Measurement of Reserve Variability

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MEASUREMENT OF RESERVE VARIABILITY

By

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Abstract

Actuaries and others have long been trying to quantify the uncertainty in reserve estimates. Attempts to address this question have led to the development of stochastic reserving methods as well as the framing of some traditional reserving methods in a stochastic setting. Stochastic methods give insight into the volatility of the forecasts or parameters for a single model and do not necessarily provide an estimate of the distribution of reserves. This paper looks at various sources of uncertainty in projections and tries to give the reader a framework in which to view different attempts to measure the distribution of reserves. Finally the author presents an approach that attempts to at least recognize the issue of model uncertainty and to see its influence on the measurement of reserve uncertainty.

Biography

Roger Hayne is a Fellow of the Casualty Actuarial Society, a Member of the American Academy of Actuaries and a Consulting Actuary in the Pasadena, California office of Milliman USA. He holds a Ph.D. in mathematics from the University of California and joined Milliman in 1977. Roger has been involved in reserve estimation for a wide range of property and liability coverages with emphasis on exposures with longer tails and in situations where full data may not be readily available. The winner of the 1995 Dorweiler Prize, he long has had interest in reserve variability and has authored several papers on the topic that have appeared in both the *Proceedings* and in the *Forum*.

MEASUREMENT OF RESERVE VARIABILITY

1. Introduction

Traditional actuarial methodologies, though not necessarily stochastically based are robust and when used as intended tend to be a holistic approach to estimating reserves. In the end the actuary using such approaches may develop a "gut feel" for the uncertainty in his or her estimates, but may not necessarily be able to quantify that "gut feel."

Conversely, more modern stochastic methods bring with them quantification of the volatility of their forecasts, but usually conditioned on a specific set of assumptions and often based on a single set of data (for example the paid loss triangle).

In this paper we will review various aspects of uncertainty. We will finish by presenting an approach that combines holistic aspects of the traditional approach with estimates of uncertainty in those estimates.

2. Reserves Are Uncertain?

If you reference an insurer's financial statement you will find a single number identified as liability for losses and another for loss adjustment expenses. There is nothing uncertain about that, it is a number printed in a financial statement. So why should we be talking about uncertainty in reserves at all?

The reason is that the number booked is an estimate of the actual liabilities. Accounting guidance tells us it must be "management's best estimate" of the amount that will be paid in the future on covered claims. We note that the guidance does not say that the reserve is an estimate of the expected or average value, it does not say that the reserve is an estimate of the mode (most likely) value, nor does it say that the reserve is an estimate of the median or middle value. The guidance only states that the reserve is management's best estimate of the amount that eventually will be paid.

The accounting guidance does not provide us with a quantitative or statistical framework to assist in setting the reserves. Actuarial guidance is similarly vague. Our statement of principles talks of actuarially sound reserves as "a provision, based on estimates derived from reasonable assumptions and appropriate actuarial methods, for the unpaid amount required to settle all claims, whether reported or not, for which liability exists."[1] That statement further comments that "[t]he uncertainty inherent in the estimation of required provision for unpaid loss or loss adjustment expenses implies that a range of reserves can be actuarially sound,"[2] and "[t]he most appropriate reserve within a range of actuarially sound estimates depends on both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented."[3]

The message that these references seem to give is that if there is greater uncertainty involved in estimating future payments, then there is likely the need for some sort of margin recognizing that uncertainty when setting the reserves. Of course, there is no mention as to the number to which the "margin" should be added.

At this point, a cynic may say that this brief discussion alone proves that the definition of reserves is itself uncertain, but we will leave that discussion to another day and another forum. A reader interested in this discussion is strongly encouraged to read Rodney Kreps' excellent paper[4] addressing this topic. In addition to a most lucid and informative review of these concepts Kreps advances a reasoned and logical answer to the question "if reserves are uncertain what is the correct amount to book in a financial statement?" Briefly he suggests minimizing the penalty of getting the reserves wrong over the entire distribution of reserves.

We hope, however, that we have made the point that there is little formal guidance as to the statistical quantity to be booked as reserves. Nevertheless, to talk in terms of statistical notions, we must know the distribution of reserves and how to estimate that distribution. That is the topic of this paper.

3. A Look at Traditional Methods

Traditional actuarial methods are generally ad-hoc, and are not originally based on specific statistical models. Probably the oldest of these traditional methods is the development factor or link ratio method. It is fairly easy to explain and has been the subject of much literature. It was not originally grounded in mathematical or statistical theory; though there is some recent work to set it into a statistical framework. In addition, it is known to be quite volatile, particularly for less mature exposure periods.

Another traditional approach is the Bornhuetter-Ferguson[5] method. Rather than being multiplicative and leveraged for less mature exposure periods, this method is additive and tends to be more stable. However, the method needs both an estimate of the loss emergence or development (as does the development factor method) but as well as an á-priori estimate of ultimate losses for each exposure year. This latter requirement can be overcome using a variant approach sometimes called the Stanard-Bühlmann or Cape Cod method. In this variant, one estimates the initial "seed" by using an approach equivalent to the development factor projection method. As with the development factor method, this method was largely developed on an ad-hoc basis.

Conceptually similar to the Bornhuetter-Ferguson method is the Frequency/Severity method presented by Berquist and Sherman.[6] Again, the method is ad-hoc and is not based on a specific statistical model. Here the focus is on incremental average cost per claim with separate selections for claim counts and trends in the incremental averages. It exhibits some of the stability of the Bornhuetter-Ferguson method for less mature exposure periods, and does not require an á-priori estimate of ultimate losses. It does exhibit some volatility due to the forecasts of ultimate claim counts, and in the selection of trends for both current leveling and forecasting into the future.

We see a common thread in these and other traditional reserving methods. The methods are generally ad-hoc, and were originally constructed without reference to an underlying statistical model; thus there is no direct way to quantify the uncertainty in their projections.

This shortcoming has been recognized by most practitioners using the traditional approaches. Rather than relying on an underlying statistical assumption to gauge the uncertainty in forecasts, practitioners using traditional techniques usually consider a range of different methods applied to different groupings of data. If the various methods tend to give reasonably consistent results, then the practitioner might get a sense of comfort with the forecasts.

If however, the estimates from the methods diverge then the practitioner might want to dig more deeply into the underlying data and situation to see whether the assumptions underlying one or another method are violated. In the end, by use of several different methods and looking into the operations underlying the data, even without specific quantification, the practitioner of traditional methods can develop a qualitative "feel" for the uncertainty. This is a significant benefit of traditional approaches that seems to be lacking from more recent statistically based methods.

This traditional approach has stood the test of time. Although not statistically sophisticated, it is a very powerful and robust approach. The variety of traditional methods has the added advantage of taking several different data elements into account including paid losses, incurred losses, open claim counts, closed claim counts, etc. By intentionally using a variety of methods the actuary has the ability to test a variety of hypotheses that may affect the final outcome of his or her projections.

4. Moving on From the Traditional

The traditional qualitative "feel" just described is often what is meant by the degree of reserve uncertainty. Though it is quite valuable to the actuary estimating reserves, it is at best subjective and difficult (impossible?) to quantify. If we wish to put numbers around this uncertainty, we probably should first specify what we are trying to measure. In this case, the author believes that the holy grail of reserve uncertainty is the distribution of the amount and timing of future payments for a particular book of policies. If we knew that distribution, we could then speak intelligently about its mean, variance, skewness, and any other characteristic we could think of. We could specifically calculate particular probability levels (amounts not to be exceeded a specific proportion of the time), as well as estimate the least painful amount to be booked given Rodney Kreps' approach.

We cannot overemphasize the importance of understanding what we seek. In particular, this holy grail can be thought of as answering the fundamental question:

> Given current knowledge, what is the distribution of possible future payments (possible reserve numbers)?

Whenever we are presented with an attempt of assigning probabilities to reserves, we should ask ourselves to what extent those probability estimates answer this fundamental question. To better frame this discussion, we will consider the basic sources of uncertainty in most statistical estimates.

5. Sources of Uncertainty

We can identify at least three sources of uncertainty that may arise in estimating the distribution of reserves:

1. Process uncertainty, the fundamental uncertainty due to the presence of randomness even when all other aspects of the distribution are known,

2. Parameter uncertainty, the uncertainty that arises due to unknown parameters of statistical models for the distribution, even if the selection of those models is perfectly correct, and

3. Model or specification uncertainty, the uncertainty that arises if the specified distributions or underlying models are unknown.

Some authors separate model and specification uncertainty; having the former relate to whether the model selected is actually the correct one for the process under review, and the latter dealing with whether the actual distributions selected in the model are correct. For example, is a gamma or lognormal the right distribution to use? For ease of discussion, we combine both here.

At this point, a brief discussion of each of these sources of uncertainty may help in understanding them and their import in the question of estimating the distribution of reserves. Suppose we throw a fair six-sided die. In this statement of the problem, the entire process generating uncertainty is known. We know that we can only observe one of six outcomes, each with equal likelihood. Even with this perfect knowledge, we do not know the outcome of the next roll of the die for certain. This is an example of process uncertainty.

The very existence of insurance depends on process uncertainty and the risk averse individual's reaction to that uncertainty. The law of large numbers implies that process uncertainty regarding the average cost per insured can be reduced to a negligible level if there are a sufficiently large number of independent insureds. A risk averse individual will pay more than his or her own expected costs if the payment amount is certain, the consequences are uncertain and there is a significant potential financial impact.

In the case of the die if it is thrown a large number of times and the result from each throw is recorded, then the sum of all throws will be rather close to 3.5 times the number of throws. Otherwise said, the average from a large number of throws will be close to the expected value of 3.5.

Alternatively, if we now do not assume that the die is fair, then there will be added uncertainty regarding the final outcome. In this case, the underlying model is the same as in the first example, the generation of numbers between one and six depends on which side lands up. However, we now lack the luxury of knowing the probability of each of the six outcomes. The model and the distributions are known with certainty, but we are uncertain about the parameters of the distribution; hence, an example of parameter uncertainty.

An example of the third source of uncertainty (model uncertainty) would arise if we try to model a series of numbers between one and six by assuming they came from the throw of a single die that may or may not be fair. If we have a more complex process, this may not be sufficient. For example, we could be observing the throw of one of many dice, each with different probabilities attached to each number. The particular die selected for a particular throw could be chosen as a function of prior throws. Here, no simple, single weighted die would be the correct model.

When evaluating methods that claim to measure uncertainty in reserves, the reader should ask which of these sources are considered, and in what way. Given the complexity of property and casualty insurance processes it is unlikely that all will or even can be completely addressed.

6. A Relatively Simple Example

Thomas Mack[7] has addressed uncertainty in development factor (chain ladder) forecasts, and has developed some fairly simple formulae based on some fairly broad, and possibly reasonable assumptions. In particular, if we are willing to assume that the development factor method is actually correct (assuming away the third source of uncertainty) and that there is a certain structure to the variance of payments at each age then Mack derives fairly simple formulae for the standard error of reserves, both by exposure year and in total. The reader is referred to the full paper, but we will attempt to provide a brief summary here.

Let C_{ij} denote cumulative payments for exposure year i at age j with I accident years and I stages of development. Mack makes the following assumptions:

1. There are age-to-age development factors f_i such that $E(C_{ij+1}| C_{i1}, C_{i2},\ldots, C_{ij})=f_jC_{ij}, 1 \leq i \leq I, 1 \leq j \leq I-1$

2. $\{C_{i1}, C_{i2},\ldots, C_{iI}\}, \{C_{j1}, C_{j2},\ldots, C_{jI}\}$ independent for $i \neq j$, and

3. There are constants σ_k such that $Var(C_{ik+1}|C_{i1}, C_{i2},\ldots, C_{ik})= C_{ik}\, \sigma_k^2$, $1 \leq k \leq I-1$

Under the first two assumptions Mack shows that the following are unbiased estimators of the development factors f_k:

(5.1)
$$\hat{f}_k = \sum_{j=1}^{I-k} C_{j,k+1} \Big/ \sum_{j=1}^{I-k} C_{j,k}$$

These are simply the volume-weighted averages of the development factors in a particular column. More importantly however, is the estimate of the variance or the reserve forecasts from the development factor method. For this set

(5.2)
$$\hat{\sigma}_k^2 = \begin{cases} \dfrac{1}{I-k-1}\sum_{i=1}^{I-k} C_{ik}\left(\dfrac{C_{ik+1}}{C_{ik}} - \hat{f}_k\right)^2, 1 \le k \le I-2 \\ \min\left(\hat{\sigma}_{I-2}^4 / \hat{\sigma}_{I-3}^2, \min\left(\hat{\sigma}_{I-2}^2, \hat{\sigma}_{I-3}^2\right)\right), k = I-1 \end{cases}$$

Mack shows that the $\hat{\sigma}_k^2$ values are unbiased estimators for $1 \le k \le I$-2. He faced the practical problem of having only one development factor from the I – 1st age to the I^{th} age and relied on a general pattern for the variances for that factor. This problem does not exist if one is willing to assume that the data presented are fully mature, thus leading one to conclude no variance in the last factor or so.

Now taking estimates of future payments from the development factor model, that is

(5.3)
$$\hat{C}_{ik} = \begin{cases} C_{iI+1-i}\hat{f}_{I+1-i}\cdots\hat{f}_{k-1}, k > I+1-i \\ C_{iI+1-i}, k = I+1-i \end{cases}$$

Mack shows that the mean squared error of the reserve forecast for one exposure year can be estimated by

(5.4) $$\widehat{mse}\left(\hat{R}_i\right) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

He further shows that the total reserve for all exposure years combined can be estimated by

(5.5) $$\widehat{mse}\left(\hat{R}\right) = \sum_{i=2}^{I} \left\{ \left(\text{s.e.}\left(\hat{R}_i\right)\right)^2 + \hat{C}_{iI} \left(\sum_{j=i+1}^{I} \hat{C}_{jI} \right) \sum_{k=I+1-i}^{I-1} \frac{2\hat{\sigma}_k^2 / \hat{f}_k^2}{\sum_{n=1}^{I-1} C_{nk}} \right\}$$

Although these formulae are a bit complicated, they are in closed form and do provide estimates of the error of development factor forecasts. One may be tempted to say that our job is done, but before we jump to that conclusion we will look at a relatively simple example.

Consider the data set shown in Exhibit 1. These hypothetical data are based on personal automobile bodily injury coverage, net of reinsurance for a rather homogeneous database. The data have been disguised, though they retain the salient features of the actual experience. The accident dates shown are real, so more than ten years later we now virtually know the ultimate losses by accident year.

Applying this approach to the paid and incurred triangles separately, and recalling that the difference between the ultimate projections and the amounts to date ("reserve" in Mack's analysis) based on an incurred triangle is actually combined provisions for incurred but not reported claims, for additional development on known claims, and claims in transit, we obtain the estimates in Table 1:

Table 1
Standard Error of Reserve Estimates

	Paid Method	Incurred Method
Case Reserve		$96,917
Estimated Future	$358,453	90,580
Total Reserve	$358,453	$187,497
s.e.(Estimate)	$41,639	$13,524

To assist in comparing these results, Exhibit 2 shows two normal distributions with means and standard deviations equal to the expected total reserve and standard error estimates respectively. As can be seen, the two distributions actually have little in common.

Obviously something is happening. The two data sets, paid and incurred development triangles, though from the same data source are telling two very different stories. What then does this tell us about the distribution of reserves?

It is likely that one or both of the paid and incurred development triangles do not satisfy Mack's hypotheses; thus, the differences are most likely due to model or specification uncertainty. This simple example highlights the importance of the third area of uncertainty. Moreover it highlights the likelihood that model or specification uncertainty can overwhelm both parameter and process uncertainty when trying to measure uncertainty in reserves rather than uncertainty in the projections of one particular model. In this case, and in many actual reserving applications, model or specification uncertainty is probably the largest single source of variability in reserve estimates, and often the source most difficult, or even impossible to quantify.

This is a key point to remember when reviewing statistically based methods applied to actuarial problems. Most statistically based methods we have seen to date deal with a single statistical model, and in most cases consider only one data set (for example a paid loss development triangle). Therefore their results would apply to the projections of a particular method and not necessarily to the final distribution of reserves.

The actuary should also be aware of what statistical element is being considered by a particular stochastic method. For example, does the distribution apply to expected forecasts for a method or to the forecasts themselves?

7. An Alternative

Rather than approaching the problem of estimating the distribution of reserves from the view of one model, we could consider the reserve distribution from a micro level. In its most simple formulation, we can assume that there is a number N of open and IBNR claims, all of which are statistically independent, and have the same probability distribution, say with mean μ and variance σ^2. Then the distribution of reserves will have mean and variance:

(6.1)
$$\begin{aligned} \mathrm{E}(R) &= N\mu \\ \mathrm{Var}(R) &= N\sigma^2 \end{aligned}$$

If the distribution for the claims X_i is known then the resulting reserve distribution will only exhibit process uncertainty. For some distributions of claim sizes, the distribution of reserves will be known and have a closed form. One simple example, though unrealistic for property and casualty reserves occurs when the claims all are drawn from the same normal distribution. In this case reserves will be normally distributed with known mean and variance.

There are few situations in property and casualty reserve applications when the number of claims is known with certainty. If the number of claims N is also random, is independent from the claim size distribution, and has mean λ and variance τ^2, then the reserves will have the following mean and variance

(6.2)
$$\begin{aligned}\mathrm{E}(R) &= \lambda\mu \\ \mathrm{Var}(R) &= \lambda\sigma^2 + \mu^2\tau^2\end{aligned}$$

Often in collective risk applications, the random variable N is assumed to have a Poisson distribution, in which case $\tau^2 = \lambda$, and we have

(6.3)
$$\mathrm{Var}(R) = \lambda\left(\sigma^2 + \mu^2\right)$$

With a Poisson claim count distribution, we see that the variance of the average reserve is:

(6.4)
$$\begin{aligned}\mathrm{Var}\left(\frac{R}{\lambda}\right) &= \frac{\lambda\left(\sigma^2 + \mu^2\right)}{\lambda^2} \\ &= \frac{\sigma^2 + \mu^2}{\lambda}\end{aligned}$$

This variance approaches zero as λ becomes arbitrarily large. Otherwise said, in the case that claim counts have a Poisson distribution, process uncertainty inherent in the average reserve will effectively disappear as the expected number of claims gets large.

A benefit of this model for estimating the distribution of reserves is that it allows us to specifically incorporate both parameter and process uncertainty and allows us to quantify the effects of each. As with most other models, estimating model or specification uncertainty is more difficult, and may not be able to be done in general reserving situations.

Heckman and Meyers[8] outline an approach that can be used to incorporate parameter uncertainty into this classical collective risk model.

The work by Heckman and Meyers referenced here presented a fundamental advance in the use of the collective risk model, essentially solving the problem of calculating aggregate distributions for collective risk models with quite weak restrictions on the claim size distribution. The solution is sufficiently straight-forward to be able to be easily programmed, in fact a copy of such a program is included as an exhibit to the paper. Their solution applies to a generalized collective risk model that includes the potential for "contagion" (the possibility that an external event could affect the frequency of claims across lines of insurance or years of coverage) and for "mixing" (the possibility of an external event could affect the size of all claims).

We will adopt their notation here. To this end, we assume that χ and β are two random variables with

$$\begin{aligned} &\mathrm{E}(\chi)=1, \mathrm{Var}(\chi)=c \\ &\mathrm{E}(1/\beta)=1, \mathrm{Var}(1/\beta)=b \end{aligned} \tag{6.5}$$

We will use χ and β to incorporate uncertainty into our collective risk model. We then consider the algorithm for generating one observation of aggregate reserves:

1. Randomly select a value for χ,

2. Randomly select the number of claims N from a Poisson distribution with expected value $\chi\lambda$,

3. Randomly select a value for β,

4. Randomly select N claims from the claim size distribution, and

5. Add the values of the N claims and divide the result by β.

Heckman and Meyers call the c parameter the "contagion" parameter and the b the "mixing" parameter. Under the assumptions that the claim count and claim size distributions are independent, and that the claim selections in step 4 are independent of each other and of the random variables χ and β, then we can calculate the expected value and variance of the aggregate reserves. These values are:

$$\begin{aligned} &\mathrm{E}(R) = \lambda\mu \\ &\mathrm{Var}(R) = \lambda(\mu^2 + \sigma^2)(1 + b) + \lambda^2\mu^2(b + c + bc) \end{aligned} \tag{6.6}$$

We see that in this formulation of the problem, the variance of the average expected reserve does not approach 0 with a large number of expected claims unless $b = c = 0$ (or in the trivial case the expected losses are zero). In the case that $b = c = 0$, the formula reduces to the case without parameter uncertainty.

The alternative approach to estimating the distribution of reserves we present here uses traditional methods in an attempt to estimate the parameters c and b.

The algorithm presented by Heckman and Meyers actually allows for the combination of aggregate loss distributions for several lines of insurance, each with its own contagion parameter c but with a global mixing parameter b. In the approach we present here, we will take advantage of this feature and have different contagion parameters for each accident year, as well as a single global mixing parameter reflecting uncertainty that affects the reserves for all accident years at once. Examples of this global uncertainty would be estimated future inflation, court decisions, and so forth.

We will first take a traditional approach to estimating reserves by accident year for the data contained in Exhibit 1. A detailed review of the data would lead the actuary to conclude that there are many changes occurring in the historical data. There appears to have been changes in the rates at which claims are being closed. There also appear to be changes in relative reserve adequacy over time. As with traditional approaches we applied a variety of different methods to both the actual data and to data adjusted for the estimated effects of changing rates of claim closure and of relative reserve adequacy. For this we used methods outlined in the paper by Berquist and Sherman.[9]

Exhibit 3 shows the reserve estimates by accident year for each method used. The development methods simply apply the usual development method to the indicated data set. By "Adjusted Incurred" we mean historical incurred losses adjusted to reflect current relative reserve adequacy. By "Severity" we mean the incremental average cost projection method described in Berquist and Sherman. The "Hindsight" method is an iterative approach that makes use of historical average future costs per open and IBNR claim to derive estimates of ultimate losses.

The bottom portion of Exhibit 3 shows the weights we assigned to each of the methods. These weights reflect our subjective view of the applicability of the particular method for a particular accident year. We will use the variation in estimates from the various methods to gauge the uncertainty of reserve estimates by accident year. In fact we will use the standard deviations in the last column of Exhibit 3 to estimate the contagion parameters for each accident year. For this we consider formula (6.6) and set $b = 0$. This then gives us the following variance estimate for reserves for accident year i:

(6.7) $$\mathrm{Var}(R_i) = \lambda_i\left(\mu_i^2 + \sigma_i^2\right) + c_i\lambda_i^2\mu_i^2$$

Here, λ_i denotes the expected number of open and IBNR claims for accident year i while μ_i and σ_i^2 are the mean and variance, respectively, of the reserves for a single claim for accident year i. We note that the two terms in this sum can be interpreted as the variance without parameter uncertainty and the contribution of parameter uncertainty to the total variance.

We note that though the derivation does not make sense, the formulae developed by Heckman and Meyers allow the contagion parameter c to be negative. In that case, the claim counts will have a binomial distribution with mean greater than its variance. In the case of a positive c value, the claims will have a negative binomial distribution with variance greater than the mean. As we have seen above, in the case of $c = 0$ claims will have a Poisson distribution.

Our analysis of the Exhibit 1 data provided us with estimates of the number of open and IBNR claims, and hence estimates of the values for μ_i. For sake of illustration we assumed that the open claims for each accident year will each have lognormal distributions with the same coefficient of variation. More sophisticated analysis of open and IBNR claims for older accident years may provide more accurate estimates of these distributions. In any case, the standard deviations for individual claims based on these distributions are also shown in Exhibit 4.

The column titled "Aggregate Process Standard Deviation" is the standard deviation implied by a collective risk model with no parameter uncertainty and a Poisson claim count distribution as described above. We can then solve equation (6.7) for c_i to obtain estimates of the contagion parameters by accident year implied by our analysis, and claim count and size distributions. That is what was done in the last column of Exhibit 4.

We now turn our attention to the mixing parameter b. In the modeling, the β random variable uniformly affects all claims in a particular iteration. In our model here we will use it as an overlay to reflect global uncertainty in the forecasts. To measure this uncertainty, we compare estimated ultimate severities against an expected smooth transition from one year to the next. Since there is volatility in the percentage of paid claims, we elected to measure this global uncertainty by reviewing the severity per ultimate claim as opposed to the average ultimate claim with payment (selected in the reserve analysis due to the fact that there is 0 probability of a lognormal claim having 0 payment).

Exhibit 5 shows our estimation of the mixing parameter b. Here, we compare the selected severities (per ultimate reported claim) to averages based on an exponential fit through all data points. We assume that observations of $1/\beta$ are ratios of the actual severity over the fitted severity. The estimate for the mixing parameter b is then the variance of the observed $1/\beta$ values.

We now have sufficient information to derive an estimate of the distribution of reserves for this sample problem. We used the algorithm discussed in Heckman and Meyers to estimate the distribution of aggregate reserves, both with parameter uncertainty (non-zero values for the c_i and b parameters) and without such parameter uncertainty. Exhibit 6 graphically compares these two distributions. As can be seen, parameter uncertainty is substantial in this case.

One striking observation from this analysis is the dispersion of the reserve distribution in this case. The distribution has a standard deviation of \$39 million on total reserves of \$202 million, for a coefficient of variation of more than 19%. The 90th and 95th percentiles for this aggregate distribution are \$250 million and about \$278 million, respectively, or 24% and 38% above the expected amount. This is a far cry from the "plus or minus 10%" that is sometimes cited in ranges for reserves. These results simply reflect the substantial uncertainty inherent in the reserve forecasts, in this case, due largely to the changes that have been occurring in the historical experience.

It is also likely that we have missed sources of uncertainty in the above analysis. We have identified methods we believed to be appropriate, and gave them subjective weights based on their relative strengths and weaknesses and the situations occurring in the experience. These weightings are subjective and potentially volatile, adding to the model or specification uncertainty, and probably not directly accounted for in the analysis.

In addition, we assumed that all random quantities are independent from one another. We attempted to take some potential correlation into account by the use of the contagion and mixing parameters. This is a crude approach at best. There has been some recent work in calculating aggregate distributions where there is some form of correlation among some of the distributions. Examples of this can be found in Wang[10] and Dhaene, et.al[11,12] The inclusion of correlation between years, should such correlation exist, would be an obvious refinement to the approach we have outlined here.

As noted above, the accident years are real for the data. As such, all accident years are now virtually completely closed. The current data would imply a December 31, 1991 reserve of approximately $170 million, outside of the "plus or minus 10%" range and at an approximate 19% probability level given the analysis discussed above. Although in hindsight our methodology was not as accurate as we would like the answer does not appear to be unreasonable given the volatility of the estimates.

8. Conclusion

We recognize this approach is far from perfect. The traditional approaches are very robust and provide the actuary with a substantial amount of valuable information, which may not be present in the application of a single statistically based approach. There is obviously more work to be done to make that information rigorous and quantifiable.

Work by Mack and others have gone a long way to putting the chain ladder or development factor approaches on statistical footing. Other traditional methods can probably also benefit from such a rigorous approach. For example, one might think of a simplified version of the incremental severity method, presented by Berquist and Sherman to be formulated by a statistical model with parameters representing an inherent trend and on-level averages for each age of development. Nonlinear statistical approaches may be helpful in gaining statistical insight to the properties of that traditional technique. A similar approach may also prove beneficial in gaining additional understanding into the Stanard-Bühlman or Bornhuetter-Ferguson approaches.

If we work with a variety of forecast methods, which is a fundamental characteristic of the traditional approach to estimating reserves, then we should also understand the correlation of results among the various methods. This understanding would also help us to better estimate the distribution of reserves.

There obviously remains much yet to be done.

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[2] Ibid.

[3] Ibid.

[4] Kreps, R.E., "Management's Best Estimate of Loss Reserves," *Casualty Actuarial Society Forum*, Fall 2002, pp. 247-258.

[5] Bornhuetter, R.L., and Ferguson, R.E., "The Actuary and IBNR," *Proceedings of the Casualty Actuarial Society*, LIX, 1972, pp. 181-195.

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EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

Cumulative Paid Losses

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$267	$1,975	$4,587	$7,375	$10,661	$15,232	$17,888	$18,541	$18,937	$19,130	$19,189	$19,209	$19,234	$19,234	$19,246	$19,246	$19,246	$19,246
1975	310	2,809	5,686	9,386	14,884	20,654	22,017	22,529	22,772	22,821	23,042	23,060	23,127	23,127	23,127	23,127	23,159	
1976	370	2,744	7,281	13,287	19,773	23,888	25,174	25,819	26,049	26,180	26,268	26,364	26,371	26,379	26,397	26,397		
1977	577	3,877	9,612	16,962	23,764	26,712	28,393	29,656	29,839	29,944	29,997	29,999	29,999	30,049	30,049			
1978	509	4,518	12,067	21,218	27,194	29,617	30,854	31,240	31,598	31,889	32,002	31,947	31,965	31,986				
1979	630	5,763	16,372	24,105	29,091	32,531	33,878	34,185	34,290	34,420	34,479	34,498	34,524					
1980	1,078	8,066	17,518	26,091	31,807	33,883	34,820	35,482	35,607	35,937	35,957	35,962						
1981	1,646	9,378	18,034	26,652	31,253	33,376	34,287	34,985	35,122	35,161	35,172							
1982	1,754	11,256	20,624	27,857	31,360	33,331	34,061	34,227	34,317	34,378								
1983	1,997	10,628	21,015	29,014	33,788	36,329	37,446	37,571	37,681									
1984	2,164	11,538	21,549	29,167	34,440	36,528	36,950	37,099										
1985	1,922	10,939	21,357	28,488	32,982	35,330	36,059											
1986	1,962	13,053	27,869	38,560	44,461	45,988												
1987	2,329	18,086	38,099	51,953	58,029													
1988	3,343	24,806	52,054	66,203														
1989	3,847	34,171	59,232															
1990	6,090	33,392																
1991	5,451																	

Claims Closed with Payment

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	268	607	858	1,090	1,333	1,743	2,000	2,076	2,113	2,129	2,137	2,141	2,143	2,143	2,145	2,145	2,145	2,145
1975	294	691	913	1,195	1,620	2,076	2,234	2,293	2,320	2,331	2,339	2,341	2,343	2,343	2,343	2,343	2,344	
1976	283	642	961	1,407	1,994	2,375	2,504	2,549	2,580	2,590	2,596	2,600	2,602	2,603	2,603	2,603		
1977	274	707	1,176	1,688	2,295	2,545	2,689	2,777	2,809	2,817	2,824	2,825	2,825	2,826	2,826			
1978	269	658	1,228	1,819	2,217	2,475	2,613	2,671	2,691	2,706	2,710	2,711	2,714	2,717				
1979	249	771	1,581	2,101	2,528	2,816	2,930	2,961	2,973	2,979	2,986	2,988	2,992					
1980	305	1,107	1,713	2,316	2,748	2,942	3,025	3,049	3,063	3,077	3,079	3,080						
1981	343	1,042	1,608	2,260	2,596	2,734	2,801	2,835	2,854	2,859	2,860							
1982	350	1,242	1,922	2,407	2,661	2,834	2,887	2,902	2,911	2,915								
1983	428	1,257	1,841	2,345	2,683	2,853	2,908	2,920	2,925									
1984	291	1,004	1,577	2,054	2,406	2,583	2,622	2,636										
1985	303	1,001	1,575	2,080	2,444	2,586	2,617											
1986	318	1,055	1,906	2,524	2,874	2,958												
1987	343	1,438	2,384	3,172	3,559													
1988	391	1,671	3,082	3,771														
1989	433	1,941	3,241															
1990	533	1,923																
1991	339																	

EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

Cumulative Reported Claims

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,912	2,854	3,350	3,945	4,057	4,104	4,149	4,155	4,164	4,167	4,169	4,169	4,169	4,170	4,170	4,170	4,170	4,170
1975	2,219	3,302	3,915	4,462	4,618	4,673	4,696	4,704	4,708	4,711	4,712	4,716	4,716	4,716	4,716	4,716	4,717	
1976	2,347	3,702	4,278	4,768	4,915	4,983	5,003	5,007	5,012	5,012	5,013	5,014	5,015	5,015	5,015	5,015		
1977	2,983	4,346	5,055	5,696	5,818	5,861	5,884	5,892	5,896	5,897	5,900	5,900	5,900	5,900	5,900			
1978	2,538	3,906	4,633	5,123	5,242	5,275	5,286	5,292	5,298	5,302	5,304	5,304	5,306	5,306				
1979	3,548	5,190	5,779	6,206	6,313	6,329	6,339	6,343	6,347	6,347	6,348	6,348	6,348					
1980	4,583	6,106	6,656	7,032	7,128	7,139	7,147	7,150	7,151	7,153	7,154	7,154						
1981	4,430	5,967	6,510	6,775	6,854	6,873	6,883	6,889	6,892	6,894	6,895							
1982	4,408	5,849	6,264	6,526	6,571	6,589	6,594	6,596	6,600	6,602								
1983	4,861	6,437	6,869	7,134	7,196	7,205	7,211	7,212	7,214									
1984	4,229	5,645	6,053	6,419	6,506	6,523	6,529	6,531										
1985	3,727	4,830	5,321	5,717	5,777	5,798	5,802											
1986	3,561	5,045	5,656	6,040	6,096	6,111												
1987	4,259	6,049	6,767	7,206	7,282													
1988	4,424	6,700	7,548	8,105														
1989	5,005	7,407	8,287															
1990	4,889	7,314																
1991	4,044																	

Outstanding Claims

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,381	1,336	1,462	1,660	1,406	772	406	191	98	57	23	13	3	4	0	0	0	0
1975	1,289	1,727	1,730	1,913	1,310	649	358	167	73	30	9	6	4	2	2	1	1	
1976	1,605	1,977	1,947	1,709	1,006	540	268	166	79	48	32	18	14	10	10	7		
1977	2,101	2,159	2,050	1,735	988	582	332	139	66	38	27	21	21	8	3			
1978	1,955	1,943	1,817	1,384	830	460	193	93	56	31	15	9	7	2				
1979	2,259	2,025	1,548	1,273	752	340	150	68	36	24	18	13	4					
1980	2,815	1,991	1,558	1,107	540	228	88	55	28	14	8	6						
1981	2,408	1,973	1,605	954	480	228	115	52	27	15	11							
1982	2,388	1,835	1,280	819	354	163	67	44	21	10								
1983	2,641	1,765	1,082	663	335	134	62	34	18									
1984	2,417	1,654	896	677	284	90	42	15										
1985	1,924	1,202	941	610	268	98	55											
1986	1,810	1,591	956	648	202	94												
1987	2,273	1,792	1,059	626	242													
1988	2,403	1,966	1,166	693														
1989	2,471	2,009	1,142															
1990	2,642	2,007																
1991	2,366																	

EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

Outstanding Losses

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$5,275	$8,867	$12,476	$11,919	$8,966	$5,367	$3,281	$1,524	$667	$348	$123	$82	$18	$40	$0	$0	$0	$0
1975	6,617	11,306	13,773	14,386	10,593	4,234	2,110	1,051	436	353	93	101	10	5	5	3	3	
1976	7,658	11,064	13,655	13,352	7,592	4,064	1,895	1,003	683	384	216	102	93	57	50	33		
1977	8,735	14,318	14,897	12,978	7,741	4,355	2,132	910	498	323	176	99	101	32	14			
1978	8,722	15,070	15,257	11,189	5,959	3,473	1,531	942	547	286	177	61	67	7				
1979	9,349	16,470	14,320	10,574	6,561	2,864	1,328	784	424	212	146	113	38					
1980	11,145	16,351	14,636	11,273	5,159	2,588	1,290	573	405	134	81	54						
1981	10,933	15,012	14,728	9,067	5,107	2,456	1,400	584	269	120	93							
1982	13,323	16,218	12,676	6,290	3,355	1,407	613	398	192	111								
1983	13,899	16,958	12,414	7,700	4,112	1,637	576	426	331									
1984	14,272	15,806	10,156	8,005	3,604	791	379	159										
1985	13,901	15,384	12,539	7,911	3,809	1,404	827											
1986	15,952	22,799	16,016	8,964	2,929	1,321												
1987	22,772	24,146	18,397	8,376	3,373													
1988	25,216	26,947	17,950	8,610														
1989	24,981	30,574	19,621															
1990	30,389	34,128																
1991	28,194																	

Accident Year	Earned Exposures
1974	11,000
1975	11,000
1976	11,000
1977	12,000
1978	12,000
1979	12,000
1980	12,000
1981	12,000
1982	11,000
1983	11,000
1984	11,000
1985	11,000
1986	12,000
1987	13,000
1988	14,000
1989	14,000
1990	14,000
1991	13,000

Distributions Based on Mack's Method

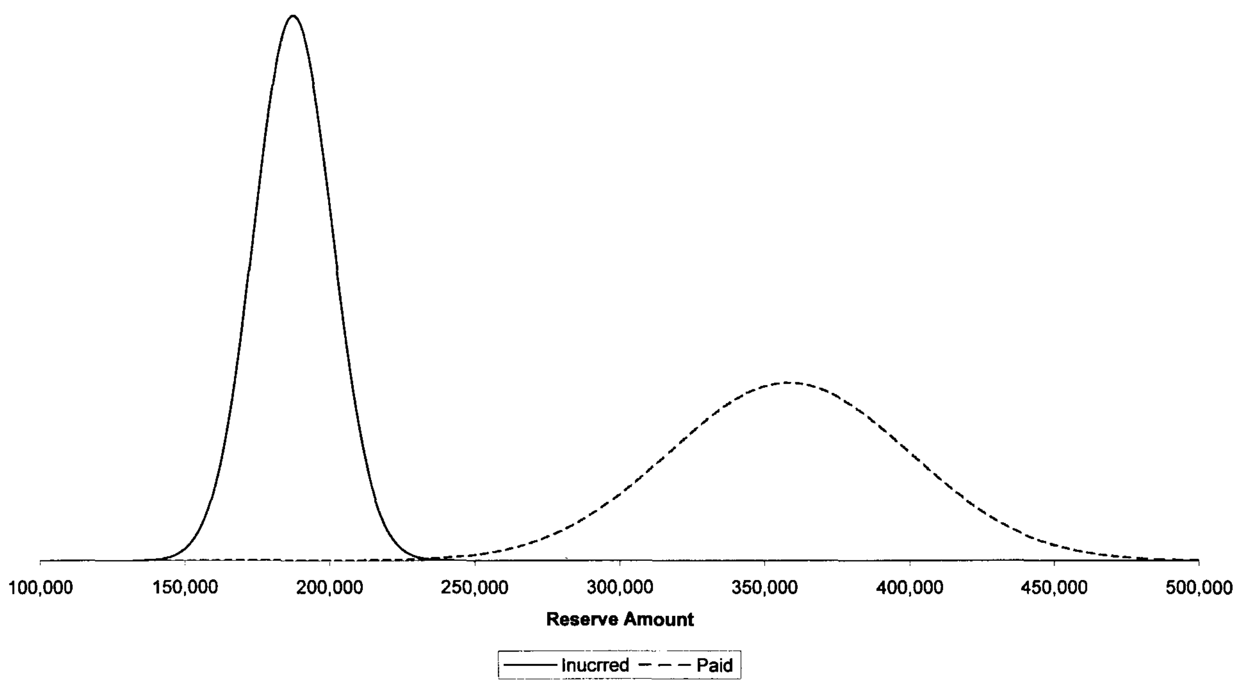

Exhibit 3

EXAMPLE RESERVE FORECASTS

Accident Year	Reserve Estimates by Ultimate Forecast Method: Incurred Development	Paid: Development	Paid: Severity	Paid: Pure Premium	Paid: Hindsight	Adjusted Incurred	Paid Adjusted for Claim Closing Changes: Development	Paid Adjusted: Severity	Paid Adjusted: Pure Premium	Paid Adjusted: Hindsight	Weighted Average	Weighted Standard Deviation
1974	$0	$0	$0	$0		$0	$0	$0	$0		0	0
1975	3	0	0	0		3	0	0	0		0	0
1976	33	0	0	0		33	21	0	0		11	14
1977	5	0	0	0		8	24	0	0		5	8
1978	-15	10	9	10		7	26	0	0		6	11
1979	-10	35	34	33		-35	28	0	0		11	24
1980	-7	54	55	50		-29	61	33	31		31	30
1981	-37	49	73	75		-20	77	47	49		39	41
1982	-41	107	136	131		-58	100	79	75		66	70
1983	114	275	297	297		-68	200	176	172		156	126
1984	-161	416	394	446		-135	352	318	351		181	258
1985	403	761	713	812		130	692	702	779		567	248
1986	744	2,143	1,760	1,909	$1,687	394	1,936	1,842	1,950	$675	1,357	637
1987	2,335	6,847	5,583	5,128	5,128	2,348	6,000	5,790	5,220	2,301	4,260	1,620
1988	8,371	19,768	16,246	13,451	14,428	10,391	17,352	16,433	13,399	8,001	12,866	3,525
1989	25,787	44,631	36,887	29,232	32,199	26,048	39,241	36,431	28,512	19,174	30,212	6,428
1990	60,211	83,760	73,987	61,846	62,974	55,734	79,667	70,246	57,192	43,286	62,516	10,198
1991	83,093	130,907	95,283	95,185	78,616	79,573	154,268	87,625	84,688	72,157	90,014	19,166
Total											202,298	

Selected Weights

Accident Year	Incurred Development	Paid: Development	Paid: Severity	Paid: Pure Premium	Paid: Hindsight	Adjusted Incurred	Paid Adjusted: Development	Paid Adjusted: Severity	Paid Adjusted: Pure Premium	Paid Adjusted: Hindsight
1974	1	1	1	1		1	1	1	1	
1975	0	1	1	1		0	1	1	1	
1976	1	1	1	1		1	1	1	1	
1977	1	1	1	1		1	1	1	1	
1978	1	1	1	1		1	1	1	1	
1979	1	1	1	1		1	1	1	1	
1980	1	1	1	1		1	1	1	1	
1981	1	1	1	1		1	1	1	1	
1982	1	1	1	1		1	1	1	1	
1983	3	1	2	2		3	1	2	2	
1984	3	1	2	2		3	1	2	2	
1985	3	1	2	2		3	1	2	2	
1986	3	1	2	2	2	3	1	2	2	2
1987	3	1	2	2	2	3	1	2	2	2
1988	3	1	2	2	2	3	1	2	2	2
1989	3	1	2	2	2	3	1	2	2	2
1990	3	1	2	2	2	3	1	2	2	2
1991	3	1	2	2	2	3	1	2	2	2

ESTIMATION OF CONTAGION PARAMETERS BY ACCIDENT YEAR

Accident Year	Estimated Reserve	Paid Claim Count Estimates: Ultimate	Closed	Open & IBNR	Average Reserve	Single Claim Standard Deviation	Aggregate Process Standard Deviation	Estimated Total Standard Deviation	Implied *c* Value
1974	$0	2,145	2,145	$0					
1975	0	2,344	2,344	0					
1976	11	2,604	2,603	1	$11,000	$16,251	$20	$14	-1.477
1977	5	2,827	2,826	1	5,000	7,387	9	8	-0.858
1978	6	2,718	2,717	1	6,000	8,864	11	11	0.092
1979	11	2,994	2,992	2	5,500	8,126	14	24	3.172
1980	31	3,083	3,080	3	10,333	15,266	32	30	-0.097
1981	39	2,865	2,860	5	7,800	11,523	31	41	0.473
1982	66	2,922	2,915	7	9,429	13,929	45	70	0.669
1983	156	2,938	2,925	13	12,000	17,728	77	126	0.405
1984	181	2,658	2,636	22	8,227	12,155	69	258	1.882
1985	567	2,661	2,617	44	12,886	19,038	152	248	0.120
1986	1,357	3,064	2,958	106	12,802	18,913	235	637	0.190
1987	4,260	3,889	3,559	330	12,909	19,072	418	1,620	0.135
1988	12,866	4,697	3,771	926	13,894	20,527	754	3,525	0.072
1989	30,212	5,135	3,241	1,894	15,951	23,566	1,238	6,428	0.044
1990	62,516	5,270	1,923	3,347	18,678	27,595	1,928	10,198	0.026
1991	90,014	4,410	339	4,071	22,111	32,666	2,517	19,166	0.045

Exhibit 5

ESTIMATE OF MIXING PARAMETER

Accident Year	Estimated Ultimate Losses	Estimated Ultimate Reported Claims	Indicated Severity	Smoothed Severity	Estimate of $1/\beta$
1974	19,246	4,170	4,615	4,165	1.108
1975	23,159	4,717	4,910	4,388	1.119
1976	26,408	5,016	5,265	4,623	1.139
1977	30,054	5,901	5,093	4,870	1.046
1978	31,992	5,307	6,028	5,130	1.175
1979	34,535	6,349	5,439	5,404	1.007
1980	35,993	7,155	5,030	5,693	0.884
1981	35,211	6,897	5,105	5,997	0.851
1982	34,444	6,605	5,215	6,317	0.825
1983	37,837	7,219	5,241	6,655	0.788
1984	37,280	6,539	5,701	7,011	0.813
1985	36,626	5,812	6,302	7,385	0.853
1986	47,345	6,130	7,723	7,780	0.993
1987	62,289	7,327	8,501	8,196	1.037
1988	79,069	8,256	9,577	8,634	1.109
1989	89,444	9,017	9,919	9,095	1.091
1990	95,908	8,931	10,739	9,581	1.121
1991	95,465	7,829	12,194	10,093	1.208
Variance (estimate of b)					0.019

Exhibit 6

ESTIMATED DISTRIBUTION OF RESERVES