The Economics of Capital Allocation

Glenn G. Meyers, FCAS, MAAA

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By

Glenn Meyers

Insurance Services Office, Inc.

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Abstract

On the surface, capital allocation sounds contradictory to the stated purpose of insurance, which is diversifying risk. In spite of that, it is commonly used as a tool by insurers to manage their underwriting risk. This paper examines the economics underlying how insurers might use capital allocation when capital is scarce and has a price. Starting from a risk-based capital framework, the paper establishes strategies for increasing the insurer's expected return on capital. It then derives capital allocation methods that are consistent with these economic strategies.

1. Introduction

The practice of allocating capital for ratemaking has long been controversial. As McClenahan [1990] said several years ago, "In essence, the method treats a multiline national company with $100 million of capital – *$1 million of which is allocated to California private passenger automobile – identically* with the California private passenger automobile carrier capitalized at $1 million." In spite of this criticism, many insurance company executives continue the practice. Their reasons for doing so are understandable. The executives are answerable to the insurance company's investors who demand a competitive return on their investment. The executive's job is to direct the managers of the various lines of business within the insurance company to achieving this goal. A seemingly straightforward way of doing this is to establish a yardstick for each line of business based on its return on allocated capital.

The problem with this management strategy lies in the particular methodology for allocating capital. That is to say, the devil is in the details. As I will show in an example below, some ways of allocating capital can lead to decisions on the part of the individual line managers that do not benefit the insurance company as a whole.

The purpose of this paper is to give some ways of allocating capital that lead to sound economic decisions. By "sound economic decisions," I mean decisions that increase the insurer's expected return on its capital investment.

In writing this paper, I do not want to imply that allocating capital is necessary. It is possible to devise a pricing methodology that is economically sound without allocating capital. But for those who choose to allocate capital, I hope to provide some useful economic advice.

An insurer operates by making pricing and underwriting decisions on the insurance policies it writes. These decisions will be evaluated according to their impact on the insurer's cost of capital. Thus our first step is to establish a yardstick for the insurer's cost of capital.

2. The Insurer's Cost of Capital

Let X be a random variable representing an insurer's loss for a particular book of business. Let $C(X)$ equal the capital needed to support the book of business with random loss X. We assume that C satisfies the following two axioms:

1. Subadditivity – For random losses X and Y, $C(X+Y)$ $C(X) + C(Y)$.
2. Positive homogeneity – For all constants k 0, $C(k \cdot X) = k \cdot C(X)$.

The subadditivity axiom means that when you pool books of business, you do not need more total capital. In fact, an efficient pooling of risk should result in needing less total capital.

Here are some examples of capital formulas that satisfy these two axioms.

<u>Example 1 – The Standard Deviation Capital Formula</u>

$$C(X) \equiv T \cdot \sigma_X$$

Generally T is in the 2 to 3 range.

$C(X)$ satisfies the two axioms above.

Proof:

$$\begin{aligned} C(X+Y) &= T \cdot \sigma_{X+Y} = T \cdot \sqrt{\sigma_X^2 + 2 \cdot \rho \cdot \sigma_X \cdot \sigma_Y + \sigma_Y^2} \le T \cdot \sqrt{\sigma_X^2 + 2 \cdot \sigma_X \cdot \sigma_Y + \sigma_Y^2} \\ &= T \cdot (\sigma_X + \sigma_Y) = C(X) + C(Y) \end{aligned}$$

$$C(k \cdot X) = T \cdot \sqrt{Var[k \cdot X]} = T \cdot \sqrt{k^2 \cdot Var[X]} = k \cdot C(X)$$

Example 2 – Capital derived from coherent measures of risk

Let $\rho(X)$ be a "measure of risk" that we assign to a random loss X. If $\rho(X)$ satisfies the following axioms:

1. Subadditivity – For all random losses X and Y,

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

2. Monotonicity – If $X \leq Y$ for each scenario, then,

$$\rho(X) \leq \rho(Y).$$

3. Positive Homogeneity – For all k 0 and random losses X,

$$\rho(k \cdot X) = k \cdot \rho(X).$$

4. Translation Invariance – For all random losses X and constants a,

$$\rho(X+a) = \rho(X) + a$$

then $\rho(X)$ is called a coherent measure of risk. These measures of risk were originated by Artzner, Delbaen, Eber and Heath [1999]. I have previously written about these measures in Meyers [2001] and Meyers [2002].

Here are some examples of coherent measures of risk.

- Let X take its values on a finite set of scenarios. Then $\rho(X) \equiv Max\{X\}$ is a coherent measure of risk.
- For a given percentile, α, let the Value-at-Risk, $VaR_\alpha(X)$, be defined as the α^{th} percentile of X. Meyers [2001] demonstrates that $VaR_\alpha(X)$ is not a coherent measure of risk; but the Tail Value-at-Risk,

$$TVaR_\alpha(X) = E[X \mid X > VaR(X)]$$

  is a coherent measure of risk. One would choose α according to their aversion to risk.

Let $\rho(X)$ be a coherent measure of risk. If we normalize $\rho(X)$ so that $\rho(0) = 0$, we can use $\rho(X)$ to denote the value of the assets held by an insurer to support its random losses X.

Let's assume that the policyholders supply $E(X)$ through their premiums. Then using a capital of

$$C(X) = \rho(X) - E[X]$$

provides the insurer with sufficient assets to support its random loss X. Also, $C(X)$ satisfies the two axioms listed at the beginning of this section.

<u>Example 3 – Transformed probability formulas</u>

Let g, mapping [0,1] to [0,1], be a nondecreasing, concave up, continuous function. Let $F(x)$ be the cumulative distribution function of X. Let U be a random variable with the cumulative distribution $g(F(u))$. Then according to Theorem 3 of Wang, Young and Panjer [1997],

$$\rho(X) \equiv E[U]$$

is a coherent measure of risk. Measures of this form have the additional property of being *co-monotonic additive*; i.e., if $(X_i - Y_i)\cdot(X_j - Y_j)$ 0 for all scenarios i and j, then $\rho(X + Y) = \rho(X) + \rho(Y)$.

Meyers [2002] gives an example of a coherent measure or risk that is not co-monotonic additive.

- If $g(u) = Max\{0, (u - \alpha)\}/(1 - \alpha)$ for $\alpha \in [0,1]$, then $E[U] = TVaR_\alpha(X)$.
- If $g(u) = \Phi(\Phi^{-1}(u) - \lambda)$, where Φ is the cumulative distribution function for the standard normal distribution, then $E[U]$ is called the Wang Transform. λ increases with risk aversion.

As these examples show, there is a good supply of capital formulas that reflect varying degrees of risk aversion.

The final step in calculating the insurer's cost of capital is to determine the expected rate of return on its capital investment. This rate of return is determined by examining the rate of return obtained by other investments of comparable risk. Security analysts have been doing this for years and I have no special insight to offer. In this paper, I take this expected rate of return as a given. Unless the insurer makes major changes in its operations, I think it is reasonable to assume that it is constant.

Once we decide how to calculate the insurer's overall cost of capital, the next step is to analyze how insurer pricing and underwriting decisions affect this overall cost of capital. This leads us to consider the insurer's marginal cost of capital.

3. The Insurer's Marginal Cost of Capital

Suppose the insurer is currently maintaining a book of business with random loss X, capital requirement C, and expected profit P. Suppose further that it is considering adding a new set of insurance policies with random loss ΔX, and expected profit ΔP, to its book of business. Define the marginal capital for the new policies as
$\Delta C = C(X + \Delta X) - C(X)$.

Proposition 1

An insurer will increase its return on capital if and only if the new business' return on marginal capital is greater than the insurer's overall return on existing capital.

Proof:

$$\frac{P+\Delta P}{C+\Delta C} > \frac{P}{C}$$
$$\Leftrightarrow PC + \Delta P \cdot C > CP + \Delta C \cdot P$$
$$\Leftrightarrow \Delta P \cdot C > \Delta C \cdot P$$
$$\Leftrightarrow \frac{\Delta P}{\Delta C} > \frac{P}{C}$$

The opposite of Proposition 1 should also be clear. An insurer will increase its return on capital by dropping business if and only if the marginal capital of the dropped business is less than the insurer's overall return on existing capital.

This proposition provides an analytic method for insurers to increase their return on capital. For a given insurance policy an insurer can follow this underwriting strategy.

Underwriting Strategy #1

1. Observe the premium that the market allows for this insurance policy.
2. Calculate the expected profit that the insurer can obtain by writing the insurance policy.
3. Calculate the marginal capital needed if it were to write the insurance policy.
4. If the expected return on the marginal capital exceeds the insurer's current expected return on capital, the insurer should increase its capital and write the new insurance policy.

At this point, I would like to introduce a fairly lengthy example.

Example 4

- An insurer writes two independent lines of business. The amount of business it writes in each line is a decision variable, and we will quantify these amounts of business by their expected claim counts ν_1 and ν_2. The amount of each claim is set equal to one.

- Let N_i be the random number of claims for line i. Each N_i will have a mixed Poisson distribution. Let χ_i be a random variable with a mean = 1 and variance = $c_i > 0$. Given χ_i, the i^{th} distribution is a Poisson with mean $\chi_i \cdot \nu_i$. Unconditionally, the mean of each claim count distribution is ν_i and the variance of each claim count distribution is:

$$E_{\chi_i}\left[Var\left[N_i \mid \chi_i\right]\right] + Var_{\chi_i}\left[E\left[N_i \mid \chi_i\right]\right] = E_{\chi_i}\left[\nu_i \cdot \chi_i\right] + Var_{\chi_i}\left[\nu_i \cdot \chi_i\right] = \nu_i + c_i \cdot \nu_i^2 .$$

Note that the variance of the loss ratio

$$Var\left[\frac{N_i}{\nu_i}\right] = \frac{1}{\nu_i} + c_i$$

decreases as the exposure increases but, as we often observe in real life, this variance is always greater than zero – no matter how much exposure the insurer writes. You can think of the random variable χ_i as an analogue to changing economic conditions.

- The insurer determines its capital by taking a multiple of two times the standard deviation of its total losses; i.e.,

$$C = 2 \cdot \sqrt{v_1 + c_1 \cdot v_1^2 + v_2 + c_2 \cdot v_2^2}\,. \tag{1}$$

- The insurer's expected profit is proportional to the expected claim counts, i.e.

$$P = r_1 \cdot v_1 + r_2 \cdot v_2. \tag{2}$$

- If the insurer makes a small change in its exposure for line i, its return on marginal capital is closely approximated by:

$$\left.\frac{\Delta P}{\Delta C}\right|_{\text{Line } i} \approx \left.\frac{dP}{dC}\right|_{\text{Line } i} \equiv \frac{\dfrac{\partial P}{\partial v_i}}{\dfrac{\partial C}{\partial v_i}} = \frac{r_i}{2 \cdot \dfrac{1 + 2 \cdot c_i \cdot v_i}{C}}. \tag{3}$$

The formulas above are easily programmed into a spreadsheet. Let's plug some numbers into these formulas. Set:

Line i	r_i	c_i
1	5%	0.02
2	2%	0.01

In this example, I view these parameters as being beyond the control of the insurer.

Let's pause for a moment and digress on how this example relates to the real world. The r_i's and the c_i's describe the economic environment in which the insurer operates. The r_i's are backed out from the premium that the market will allow the insurer to charge. The c_i's are analogous to a measure of the inherent volatility of a line of insurance.

What the insurer can control is how much business it writes in each line of insurance. In this example, the control parameters are the v_i's. The quantities for Equations 1-3 for various v_i's are given in the following table.

Table 1

v_1	v_2	C	P	P/C	Line 1 dP/dC	Line 2 dP/dC
100.00	100.00	44.72	7.00	15.65%	22.36%	14.91%
117.69	64.15	44.72	7.17	16.03%	19.59%	19.59%
133.41	76.73	50.00	8.21	16.41%	19.73%	19.73%
163.44	100.76	60.00	10.19	16.98%	19.90%	19.90%
208.85	137.08	75.00	13.18	17.58%	20.04%	20.04%
285.03	198.02	100.00	18.21	18.21%	20.16%	20.16%
3,052.52	2,412.01	1,000.00	200.87	20.09%	20.31%	20.31%
30,747.90	24,568.32	10,000.00	2,028.76	20.29%	20.31%	20.31%
307,703.73	246,132.98	100,000.00	20,307.85	20.31%	20.31%	20.31%

There are several points that can be made with this table.

1. First let's consider the case $v_1 = v_2 = 100$. Note that the return on marginal capital for Line 1 is greater than the overall return, and the return on marginal capital for Line 2 is less than the overall return. By Proposition 1, the insurer can increase its overall rate of return by adding business in Line 1, and reducing business in Line 2.

2. Increasing v_1 to 117.69 and decreasing v_2 to 64.15 gives a higher return for the same amount of capital. These numbers can be derived by choosing v_1 and v_2 so the total profit, P, is maximized subject to a constraint on the capital, $C(X)$ I, using the method of Lagrange multipliers. This problem is solved in a more general setting in Equation 5.1 of Meyers [1991]. Here is the solution for this special case[1].

$$v_i^* = \frac{\dfrac{I \cdot r_i}{2 \cdot \lambda^*} - 1}{2 \cdot c_i}, \text{ where } \lambda^* = \frac{I}{2} \cdot \sqrt{\frac{\dfrac{r_1^2}{c_1} + \dfrac{r_2^2}{c_2}}{I^2 + \dfrac{1}{c_1} + \dfrac{1}{c_2}}}. \tag{4}$$

[1] Meyers [1991] had a variance constraint rather than a capital constraint. Since in this example our capital is a function of the variance, the solutions are equivalent. But the Lagrange multiplier, λ^*, for the capital constraint is $I/2$ times the Lagrange multiplier, λ^*, for the variance constraint. This change is cancelled out in the expression for v_i^*.

3. For the choice of v_i's immediately above, the return on marginal capital for both lines is higher than the overall return on capital. This means that we can add exposure to both lines and increase the overall rate of return. Successive lines in Table 1 are calculated by first increasing the constraint, I, on the capital, C, and setting $v_1 = v_1^*$ and $v_2 = v_2^*$ using Equation 4.

4. When we use Equation 4 to choose the v_i's, note that the returns on the marginal capitals for each line are equal. It turns out that that property of this example can be generalized.

Proposition 2

Suppose:

i. An insurer can write business in any of n lines of insurance.
ii. The amount of business in Line i, is quantified by an exposure amount e_i. The random loss, X_i, in Line i, does not decrease as e_i increases.
iii. The insurer's expected profit, $P(e_1,\ldots,e_n)$ is a differentiable function of each e_i.
iv. The insurer's total capital, $C_E\left(e_1,\ldots,e_n\right) \equiv C\left(\sum_{i=1}^{n} X_i \mid e_1,\ldots,e_n\right)$, is a differentiable function of each e_i.

The insurer wishes to choose exposure amounts, e_i^* for $i = 1, \ldots n$, in such a way as to maximize its expected profit, P, subject to a limitation, I, on its capital investment. Then for all lines, the return on marginal capitals:

$$\left.\frac{dP}{dC_E}\right|_{\text{Line } i} \equiv \left.\frac{\frac{\partial P}{\partial e_i}}{\frac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} \tag{5}$$

are all equal to each other.

I provide two proofs of this proposition.

Proof #1:

We solve for the exposures $\{e_i^*\}$ by the method of Lagrange multipliers. Set:

$$L = P(e_1,\ldots,e_n) + \lambda\cdot(I - C_E(e_1,\ldots,e_n)).$$

The method works by solving the $n+1$ equations for $\{e_i^*\}$ and λ^*:

$$\left.\frac{\partial L}{\partial e_i}\right|_{e_i=e_i^*} = 0 \text{ for } i = 1,\mathrm{K}\ ,n \text{ and } \left.\frac{\partial L}{\partial \lambda}\right|_{\lambda=\lambda^*} = 0\ .$$

The first n equations give:

$$\left.\frac{\partial L}{\partial e_i}\right|_{e_i=e_i^*} = \left.\frac{\partial P}{\partial e_i}\right|_{e_i=e_i^*} - \lambda^*\cdot\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} = 0 \Rightarrow \left.\frac{\dfrac{\partial P}{\partial e_i}}{\dfrac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} = \lambda^*, \tag{6}$$

which is what we need to prove. It turns out that the Lagrange multiplier, λ^*, is equal to the return on marginal capital that is common for all the lines of insurance.

Proof #2

Suppose we have chosen exposure amounts, e_i^* for $i = 1, \ldots n$, in such a way as to maximize expected profit, P, subject to a limitation, I, on the capital investment. Suppose further that for two lines i and j:

$$\left.\frac{dP}{dC_E}\right|_{\text{Line } i} > \left.\frac{dP}{dC_E}\right|_{\text{Line } j}$$

Then rewriting Equation 5 as a differential[2] we have:

$$dP = \left.\frac{\partial P}{\partial C_E}\right|_{Line\,i}\cdot\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}\cdot de_i \text{ and } dP = \left.\frac{\partial P}{\partial C_E}\right|_{Line\,j}\cdot\left.\frac{\partial C_E}{\partial e_j}\right|_{e_j=e_j^*}\cdot de_j \tag{7}$$

[2] I am using differential notation to express the approximate effect of a "small" change in one variable with a "small" change in another variable. If $y = f(x)$ the differential dy is equal to $f'(x)dx$.

Adjust the incremental exposures de_i and de_j so that:

$$dC_E = \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} \cdot de_i = \left.\frac{\partial C_E}{\partial e_j}\right|_{e_j=e_j^*} \cdot de_j$$

Then reduce the exposure in Line j by de_j and increase the exposure in Line i by de_i without changing C_E. But according to Equation 7, doing so would increase P and lead to a contradiction.

The first proof provides the means, at least in principle, to explicitly solve for the optimal exposures, $\{e_i^*\}$. The second proof shows how to increase profitability when you are not at the optimal level of exposure. This is related to the following strategy which is a continuous analogue of Underwriting Strategy #1.

Underwriting Strategy #2

1. Observe the premium that the market allows for insurance policies in each line of insurance.
2. Calculate the insurer's profitability, P, as a function of its exposure, e_i, in each line of insurance i.
3. Calculate the insurer's needed capital, C_E, as a function of its exposure, e_i, in each line of insurance i.
4. If the expected return on marginal capital for line i (given by Equation 5), is greater than the insurer's current expected return on capital, increase the exposure in line i. Conversely, if the expected return on marginal capital for line i is less the insurer's current expected return on capital, decrease the exposure in line i. Adjust capital accordingly.

4. Allocating Capital

So far, we have not addressed the main topic of this paper – allocating capital. My reason for organizing the paper in this way was to emphasize the point that economically sound insurance decisions (defined in this paper as underwriting decisions that increase the insurer's expected return on capital) can be made without allocating capital.

Those who choose to allocate capital usually adopt an underwriting strategy similar to the following.

Underwriting Strategy #3

1. Establish a target rate of return for the insurance company.
2. Observe the premium that the market allows for a given insurance policy.
3. Calculate the expected profit that the insurer can obtain by writing this insurance policy.
4. Calculate the amount of capital that would be allocated to this insurance policy if it were written.
5. If the expected return on the allocated capital exceeds the insurer's target return on capital, the insurer should write the new insurance policy.

There are two differences between Underwriting Strategies #1 and #3. The first difference is the introduction of a "target" rate of return. According to Proposition 1, an insurer can increase its rate of return by adding policies where the market price allows an expected return on marginal capital greater than its current expected return on capital. As long as the insurer can raise the necessary capital, this is fine. But for a host of "practical" reasons there are limits on how much capital an insurer can raise. Thus, the insurer's board of directors will set the target rate of return based on what it feels is attainable with its scarce capital resources. Under these conditions, the insurer should be more selective and choose to underwrite the policies that yield the greatest expected return on marginal capital. Proposition 2 shows that when we can continuously adjust the exposure, this strategy of more selective underwriting leads to insurance policies each with an equally high expected return on marginal capital.

The second difference between Underwriting Strategy #1 and #3 is the substitution of the words "allocated capital" for the words "marginal capital." This is an important distinction because of the following proposition.

Proposition 3

The sum of the marginal capital for all exposures is less than or equal to the total capital.

Proof:

We prove the proposition when there are two distinct exposures. The general statement follows by induction.

The sum of the marginal capitals is equal to

$C(X+Y) - C(X) + C(X+Y) - C(Y) = 2\cdot C(X+Y) - (C(X) + C(Y))$

$2\cdot C(X+Y) - C(X+Y)$ (by the subadditivity axiom)

$= C(X+Y)$ which is the total capital.

Here is the general principle I use to allocate capital[3].

Back-Out Allocation Method

1. Establish an underwriting strategy in accordance with the economic principles described in Section 3 above.
2. Back out the method of allocating capital that is consistent with this strategy.

The only prior constraint on the method of allocating capital is that if the insurer properly executes Underwriting Strategy #3, it expects to achieve its target rate of return on its capital investment. That is to say, if A is the total allocated capital then:

$$P = r\cdot A. \tag{8}$$

I will first apply this general principle in the case where we can continuously adjust the exposure. The continuous analogue to Underwriting Strategy #3 is as follows.

[3] Gary Venter [2002] makes a similar tongue in cheek suggestion at the end of his article titled "Allocating Surplus – Not." But here, I am serious.

Underwriting Strategy #4

1. Establish a target rate of return for the insurance company.
2. Observe the premium that the market allows for insurance policies in each line of insurance.
3. Calculate the insurer's profitability, P, as a function of its exposure, e_i, in each line of insurance i.
4. Calculate the differential for the allocated capital, A_i, that accompanies a small change in exposure, e_i, in each line of insurance i.
5. If the expected return on allocated capital for line i is greater than the insurer's target rate of return on capital, increase the exposure in line i. Conversely, if the expected return on allocated capital for line i is less the insurer's target rate of return on capital, decrease the exposure in line i.

The question we now address is: What formula for allocating capital makes economic sense for this strategy? The answer is given by the following proposition.

Proposition 4

Assume the conditions of Proposition 2 hold and that the insurer has chosen exposure amounts, e_i^* for lines $i = 1, \ldots, n$, in such a way as to maximize its expected profit, P, subject to a limitation, I, on its capital investment. This strategy is equivalent to Underwriting Strategy #4 when allocating capital in proportion to the marginal capital.

Proof:

Using Equation 8 we have:

$$r \cdot \partial A\big|_{e_i=e_i^*} = \partial P\big|_{e_i=e_i^*} = \left.\frac{\partial P}{\partial C_E}\right|_{e_i=e_i^*} \cdot \partial C_E\big|_{e_i=e_i^*} = \left.\frac{\dfrac{\partial P}{\partial e_i}}{\dfrac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} \cdot \partial C_E\big|_{e_i=e_i^*}.$$

Since the insurer is maximizing its expected profit P, we can apply Equation 6 to the above with the result that:

$$r \cdot \partial A\big|_{e_i=e_i^*} = \lambda^* \cdot \partial C_E\big|_{e_i=e_i^*} \text{ or } \partial A\big|_{e_i=e_i^*} = \frac{\lambda^*}{r} \cdot \partial C_E\big|_{e_i=e_i^*}.$$

In words, this says that a small change in the exposure when $e_i = e_i^*$ causes the capital allocated to this change in exposure, $\partial A\big|_{e_i=e_i^*}$, to be equal to λ^*/r times the marginal capital, $\partial C_E\big|_{e_i=e_i^*}$. Furthermore, each small increment of exposure, $\partial e_i\big|_{e_i=e_i^*}$ adds $r \cdot \partial A\big|_{e_i=e_i^*}$ to the insurer's expected profit. Since the marginal expected profit is the same for all exposures in a given line of insurance the total expected profit, P, is equal to $r \cdot \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial A}{\partial e_i}\right|_{e_i=e_i^*} = r \cdot \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}$, which is also equal to $r \cdot C_E$. It then follows that:

$$C_E = \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}. \tag{9}$$

For reasons that will become clear in the next section, we call the ratio, λ^*/r, the ***heterogeneity multiplier***. So the allocated capital will be equal to marginal capital times the heterogeneity multiplier.

As a consequence of Proposition 3, the heterogeneity multiplier has a theoretical minimum of one.

I now illustrate the use of Proposition 4 by continuing Example 4. The results of applying the following equations are in Table 2 below. I think it will help your understanding of the results if you try to reproduce some of the numbers yourself.

Example 4 – Continued

- Using $\underline{e_i} = v_i$ as our measure of exposure and Equation 1 we find that:

$$\partial C_{\mathrm{N}}\Big|_{v_i = v_i^*} = \frac{2 \cdot \left(1 + 2 \cdot v_i^* \cdot c_i\right)}{C_{\mathrm{N}}} \cdot \partial v_i .$$

- The total marginal capital for line i is the sum over all the v_i's and is given by:

$$\partial C_{\mathrm{N}}\Big|_{v_i = v_i^*} \cdot v_i^* = \frac{2 \cdot \left(1 + 2 \cdot v_i^* \cdot c_i\right)}{C_{\mathrm{N}}} \cdot v_i^* .$$

- In the various cases illustrated in Table 2, note that the total marginal capital over both lines is less than the total capital. This is predicted by Proposition 4.
- Set the target rate of return, r, equal to the maximum rate attainable subject to a given constraint on capital. This is equal to *P/C* in Table 1. Then use Equation 4 to calculate the Lagrange multiplier, λ^*. Finally, calculate the heterogeneity multiplier, λ^*/r.
- The total allocated capital for line i is the total marginal capital for line i times the heterogeneity multiplier. Note that the total allocated capital is equal to the original capital.

Table 2

C_E	v_1^*	v_2^*	Total Marginal Capital Line 1	Total Marginal Capital Line 2	Heterogeneity Multiplier	Total Allocated Capital Line 1	Total Allocated Capital Line 2
50.00	133.41	76.73	33.81	7.78	1.2021	40.65	9.35
60.00	163.44	100.76	41.07	10.13	1.1720	48.13	11.87
75.00	208.85	137.08	52.10	13.68	1.1403	59.40	15.60
100.00	285.03	198.02	70.69	19.65	1.1069	78.25	21.75
1,000.00	3,052.52	2,412.01	751.53	237.54	1.0110	759.84	240.16
10,000.00	30,747.90	24,568.32	7,569.61	2,419.32	1.0011	7,578.00	2,422.00
100,000.00	307,703.73	246,132.98	75,751.42	24,237.50	1.0001	75,759.81	24,240.19

Note that the heterogeneity multiplier approaches one as the capital constraint increases. Understanding the reason for this is not central to allocating capital, but I do regard it as an important curiosity. Let's look into this.

5. Allocating Capital with Homogeneous Loss Distributions

Suppose for a line of insurance i, the random losses, X_i, for the line are equal to a random number, U_i, times the exposure measure, e_i, for all possible values of e_i. Then, following Mildenhall [2002], the distribution of X_i is said to be ***homogeneous*** with respect to the exposure measure, e_i.

Lemma 1

Let the distribution of X_i be homogeneous with respect to the exposure measure, e_i. Then the sum of all marginal capital,

$$\sum_{i=1}^{n} e_i \cdot \frac{\partial C_E}{\partial e_i}$$

is equal to C_E.

Proof:

Since each X_i is homogeneous with respect to e_i we have:

$$C_E(X) = C_E\left(\sum_{i=1}^{n} X_i\right) = C_E\left(\sum_{i=1}^{n} e_i \cdot U_i\right).$$

Since the measure of capital, C_E, satisfies the positive homogeneity axiom, we can write:

$$C_E\left(\sum_{i=1}^{n} e_i \cdot U_i\right) = e_1 \cdot C_E\left(\sum_{i=1}^{n} \frac{e_i}{e_1} U_i\right) \equiv e_1 \cdot \tilde{C}_E\left(\frac{e_2}{e_1}, \ldots, \frac{e_n}{e_1}\right)$$

and similarly for $e_2, \ldots, e_n$.

The result follows from Lemma 2 in Mildenhall [2002].

Proposition 5

Assume the conditions of Proposition 2 hold and that the insurer has chosen exposure amounts, e_i^*, for lines $i = 1, \ldots, n$, in such a way as to maximize its expected profit, P, subject to a limitation, I, on its capital investment. Suppose further that the distribution of X_i is homogeneous with respect to the exposure measure, e_i. Then the heterogeneity multiplier is equal to one, $\partial A\big|_{e_i = e_i^*} = \partial C_E\big|_{e_i = e_i^*}$, and the total allocated capital is equal to the total marginal capital; i.e.,

$$C_E = \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

Proof:

From Equation 9 we have that:

$$C_E = \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

From Lemma 1 we have that:

$$C_E = \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

The conclusions follow.

Thus if the distribution of X_i is homogeneous with respect to e_i, the heterogeneity multiplier is equal to one and has no need to exist. But if the distribution of X_i is not homogeneous with respect to e_i, then we need the multiplier and hence the name "heterogeneity multiplier."

Example 4 – Continued

Recall in Example 4, the loss, N_i, in Line i has a mixed Poisson distribution. Let χ_i be a random variable with a mean = 1 and variance = $c_i > 0$. Given χ_i, the i^{th} distribution is a Poisson with mean $\chi_i \cdot v_i$. Now let's compare these distributions with distributions of the form $\chi_i \cdot v_i$, which are by definition homogeneous with respect to v_i. As we compare the mixed and the homogeneous distributions for the same lines of insurance, we find that they closely approximate each other if v_i is large. This explains why the heterogeneity multiplier is close to one for large values of v_i.

Let's look at a pure homogeneous example.

Example 5

This example is the same as Example 4, with one key change. Instead of a loss of 1 per claim, the loss in line i is b_i per claim. For fixed v_i's the b_i's are proportional to the expected loss and can serve as a measure of exposure. You can think of varying the b_i's by choosing a share of the loss whose claim sizes are much bigger than any b_i we may choose. This changes a number of the equations that describe Example 4. What follows are the equations that are analogous to those of Example 4.

- The capital is given by:

$$C_B = 2 \cdot \sqrt{b_1^2 \cdot \left(v_1 + c_1 \cdot v_1^2\right) + b_2^2 \cdot \left(v_2 + c_2 \cdot v_2^2\right)}. \tag{1'}$$

- The expected profit is given by:

$$P = r_1 \cdot b_1 \cdot v_1 + r_2 \cdot b_2 \cdot v_2. \tag{2'}$$

- Select the claim sizes, b_1 and b_2, so that the total profit, P, is maximized subject to a constraint on the capital, $C_B(X)$ I, using the method of Lagrange multipliers. The equations for the b_i's are given by:

$$b_i = \frac{r_i \cdot v_i \cdot I}{4 \cdot \lambda \cdot \left(v_i + c_i \cdot v_i^2\right)}, \text{ where } \lambda = \frac{1}{2}\sqrt{\frac{r_1^2 \cdot v_1^2}{v_1 + c_1 \cdot v_1^2} + \frac{r_2^2 \cdot v_2^2}{v_2 + c_2 \cdot v_2^2}}. \tag{4'}$$

Table 2′ gives sample calculations for various choices of v_1, v_2 and I that are comparable to those of Example 4.

Table 2′

							Total Marginal Capital	
C_B	v_1	v_2	λ^*	B_1	b_2	P	**Line 1**	**Line 2**
100.00	250.00	250.00	0.1822	1.1436	0.7842	18.22	78.48	21.52
100.00	285.03	198.02	0.1823	1.0234	0.9203	18.23	80.00	20.00
1000.00	2500.00	2500.00	0.2006	1.2216	0.9585	200.63	761.12	238.88
1000.00	3052.52	2412.01	0.2009	1.0029	0.9909	200.87	762.02	237.98

Some observations:

- This table illustrates the results of Proposition 5. The heterogeneity multiplier λ^*/r (= $C_B \cdot \lambda^*/P$) is equal to one, and the total marginal capital is equal to C_B for all cases.
- The expected profit, P, varies with the choice of the v_i's, which remain fixed as you find the optimal b_i's.
- In two of the cases, I put in the same v_i's that maximized the expected profit in Example 4 (where the b_i's = 1). With those v_i's, the optimal b_i's did not equal one. This shows that the result of a capital allocation exercise depends upon the applicable exposure base.

Myers and Read [2001] prove a result that is similar to Proposition 5. Their use of the homogeneity assumption has generated some controversy. The justification for this assumption appears to follow from their statement: "The only requirement is frictionless financial markets and fixed state-contingent prices for all relevant outcomes."

Mildenhall [2002] illustrates that many commonly used actuarial loss models do not satisfy the homogeneity assumption. He further shows that the Myers/Read homogeneity assumption is both a necessary and sufficient condition to prove their analogue to Proposition 5.

Example 4 is one example where the homogeneity assumption is not met. I regard Proposition 4 as a generalization of Proposition 5 that applies when the homogeneity assumption is not met.

6. Allocating Capital with Discrete Exposure Changes

Strictly speaking, the continuity assumption underlying Propositions 4 and 5 is almost never met. For example, when an insurer increases its exposure in auto insurance, it typically writes an entire auto policy and increases its exposure by at least one car year. In cases like auto insurance, the discrete exposure environment is closely approximated by the continuous exposure environment, that is:

$$\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} \approx \frac{C_E\left(e_1^*,\mathrm{K}\,,e_i^*,\mathrm{K}\,,e_n^*\right)-C_E\left(e_1^*,\mathrm{K}\,,e_i^*-\Delta e_i,\mathrm{K}\,,e_n^*\right)}{\Delta e_i}. \tag{10}$$

If this approximation is good then you can estimate the heterogeneity multiplier and allocate capital as follows.

Gross-Up Allocation Method

1. Calculate the marginal capital required for each insurance policy in the current portfolio by calculating the capital needed when it is removed from the current portfolio and subtracting that from the current capital.
2. Calculate the heterogeneity multiplier by dividing the total required capital by the sum of the marginal capitals over all insurance policies.
3. The capital allocated to a given insurance policy is equal to its marginal capital times this heterogeneity multiplier.

To illustrate how well this can work, I calculated the heterogeneity multiplier by the gross-up method for the first line in Table 2 (Capital = 50) by dividing the total exposure by line into a varying number of insurance policies, with the following results.

Table 3

Number of Policies in Each Line	Gross-Up Heterogeneity Multiplier
(NA – Continuous)	1.2021
1000	1.2022
100	1.2034
10	1.2161
5	1.2320
1	1.5401

One can apply Underwriting Strategy #3 with the gross-up allocation method under any circumstance. Proposition 4 says that if you can continuously adjust the exposure, the strategy should lead to the optimal result. If Equation 10 provides a good approximation, the strategy should also get close to the optimal result with discrete exposures.

Quite often, insurers make bigger decisions such as adding or dropping entire lines of business. Consider the following example.

<u>Example 6</u>

- The insurer writes in two lines of insurance Line A and Line B. The insurer's only choices are to write all of Line A, all of Line B or both Lines A and B.
- There are no transaction costs and no interest is earned on invested assets.
- The "market" price provides an expected loss ratio of 60%. This means that the expected profit is equal to two thirds of the expected loss.
- The insurer must have capital equal to the maximum loss minus the expected loss.

- For losses payable in one year, there three possible scenarios.

Table 4

Scenario	Probability	Line A	Line B	Line A + Line B
1	2/39	60	135	195
2	7/39	150	45	195
3	30/39	0	0	0
Average Loss		30	15	45
Expected Profit		20	10	30
Required Capital		120	120	150
Marginal Capital		30	30	

- If the insurer writes Line A, it needs capital of 120 and has an expected profit of 20, which implies an expected return on its capital investment of 16.7%.

- If the insurer writes Line B, it needs capital of 120 and has an expected profit of 10, which implies an expected return on its capital investment of 8.3%.

- If the insurer writes both lines, it needs capital of 150 and has an expected profit of 30, which implies an expected return on its capital investment of 20%.

- Thus the best strategy is to write both lines since it yields the greatest return on capital.

- The return on marginal capital for Line A = 20/30 = 66.7%. The expected return on marginal capital for Line B is 10/30 = 33.3%. Both returns on marginal capital are higher than the 20% return on capital obtained by combining the two contracts.

Now let's apply Underwriting Strategy #3 using the gross-up method to allocate capital.

- The total capital is 150 and the sum of the marginal capitals is 60. Thus heterogeneity multiplier is equal to 2.5.

- The expected return on allocated capital for Line A is $20/(2.5 \cdot 30) = 26.7\%$. The expected return on allocated capital for Line B is $10/(2.5 \cdot 30) = 13.3\%$

- If the insurer followed Underwriting Strategy #3 it would write Line A since its expected return on allocated capital is higher than the 20% target. It would not write Line B since its expected return on allocated is below the 20% target. *This contradicts the fact that the insurer gets a higher expected ROE by writing both lines!*

This example gives a case where the gross-up capital allocation formula is not optimal. Note that if the insurer applies the back-out allocation method, it will allocate 100 to Line A, and 50 to Line B, yielding the target expected return on allocated capital of 20% for both lines.

7. Summary and Conclusions

Proposition 1 shows that if an insurer can obtain an expected rate of return on marginal capital on a given insurance policy that is greater than its current expected return on capital, then it can increase its rate of return by raising more capital and writing the policy.

If insurer capital is not a scarce resource, there is no need to allocate capital. But if there is a limit on the amount of capital that the insurer can raise, the insurer should be more selective in its underwriting and concentrate on the business that yields the greatest return on marginal capital.

If the insurer can make "small" adjustments in its exposure over time, Proposition 4 shows that the optimal result is obtained by:

1. Setting a high, but attainable target rate of return, r, on its capital.
2. Allocating capital in proportion to marginal capital using the gross-up allocation method.
3. Accepting only those policies for which the expected return on allocated capital is at least as high as r.

While I made use of Lagrange multipliers to illustrate this strategy, it is not necessary to resort to this mathematical technique. A simple trial and error analysis on the expected return on marginal capital for several lines of insurance should indicate what a realistic value of r should be. The proper execution of this strategy should incrementally move the insurer's expected return on capital toward the optimal result.

The strategy above should not be applied blindly for large scale underwriting decisions such as adding or dropping entire lines of insurance. Normally, the number of possible decisions on this scale is small, and so one can analyze each decision individually.

8. Additional Comments

For the interested reader, I would like to go a little beyond the scope of this paper with the following comments.

- The underwriting strategies discussed in this paper apply for all coherent measures of risk. I did not use the common coherent measures (such as the tail value-at-risk) in the examples because the ones I did use are more easily implemented on a spreadsheet. But for practical situations (especially if catastrophes are involved), I favor these other measures. See Meyers [2001] for an example where the choice of risk measure makes a noticeable difference in the ultimate conclusions.

- To keep the presentation as simple as possible, I ignored the time value of money. In practice, we should take it into account. Insurance policies covering natural disasters can be very risky; but once the policy has expired, the insurer can release some capital for other uses. For liability lines of insurance, the ultimate loss may not be known for some time. The insurer must hold capital until the loss is certain, and the cost of holding that capital must be considered in the underwriting strategy. Meyers [2001] discusses this also.

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