A Characterization of Life Expectancy with Applications to Loss Models

Daniel R. Corro

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Dan Corro
National Council on Compensation Insurance, Inc.
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Abstract:

In its usual (one-dimensional) form, a loss model is just a distribution of nonnegative real numbers $[0,\infty)$. *This note establishes necessary and sufficient conditions for a differentiable function to equal the life expectancy of some loss model. Examples are provided to illustrate the shape of the life expectancy function of several common loss models. The characterization is used to define a general class of loss models flexible enough to cover the Pareto, Lognormal, Weibull, and Gamma densities. Finally, the approach is extended to model multi-dimensional survivorship.*

I. Introduction

In general, life expectancy can be expressed as a simple descriptive statistic. The usual functional forms used to describe loss distributions, namely cumulative density functions [CDFs], probability density functions [PDFs], and hazard rate functions generally demand some processing to visualize and often require fitting parameters to an assumed form for calculation purposes.

On the other hand, the formal nature of CDFs and PDFs and hazard rates are apparent. A differentiable function $F(t)$ on $[0,\infty)$ is a CDF of a loss model exactly when:

$$F(0)=0, \quad \frac{dF}{dt}\geq 0, \quad and \quad \lim_{t\to\infty} F(t)=1.$$

An integrable function, $f(t)$, on $[0,\infty)$ is a PDF of a loss model exactly when:

$$f(t)\geq 0, \quad and \quad \int_0^\infty f(t)dt=1.$$

Similarly, an integrable function, $h(t)$, on $[0,\infty)$ is a hazard rate function of a loss model exactly when:

$$h(t)\geq 0, \quad and \quad \int_0^\infty h(t)dt=\infty.$$

The main result is that a differentiable function $\rho(t)>0$ on $[0,\infty)$ is a life expectancy function [LEF] of a loss model exactly when:

$$\frac{d\rho}{dt}\geq -1, \quad and \quad \int_0^\infty \frac{1}{\rho(t)}dt=\infty.$$

When working with insurance data, "claim life expectancy" can often be regarded as a reserve and conversely a reserve as a life expectancy (c.f. [4]). In practice, reserves may be related with claim survival data to the extent that closed, i.e. "dead", cases are characterized by having no reserves.

It is evident from the discussion below how a life expectancy function completely determines the loss model. Because life expectancy is often easier to determine than the CDF, PDF or hazard function, being able to recognize such functions may come in handy. Examples show that the graph of the life expectancy function is simpler than those of the CDF or PDF functional forms used to define some popular loss models. Also, bivariate loss models pose many technical difficulties; however, these observations on life expectancy are readily extended to higher dimensions (c.f. [5]).

II. Notation and Background

Let $f(t)$ denote an integrable function on the nonnegative real numbers $[0,\infty)$ satisfying:

$$\int_0^\infty f(t)dt = 1$$

Regard $f(t)$ as a probability density of failure times and define the function:

$$S(t) = 1 - \int_0^t f(s)ds = \int_t^\infty f(s)ds$$

As is customary, we refer to $S(t)$ as the *survival function*, $f(t)$ as the *probability density function [PDF]*, *F(t)*=1-*S(t)* as the *cumulative density function[CDF]*, and t as "time." We also let *T* denote the random variable for the distribution of survival times and $\mu = E(T)$ the mean duration or life expectancy, which we assume throughout to be finite and nonzero. Survival analysis refers to the following function:

$$h(t) = \frac{f(t)}{S(t)}$$

as the *hazard rate function* or sometimes as the *force of mortality*. The hazard rate function measures the instantaneous rate of failure at time *t* and can be expressed as a limit of conditional probabilities:

$$h(t) = \lim_{\Delta t \to 0} \frac{\Pr\{t \le T < t + \Delta t \mid T \ge t\}}{\Delta t}$$

There are many well-known relationships and interpretations of hazard rate functions (refer to Allison[1] for a particularly succinct discussion).

It is convenient to recall that if we set

$$g(t) = \int_0^t h(s)ds, \text{then } S(t) = e^{-g(t)}.$$

Let's fix *t* and restrict our attention to values of time $w > t$. The conditional probability of survival to *w*, given survival to *t*, is $S_t(w) = \dfrac{S(w)}{S(t)}$. In this context (see [2]), the *expectation of life at time t*, given survival to time *t*, is just:

$$\rho(t) = \frac{\int_t^\infty (w-t)f(w)dw}{\int_t^\infty f(w)dw} = \int_t^\infty S_t(w)dw = \int_t^\infty \frac{S(w)}{S(t)}dw$$

Observe that under our assumptions, $\rho(0) = \mu > 0$ and the function $\rho(t)$ is well defined for all $t>0$. We also observe that for any $a<b$ with $S(a) > 0$, we have the relation:

$$\rho(a)S(a) = \int_a^\infty S(t)dt = \int_a^b S(t)dt + \int_b^\infty S(t)dt$$

$$\leq \int_a^b S(a)dt + \int_b^\infty S(t)dt = S(a)(b-a) + \rho(b)S(b)$$

$$\Rightarrow \text{(read "implies")} \qquad a + \rho(a) \leq b + \frac{\rho(b)S(b)}{S(a)} \leq b + \rho(b)\,,$$

with strict inequality exactly when $S(b) < S(a)$.

Not surprisingly, there are formal relationships between hazard, $h(t)$, and life expectancy, $\rho(t)$, as in:

Proposition 1:

$$1 + \frac{d\rho}{dt} = h(t)\rho(t)$$

Proof: This is straightforward from the above definitions--see [2].

Proposition 2: *For any differentiable function, $\varphi(t)$, on $[0,\infty)$, the following are equivalent:*

i) $\quad a,b \in [0,\infty), a \leq b \quad \Rightarrow \quad a + \varphi(a) \leq b + \varphi(b)$

ii) $\quad \dfrac{d\varphi}{dt} \geq -1 \quad on\,[0,\infty)$

Proof: Consider the function $\psi(t) = \varphi(t) + t$, then ψ is non-decreasing on $[0,\infty)$ if and only $\dfrac{d\psi}{dt} = \dfrac{d\varphi}{dt} + 1 \geq 0$ on $[0,\infty)$; the result follows.

So we now let $\varphi(t) > 0$ be a differentiable function on $[0,\infty)$ such that $\frac{d\varphi}{dt} \geq -1$ on $[0,\infty)$. From Proposition 1, it is natural to consider the loss model defined via its hazard function, as above, by:

$$h(t) = h_\varphi(t) = \frac{1 + \dfrac{d\varphi}{dt}}{\varphi(t)} \geq 0 \; on\,[0,\infty)$$

Keeping the above notation, we have:

$$h(t) = \frac{1}{\varphi(t)} + \frac{d\ln\varphi(t)}{dt} \Rightarrow g(t) = \int_0^t h(w)dw = \int_0^t \frac{dw}{\varphi(w)} + \ln\left(\frac{\varphi(t)}{\varphi(0)}\right)$$

$$\Rightarrow S(t) = e^{-g(t)} = \frac{\varphi(0)e^{-\int_0^t \frac{dw}{\varphi(w)}}}{\varphi(t)}$$

$$\Rightarrow \rho(t) = \int_t^\infty \frac{S(v)}{S(t)}dv = \varphi(t)\int_t^\infty \frac{e^{-\int_t^v \frac{dw}{\varphi(w)}}}{\varphi(v)}dv$$

Regard t as fixed and use the change of variable:

$$u(v) = \int_t^v \frac{dw}{\varphi(w)} \Rightarrow du = \frac{dv}{\varphi(v)}$$

At the limits of integration we have

$v = t$ corresponds to $u = 0$ and $v = \infty$ corresponds to $u = \int_t^\infty \frac{dw}{\varphi(w)}$.

It follows that:

$$\rho(t) = \varphi(t)\int_0^{\int_t^\infty \frac{dw}{\varphi(w)}} e^{-u}du = \varphi(t)\left(1 - e^{-\int_t^\infty \frac{dw}{\varphi(w)}}\right) \le \varphi(t)$$

Which means that the life expectancy function, or, can be characterized as the smallest solution to the differential equation (Proposition 1) that relates hazard with life expectancy.

Since clearly

$$\int_0^\infty \frac{dw}{\varphi(w)} = \infty$$

$$\Leftrightarrow \text{(read "if and only if")} \int_t^\infty \frac{dw}{\varphi(w)} = \infty \text{ for all } t \in [0,\infty),$$

it follows that:

$$\rho(t) = \varphi(t) \Leftrightarrow \int_0^\infty \frac{dw}{\varphi(w)} = \infty$$

and we have established the main result of this paper, which is stated as the following Proposition:

Proposition 3: *A differentiable function* $\rho(t) > 0$ *on* $[0,\infty)$ *is a life expectancy of a loss model exactly when:*

$$\frac{d\rho}{dt} \geq -1, \quad and \quad \int_0^\infty \frac{1}{\rho(t)}dt = \infty$$

In this paper we will refer to a function $\rho(t)$ as an LEF exactly when it is a life expectancy of a loss model. The remainder of this paper consists primarily of applying the Proposition 3 characterization of LEFs. Conceptually, the "local" derivative constraint relates to a limitation that at any time no more "deaths" can occur than the number then "living" while the "global" integral constraint requires the model to account for all lives.

Example: Suppose $\varphi(t) = t^2 + 1$. Then $\frac{d\varphi}{dt} = 2t \geq -1$ when $t \in [0,\infty)$ and we can define

$$h(t) = h_\varphi(t) = \frac{1 + \frac{d\varphi}{dt}}{\varphi(t)} = \frac{2t+1}{t^2+1}$$

The reader can readily verify that in this case we have:

$$g(t) = \ln(t^2+1) + \tan^{-1}(t)$$

$$S(t) = \frac{1}{(t^2+1)e^{\tan^{-1}(t)}}$$

$$\rho(t) = (t^2+1)\left(1 - e^{\tan^{-1}(t) - \frac{\pi}{2}}\right) = \varphi(t)\left(1 - e^{\tan^{-1}(t) - \frac{\pi}{2}}\right) < \varphi(t)$$

We see that $\varphi(t) = t^2 + 1$ is **not** the life expectancy of any loss model.

III. Examples of Life Expectancy Functions

In this section we show what the life expectancy looks like for several of the most commonly used loss models.

Example III.1. *Pareto density with parameters* $a > 1, b > 0$. In this example, define

$$f(a,b;t) = ab^a(b+t)^{-a-1}.$$

Then (see, e.g. [6], pp. 222-223)

$$S(t) = \left(\frac{b}{b+t}\right)^a, \; h(t) = \frac{a}{b+t} \quad \text{and} \quad \rho(t) = \frac{b+t}{a-1}.$$

The Pareto density is characterized by a linear LEF. Note that for the Pareto loss model:

$$\frac{d\rho}{dt} \equiv \frac{1}{a-1} \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \infty .$$

Example III.2. *Lognormal density with parameters* $\mu, \sigma > 0$. In this example, define

$$f(\mu,\sigma;t) = \frac{e^{\frac{-1}{2\sigma^2}(\ln t-\mu)^2}}{t\sigma\sqrt{2\pi}}$$

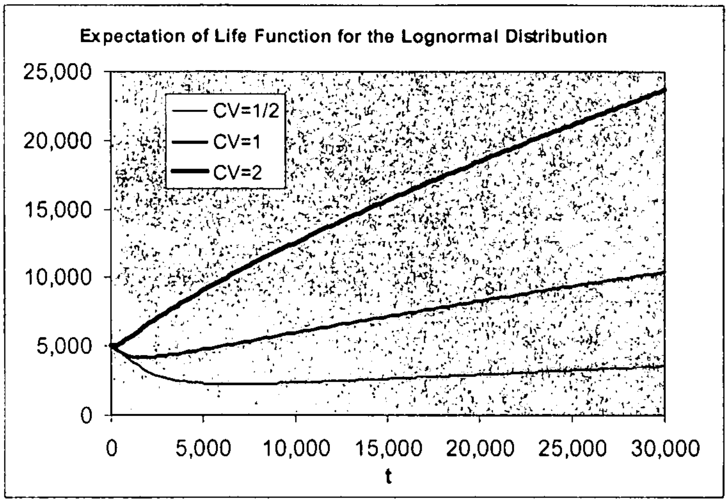
(see, e.g. [6], pp. 229-230). It can be shown that for a Lognormal loss model:

$$\lim_{t \to \infty} h(t) = 0; \qquad \lim_{t \to \infty} \rho(t) = \infty; \qquad \lim_{t \to \infty} \frac{d\rho}{dt} = \infty .$$

The coefficient of variation, CV, is defined as the ratio of the standard deviation to the mean; it is a convenient and dimensionless measure of variation. We leave to the reader the verification that the parameters for a Lognormal density with mean M and coefficient of variation C can be determined from:

$$\sigma = \sqrt{\ln(C^2+1)} \qquad \mu = \ln(M) - \frac{\sigma^2}{2}$$

The following chart shows the LEF's for a Lognormal loss model, expressed as above as a function $\varphi(t)$ of "time" t and with a constant mean, $\varphi(0) = 5{,}000$, and for CV = ½, 1, and 2, respectively.



Example III.3. *Weibull density with parameters* $a, b > 0$. In this example, define

$$f(a,b;t) = abt^{b-1}e^{-at^b}.$$

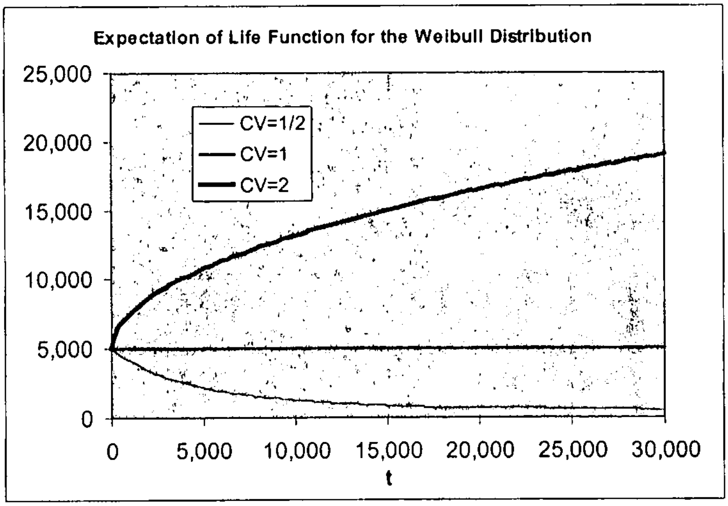
Then (see, e.g. [6] pp. 231-232)

$$S(t) = e^{-at^b};\ h(t) = abt^{b-1};\quad \text{and} \qquad \mu = \frac{\Gamma\left(\frac{1}{b}\right)}{ba^{\frac{1}{b}}}$$

For a Weibull density we have:

$$\lim_{t\to\infty} \rho(t) = \frac{1}{\lim_{t\to\infty} h(t)} = \begin{cases} \infty & b < 1 \\ \frac{1}{a} & b = 1 \\ 0 & b > 1 \end{cases}$$

The following chart shows the LEF's for a Weibull loss model with mean of 5,000 and coefficients of variation = ½, 1, and 2, respectively. Note that a Weibull loss model with CV = 1 is an exponential density (case b=1), characterized by a constant LEF.



Example III.4. *Gamma density with parameters* $a, b > 0$. In this example, define

$$f(a,b;t) = \frac{a^b t^{b-1} e^{-at}}{\Gamma(b)}.$$

Then (see, e.g. [6] pp. 226-227)

$$S(t) = 1 - \Gamma(b;at); \quad \text{and} \quad \mu = \frac{b}{a}$$

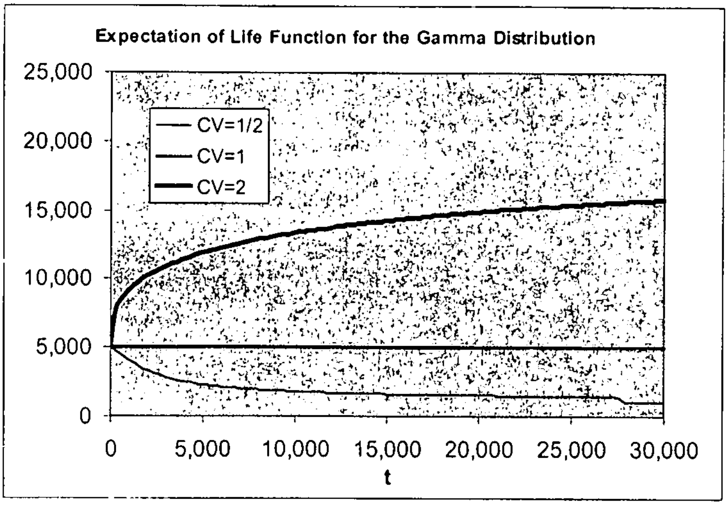
It can be shown (see, e.g. [7] pp. 86-87) that for a Gamma loss model:

$$\lim_{t \to \infty} h(t) = a \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \frac{1}{a}.$$

We leave to the reader the verification that the parameters for a Gamma density with mean M and coefficient of variation C can be determined from:

$$a = \frac{1}{C^2 M} \quad \text{and} \quad b = \frac{1}{C^2}$$

The following chart shows the expectation of life for a Gamma loss model with mean of 5,000 and coefficients of variation = ½, 1, and 2, respectively. Note that a Gamma loss model with CV = 1 is again an exponential density (case b=1).



This section concludes with two general examples.

Example III.5. *Piecewise linear functions*

In each of the above loss models the graph of the expectation of life function is rather flat, exhibiting at most one relative maximum or minimum. This suggests that such curves can be successfully approximated by fairly simple functions, e.g. by piecewise linear functions with rather few pieces. Consider any positive, continuous, piecewise linear function on $[0,\infty)$ with finitely many pieces. Then the rightmost slope must be nonnegative, so the integral over $[0,\infty)$ diverges. It is intuitively clear (and easy to prove) each of the "corners" of such a function can be approximated to any desired tolerance by a smooth curve that matches the slopes of the corner's two sides while keeping its derivative within the range of those two slopes. It follows from our findings that a positive, continuous piecewise linear function on $[0,\infty)$ represents the expectation of life of a loss model exactly when all its (finitely many) slopes are ≥ -1. This is a very simple criterion to accommodate when fitting empirical data to a piecewise linear representation.

Example III.6. *Rational functions*

Another natural choice of "simple" functions, these differentiable, is the set of rational functions. We consider first the case of a ratio of two first degree polynomials:

$$\varphi(t) = \varphi(b,c,d;t) = \frac{bt+c}{t+d} \qquad t \in [0,\infty)$$

We claim that the following are necessary and sufficient conditions for $\varphi(t)$ to be LEF of a loss model on $[0,\infty)$ with positive mean:

$$(RF1) \quad c > 0,\, d > 0,\, b \geq Max\left(\frac{c}{d} - d, 0\right).$$

We will abuse notation somewhat and use RF1 to denote both these conditions and the class of functions they determine. To verify the claim, observe first that:

$$-1 \leq \frac{d\varphi}{dt} = \frac{(t+d)b-(bt+c)}{(t+d)^2} \Leftrightarrow (t+d)^2 \geq c - db$$

which holds for all $t \geq 0$ exactly when $d^2 \geq c - db$.

Assume first that $\varphi(t)$ satisfies conditions RF1, then clearly $\varphi(t)$ is differentiable and positive on $[0,\infty)$ and we have just verified that its derivative is ≥ -1. We also have:

$$b = 0 \Rightarrow \frac{1}{\varphi(t)} = \frac{t+d}{c} \Rightarrow \int_0^\infty \frac{dt}{\varphi(t)} = \frac{1}{c}\left[\frac{t^2}{2} + dt\right]_0^\infty = \infty$$

$$b > 0 \Rightarrow \lim_{t\to\infty} \frac{1}{\varphi(t)} = \lim_{t\to\infty} \frac{t+d}{bt+c} = \frac{1}{b} > 0 \Rightarrow \int_0^\infty \frac{dt}{\varphi(t)} = \infty$$

and so conditions RF1 suffice to make $\varphi(t)$ an LEF.

Convesely, if $\varphi(t)$ is an LEF, then being well defined on $[0,\infty)$ forces d>0 and clearly:

$$0 < \mu = \varphi(0) = \frac{c}{d} \Rightarrow c > 0$$
$$\lim_{t\to\infty} \varphi(t) = b \Rightarrow b \geq 0$$

and the observation on $\frac{d\varphi}{dt}$ implies that conditions RF1 hold.

Ratios of linear terms are a rather restricted class of functions, not even including linear functions. So we consider next the case of a second-degree polynomial divided by a linear term:

$$\varphi(t) = \varphi(a,b,c,d;t) = \frac{at^2 + bt + c}{t + d} \qquad t \in [0,\infty)$$

Two simple lemmas are useful here:

Lemma: *For a>0, b>0 the quadratic* $at^2 + bt + c$ *has a positive root if and only if* $b \leq -2\sqrt{ac}$.

Proof: Assume first that $b \leq -2\sqrt{ac}$, then $b^2 - 4ac \geq 0$, and from the quadratic formula,

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{|b| + \sqrt{b^2 - 4ac}}{2a} > 0$$

is a positive root. Conversely, if there is a positive root, the quadratic formula implies that

$$-b + \sqrt{b^2 - 4ac} > 0 \Rightarrow |b| = \sqrt{b^2} > \sqrt{b^2 - 4ac} > b \Rightarrow b < 0,$$

and it follows that

$$b^2 - 4ac \geq 0 \Rightarrow b^2 \geq 4ac \Rightarrow |b| \geq 2\sqrt{ac} \Rightarrow b = -|b| \leq -2\sqrt{ac}$$

and the lemma is established.

Lemma: $\frac{d\varphi}{dt} = a + \frac{bd - ad^2 - c}{(t + d)^2} \qquad \frac{d^2\varphi}{dt^2} = -2\frac{bd - ad^2 - c}{(t + d)^3}$

Proof: This is just a straightforward calculation:

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{(t + d)(2at + b) - (at^2 + bt + c)}{(t + d)^2} \\ &= \frac{at^2 + 2adt + ad^2 - ad^2 + bd - c}{t^2 + 2dt + d^2} \\ &= a + \frac{bd - ad^2 - c}{(t + d)^2}\end{aligned}$$

and the lemma is clear.

We claim that the following are necessary and sufficient conditions for $\varphi(t)$ to be the LEF of a loss model on $[0,\infty)$ with positive mean:

$$(RF2) \quad a \geq 0, c > 0, d > 0, b \geq Max\left(\frac{c}{d} - d, -2\sqrt{ac}\right)$$

We have already verified this for $a = 0$, so we assume $a \neq 0$.

Assume first that $\varphi(t)$ satisfies conditions (*RF2*), then clearly $\varphi(t)$ is differentiable and the above observations assure that $\varphi(t)$ is positive on $[0,\infty)$ with derivative ≥ -1. Also,

$$\int_0^\infty \frac{dt}{\varphi(t)} = \int_0^\infty \frac{t+d}{at^2+bt+c}dt = \left(\frac{1}{2a}\right)\int_0^\infty \frac{2at+b}{at^2+bt+c}dt + \left(\frac{2ad-b}{2a}\right)\int_0^\infty \frac{1}{at^2+bt+c}dt.$$

We observe that the first integral diverges:

$$\int_0^\infty \frac{2at+b}{at^2+bt+c}dt = \left[\ln(at^2+bt+c)\right]_0^\infty = \infty,$$

while the right hand integral is finite:

$$at^2+bt+c \geq at^2 - 2\sqrt{ac}\,t + c = \left(\sqrt{a}\,t - \sqrt{c}\right)^2 = u(t)^2$$

$$\Rightarrow$$

$$\int_0^\infty \frac{1}{at^2+bt+c}dt = \int_0^{\frac{1+\sqrt{c}}{\sqrt{a}}} \frac{1}{at^2+bt+c}dt + \int_{\frac{1+\sqrt{c}}{\sqrt{a}}}^\infty \frac{1}{at^2+bt+c}dt$$

$$\leq \int_0^{\frac{1+\sqrt{c}}{\sqrt{a}}} \frac{1}{at^2+bt+c}dt + \frac{1}{\sqrt{a}}\int_1^\infty \frac{du}{u^2} = \int_0^{\frac{1+\sqrt{c}}{\sqrt{a}}} \frac{1}{at^2+bt+c}dt + \frac{1}{\sqrt{a}} < \infty$$

$$\Rightarrow \int_0^\infty \frac{dt}{\varphi(t)} = \infty$$

and (*RF2*) is sufficient to make $\varphi(t)$ an LEF.

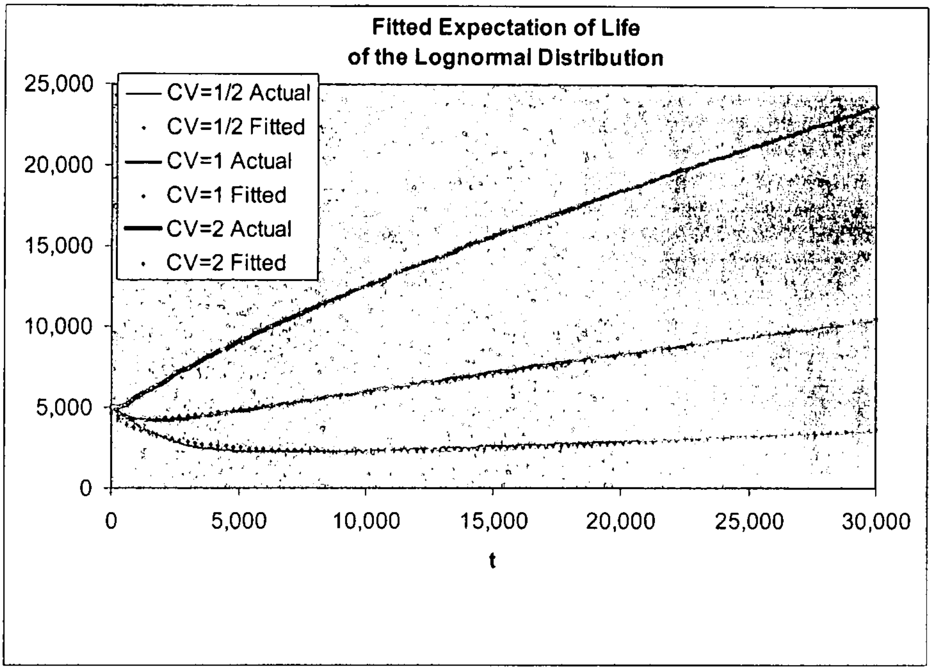
Conversely, if $\varphi(t)$ is an LEF, then being well defined on $[0,\infty)$ again forces d>0 and clearly:

$$0 < \mu = \varphi(0) = \frac{c}{d} \Rightarrow c > 0$$

$$0 \leq \lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty}\frac{at^2+bt+c}{t+d} = \lim_{t\to\infty}\frac{2at+b}{1} \Rightarrow a \geq 0$$

and the lemmas imply that conditions (*RF2*) hold.

These constraints can be imposed when fitting data. Since this class of functions includes any linear expectancy function, it covers the Pareto and exponential cases. The following graphs show how the RF2 class of functions approximates the Lognormal Example III.2:



$RF2$ Fit to Lognormal				
Parameters	a	b	c	d
CV=½	0.1079	332	18,814,530	4,175
CV=1	0.2337	3,739	4,388,204	938
CV=2	0.4245	22,265.8	85,152,045	18013.2

Observe that while the tail behavior seems closely fit, the *RF2* approximation is not particularly good for CV=2 near t=0. This is because the *RF2* class of functions is not adept at fitting a slope at or near -1 over an interval. The Lognormal density shows few failures near t=0, corresponding to the thin right-hand tail of the corresponding normal density. There are various approaches to dealing with this (the next section illustrates restricting or renormalizing the loss interval); we conclude this section with a refinement of the formula. Consider broadening *RF2* by eliminating the derivative constraint:

$$(\overline{RF2}) \quad a \geq 0, c > 0, d > 0, b \geq -2\sqrt{ac}\,.$$

Let

$$\alpha = \alpha(a,b,c,d) = \begin{cases} \sqrt{\dfrac{ad^2 - bd + c}{a+1}} - d & ad^2 - bd + c \geq 0 \\ -d & ad^2 - bd + c \leq 0 \end{cases}$$

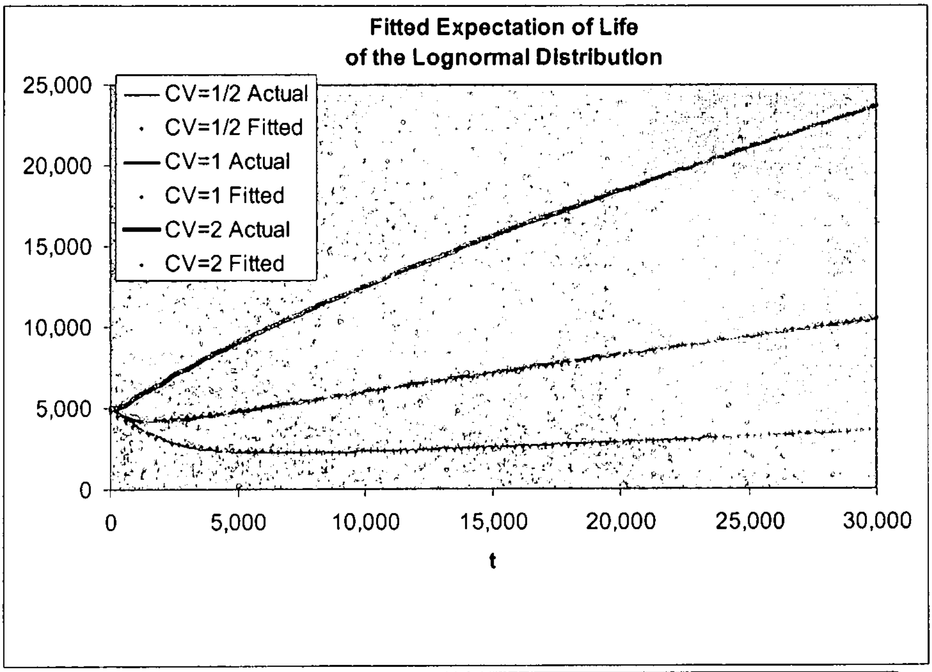
From an earlier lemma, $\dfrac{d^2\varphi}{dt^2}$ has the same sign as the constant $ad^2 - bd + c$, which implies that α is the largest value of t, if any, for which $\dfrac{d\varphi}{dt} = -1$. We can now define:

$$\overline{\varphi}(t) = \overline{\varphi}(a,b,c,d;t) = \begin{cases} \varphi(\alpha) + (\alpha - t) & t \leq \alpha \\ \varphi(t) & t \geq \alpha \end{cases}$$

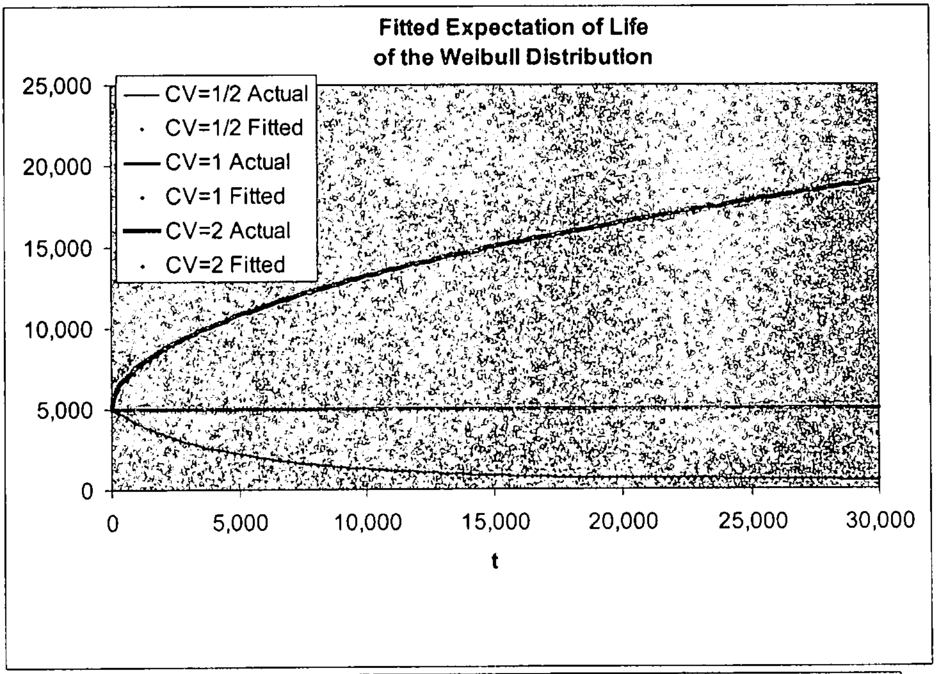
Then $\overline{\varphi}(t)$ is a differentiable function and our observations show $\overline{\varphi}(t)$ is an LEF. As no surprise, we note that:

$$\begin{aligned}
(\overline{RF2})\ \&\ (\overline{\varphi} = \varphi) &\Leftrightarrow (\overline{RF2})\ \&\ (\alpha \leq 0) \\
&\Leftrightarrow (\overline{RF2})\ \&\left(\sqrt{\frac{ad^2 - bd + c}{a+1}} \leq d\right) \\
&\Leftrightarrow (\overline{RF2})\ \&\left(ad^2 - bd + c \leq ad^2 + d^2\right) \\
&\Leftrightarrow (\overline{RF2})\ \&\left(\frac{c}{d} - d \leq b\right) \Leftrightarrow (RF2)
\end{aligned}$$

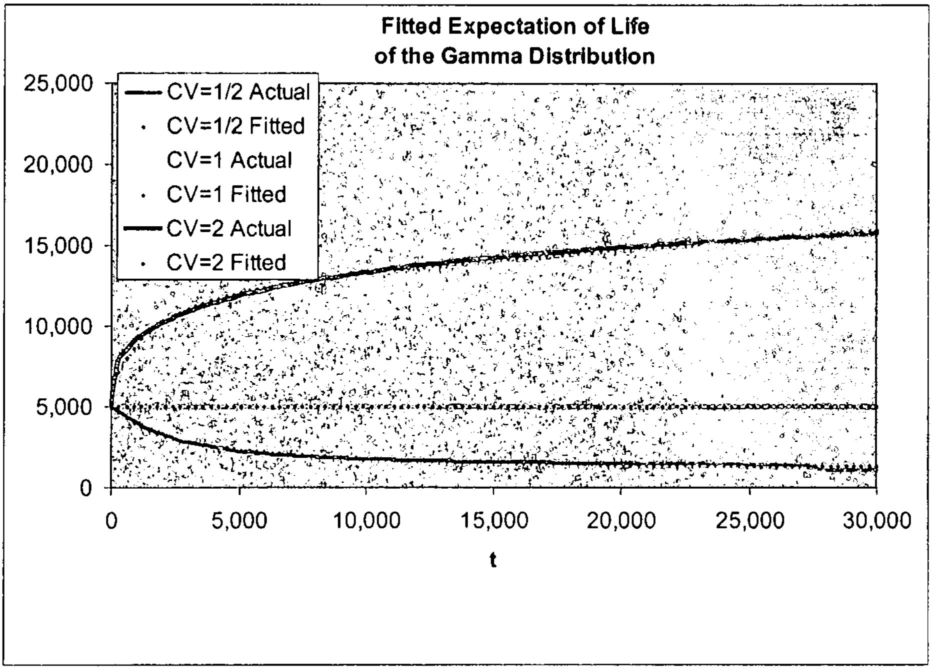
We conclude this section with charts illustrating how well the class of functions $\overline{RF2}$ is able to approximate Examples III.2, III.3, and III.4. We arrived at these estimates by first fitting the form $\varphi(t)$ without the derivative constraint on parameter b (using the SAS PROC NLIN procedure) and then using $\overline{\varphi}(t)$ as the fitted LEF.



$\overline{RF2}$ **Fit to Lognormal**				
Parameters	a	b	c	d
CV=½	0.0867	919	9,728,662	1,845.1
CV=1	0.2313	3,685.7	1,968,510	387.5
CV=2	0.4245	22,265.8	85,152,045	18,013.2



$\overline{RF2}$ Fit to Weibull				
Parameters	a	b	c	d
CV=½	0.0289	-1,081.5	27,621,056	5,479
CV=1	0	5,000	5,000	1
CV=2	0.2339	13,948.5	22,940,996	4,040.4



$\overline{RF2}$ Fit to Gamma				
Parameters	a	b	c	d
CV=½	0	998.9	11,943,663	2,306.2
CV=1	0	5,000	5,000	1
CV=2	0.098	13,558.6	9,600,852	1,636.5

IV. Limited Loss Models

In the previous discussion we have referred to loss models as essentially equivalent to continuous probability densities on $[0,\infty)$. The astute reader will have noticed a rather clumsy slight of hand as regards loss models of finite support, i.e., for which there is an upper loss limit L such that $f(t)=0$ for $t>L$. We have implicitly assumed that life expectancy $\rho(t)>0$ on $[0,\infty)$, which in effect means that there is no maximum loss. Consider, then, any probability density on $[0,\infty)$ with survival function, S(t), and expectation of life function $\rho(t)$. We have:

$$\frac{dS}{dt}=-f(t)\le 0 \Rightarrow S \text{ nonincreasing} \Rightarrow \mathrm{S}^{-1}(\{0\})=[L,\infty)$$

for some L. Note that we may have $L = \infty$ (and $[\infty,\infty) = \phi$ is empty). We find, therefore, from the definition of the expectation of life function that:

$$\rho(t) = \int_t^\infty \frac{S(v)}{S(t)} dv \Rightarrow \{t \mid \rho(t) = 0\} = S^{-1}(\{0\}) = [L,\infty)$$

and the reader can easily verify that this observation, together with our previous arguments, enables us to refine our main result somewhat:

Proposition 3: *A differentiable function $\rho(t)$ on $[0,\infty)$ is a life expectancy of a loss model exactly when:*

$$\{t \mid \rho(t) = 0\} = [L,\infty), \quad \frac{d\rho}{dt} \geq -1, \quad \textit{and} \quad \int_0^L \frac{1}{\rho(t)} dt = \infty .$$

When $L < \infty$ is finite, the (continuous) loss models we have considered still demand that the probability of meeting or exceeding L is 0. It is more convenient when dealing with limited losses to consider an alternative formulation. By a limited loss model, we mean a probability density on $[0,1]$ that is a combination of a continuous density on $[0,1)$ and a point mass at $\{1\}$ that may have a positive probability. This corresponds to the case when all losses may not exceed a particular maximum value. It is convenient to use that maximum value as the unit for expressing loss amounts. In effect, this amounts to a change of variable $x = \frac{t}{L}$ and the point mass at $\{1\}$ corresponds to the probability that a loss hits the per occurrence loss limit. For convenience, we further require that $S(t) > 0$ on $[0,1)$ (see [3] for a more complete discussion, where these models are related to "hazard functions with finite support").

In this case, some of the arithmetic is simplified, as we have fewer improper integrals to worry about.

A differentiable function, $F(t)$, on $[0,1)$ is a CDF of a limited loss model exactly when:

$$F(0) = 0, \quad \frac{dF}{dt} \geq 0, \quad \text{and} \quad \lim_{t \to 1} F(t) \leq 1$$

An integrable function $f(t)$ on $[0,1]$ is a PDF of a limited loss model exactly when:

$$f(t) \geq 0, \quad \text{and} \quad \int_0^1 f(t) dt = 1 - f(1)$$

Similarly, any nonnegative integrable function $h(t)$ on $[0,1)$ is a hazard rate function of a limited loss model. Observe that Propositions 1 and 2 apply in this context, when restricted to the open interval $(0,1)$, and we have:

Proposition 3A: *A differentiable function $\rho(t)$ on $[0,1)$ is a life expectancy of a limited loss model exactly when:*

$$\rho(t) > 0, \quad \frac{d\rho}{dt} \geq -1, \quad and \quad \int_0^1 \frac{1}{\rho(t)} dt = \infty$$

Proof: Let $\varphi(t) > 0$ be a differentiable function on $[0,1)$ such that $\frac{d\varphi}{dt} \geq -1$ on $[0,1)$ and consider the limited loss model determined via its hazard function, as above, by:

$$h(t) = h_\varphi(t) = \frac{1 + \frac{d\varphi}{dt}}{\varphi(t)} \geq 0 \; on \; [0,1)$$

Keeping the above notation, we have, just as before:

$$h(t) = \frac{1}{\varphi(t)} + \frac{d \ln \varphi}{dt} \Rightarrow g(t) = \int_0^t h(w) dw = \int_0^t \frac{dw}{\varphi(w)} + \ln\left(\frac{\varphi(t)}{\varphi(0)}\right)$$

$$\Rightarrow S(t) = e^{-g(t)} = \frac{\varphi(0) e^{-\int_0^t \frac{dw}{\varphi(w)}}}{\varphi(t)}$$

$$\Rightarrow \rho(t) = \int_t^1 \frac{S(v)}{S(t)} dv = \varphi(t) \int_t^1 \frac{e^{-\int_t^v \frac{dw}{\varphi(w)}}}{\varphi(v)} dv$$

Similar to before, using the change of variable

$$u(v) = \int_t^v \frac{dw}{\varphi(w)} \Rightarrow du = \frac{dv}{\varphi(v)}$$

$v = t$ corresponds to $u = 0$ and $v = 1$ corresponds to $u = \int_t^1 \frac{dw}{\varphi(w)}$

$$\Rightarrow \rho(t) = \varphi(t) \int_0^{\int_t^1 \frac{dw}{\varphi(w)}} e^{-u} du = \varphi(t)\left(1 - e^{-\int_t^1 \frac{dw}{\varphi(w)}}\right) \leq \varphi(t)$$

Which means that here too the LEF is the smallest solution to the differential equation (Proposition 1) that relates hazard with life expectancy and it follows that:

$$\rho(t) = \varphi(t) \Leftrightarrow \int_0^1 \frac{dw}{\varphi(w)} = \infty$$

and we have established the sufficiency of the conditions to be a LEF. For the necessity, it only remains to observe that $S(t) > 0$ on $[0,1)$ implies that $\rho(t) > 0$ on $[0,1)$, completing the proof.

Note that evidently:

$$\lim_{t \to 1} \rho(t) = 0$$

for the LEF of any limited loss model, even though we did not need to make that an explicit requirement in the statement of Proposition 3A.

V. Application to Multi-Dimensional Loss Models

One significant advantage of expectation of life is that it is rather simple to generate empirical data in multi-dimensional contexts. Given a database of individual claim information, it would be reasonable to expect to be able to identify closed cases and to be able to identify claims whose paid costs exceed a fixed amount x and whose ALAE exceeds a fixed amount y. Taking the average benefits =MeanCost(x,y) and average ALAE costs =MeanALAE(x,y) over that set of claims leads to another pair of positive numbers (U,V) = (U(x,y),V(x,y)) = (MeanCost(x,y)-x,MeanALAE(x,y)-y). Because we are considering closed cases, (U,V) can be regarded as a life expectancy or "reserve" vector. The association of (x,y) with (U,V) is a vector field which is termed an "expected survival" vector field in [5]. The correlation between claim costs, ALAE, and claim closure is all captured in that vector field.

Similarly, we could let x represent indemnity benefits and y medical benefits on Workers' Compensation claims. A good model of the survival vector field might help in the determination of case reserves or in modeling loss development.

It follows that an understanding of what type of functions can reasonably model life expectancy can be helpful in producing multi-dimensional survival models. It can be shown that these models are more flexible than traditional multi-variate loss models (see [5]). The use of piecewise linear functions to approximate life expectancy is straightforward, just noting, as above, the condition that the partial derivatives (where they exist) exceed or equal -1 and that the function be nonnegative sufficiently far from the origin.

To illustrate, we conclude this paper by presenting a model for using rational functions, as above, to approximate life expectancy in two dimensions. Begin with the observation that, formally:

$$\frac{a_{11}x^2 + \beta_1 xy + a_{12}y^2 + b_{11}x + b_{12}y + c_{11}}{x + y + d_{11}} = \frac{a_{11}x^2 + (b_{11} + \beta_1 y)x + (a_{12}y^2 + b_{12}y + c_{11})}{x + (y + d_{11})}$$

$$= \varphi(a_{11}, b_{11} + \beta_1 y, (d_{11} + y)\varphi(a_{12}, b_{12}, c_{11}, d_{11}; y), d_{11} + y; x)$$

Consider, therefore, the following two-dimensional vector field on the positive quadrant in the xy-plane:

$$U(x,y)=\hat{\varphi}(a_{11},b_{11}+\beta_1 y,(d_{11}+y)\varphi(a_{12},b_{12},c_{11},d_{11};y),d_{11}+y;x)$$
$$V(x,y)=\hat{\varphi}(a_{21},b_{21}+\beta_2 x,(d_{21}+x)\varphi(a_{22},b_{22},c_{21},d_{21};x),d_{21}+x;y) \quad x,y\in[0,\infty)$$

and then the vector field defined by:

$$\eta(x,y)=\left(\frac{-1+\frac{\partial U}{\partial x}}{U(x,y)}\ ,\ \frac{-1+\frac{\partial V}{\partial y}}{V(x,y)}\right) \qquad x,y\in[0,\infty)$$

We claim that the following conditions suffice to assure that η is a hazard vector field as defined in [5] (or what amounts to the same, that (U,V) is an expected survival vector field as defined there):

$$a_{j1}\geq 0,\ b_{j1}\geq -2\sqrt{a_{j1}c_{j1}},\ c_{j1}>0,\ d_{j1}>0,$$
$$a_{j2}\geq 0, b_{j2}\geq 2\sqrt{a_{j2}c_{j1}}, \beta_j\geq -2\sqrt{a_{j1}a_{j2}} \qquad j=1,2$$

From the above, and the obvious symmetry in x and y, all is clear except to verify that these conditions assure that:

$$b_{11}+\beta_1 y\geq -2\sqrt{a_{11}(a_{12}y^2+b_{12}y+c_{11})} \quad \text{for all } y\geq 0$$

To see this, first note that

$$a_{12}y^2+b_{12}y+c_{11}\geq a_{12}y^2+2\sqrt{a_{12}c_{11}}\,y+c_{11}=\left(\sqrt{a_{12}}+\sqrt{c_{11}}\right)^2$$

And it follows that:

$$\begin{aligned}
&b_{11}+\beta_1 y+2\sqrt{a_{11}(a_{12}y^2+b_{12}y+c_{11})}\\
&\geq b_{11}+\beta_1 y+2\sqrt{a_{11}}\sqrt{(\sqrt{a_{12}}\,y+\sqrt{c_{11}})^2}\\
&=\ b_{11}+\beta_1 y+2\sqrt{a_{11}}(\sqrt{a_{12}}\,y+\sqrt{c_{11}})\\
&=\left(b_{11}+2\sqrt{a_{11}c_{11}}\right)+\left(\beta_1+2\sqrt{a_{11}a_{12}}\right)y\geq 0
\end{aligned}$$

and the result follows. In practice, however, the recommendation is to fit data without constraints and then make any ad hoc adjustments needed to assure the use of a valid LEF.

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