likely to fall within a desired tolerance. A binomial test can be applied using the normal approximation to the binomial in order to estimate this requisite sample size. Simulations run with that sample size will still typically yield age-to-age patterns with small statistical fluctuations. To get a smoother curve requires a larger sample size.

In practice, if we have a model for the Process Lag that generates simulated factors that closely match given accident year factors, we can reuse the simulated values of the process variable to generate the factors for another exposure period. To do this we simply add each previously simulated process lag to a simulated accident lag for the other exposure period. Since the simulated Total Lag for the accident year already fits the accident year pattern, the simulated Total Lag for the other exposure period should also be reasonably close to its true value.

Simulation provides a powerful all-purpose tool for solving problems using the exposure dependent model. It may be especially useful when trying to estimate development patters for an irregular exposure period. For example, we could use simulation to estimate development patterns on a risks attaching reinsurance contract covering a mix of 3 month and 12 month term policies where the contract was cut-off so that it only covers accidents occurring during the first 12 months. We will next derive a formula for accident period development and use it as the basis for other application techniques.

6. A SIMPLE ACCIDENT PERIOD DEVELOPMENT FORMULA

Though the convolution integral formula 4.5 may initially look forbidding, it reduces to a quite tractable formula when applied to accident period exposures. For a uniform accident period of duration, D, the cumulative distribution and density of the accident lag variable, W, are given as:

$$F_{A(D)}(w) = \frac{w}{D} \text{ for } 0 < w < D \tag{6.1a}$$

$$f_{A(D)}(w) = \frac{1}{D} \text{ for } 0 < w < D \tag{6.1b}$$

Here for clarity we have written A(D) instead of W when subscripting the cumulative distribution and density. The cumulative distribution for the loss development pattern generated from a uniform accident period is thus given as:

$$F_{T_{A(D)}}(t) = \int_0^{\min(t,D)} dw \, \frac{1}{D} \cdot F_S(t-w) = \int_0^{\min(t,D)} dw \, \frac{1}{D} \cdot (1 - G_S(t-w)) \tag{6.2}$$

where G denotes the tail probability.

We simplify this percent of ultimate formula using the fact that the integral of the tail probability is the limited expected value:

If t<D:

$$F_{T_{A(D)}}(t) = \frac{t}{D} - \int_0^t dw \frac{1}{D} \cdot G_S(t-w) = \frac{t}{D} - \frac{1}{D}\int_0^t du G_S(u) = \frac{t}{D} - \frac{E[S;t]}{D} \qquad (6.3a)$$

If t>D:

$$F_{T_{A(D)}}(t) = \frac{D}{D} - \int_0^D dw \frac{1}{D} \cdot G_S(t-w) = 1 - \frac{1}{D}\int_{t-D}^t du G_S(u)$$

$$= \quad 1 - \frac{E[S;t] - E[S;t-D]}{D} \qquad (6.3b)$$

In these formulas, E[S;s] is the limited expected value of S at s. Limited expected value formulas for many distributions are given in various books on loss distributions and statistics [2]. With 6.3, we can then use any one of these to generate consistent accident period curves for accident periods of different duration.

7. CURVE FITTING AND CONVERSION

The accident period development formula can be readily applied to fitting accident year-by-year data. After fitting some data, we will then use the formula to generate the associated accident quarter-by-quarter development pattern.

We will fit age-to-age factors using three different parametric distributions: the Pareto, the Gamma and a two-parameter form of the Burr. The limited expected value functions are as follows:

$$\text{Pareto:}\quad E[S;s] = \mu \cdot \left(1 - \left(\frac{\mu(\alpha-1)}{\mu(\alpha-1)+s}\right)^{\alpha-1}\right) \tag{7.1}$$

$$\text{Gamma:}\quad E[S;s] = \mu \cdot \Gamma(s \mid \alpha+1, \frac{\mu}{\alpha}) + s \cdot \left(1 - \Gamma(s \mid \alpha, \frac{\mu}{\alpha})\right) \tag{7.2}$$

$$\text{Two Parameter Burr:}\quad E[S;s] = s \cdot \left(1 + \left(\frac{s}{\mu}\right)^{\alpha}\right)^{-1/\alpha} \tag{7.3}$$

We have parameterized all of these so they have two parameters: μ, the mean, and α, the shape. It is the experience of the author that numerical fitting routines often work better if the mean is isolated as a single parameter. The reader can find sources (see Hogg and Klugman, [2]) for all of these except for the two-parameter form of the Burr. To illustrate how accident year percent of ultimate values would be derived for this modified Burr distribution, let $\alpha = 1$ and $\mu = 2$. We compute E[S;1] = 2/3, E[S;2] = 1, and E[S;3] = 3*(1+3/2) $^{-1}$ = 6/5. Using formula 7.3, this yields percent of ultimate values of 33.10%, 66.7%, and 80.0% at the end of the first three years respectively.

Next, we use these limited expected value formulas to derive age-to-age factors and fit them to one set of age-to-age factor data shown in the Sherman's paper [5]. The results are shown in Exhibit 3. The fits were obtained so as to minimize the sum of square errors in the running back-products of the age-to-age factors. Other fitting criteria could be used, but this one is easy to program. Also, it naturally assigns more weight to the shape of the tail of the available data and seems more forgiving if there happens to be a strange factor or two in the data. Sherman's fit with a power curve is shown for comparison. Reviewing our results, we see the Burr fit is good, the Pareto fit is fair, and the Gamma fit is not good. Perhaps the Gamma would fare better with a different fitting criterion, or perhaps this curve form just does not fit the data. In any event, the Burr fit is arguably as good as that obtained by Sherman using the power curve. However, the conclusion from the example is <u>not</u> that the Two-parameter Burr fits better than the power curve or that the exposure dependent percent of ultimate model does a better job of fitting the factors. It merely demonstrates that the exposure dependent model is practical and can produce good fits. In real applications, it would be advisable to look at more than three curves and to try different fitting criteria.

While the exposure dependent model has no advantage over pure curve formulas in fitting a given set of development factors, some advantages come to light after the fit is obtained. Suppose we have just fitted accident year-by-year age-to-age factors. With our model, we automatically get the resulting age-to-ultimate factors. With a power curve or other age-to-age formula, one may have to posit an arbitrary cut-off age. This difficulty arises because the product of the infinite series of formula generated age-to-age factors may be infinite or at least difficult to compute. The root of the problem stems from viewing the age-to-age factors as a series of numbers, instead of deriving them from a percent of ultimate curve, as was done in our model. Second, with our model, interpolation is easy. One can simply compute limited expected values at requisite intermediate ages and use them to compute the percent of ultimate curve at the desired evaluation ages. With the pure curve fitting approaches, interpolation may entail rebalancing and refitting procedures [5]. Another advantage of our model is that we can quickly generate the associated accident quarter-by-quarter factors. These are shown in Exhibit 4 for the Burr fit in our example. The pure curve fitting methods run into difficulty with this problem [5], whereas our model handles it with ease precisely because dependence on the exposure period is built in from the start.

8. AVERAGE MATURITY OF LOSS APPROXIMATION

Next we generalize and extend the usual average date of loss approximation so that it handles immature evaluation ages. We call the generalization the Average Maturity of Loss Approximation. Under the average date of loss approximation, loss development for one exposure period as of a given evaluation age is estimated by the development for another exposure period at an adjusted age. The adjustment is equal to the difference between the average dates of loss for the two exposure periods.

To express this mathematically, let W be an exposure random variable and define $\mu_W = E[W]$ as its average date of loss. Given another exposure random variable, W^*, we define the average date of loss approximation of W^* using W via:

$$PCT^*(t^*) \approx PCT(t^* + \mu_W - \mu_{W^*}) \tag{8.1}$$

Here PCT denotes the percent of ultimate loss.

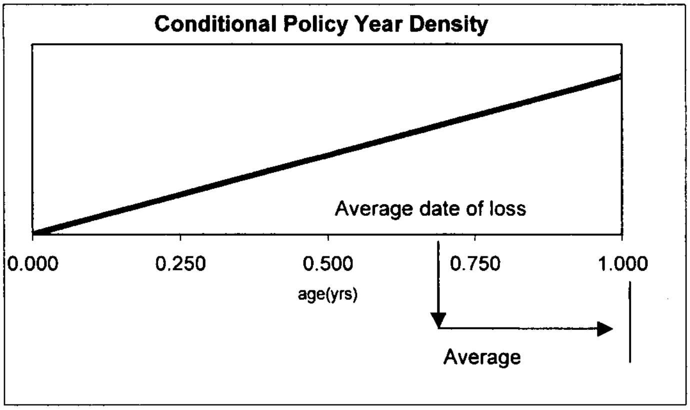
For example, if W represents uniform accident year exposure and W^* is the exposure variable for a policy year, then μ_W =6 months, μ_{W^*} = 12 months and we approximate the policy year using the accident year factor at the age six months earlier. For evaluation ages greater than two years, the approximation has some error but is not unreasonable. It becomes fairly accurate at ages above three years. However, for ages less than two years, the logic of the fixed six-month shift breaks down and for ages below six months the shift fails to yield an answer at all.

Following Robbin and Homer [4], we extend the approximation so that it works at immature ages by first defining the conditional average date of loss, $\mu_W(t)$ =E[W| W< t]. We next the define the average maturity of loss, $m_W(t)$, via:

$$m_W(t) = t - \mu_W(t). \tag{8.2}$$

A loss that occurred at the average date of loss has developed, as of time t, for a period equal to the average maturity. For example, an accident year as of 8 months has a conditional average date of loss equal to 4 months and an average maturity of loss also equal to 4 months. Using 2.2 we can show a policy year as of 12 months has an average date of loss equal to 8 months and an average maturity equal to 4 months. This can be seen geometrically by observing that the policy year density forms an upward sloping triangle over the first 12 months. The average for a triangle occurs 2/3 of the way along its base. The picture is shown in Figure 8.

Figure 8



In general, we approximate:

$$PCT^*(t^*) \approx PCT(t)\frac{ETD^*(t^*)}{ETD(t)} \qquad (8.3)$$

where $m_W(t) = m_{W^*}(t^*)$.

In words, we first find the date, t, at which W has the same average maturity as W* does at t*. We call "t" the evaluation age of equivalent maturity. The percent of ultimate loss curve for W at the evaluation age of equivalent maturity is then used to approximate the percent of ultimate for W* at t*, where the denominators in the formula adjust for differences in the exposures earned to date. Applying 8.3, we would for example approximate the policy year as of 12 months using the using the accident year as 8 months and the multiplying by ½ and dividing by 8/12. So if the accident year percent of ultimate as of 8 months was 40%, the policy year percent of ultimate as of 12 months would be estimated as 40%*(3/4) = 30%. The corresponding age-to-ultimate factors would be 2.5 and 3.3.

To show why this approximation works, we first follow Robbin and Homer [4] and expand the percent of ultimate convolution formula, 4.5, using the Taylor series expansion of the process distribution. For notational brevity, we will drop subscripts at times during the derivations; for instance writing $\mu(t)$ in place of $\mu_W(t)$. We expand up to second order as follows:

$$\begin{aligned} &P(t-w) = P(t-\mu(t)+\mu(t)-w) = P(m(t)+\mu(t)-w) = \\ &P(m(t)) + (-1)\cdot(\mu(t)-w)\cdot P'(m(t)) + \tfrac{1}{2}(\mu(t)-w)^2 \cdot P''(\delta) \end{aligned} \tag{8.4}$$

where $0<\delta<m(t)$

Note that $\mu(t)$ has been defined so that the integral of the first order term times the exposure density vanishes over the interval from 0 to t. If we now only use the expansion up to first order and plug 8.4 into 4.5, we obtain the approximation:

$$PCT(t) = F_{T_W}(t) = ETD_W(t)\cdot P(m(t)) \tag{8.5}$$

The approximation says that the percent of ultimate loss pattern as of time t for exposures given by W is equal to the percent earned to date times the cumulative distribution of the process distribution as of the conditional average maturity. We see that P(m(t)) approximates the percent of ultimate for the exposures earned to date. If we now write the approximation 8.5 for W* and have t such that $m_W(t) = m_{W^*}(t^*)$, it is then a small rearrangement of terms to arrive at our average maturity of loss approximation as shown in 8.3.

In Exhibit 5 an average maturity of loss approximation for policy year development is computed based on accident year factors. The first sheet of the exhibit shows the derivation of the conditional policy year average date of loss and average maturity of loss at quarterly evaluations. To simplify the calculations, the derivation is done using the exposure growth and decay triangles for the policy year density. The first sheet also shows the accident year evaluation age of equivalent maturity. Then in the second sheet the accident year percent of ultimate and age-to-ultimate factors at the original evaluation ages are shown. This is for information and comparison purposes only. The subsequent derivation of the average maturity approximation makes no use of them. As shown in the second sheet of Exhibit 5, accident year factors are posted for the ages of equivalent maturity. These are then multiplied by the appropriate Earned to Date ratios to obtain the Average Maturity Approximation. Finally, the approximation is compared against the true policy year factors. The accident year factors and the true policy year factors were generated used a Pareto Process with shape equal to 2.0. In actual applications, one should not develop policy year losses evaluated at ages below one year as the data is too immature and the corresponding factors are so large that results are too unstable to be reliable. Note that after two years the approximation reduces to a six-month shift as per the usual Average Date of Loss Approximation.

To summarize, there are two key aspects of the Average Maturity Approximation. First, it adjusts evaluation dates so losses for the two exposure periods have the same conditional average maturity. Second, it adjusts for differences in exposures earned to date. This second adjustment is critical when dealing with immature exposures. Because exposure dependence is built into our model, this earned to date exposure adjustment falls out naturally from the basic equations.

Next we turn to another approximation techniques in which a desired pattern is estimated using a weighted average of the shifted accident period patterns.

9. MULTI-SHIFTED ACCIDENT PERIOD APPROXIMATION

The idea here is that if we can approximate an exposure period random variable as the weighted average of shifted accident period distributions, then we could approximate its development pattern as a weighted average of shifted accident period patterns. Since we have a convenient formula that allows us to evaluate an accident period pattern at arbitrary ages, we will then arrive at a practical way to approximate the development pattern for the original exposure period. After explaining the technique in mathematical terms, we will use it to approximate a policy year pattern as a weighted sum of shifted accident quarter patterns.

Let $A(D_i, c_i)$ be the exposure random variable for an accident period of duration, D_i, which begins at time c_i. Given a process random variable, S, we can write the resulting percent of ultimate, T, as:

$$F_{T(A(D,c))}(t) = \begin{cases} \dfrac{\max(0,(t-c)) - E[S;t-c]}{D} & \text{if } t-c \le D \\ 1 - \dfrac{E[S;t-c] - E[S;t-D-c]}{D} & \text{if } t-c > D \end{cases} \tag{9.1}$$

Now take a finite sequence, $(A(D_1,c_1), A(D_2, c_2), \ldots, A(D_m, c_m))$ of such shifted uniform random variables, and corresponding weights, $(p_1, p_2, \ldots, p_m)$ that sum to unity. Define the mixed multi-shifted exposure random variable, W as follows:

$$F_W(w) = \sum_{i=1}^{m} p_i \cdot \min(1, \max(0, w - c_i)) / D_i \tag{9.2}$$

Given a process random variable, S, the percent of ultimate, T, based on the mixed exposures, W, can be written as:

$$F_{T|\tilde{A}(\vec{D},\vec{c}),\vec{p}}(t) = \sum_{i=1}^{m} p_i \cdot F_{T(A(D_i,c_i))}(t) \tag{9.3}$$

While these formulas may look terribly complicated, they are very easy to apply in practice. When the durations are all the same and the shifts follow a simple pattern, one can typically generate the pattern for the common duration and then "copy and paste" to apply (9.3). Generating the basic pattern involves taking limited expected values; so that step is not too difficult either.

The conclusion is that if we can approximate a given exposure random variable as a weighted average of shifted accident period variables, then we can approximate the loss development pattern for the given exposures. In Exhibit 6 we approximate a policy year as the weighted average of five shifted accident year patterns. The weights are: (1/8,1/4, 1/4, 1/4,1/8) and the shifts are: (0,1,2,3,4,5) quarters. While the fit against the true pattern is not exact, it is nonetheless fairly good and we could refine it further by using thirteen accident months with monthly shifts. Note the multi-shift approximation does not inherently fall apart at early ages.

10. MIXING AND NEGATIVE DEVELOPMENT

So far we have used a single Process to describe the underlying multi-step development of claims. While we have proved such a single summary process exists when all the amount changes are non-negative, in practice it may still be useful to regard the single process as a mix of two or more processes. For example, if we know there are two types of claims in our data, one type that develops quickly and the other type more slowly, it may be best to try a model with two processes. Exhibit 7 shows the accident year pattern resulting from a mix of two Gammas, one short-tailed and the other long-tailed.

Also, in all we have done so far, it has been assumed that incremental development must always be non-negative. We now extend the model to handle negative development. For clarity, we will consider a model for the development of the number of non-zero claims. Negative development occurs when a claim is closed without payment. We count the number of non-zero claims as the difference between the total number of claims reported less the number closed without payment.

Let N be the ultimate total number reported, CNP the number closed without payment, and define M as the ultimate number of non-zero claims. Thus M=N-

CNP. For each of the CNP claims, we define a closing lag, U, as the difference between when the claim was reported and when it was closed without payment. Given values of the exposure lag, W, the reporting process lag, S, and the closed without pay lag, U, a claim will be counted as a non-zero claim as of time t if t is between W+S and W+S+U. .

The percent of ultimate for the number of non-zero claims is given as:

$$PCT_M(t) = E\left[\frac{M(t)}{M}\right] = E\left[\frac{N(t) - CNP(t)}{N - CNP}\right] \qquad (10.1)$$

If N and CNP are assumed fixed for the moment, it follows that N(t)-CNP(t) is the sum of two binomially distributed random variables with parameters :

- $(N\text{-}CNP,\ F_{W+S}(t))$ (10.2a)
- $(CNP,\ F_{W+S}(t) - F_{W+S+U}(t))$ (10.2b)

Thus

$$\begin{aligned} E[M(t)] &= (N - CNP)\, F_{W+S}(t)) + CNP(F_{W+S}(t) - F_{W+S+U}(t)) \\ &= N\, F_{W+S}(t) - CNP\, F_{W+S+U}(t) \qquad (10.3) \end{aligned}$$

Let r denote the expected ratio at ultimate of the number of claims closed without over the total number of claims ever reported. Then for any reasonably large number of claims we can approximate the percent of ultimate curve as follows:

$$PCT_M(t) = E\left[\frac{M(t)}{M}\right] = E\left[\frac{N(t) - CNP(t)}{N - CNP}\right] \approx \frac{F_{W+S}(t) - rF_{W+S+U}(t)}{1-r} \tag{10.4}$$

In Exhibit 8 we use patterns based on a Gamma base process with a Gamma decrementing process.

Finally, in Exhibit 9 we generate patterns from a mix of two processes, one of which undergoes negative development. The resulting shape of the development curve is fairly complex with age-to-age factors above unity, then below unity, then back above unity till they taper off in the tail. Yet loss data sometimes exhibits this type of behavior. This could happen when reserves on some claims are taken down as quick settlements are made, but the remaining claims slowly develop upwards over many years.

11. CONCLUSION

It is useful to end with a brief review of what we have done. First we have established a conceptual foundation by identifying exposure with a distribution of accident date lags and then viewing total lag as the sum of exposure and process lag random variables. The single process lag was obtained as a simplified summary of a more general multi-step model of non-negative amount changes and step lags. We were able to connect our model with standard actuarial descriptions of loss development by proving the percent of ultimate development curve is synonymous with the cumulative distribution of the total lag random variable. Having separated exposure from process, we were able to vary the exposure to obtain exposure dependent development curves. Just having a random variable model of loss development was shown to be useful, because it allowed us to simulate loss development patterns. A key result was the derivation of an accident period loss development formula in terms of limited expected values. Because the formula is readily programmable for a large number of distributions, we were able to use it in fitting accident year factors, generating accident quarter factors, and computing multi-shift approximations. Adding in mixed processes and negative development allowed us to structure a model that can reflect our knowledge of the claims process and capture more complex patterns of development.

Hopefully, the reader now has a solid understanding of the conceptual foundation of exposure dependent modeling of loss development patterns and has seen that it may be put to good practical use. Future research along these lines will likely yield new insights and techniques.

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5. Sherman, Richard E., "Extrapolating, Smoothing and Interpolating Development Factors", PCAS, 1984, p122-155
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APPENDIX

Accident Year and Policy Year Percent of Ultimate Formulas

for a Pareto Process with Shape Equal to 2.0

Let S be a Pareto distribution with scale parameter, λ, and shape parameter equal to 2.0. Then

$$E[S;s]=\frac{\lambda}{2-1}\left(1-\left(\frac{\lambda}{\lambda+s}\right)^{2-1}\right)=\lambda\frac{s}{\lambda+s} \tag{A.1}$$

It follows that:

For t<D

$$F_{T_{A(D)}}(t)=\frac{t-E[S;t]}{D}=\frac{1}{D}\left(t-\frac{\lambda t}{\lambda+t}\right)=\frac{t}{D}\left(\frac{(\lambda+t)-\lambda}{\lambda+t}\right)=\frac{1}{D}\left(\frac{t^2}{\lambda+t}\right) \tag{A.2a}$$

For t >D

$$\begin{aligned} F_{T_{A(D)}}(t) &= 1-\frac{E[s;t]-E[S;t-D]}{D} \\ &= 1-\frac{1}{D}\left(\frac{\lambda t}{\lambda+t}-\frac{\lambda(t-D)}{\lambda+t-D}\right)=1-\left(\frac{\lambda^2}{(\lambda+t)(\lambda+t-D)}\right) \end{aligned} \tag{A.2b}$$

For an accident year, D=1, and we get:

For t<1,

$$F_{T_{AY}}(t) = \left(\frac{t^2}{\lambda + t}\right) \qquad \text{(A.3a)}$$

For t >1,

$$F_{T_{AY}}(t) = 1 - \left(\frac{\lambda^2}{(\lambda + t)(\lambda + t - 1)}\right) \qquad \text{(A.3b)}$$

For a policy year, we first consider the exposures from the first calendar year.

We derive:

For t<1,

$$F_{T_{PY1}}(t) = \int_0^t dw\, w \cdot \left(1 - \left(\frac{\lambda}{\lambda + t - w}\right)^2\right) = \frac{t^2}{2} - \int_0^t du(t-u) \cdot \left(\frac{\lambda}{\lambda + u}\right)^2 \qquad \text{(A.4)}$$

After several integrations by parts and various standard manipulations, this reduces to:

$$F_{T_{PY1}}(t) = \frac{t^2}{2} + \lambda^2 \ln(1 + t/\lambda) - \lambda t \qquad \text{(A.5)}$$

Again restricting our attention for the moment to only those exposures earned in the first year of the policy year but now looking at evaluations exceeding unity, we derive the percent of ultimate:

For t>1,

$$F_{T_{PY1}}(t) = \int_0^1 dw\, w \cdot \left(1 - \left(\frac{\lambda}{\lambda + t - w}\right)^2\right) = \frac{1}{2} - \int_{t-1}^t du(t-u) \cdot \left(\frac{\lambda}{\lambda + u}\right)^2 \qquad \text{(A.6)}$$

After various standard manipulations, this simplifies to:

For t>1,

$$F_{T_{PY1}}(t) = \frac{1}{2} + \lambda^2 \ln\left(\frac{\lambda + t}{\lambda + t - 1}\right) - \frac{\lambda^2}{\lambda + t - 1} \tag{A.7}$$

Now we turn our attention to policy exposures earned in the second calendar year. We first consider evaluation dates in the second year and derive:

For 1<t<2

$$\begin{aligned} F_{T_{PY2}}(t) &= \int_1^t dw\,(2-w)\cdot\left(1-\left(\frac{\lambda}{\lambda+t-w}\right)^2\right) \\ &= \frac{1-(2-t)^2}{2} - \int_0^{t-1} du(2+u-t)\cdot\left(\frac{\lambda}{\lambda+u}\right)^2 \end{aligned} \tag{A.8}$$

This reduces to:

For 1<t<2,

$$\begin{aligned} F_{T_{PY2}}(t) &= \int_1^t dw\,(2-w)\cdot\left(1-\left(\frac{\lambda}{\lambda+t-w}\right)^2\right) \\ &= \frac{1-(2-t)^2}{2} - \lambda^2 \ln\left(\frac{\lambda+t-1}{\lambda}\right) - \frac{(\lambda+t-2)\lambda(t-1)}{\lambda+t-1} \end{aligned} \tag{A.9}$$

Again considering only policy year exposures earned in the second year, but now looking at evaluation dates beyond two years, we derive:

For t>2,

$$\begin{aligned} F_{T_{PY2}}(t) &= \int_1^2 dw\,(2-w)\cdot\left(1-\left(\frac{\lambda}{\lambda+t-w}\right)^2\right) \\ &= \frac{1}{2} - \int_{t-2}^{t-1} du(2+u-t)\cdot\left(\frac{\lambda}{\lambda+u}\right)^2 \end{aligned} \tag{A.10}$$

After some rather tedious but straightforward manipulations, this simplifies to:

For t>2,

$$F_{T_{PY2}}(t) = \frac{1}{2} - \frac{\lambda^2}{\lambda + t - 1} - \lambda^2 \ln\left(\frac{\lambda + t - 1}{\lambda + t - 2}\right) \tag{A.11}$$

Now we finally have all the pieces to evaluate the policy year percent of ultimate.

For example, at t=3, we would add together A.7 plus A.11 to get:

$$\begin{aligned} F_{T_{PY1}}(3) + F_{T_{PY2}}(3) &= \frac{1}{2} + \lambda^2 \ln\left(\frac{\lambda + 3}{\lambda + 2}\right) - \frac{\lambda^2}{\lambda + 2} + \frac{1}{2} - \frac{\lambda^2}{\lambda + 2} - \lambda^2 \ln\left(\frac{\lambda + 2}{\lambda + 1}\right) \\ &= 1 - \lambda^2 \ln\left(\frac{(\lambda + 2)^2}{(\lambda + 2)^2 - 1}\right) \end{aligned} \tag{A.12}$$

With the scale equal to 1.5, we obtain:

$$F_{T_{PY}}(3) = 1 - 2.25 \ln\left(\frac{(3.5)^2}{(3.5)^2 - 1}\right) = .80839 \tag{A.13}$$

Exhibit 1
Sheet 1

Sample Simulation of Exposure Dependent Development
Development Patterns Generated from Random Trails
20 = Number of Random Trials

	Exposure: W Accident Year				*Exposure: W* Policy Year			
Evaluation Age	*Formula Pct of Ultimate*	*Formula AU LDF*	*Simulated Pct of Ultimate*	*Simulated AU LDF*	*Formula Pct of Ultimate*	*Formula AU LDF*	*Simulated Pct of Ultimate*	*Simulated AU LDF*
0.250	3.57%	28.000	0.00%	#DIV/0!	0.31%	323.726	0.00%	#DIV/0!
0.500	12.50%	8.000	5.00%	20.000	2.23%	44.874	0.00%	#DIV/0!
0.750	25.00%	4.000	15.00%	6.667	6.85%	14.589	5.00%	20.000
1.000	40.00%	2.500	25.00%	4.000	14.94%	6.695	5.00%	20.000
1.250	53.25%	1.878	55.00%	1.818	26.39%	3.790	15.00%	6.667
1.500	62.50%	1.600	65.00%	1.538	39.00%	2.564	25.00%	4.000
1.750	69.23%	1.444	65.00%	1.538	50.88%	1.965	45.00%	2.222
2.000	74.29%	1.346	70.00%	1.429	60.77%	1.646	65.00%	1.538
2.250	78.18%	1.279	70.00%	1.429	68.09%	1.469	65.00%	1.538
2.500	81.25%	1.231	75.00%	1.333	73.50%	1.361	70.00%	1.429
2.750	83.71%	1.195	75.00%	1.333	77.62%	1.288	70.00%	1.429
3.000	85.71%	1.167	80.00%	1.250	80.84%	1.237	70.00%	1.429
3.250	87.37%	1.145	95.00%	1.053	83.40%	1.199	75.00%	1.333
3.500	88.75%	1.127	95.00%	1.053	85.48%	1.170	85.00%	1.176
3.750	89.92%	1.112	95.00%	1.053	87.19%	1.147	95.00%	1.053
4.000	90.91%	1.100	95.00%	1.053	88.61%	1.129	95.00%	1.053
4.250	91.76%	1.090	100.00%	1.000	89.80%	1.114	95.00%	1.053
4.500	92.50%	1.081	100.00%	1.000	90.82%	1.101	95.00%	1.053
4.750	93.14%	1.074	100.00%	1.000	91.68%	1.091	100.00%	1.000
5.000	93.71%	1.067	100.00%	1.000	92.44%	1.082	100.00%	1.000

Exhibit 1
Sheet 2

Sample Simulation of Exposure Dependent Development- Trial Listing
20 Random trials

	Process: S Pareto		*Exposure Generator*	*Exposure: W* Accident Year		*Exposure: W* Policy Year	
	Mean **1.5000** Shape **2.0000**						
Trial	***Process Random Number***	***Simulated S***	***Exposure Random Number***	***Simulated W***	***Total Lag T=S+W***	***Simulated W***	***Total Lag T=S+W***
1	0.9203	3.8131	0.4196	0.4196	4.2326	0.9160	4.7291
2	0.7438	1.4634	0.0127	0.0127	1.4761	0.1594	1.6228
3	0.8923	3.0703	0.0396	0.0396	3.1099	0.2813	3.3516
4	0.3144	0.3116	0.8249	0.8249	1.1366	1.4083	1.7199
5	0.3636	0.3803	0.7865	0.7865	1.1668	1.3465	1.7268
6	0.3508	0.3617	0.0799	0.0799	0.4416	0.3997	0.7614
7	0.3905	0.4213	0.8603	0.8603	1.2816	1.4714	1.8927
8	0.8827	2.8795	0.1994	0.1994	3.0790	0.6316	3.5111
9	0.1185	0.0976	0.9674	0.9674	1.0650	1.7445	1.8422
10	0.8309	2.1480	0.7681	0.7681	2.9162	1.3190	3.4671
11	0.7886	1.7623	0.1612	0.1612	1.9235	0.5679	2.3301
12	0.8543	2.4298	0.7724	0.7724	3.2022	1.3254	3.7551
13	0.3334	0.3372	0.2267	0.2267	0.5639	0.6733	1.0105
14	0.3494	0.3597	0.4011	0.4011	0.7608	0.8957	1.2554
15	0.7848	1.7332	0.7657	0.7657	2.4989	1.3154	3.0486
16	0.4073	0.4484	0.8111	0.8111	1.2594	1.3853	1.8337
17	0.2008	0.1779	0.8887	0.8887	1.0666	1.5283	1.7062
18	0.3040	0.2980	0.5561	0.5561	0.8541	1.0578	1.3557
19	0.5301	0.6882	0.6978	0.6978	1.3860	1.2226	1.9108
20	0.1742	0.1506	0.7604	0.7604	0.9110	1.3077	1.4584
Average	0.5267	1.1666	0.5500	0.5500	1.7166	1.0479	2.2145

Simulation of Exposure Dependent Development

Development Patterns Generated from Random Trails

2000 = Number of Random Trials

	Exposure: W Accident Year				*Exposure: W* Policy Year			
Evaluation Age	*Formula Pct of Ultimate*	*Formula AU LDF*	*Simulated Pct of Ultimate*	*Simulated AU LDF*	*Formula Pct of Ultimate*	*Formula AU LDF*	*Simulated Pct of Ultimate*	*Simulated AU LDF*
0.250	3.57%	28.000	3.95%	25.316	0.31%	323.726	0.40%	250.000
0.500	12.50%	8.000	12.50%	8.000	2.23%	44.874	2.60%	38.462
0.750	25.00%	4.000	24.70%	4.049	6.85%	14.589	7.60%	13.158
1.000	40.00%	2.500	40.35%	2.478	14.94%	6.695	15.15%	6.601
1.250	53.25%	1.878	52.55%	1.903	26.39%	3.790	25.95%	3.854
1.500	62.50%	1.600	62.30%	1.605	39.00%	2.564	39.70%	2.519
1.750	69.23%	1.444	69.35%	1.442	50.88%	1.965	50.65%	1.974
2.000	74.29%	1.346	74.15%	1.349	60.77%	1.646	60.20%	1.661
2.250	78.18%	1.279	78.05%	1.281	68.09%	1.469	67.60%	1.479
2.500	81.25%	1.231	81.15%	1.232	73.50%	1.361	73.40%	1.362
2.750	83.71%	1.195	83.85%	1.193	77.62%	1.288	77.80%	1.285
3.000	85.71%	1.167	85.45%	1.170	80.84%	1.237	80.90%	1.236
3.250	87.37%	1.145	87.05%	1.149	83.40%	1.199	83.40%	1.199
3.500	88.75%	1.127	88.60%	1.129	85.48%	1.170	84.85%	1.179
3.750	89.92%	1.112	90.00%	1.111	87.19%	1.147	87.15%	1.147
4.000	90.91%	1.100	90.75%	1.102	88.61%	1.129	88.40%	1.131
4.250	91.76%	1.090	91.60%	1.092	89.80%	1.114	89.75%	1.114
4.500	92.50%	1.081	92.45%	1.082	90.82%	1.101	90.75%	1.102
4.750	93.14%	1.074	93.15%	1.074	91.68%	1.091	91.50%	1.093
5.000	93.71%	1.067	93.80%	1.066	92.44%	1.082	92.40%	1.082

Exhibit 3
Sheet 1

Accident Year - AA LDF Fitting Summary

		Fitted AA LDF			
Age (year)	*Given AA LDF*	*Gamma*	*Pareto*	*Burr*	*Sherman Power Curve*
1	**1.920**	1.884	1.960	1.924	1.889
2	**1.228**	1.238	1.205	1.216	1.224
3	**1.098**	1.115	1.094	1.101	1.100
4	**1.051**	1.064	1.054	1.059	1.056
5	**1.036**	1.039	1.036	1.038	1.036
6	**1.025**	1.024	1.025	1.026	1.025
7	**1.019**	1.016	1.019	1.019	1.018
8	**1.014**	1.010	1.015	1.014	1.014
9	**1.011**	1.007	1.012	1.011	1.011
10	**1.009**	1.004	1.009	1.009	1.009
11	**1.008**	1.003	1.008	1.007	1.008

Exhibit 3
Sheet 2

Accident Year - AA LDF fitting

Process Distribution	Gamma
Mean	1.7731
Shape	0.6416
Scale	2.7636

Fitting Criteria	
Minimize Square Error	
Error	Difference in AALDF Back Product
Square Error	**0.0030**

Fitting

Age (year)	*Given AA LDF*	*Fitted LEV*	*Fitted % of Ult*	*Fitted AU LDF*	*Fitted AA LDF*	*Error in AA LDF*	*Back Product*	*Fitted Back Product*	*Error*
1	**1.9200**	0.6757	32.43%	3.0831	1.8843	-0.0357	3.0698	3.0638	-0.0060
2	**1.2280**	1.0645	61.12%	1.6362	1.2379	0.0099	1.5988	1.6259	0.0271
3	**1.0980**	1.3079	75.66%	1.3218	1.1145	0.0165	1.3020	1.3135	0.0115
4	**1.0510**	1.4647	84.32%	1.1859	1.0643	0.0133	1.1858	1.1785	-0.0073
5	**1.0360**	1.5672	89.74%	1.1143	1.0388	0.0028	1.1282	1.1073	-0.0209
6	**1.0250**	1.6350	93.22%	1.0727	1.0243	-0.0007	1.0890	1.0660	-0.0231
7	**1.0190**	1.6801	95.49%	1.0472	1.0156	-0.0034	1.0625	1.0407	-0.0218
8	**1.0140**	1.7103	96.98%	1.0311	1.0102	-0.0038	1.0427	1.0247	-0.0180
9	**1.0110**	1.7306	97.97%	1.0207	1.0067	-0.0043	1.0283	1.0143	-0.0140
10	**1.0090**	1.7442	98.63%	1.0139	1.0045	-0.0045	1.0171	1.0075	-0.0096
11	**1.0080**	1.7535	99.08%	1.0093	1.0030	-0.0050	1.0080	1.0030	-0.0050
12		1.7598	99.37%						

Exhibit 3
Sheet 3

Accident Year - AA LDF Fitting

Process Distribution	Pareto
Mean	64.8752
Shape	1.0164

Fitting Criteria	
Minimize Square Error	
Error	Difference in AALDF Back Product
Square Error	**0.00094**

Fitting

Age (year)	*Given AA LDF*	*Fitted LEV*	*Fitted % of Ult*	*Fitted AU LDF*	*Fitted AA LDF*	*Error in AA LDF*	*Back Product*	*Fitted Back Product*	*Error*
1	**1.9200**	0.7015	29.85%	3.3506	1.9604	0.0404	3.0698	3.0775	0.0078
2	**1.2280**	1.1165	58.51%	1.7091	1.2049	-0.0231	1.5988	1.5698	-0.0290
3	**1.0980**	1.4115	70.50%	1.4185	1.0939	-0.0041	1.3020	1.3029	0.0009
4	**1.0510**	1.6404	77.11%	1.2968	1.0544	0.0034	1.1858	1.1911	0.0054
5	**1.0360**	1.8273	81.31%	1.2299	1.0357	-0.0003	1.1282	1.1296	0.0014
6	**1.0250**	1.9852	84.21%	1.1875	1.0252	0.0002	1.0890	1.0907	0.0017
7	**1.0190**	2.1218	86.34%	1.1583	1.0188	-0.0002	1.0625	1.0638	0.0014
8	**1.0140**	2.2422	87.96%	1.1369	1.0146	0.0006	1.0427	1.0442	0.0016
9	**1.0110**	2.3498	89.24%	1.1206	1.0116	0.0006	1.0283	1.0292	0.0010
10	**1.0090**	2.4470	90.28%	1.1077	1.0095	0.0005	1.0171	1.0174	0.0003
11	**1.0080**	2.5357	91.13%	1.0973	1.0079	-0.0001	1.0080	1.0079	-0.0001
12		2.6172	91.85%						

Accident Year - AA LDF fitting

Process Distribution	**Burr**
Mean	3.2549
Shape	0.8505

Fitting Criteria	
Minimize Square Error	
Error	Difference in AALDF Back Product
Square Error	**0.00082**

Fitting

Age (year)	*Given AA LDF*	*Fitted LEV*	*Fitted % of Ult*	*Fitted AU LDF*	*Fitted AA LDF*	*Error in AA LDF*	*Back Product*	*Fitted Back Product*	*Error*
1	**1.9200**	0.6927	30.73%	3.2540	1.9240	0.0040	3.0698	3.0878	0.0180
2	**1.2280**	1.1014	59.13%	1.6913	1.2164	-0.0116	1.5988	1.6049	0.0061
3	**1.0980**	1.3822	71.92%	1.3904	1.1015	0.0035	1.3020	1.3193	0.0174
4	**1.0510**	1.5899	79.22%	1.2622	1.0587	0.0077	1.1858	1.1978	0.0120
5	**1.0360**	1.7512	83.87%	1.1923	1.0379	0.0019	1.1282	1.1314	0.0031
6	**1.0250**	1.8806	87.05%	1.1487	1.0263	0.0013	1.0890	1.0900	0.0010
7	**1.0190**	1.9872	89.34%	1.1193	1.0191	0.0001	1.0625	1.0622	-0.0003
8	**1.0140**	2.0768	91.05%	1.0984	1.0144	0.0004	1.0427	1.0423	-0.0004
9	**1.0110**	2.1532	92.36%	1.0828	1.0112	0.0002	1.0283	1.0275	-0.0008
10	**1.0090**	2.2194	93.39%	1.0708	1.0089	-0.0001	1.0171	1.0161	-0.0010
11	**1.0080**	2.2772	94.22%	1.0614	1.0072	-0.0008	1.0080	1.0072	-0.0008
12		2.3283	94.89%						

Exhibit 4

Accident Quarter by Quarter - LDF Generation

Process Distribution	Burr
Mean	3.2549
Shape	0.8505

AQ by Q LDF Generated by LEV Formula

Age (year)	LEV	% of Ult	AU LDF	AA LDF
0.250	0.2205	11.80%	8.4716	2.3132
0.500	0.4022	27.30%	3.6624	1.3881
0.750	0.5575	37.90%	2.6384	1.2114
1.000	0.6927	45.92%	2.1779	1.1378
1.250	0.8121	52.24%	1.9141	1.0984
1.500	0.9186	57.38%	1.7426	1.0743
1.750	1.0145	61.65%	1.6222	1.0582
2.000	1.1014	65.23%	1.5329	1.0469
2.250	1.1807	68.29%	1.4642	1.0386
2.500	1.2533	70.93%	1.4098	1.0324
2.750	1.3203	73.23%	1.3656	1.0275
3.000	1.3822	75.24%	1.3291	1.0236
3.250	1.4396	77.02%	1.2984	1.0205
3.500	1.4931	78.60%	1.2723	1.0179
3.750	1.5431	80.01%	1.2499	1.0158
4.000	1.5899	81.27%	1.2304	1.0140
4.250	1.6339	82.41%	1.2134	1.0125
4.500	1.6753	83.45%	1.1983	1.0113
4.750	1.7143	84.39%	1.1850	1.0102
5.000	1.7512	85.25%	1.1731	1.0092
5.250	1.7861	86.03%	1.1624	1.0084
5.500	1.8192	86.75%	1.1527	1.0076
5.750	1.8507	87.41%	1.1440	1.0070
6.000	1.8806	88.02%	1.1361	

Exhibit 5
Sheet 1

Average Maturity of Loss Approximation of Policy Year Development Based on Accident Year

Derivation of PY Average Maturity and AY Age of Equivalent Maturity

Evaluation Age t*	***Exposure Growth Triangle Prob***	***Exposure Growth Triangle Avg Date***	***Exposure Decay Triangle Prob***	***Exposure Decay Triangle Avg Date***	***PY ETD ETD*(t*)***	***PY Average Date of Loss***	***PY Average Maturity***	***AY Age of Equivalent Maturity t***	***AY ETD ETD(t)***
0.250	3.125%	0.167	0.000%	1.000	3.125%	0.167	0.083	0.167	16.667%
0.500	12.500%	0.333	0.000%	1.000	12.500%	0.333	0.167	0.333	33.333%
0.750	28.125%	0.500	0.000%	1.000	28.125%	0.500	0.250	0.500	50.000%
1.000	50.000%	0.667	0.000%	1.000	50.000%	0.667	0.333	0.667	66.667%
1.250	50.000%	0.667	21.875%	1.119	71.875%	0.804	0.446	0.891	89.130%
1.500	50.000%	0.667	37.500%	1.222	87.500%	0.905	0.595	1.095	100.000%
1.750	50.000%	0.667	46.875%	1.300	96.875%	0.973	0.777	1.277	100.000%
2.000	50.000%	0.667	50.000%	1.333	100.000%	1.000	1.000	1.500	100.000%
2.250	50.000%	0.667	50.000%	1.333	100.000%	1.000	1.250	1.750	100.000%
2.500	50.000%	0.667	50.000%	1.333	100.000%	1.000	1.500	2.000	100.000%
2.750	50.000%	0.667	50.000%	1.333	100.000%	1.000	1.750	2.250	100.000%
3.000	50.000%	0.667	50.000%	1.333	100.000%	1.000	2.000	2.500	100.000%
3.250	50.000%	0.667	50.000%	1.333	100.000%	1.000	2.250	2.750	100.000%
3.500	50.000%	0.667	50.000%	1.333	100.000%	1.000	2.500	3.000	100.000%
3.750	50.000%	0.667	50.000%	1.333	100.000%	1.000	2.750	3.250	100.000%
4.000	50.000%	0.667	50.000%	1.333	100.000%	1.000	3.000	3.500	100.000%
4.250	50.000%	0.667	50.000%	1.333	100.000%	1.000	3.250	3.750	100.000%
4.500	50.000%	0.667	50.000%	1.333	100.000%	1.000	3.500	4.000	100.000%
4.750	50.000%	0.667	50.000%	1.333	100.000%	1.000	3.750	4.250	100.000%
5.000	50.000%	0.667	50.000%	1.333	100.000%	1.000	4.000	4.500	100.000%

Exhibit 5
Sheet 2

Average Maturity of Loss Approximation of Policy Year Development Based on Accident Year

Derivation of PY AU LDF Approximation

Evaluation Age t^*	PY ETD $ETD^*(t^*)$	AY PCT of ULT $PCT(t^*)$	AY AU $LDF(t^*)$	AY Age of Equivalent Maturity t	AY ETD $ETD(t)$	AY PCT of ULT $PCT(t)$	AY AU $LDF(t)$	PY Avg Maturity Approx PCT of ULT	PY Avg Maturity Approx AU LDF	PY True AU LDF	Error
0.250	3.125%	3.57%	28.000	0.167	16.667%	1.67%	60.000	0.31%	320.000	323.726	-3.726
0.500	12.500%	12.50%	8.000	0.333	33.333%	6.06%	16.500	2.27%	44.000	44.874	-0.874
0.750	28.125%	25.00%	4.000	0.500	50.000%	12.50%	8.000	7.03%	14.222	14.589	-0.366
1.000	50.000%	40.00%	2.500	0.667	66.667%	20.51%	4.875	15.38%	6.500	6.695	-0.195
1.250	71.875%	53.25%	1.878	0.891	89.130%	33.22%	3.010	26.79%	3.733	3.790	-0.057
1.500	87.500%	62.50%	1.600	1.095	100.000%	45.65%	2.190	39.95%	2.503	2.564	-0.061
1.750	96.875%	69.23%	1.444	1.277	100.000%	54.40%	1.838	52.70%	1.898	1.965	-0.068
2.000	100.000%	74.29%	1.346	1.500	100.000%	62.50%	1.600	62.50%	1.600	1.646	-0.046
2.250	100.000%	78.18%	1.279	1.750	100.000%	69.23%	1.444	69.23%	1.444	1.469	-0.024
2.500	100.000%	81.25%	1.231	2.000	100.000%	74.29%	1.346	74.29%	1.346	1.361	-0.014
2.750	100.000%	83.71%	1.195	2.250	100.000%	78.18%	1.279	78.18%	1.279	1.288	-0.009
3.000	100.000%	85.71%	1.167	2.500	100.000%	81.25%	1.231	81.25%	1.231	1.237	-0.006
3.250	100.000%	87.37%	1.145	2.750	100.000%	83.71%	1.195	83.71%	1.195	1.199	-0.004
3.500	100.000%	88.75%	1.127	3.000	100.000%	85.71%	1.167	85.71%	1.167	1.170	-0.003
3.750	100.000%	89.92%	1.112	3.250	100.000%	87.37%	1.145	87.37%	1.145	1.147	-0.002
4.000	100.000%	90.91%	1.100	3.500	100.000%	88.75%	1.127	88.75%	1.127	1.129	-0.002
4.250	100.000%	91.76%	1.090	3.750	100.000%	89.92%	1.112	89.92%	1.112	1.114	-0.001
4.500	100.000%	92.50%	1.081	4.000	100.000%	90.91%	1.100	90.91%	1.100	1.101	-0.001
4.750	100.000%	93.14%	1.074	4.250	100.000%	91.76%	1.090	91.76%	1.090	1.091	-0.001
5.000	100.000%	93.71%	1.067	4.500	100.000%	92.50%	1.081	92.50%	1.081	1.082	-0.001

Multi-shifted Approximation of Policy Year
Using 5 Shifted Accident Years

	Exposure: W Accident Year		*Exposure: W* Policy Year		*Policy Year* *Multi-shifted Approximation*						
						Shift	0.000	0.250	0.500	0.750	1.000
						Weight	0.125	0.250	0.250	0.250	0.125
Evaluation Age	***Formula Pct of Ultimate***	***Formula AU LDF***	***Formula Pct of Ultimate***	***Formula AU LDF***	***Pct of Ult***	***AU LDF***	***Pct of Ultimate***	***Pct of Ultimate***	***Pct of Ultimate***	***Pct of Ultimate***	***Pct of Ultimate***
0.250	3.57%	28.000	0.31%	323.726	0.45%	224.000	3.57%				
0.500	12.50%	8.000	2.23%	44.874	2.46%	40.727	12.50%	3.57%			
0.750	25.00%	4.000	6.85%	14.589	7.14%	14.000	25.00%	12.50%	3.57%		
1.000	40.00%	2.500	14.94%	6.695	15.27%	6.550	40.00%	25.00%	12.50%	3.57%	
1.250	53.25%	1.878	26.39%	3.790	26.48%	3.777	53.25%	40.00%	25.00%	12.50%	3.57%
1.500	62.50%	1.600	39.00%	2.564	38.94%	2.568	62.50%	53.25%	40.00%	25.00%	12.50%
1.750	69.23%	1.444	50.88%	1.965	50.72%	1.972	69.23%	62.50%	53.25%	40.00%	25.00%
2.000	74.29%	1.346	60.77%	1.646	60.53%	1.652	74.29%	69.23%	62.50%	53.25%	40.00%
2.250	78.18%	1.279	68.09%	1.469	67.93%	1.472	78.18%	74.29%	69.23%	62.50%	53.25%
2.500	81.25%	1.231	73.50%	1.361	73.39%	1.363	81.25%	78.18%	74.29%	69.23%	62.50%
2.750	83.71%	1.195	77.62%	1.288	77.55%	1.290	83.71%	81.25%	78.18%	74.29%	69.23%
3.000	85.71%	1.167	80.84%	1.237	80.79%	1.238	85.71%	83.71%	81.25%	78.18%	74.29%
3.250	87.37%	1.145	83.40%	1.199	83.36%	1.200	87.37%	85.71%	83.71%	81.25%	78.18%
3.500	88.75%	1.127	85.48%	1.170	85.45%	1.170	88.75%	87.37%	85.71%	83.71%	81.25%
3.750	89.92%	1.112	87.19%	1.147	87.16%	1.147	89.92%	88.75%	87.37%	85.71%	83.71%
4.000	90.91%	1.100	88.61%	1.129	88.59%	1.129	90.91%	89.92%	88.75%	87.37%	85.71%
4.250	91.76%	1.090	89.80%	1.114	89.79%	1.114	91.76%	90.91%	89.92%	88.75%	87.37%
4.500	92.50%	1.081	90.82%	1.101	90.80%	1.101	92.50%	91.76%	90.91%	89.92%	88.75%
4.750	93.14%	1.074	91.68%	1.091	91.68%	1.091	93.14%	92.50%	91.76%	90.91%	89.92%
5.000	93.71%	1.067	92.44%	1.082	92.43%	1.082	93.71%	93.14%	92.50%	91.76%	90.91%

Exhibit 7

Accident Year - LDF Generation
Mixed Processes

Process1	
Distribution	**Gamma**
Mean	1.0000
Shape	1.0000
Scale	1.0000

Process2	
Distribution	**Gamma**
Mean	8.0000
Shape	2.0000
Scale	4.0000
Weight	10.00%

Derivation of Development Pattern for Total Mixed Process

Age (year)	*Process1 LEV*	*Process1 % of Ult*	*Process2 LEV*	*Process2 % of Ult*	*Total Process % of Ult*	*Total Process AU LDF*	*Total Process AA LDF*
1	0.6321	36.79%	0.9908	0.92%	33.20%	3.0119	2.0973
2	0.8647	76.75%	1.9347	5.61%	69.63%	1.4361	1.2007
3	0.9502	91.45%	2.8040	13.07%	83.61%	1.1961	1.0687
4	0.9817	96.85%	3.5854	21.85%	89.35%	1.1192	1.0303
5	0.9933	98.84%	4.2754	31.00%	92.06%	1.0863	1.0168
6	0.9975	99.57%	4.8762	39.93%	93.61%	1.0683	1.0115
7	0.9991	99.84%	5.3934	48.28%	94.69%	1.0561	1.0090
8	0.9997	99.94%	5.8346	55.88%	95.54%	1.0467	1.0074
9	0.9999	99.98%	6.2082	62.64%	96.25%	1.0390	1.0063
10	1.0000	99.99%	6.5225	68.57%	96.85%	1.0325	1.0053
11	1.0000	100.00%	6.7854	73.71%	97.37%	1.0270	1.0045
12	1.0000	100.00%	7.0043	78.11%	97.81%	1.0224	1.0038
13	1.0000	100.00%	7.1857	81.85%	98.18%	1.0185	1.0032
14	1.0000	100.00%	7.3357	85.01%	98.50%	1.0152	1.0027
15	1.0000	100.00%	7.4591	87.66%	98.77%	1.0125	

Accident Year - LDF Generation
Negative Development

Process Distribution	Gamma
Mean	1.0000
Shape	1.0000
Scale	1.0000

Decrement Lag Distribution	Gamma
Mean	2.0000
Shape	2.0000
Scale	1.0000
Weight	**30.00%**

Process +Decrement Lag	Gamma
Mean	3.0000
Shape	3.0000
Scale	1.0000

Derivation of Development Pattern

Age (year)	*Process LEV*	*Process % of Ult*	*Process +Decrement LEV*	*Process + Decrement % of Ult*	*% of Ult*	*AU LDF*	*AA LDF*
1	0.6321	36.79%	0.9767	2.33%	51.55%	1.9397	1.9648
2	0.8647	76.75%	1.7820	19.47%	101.29%	0.9872	1.0975
3	0.9502	91.45%	2.3279	45.41%	111.17%	0.8995	0.9840
4	0.9817	96.85%	2.6520	67.59%	109.40%	0.9141	0.9680
5	0.9933	98.84%	2.8282	82.38%	105.90%	0.9443	0.9750
6	0.9975	99.57%	2.9182	91.00%	103.25%	0.9685	0.9846
7	0.9991	99.84%	2.9622	95.60%	101.66%	0.9837	0.9916
8	0.9997	99.94%	2.9829	97.93%	100.81%	0.9920	0.9957
9	0.9999	99.98%	2.9924	99.05%	100.38%	0.9962	0.9980
10	1.0000	99.99%	2.9967	99.57%	100.17%	0.9983	0.9990
11	1.0000	100.00%	2.9986	99.81%	100.08%	0.9992	0.9996
12	1.0000	100.00%	2.9994	99.92%	100.03%	0.9997	0.9998
13	1.0000	100.00%	2.9997	99.96%	100.01%	0.9999	0.9999
14	1.0000	100.00%	2.9999	99.99%	100.01%	0.9999	1.0000
15	1.0000	100.00%	3.0000	99.99%	100.00%	1.0000	

Accident Year - LDF Generation
Mixed Processes- One with Negative Development

Process1 Distribution	Gamma
Mean	1.0000
Shape	1.0000
Scale	1.0000

Decrement Lag Distribution	Gamma
Mean	2.0000
Shape	2.0000
Scale	1.0000
Weight	**30.00%**

Process1 +Decrement Lag	Gamma
Mean	3.0000
Shape	3.0000
Scale	1.0000

Process2 Distribution	Gamma
Mean	8.0000
Shape	2.0000
Scale	4.0000
Weight	**20.00%**

Derivation of Development Pattern

Age (year)	*Process1 LEV*	*Process1 % of Ult*	*Process1 +Decrement LEV*	*Process1 + Decrement % of Ult*	*Process1 After Decrement % of Ult*	*Process2 LEV*	*Process2 % of Ult*	*Total % of Ult*	*Total AU LDF*	*Total AA LDF*
1	0.6321	36.79%	0.9767	2.33%	51.55%	0.9908	0.92%	41.43%	2.4139	1.9831
2	0.8647	76.75%	1.7820	19.47%	101.29%	1.9347	5.61%	82.16%	1.2172	1.1144
3	0.9502	91.45%	2.3279	45.41%	111.17%	2.8040	13.07%	91.55%	1.0923	1.0036
4	0.9817	96.85%	2.6520	67.59%	109.40%	3.5854	21.85%	91.89%	1.0883	0.9895
5	0.9933	98.84%	2.8282	82.38%	105.90%	4.2754	31.00%	90.92%	1.0999	0.9963
6	0.9975	99.57%	2.9182	91.00%	103.25%	4.8762	39.93%	90.58%	1.1039	1.0044
7	0.9991	99.84%	2.9622	95.60%	101.66%	5.3934	48.28%	90.98%	1.0991	1.0092
8	0.9997	99.94%	2.9829	97.93%	100.81%	5.8346	55.88%	91.82%	1.0891	1.0110
9	0.9999	99.98%	2.9924	99.05%	100.38%	6.2082	62.64%	92.83%	1.0772	1.0110
10	1.0000	99.99%	2.9967	99.57%	100.17%	6.5225	68.57%	93.85%	1.0655	1.0101
11	1.0000	100.00%	2.9986	99.81%	100.08%	6.7854	73.71%	94.80%	1.0548	1.0089
12	1.0000	100.00%	2.9994	99.92%	100.03%	7.0043	78.11%	95.65%	1.0455	1.0077
13	1.0000	100.00%	2.9997	99.96%	100.01%	7.1857	81.85%	96.38%	1.0375	1.0065
14	1.0000	100.00%	2.9999	99.99%	100.01%	7.3357	85.01%	97.01%	1.0309	1.0054
15	1.0000	100.00%	3.0000	99.99%	100.00%	7.4591	87.66%	97.53%	1.0253	