On the Optimality of Multiline Excess of Loss Covers

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Abstract

It is well known that diversifying the risk between independent policies reduces the total risk in the sense that less deviations around the aggregate mean loss are expected. In other words, less capital has to be allocated due to the diversification effect.
The same effect can be obtained when an insurance company buys an excess of loss cover. Instead of buying independently covers for different lines of business, it is intuitively acceptable to believe that the insurance company has interest in diversifying by buying a multiline excess of loss cover.
In the present paper I show how to deal with the dependencies induced by such a model and using some risk measures we show on a numerical example the optimality of the multiline agreement.

Keywords

Multivariate Panjer's algorithm, multiline excess of loss cover, standard deviation, Wang Transform, optimal reinsurance.

Biography

Jean-François Walhin is R&D Manager at Secura Belgian Re which he joined in 1999 after having worked for three years for a primary insurer. He is also a visiting Professor at the Catholic University of Louvain-la-Neuve (UCL). Jean-François is a civil engineer and an actuary and holds a PhD in science from the UCL. He is the author of 20 papers that have appeared in actuarial journals. Jean-François is a Fellow of the ARAB-KVBA, the Belgian Association of Actuaries.

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1. INTRODUCTION

Multiline excess of loss covers are introduced in Ribeaud (2000).
Walhin (2002) introduced the practical pricing of multiline excess of loss covers. To keep things simple, we will assume that we have two lines of business : Fire and MTPL. Let us define

- X_i^{Fire} as the i^{th} claim amount of type Fire,
- X_i^{MTPL} as the i^{th} claim amount of type Motor Third Part Liability (MTPL in short).

It is assumed that the X_i^{Fire}'s are independent and identically distributed as well as the X_i^{MTPL}'s. X_i^{Fire}'s and X_i^{MTPL}'s are assumed to be mutually independent. We also define

- N as the number of claims of type Fire,
- M as the number of claims of type MTPL.

We assume that N and M are independent and that N and the X_i^{Fire}'s on the one hand and M and the X_i^{MTPL}'s on the other hand are also independent.
Let us define the liability of the excess of loss reinsurer for each claim :

$$\begin{aligned} R_i^{Fire} &= \min(L^{Fire}, \max(0, X_i^{Fire} - D^{Fire})), \\ R_i^{MTPL} &= \min(L^{MTPL}, \max(0, X_i^{MTPL} - D^{MTPL})). \end{aligned}$$

where

1. D^{Fire} is the deductible for fire claims
2. L^{Fire} is the limit for fire claims
3. D^{MTPL} is the deductible for MTPL claims
4. L^{MTPL} is the limit for MTPL claims

Let us define the liability of the ceding company for each claim :

$$\begin{aligned} C_i^{Fire} &= X_i^{Fire} - R_i^{Fire}, \\ C_i^{MTPL} &= X_i^{MTPL} - R_i^{MTPL}. \end{aligned}$$

Let us define the aggregate liability of the reinsurer for each line :

$$\begin{aligned} S^{Fire} &= \sum_{i=1}^{N} R_i^{Fire}, \\ S^{MTPL} &= \sum_{i=1}^{M} R_i^{MTPL}. \end{aligned}$$

Let us define the aggregate liability of the ceding company for each line :

$$\begin{aligned} T^{Fire} &= \sum_{i=1}^{N} C_i^{Fire}, \\ T^{MTPL} &= \sum_{i=1}^{M} C_i^{MTPL}. \end{aligned}$$

Now let us assume that the ceding company buys a multiline excess of loss cover of the form

$$Cover = \max(0, S^{Fire} + S^{MTPL} - GAAD)$$

where $GAAD$ is a global annual aggregate deductible playing on both lines of business.
In this paper we are interested in analysing the retention's risk of the ceding company :

$$Retention = T^{Fire} + T^{MTPL} + \min(S^{Fire} + S^{MTPL}, GAAD).$$

2. Dependencies generated by the model

Analysing and modelling dependencies is a subject that received great attention during the last few years. Different methods have been proposed to tackle that problem, e.g. the use of Fréchet bounds (see e.g. Dhaene et al. (2001)) or the use of copulas (see e.g. Frees and Valdez (1998). These methods do not recognize the exact dependency structure because it is often not possible to model it.
In our case, there is clearly some dependency which does not allow an easy analysis of the problem. However the dependency in our model is induced by the model itself. We then have the chance to model the dependency exactly and possibly obtain exact calculations.
The fact that reinsurance induces dependencies has been observed by Walhin and Paris (2000) for the analysis of the cedent's retention's risk when there are paid reinstatements, by Walhin and Denuit (2003) for the practical pricing of Top & Drop covers, by Walhin (2003) for the pricing of exotic excess of loss covers. The present paper shows another dependency induced by the model.
Fortunately it is easy to make a modelization of our dependency : the random variables R_i^{Fire}, C_i^{Fire} depend on X_i^{Fire} whereas R_i^{MTPL}, C_i^{MTPL} depend on X_i^{MTPL}. This means that even though N, M, X^{Fire}, and Y^{Fire} are mutually independent, S^{Fire}, S^{MTPL}, T^{Fire}, T^{MTPL} are not which makes the calculation of the distribution of $Retention$ difficult. We need to obtain the joint distribution of

$$(S^{Fire}, S^{MTPL}, T^{Fire}, T^{MTPL}).$$

In fact if we obtain the joint distributions of (S^{Fire}, T^{Fire}) and (S^{MTPL}, T^{MTPL}), we have a solution to our problem because these random vectors are independent thanks to the mutual independence hypotheses we made.
An easy solution is available and is described in the next section.

3. Bivariate Panjer's algorithm

Our problem fits exactly within the framework of the multivariate Panjer's algorithm, described in Walhin and Paris (2000), or in Sundt (1999).
We just need the bivariate setting in order to obtain the joint distributions we need.
Let us define :

$$\begin{aligned} f^i(x,y) &= \mathbb{P}[R^i = x, C^i = y] \quad , \quad i = Fire, MTPL, \\ g^i(s,t) &= \mathbb{P}[S^i = s, T^i = t] \quad , \quad i = Fire, MTPL. \end{aligned}$$

From now on we will not use the superscript anymore.
Let us assume that N belongs to the Panjer's family of counting distributions :

$$\frac{\mathbb{P}[N=n]}{\mathbb{P}[N=n-1]} = a + \frac{b}{n} \quad , \quad n \geq 1.$$

We have :

$$\begin{aligned} g(0,0) &= \Psi_N(f(0,0)), \\ g(s,t) &= \frac{1}{(1-af(0,0))} \sum_{x,y}^{s,t} [a + b\frac{x}{s}] g(s-x,t-y) f(x,y) \quad , \quad s \geq 1, \\ g(s,t) &= \frac{1}{(1-af(0,0))} \sum_{x,y}^{s,t} [a + b\frac{y}{t}] g(s-x,t-y) f(x,y) \quad , \quad t \geq 1, \end{aligned}$$

where

$$\begin{aligned} \sum_{x,y}^{s,t} g(x,y) &= \sum_{x=0}^{\min(s,m)} \sum_{y=0}^{\min(t,n)} g(x,y) - g(0,0), \\ m &= \max(x | f(x,y) > 0), \\ n &= \max(y | f(x,y) > 0). \end{aligned}$$

and $\Psi_N(u)$ denotes the probability generating function of N : $\Psi_N(u) = \mathbb{E}[u^N]$.

It is clear that the above-mentioned algorithm is time-consuming. However we will take advantage of the specific dependence structure in order to minimize the computing time. Indeed the random vector (R, C) has positive masses only along an S-shape. So we may adapt the formula as :

$$
\begin{aligned}
g(0,0) &= \Psi_N(f(0,0)),\\
g(0,t) &= 0 \quad , \quad t \geq 1,\\
g(s,0) &= \frac{1}{(1-af(0,0))}\sum_{x=1}^{s}[a+b\frac{x}{s}]g(s-x,0)f(x,0) \quad , \quad 1 \leq s \leq D,\\
g(s,t) &= \frac{1}{(1-af(0,0))}\times\\
&\left[\sum_{x=1}^{s}[a+b\frac{x}{s}]g(s-x,0)f(x,0)+\sum_{y=1}^{t}[a+b\frac{D}{s}]g(s-D,t-y)f(D,y)\right] \quad ,\\
&1 \leq s \leq D \quad , \quad 1 \leq t \leq L,\\
g(s,t) &= \frac{1}{(1-af(0,0)))}\times\\
&\left[\sum_{x=1}^{s}[a+b\frac{x}{s}]g(s-x,0)f(x,0)+\sum_{y=1}^{t}[a+b\frac{D}{s}]g(s-D,t-y)f(D,y)+\right.\\
&\left.\sum_{x=D+1}^{s}[a+b\frac{x}{s}]g(s-x,t-L)f(x,L)\right] \quad , \quad s > D \quad , \quad t \geq L.
\end{aligned}
$$

4. Numerical application

Let us make the following hypotheses for our numerical example :

- the distribution of the fire claim amounts, X^{Fire}, is limited Pareto with parameters $A = 400$, $B = 2000$ and $\alpha = 1.50$. The distribution of the MTPL claim amounts, X^{MTPL} is limited Pareto with parameters $A = 700$, $B = 2000$ and $\alpha = 2.50$. Let us recall the cumulative density distribution of a limited Pareto distribution ($X \sim Pa(A,B,\alpha)$) :

$$
\begin{aligned}
F_X(x) &= 0 \quad \text{if } x \leq A,\\
&= \frac{A^{-\alpha}-x^{-\alpha}}{A^{-\alpha}-B^{-\alpha}} \quad \text{if } A < x \leq B,\\
&= 1 \quad , \quad x > B.
\end{aligned}
$$

- the distribution of the fire claim numbers, N is Poisson with parameter $\lambda = 2.5$. The distribution of the MTPL claim numbers, M is Poisson with parameter $\lambda = 5$. Let us recall the probability function of a Poisson distribution ($N \sim Po(\lambda)$) :

$$
\mathbb{P}[N=n] = p(n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad , \quad n = 0,1,\ldots
$$

Working with Poisson distributions allows us to work with the bivariate Panjer's algorithm. Moreover, it simplifies the use of the algorithm as $a = 0$ in the Poisson case.

We note that the limited Pareto distribution is a continuous distribution whereas we need a discrete distribution in order to use the bivariate Panjer's algorithm. We therefore choose to obtain a discretization of our limited Pareto distributions by using the local moment matching method with one moment (see Gerber (1982)). It is not difficult to show that the discrete version of a limited Pareto distribution is given by

$$
\begin{aligned}
f_{X_{dis}}(A) &= 1 - \frac{\frac{(A+h)^{1-\alpha}}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} - B^{-\alpha}h}{h(A^{-\alpha} - B^{-\alpha})}, \\
f_{X_{dis}}(A + jh) &= \frac{2(A+jh)^{1-\alpha} - (A+(j-1)h)^{1-\alpha} - (A+(j+1)h)^{1-\alpha}}{h(1-\alpha)(A^{-\alpha} - B^{-\alpha})}, \\
& \quad j = 1, \ldots, \frac{B-A}{h} - 1, \\
f_{X_{dis}}(B) &= 1 - f_{X_{dis}}(A) - f_{X_{dis}}(A+h) - \cdots - f_{X_{dis}}(B-h).
\end{aligned}
$$

where h is chosen such that $\frac{B-A}{h}$ is an integer.
Obtaining the expected retained loss is easily given by

$$\mathbb{E}Retention = \mathbb{E}T^{Fire} + \mathbb{E}T^{MTPL} + \mathbb{E}\min(S^{Fire} + S^{MTPL}, GAAD).$$

As S^{Fire} and S^{MTPL} are independent, we do not need to apply the bivariate Panjer's algorithm. However as we will compute standard deviation and Wang Transforms of *Retention*, we will need the distribution of *Retention* and thus we will have to apply the bivariate Panjer's algorithm. We will also make the calculations with the false assumption of independence between S^{Fire} and T^{Fire} on the one hand and S^{MTPL} and T^{MTPL} on the other hand.
Our aim is now to analyse different reinsurance structures in order to find optimal reinsurance agreements. We will therefore let the deductibles and limits vary as well as the global annual aggregate deductible.
For each situation we are going to compute the following elements :

1. $\mathbb{E}Retention$
2. $\sigma(Retention)$
3. $WT_{0.90}(Retention)$
4. $WT_{0.95}(Retention)$
5. $WT_{0.99}(Retention)$

where $WT_{1-\alpha}(Retention)$ denotes the Wang Transform of level α of the random variable *Retention* (see Wang (2002)). Let us define

1. F the cumulative density function of the random variable *Retention*
2. $\Phi(.)$ the cumulative density function of the standard normal distribution
3. α a security level
4. $\lambda = \Phi^{-1}(\alpha)$

5. $F^*(x) = \Phi[\Phi^{-1}(F(x)) - \lambda]$

Then the Wang Transform of level α is given by the expectation of *Retention* under the measure F^* :

$$WT_{1-\alpha}(Retention) = \mathbb{E}^*(Retention).$$

A good situation for the insurer is when $\mathbb{E}Retention$ is as high as possible (in such a case, it means that the cession to the reinsurer is small which means in other words that the smallest expected profit is ceded to the reinsurer) and when the risk measure (either the standard deviation or the Wang Transform) is as small as possible (which means that few capital has to be allocated).

Let us first analyse the following case, which we denote Treaty 1 :

$$\begin{aligned} D^{Fire} &= 500, \\ L^{Fire} &= 1500, \\ D^{MTPL} &= 800, \\ L^{MTPL} &= 1200, \\ GAAD &= 0. \end{aligned}$$

Table 1: Treaty 1

We obtain the following quantities of interest :

$$\begin{aligned} \mathbb{E}(Retention) &= 3949.617, \\ \sigma(Retention) &= 1655.303, \\ WT_{0.90}(Retention) &= 6252.296, \\ WT_{0.95}(Retention) &= 6971.925, \\ WT_{0.99}(Retention) &= 8394.352. \end{aligned}$$

Table 2: Retained risk for Treaty 1

Assume that the ceding company does not agree with such a large cession. Then a natural solution is to increase the priorities of the treaties. We then move to Treaty 2 :

$$\begin{aligned} D^{Fire} &= 800, \\ L^{Fire} &= 1200, \\ D^{MTPL} &= 1000, \\ L^{MTPL} &= 1000, \\ GAAD &= 0. \end{aligned}$$

Table 3: Treaty 2

We obtain the following quantities of interest :

$$\begin{aligned} \mathbb{E}(Retention) &= 4642.687 \\ \sigma(Retention) &= 1949.410 \\ WT_{0.90}(Retention) &= 7355.088 \\ WT_{0.95}(Retention) &= 8202.904 \\ WT_{0.99}(Retention) &= 9878.696 \end{aligned}$$

Table 4: Retained risk for Treaty 2

Obviously the objective is attained : the cession is now smaller. However, on the other hand the risk level is higher (larger standard deviation and larger Wang Transforms). This behaviour is obvious. Now let us move to Treaty 3 which is the same than Treaty 1 but with a global annual aggregate deductible :

$$\begin{aligned} D^{Fire} &= 500, \\ L^{Fire} &= 1500, \\ D^{MTPL} &= 800, \\ L^{MTPL} &= 1200, \\ GAAD &= 1000. \end{aligned}$$

Table 5: Treaty 3

We obtain the following quantities of interest :

$$
\begin{aligned}
\mathbb{E}(Retention) &= 4756.575 \\
\sigma(Retention) &= 1822.765 \\
WT_{0.90}(Retention) &= 7202.147 \\
WT_{0.95}(Retention) &= 7939.854 \\
WT_{0.99}(Retention) &= 9381.442
\end{aligned}
$$

Table 6: Retained risk for Treaty 3

We immediately observe that this treaty is optimal with respect to Treaty 2 : the cession is smaller and the retained risk is also smaller. So clearly Treaty 3 is a better choice than Treaty 2.

Other situations may be described. For example, let us compare Treaty 4 with Treaty 5 :

	Treaty 4	Treaty 5
D^{Fire}	1000	500
L^{Fire}	1000	1500
D^{MTPL}	1200	800
L^{MTPL}	800	1200
$GAAD$	0	2000

Table 7: Treaties 4 and 5

We obtain the following quantities of interest :

	Treaty 4	Treaty 5
$\mathbb{E}(Retention)$	4946.616	5150.214
$\sigma(Retention)$	2103.647	2093.537
$WT_{0.90}(Retention)$	7884.110	7921.404
$WT_{0.95}(Retention)$	8804.185	8729.225
$WT_{0.99}(Retention)$	10626.00	10266.98

Table 8: Retained risk for Treaties 4 and 5

Here again, we observe that Treaty 5 is optimal with respect with Treaty 4 : smaller cession and smaller retained risk.

Now let us compute the quantities of interest of Treaty 5 with the wrong assumption of independence. We obtain :

$$
\begin{aligned}
\mathbb{E}(Retention) &= 5150.214 \\
\sigma(Retention) &= 1777.361 \\
WT_{0.90}(Retention) &= 7584.320 \\
WT_{0.95}(Retention) &= 8332.368 \\
WT_{0.99}(Retention) &= 9800.117
\end{aligned}
$$

Table 9: Retained risk for Treaty 5 with wrong assumption of independence

As explained above, the expected retention is the same as in the exact model. However the risk measures are smaller in the wrong model which is logical because the wrong model ignores the positive dependence that is present in the model.

Using the standard deviation of the $WT_{0.99}$ as the risk criterion, we may conclude that Treaty 5 is optimal with respect to Treaty 2. In fact, using the exact model, we immediately see that we are not allowed to give such a conclusion. This shows the danger of working with the model ignoring the dependencies.

5. Conclusion

We have analysed an actuarial situation where dependence is induced by the model. This kind of dependence was tractable by using the multivariate Panjer's algorithm. We have been able to show, on our numerical example the danger of working within a wrongly assumed model where there is no dependence and we also have shown the optimality of the multiline excess of loss cover. Some practical considerations are

1. It may be the case that the loading of the insurer and reinsurer are very different. Then the optimality should be studied with respect to the expected gain and not with respect to the expected retention.

2. We here have analysed the large claims, that are reinsured through an excess of loss treaty. Obviously we should account for the small claims in order to compute the risk measures.

3. This paper says that it is better for the ceding company to buy excess of loss treaties with small priorities and with a global annual aggregate deductible. Administrative reasons may go against these solutions. Indeed, small priorities means that large number of claims are expected to hit the layers, which makes lots of administration for both the insurer and the reinsurer. This is in particular true for long-tailed business like MTPL where a stability clause is generally in use.

4. In practice, the reinsurer would limit its annual liability through a global annual aggregate limit.

5. When the priorities of the treaties tend to 0 and the limits tend to infinity, then the cover becomes a multiline stop-loss treaty.

The hypothesis of independence between the lines of business may be relaxed for the case of umbrella covers where correlations exist between the covered lines of business. In such a case, copulas may help in order to price the cover. However, in order to analyse the retained risk of the cedant, only simulations would help and one should be aware of the fact that a huge number of simulations would be necessary in order to correctly catch the dependencies in the tails.

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