On the Practical Multiline Excess of Loss Pricing

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ON THE PRACTICAL MULTILINE EXCESS OF LOSS PRICING

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Abstract

More and more ceding companies are asking for global protections of their portfolios. One example is the protection by the reinsurer of two (or more) lines, e.g. fire and motor third party liability. Clearly this allows the insurance company to optimally balance its portfolio and to pay the lowest reinsurance premium. In this paper we analyse how to price an excess of loss treaty covering multiple lines.

Keywords

Excess of loss, multiple cover, payment pattern, stability clause, capital allocation, cost of capital, cash flow model, multiline aggregate deductible, sliding scale, profit commission.

1. Introduction

Insurance companies are corporations and, as such, they are willing to buy reinsurance for the same reasons that corporations buy insurance. These reasons include the fact that entities are not able to diversify insurable risks. They will therefore demand some compensation for their risk-averseness. This compensation may take different forms :

- higher wages for employees and managers
- lower rates for clients
- more allocated capital by the shareholders
- buying some (re)insurance.

The latter is observed on the market and we will discuss in this paper the pricing of some particular reinsurance treaties.
More and more insurance companies are trying to optimize their reinsurance structure. They are looking for a global protection with their reinsurers. One of these global solutions is to cover two lines simultaneously. Clearly this allows to take better advantage of the diversification of an insurance portfolio. Thus a better reinsurance cover follows.
Let us take an example. Assume a fire treaty existing of three layers :

- Layer 1 (Fire) : 2500 xs 1000 with three reinstatements at 100%.
- Layer 2 (Fire) : 3000 xs 3000 with two reinstatements at 100%.
- Layer 3 (Fire) : 4000 xs 6000 with one reinstatement at 100%.

Assume a MTPL (Motor Third Party Liability) treaty existing of three layers :

- Layer 1 (MTPL) : 3000 xs 2000 with unlimited free reinstatements.
- Layer 2 (MTPL) : 5000 xs 5000 with unlimited free reinstatements.
- Layer 3 (MTPL) : ∞ xs 10000 with unlimited free reinstatements.

North American readers may be surprised to see layers with unlimited free reinstatements, as well as an unlimited layer. This is in fact common practice in Europe, and in particular in Belgium, at least for Motor Third Party Liability covers. Property covers are always limited and General Liability covers are usually limited.
An alternative solution might be to keep Layers 2 and 3 for Fire and MTPL and to create a global treaty with alternative Layer 1bis (Fire) and Layer 1bis (MTPL) :

- Layer 1bis (Fire) : 2500 xs 500 with unlimited free reinstatements,
- Layer 1bis (MPTL) : 4000 xs 1000 with unlimited free reinstatements.

with a global annual aggregate deductible of, say, 1000 (Ribeaud (2000) calls it a multiline aggregate deductible). So, for the working layer we combine Fire and MTPL and, as it is a working layer, we impose a large annual aggregate deductible in order to avoid a huge amount of claims to be paid by the reinsurer and high premiums to be paid by the insurer. Note that Layer 1bis (Fire) and Layer 1bis (MTPL) are one treaty. One global premium is asked for that cover. We now have three treaties :

- Fire with two layers : 3000 xs 3000 and 4000 xs 6000.
- MTPL with two layers : 5000 xs 5000 and ∞ xs 10000.
- Global, which is affected by claims hitting Layer 1bis (Fire) and Layer 1bis (MTPL) with a global (multiline) annual aggregate deductible of 1000.

This global treaty is exactly the kind of treaty we want to price in this paper.
Throughout the paper we will use a numerical example in order to apply the models and formulae that will be derived.
The rest of the paper is organized as follows. Section 2 presents the general model we will work with as well as the particular distributions that will be used in the numerical example. Section 3 recalls the use of the Panjer's algorithm as well as the use of lattice distributions. Section 4 presents the detailed model we will work with, i.e. reinsurance liabilities with potential clauses. Section 5 shows how to mix both lines and obtains expected values required for the cash flow model that is presented in section 6. Section 7 discusses the use of clauses making the reinsurance premium random. Section 8 gives the conclusion.

2. General model

From now on we will adopt the traditional convention that treaties are yearly based, which is common practice.
We will work within the collective risk model. In this model, claims arise anonymously from the portfolio. It is assumed that the losses are identically distributed and mutually independent. It is also assumed that they are independent of the number of claims, which is a random variable (typically a Poisson distribution).
Working with the collective risk model is not a limitation, as other models may be used, e.g. the individual risk model. In this model it is assumed that each risk has a (known) chance to produce at least one claim during the coverage period. It is also assumed that the loss distribution, in case of a claim, is known for each risk.
Let us define

- X_i as the i^{th} claim amount of type Fire,
- Y_i as the i^{th} claim amount of type MTPL.

It is assumed that the X_i's are independent and identically distributed as well as the Y_i's. X_i's and Y_i's are assumed to be mutually independent. We also define

- N as the number of claims of type Fire,
- M as the number of claims of type MTPL.

We assume that N and M are independent and that N and the X_i's on the one hand and M and the Y_i's on the other hand are also independent.
We are then able to build two collective risk models :

$$\begin{aligned} S &= X_1 + \cdots + X_N, \\ T &= Y_1 + \cdots + Y_M, \end{aligned}$$

where S denotes the aggregate fire claims and T denotes the aggregate MTPL claims.
Let us assume that the distributions of X, Y, N and M have been estimated, possibly based on past data, as follows

- the distribution of the fire claim amounts, X, is Pareto with parameters $A = 400$ and $\alpha = 1.50$. The distribution of the MTPL claim amounts, Y is Pareto with parameters $A = 700$ and $\alpha = 2.50$. Let us recall the cumulative density distribution of a Pareto distribution ($X \sim Pa(A, \alpha)$) :

$$\begin{aligned} F_X(x) &= 0 \quad \text{if } x \leq A, \\ &= 1 - \left(\frac{x}{A}\right)^{-\alpha} \quad \text{if } x > A. \end{aligned}$$

- the distribution of the fire claim numbers, N is Poisson with parameter $\lambda = 2.5$. The distribution of the MTPL claim numbers, M is Poisson with parameter $\lambda = 5$. Let us recall the probability function of a Poisson distribution ($N \sim Po(\lambda)$) :

$$\mathbb{P}[N = n] = p(n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad , \quad n = 0, 1, \ldots$$

3. Practical calculations for the required distributions

In general, the actuary knows the behaviour of the claims losses. He has fitted, based on past data, a continuous distribution for X and Y. Furthermore, he assumes that N and M are Poisson distributed because he chose to work within the collective risk model.
First we have to obtain a discretization of the claims distributions. Indeed we will use Panjer's algorithm (see Panjer (1981)) that works with lattice distributions. For the distribution of S, we have :

$$\begin{aligned} f_S(0) &= e^{-\lambda(1 - f_X(0))}, \\ f_S(s) &= \lambda \sum_{i=1}^{s} \frac{i}{s} f_X(i) f_S(s-i) \quad , \quad s = 1, 2, \ldots \end{aligned}$$

where f_X (resp. f_S) denotes the probability density function of X (resp. S) and λ is the parameter of the distribution of N. We observe that the Panjer's algorithm needs a discrete distribution. Therefore a continuous distribution may not be used as such and has to be discretized. Moreover it will be most convenient to obtain a discrete version of the continuous distribution which will be of lattice type, that is with non-negative masses on points of the type $x = kh$, $k = 0, 1, \ldots$ with $h > 0$. h is called the span. When the span is different from 1, a simple change of money (divide losses by h) allows to use the Panjer's algorithm optimally with respect to computing-time.
We immediately observe that the smaller the span, the better the precision of the discretization. However, the smaller the span, the longer the computing-time. The user should make a choice regarding the step in order to obtain a good precision and a sufficiently low computing-time. There are various methods for obtaining a lattice distribution from a general distribution. I choose to work with the easiest method : the rounding method (see Gerber and Jones (1976)). Let us choose a span h. The rounding method simply accumulates the original mass of a random variable X around the mass points of the lattice distribution (X_{dis}) as follows :

$$\begin{aligned} f_{X_{dis}}(0) &= F_X(\frac{h}{2} - 0), \\ f_{X_{dis}}(xh) &= F_X(xh + \frac{h}{2} - 0) - F_X(xh - \frac{h}{2} - 0) \quad , \quad x = 1, 2, \ldots \end{aligned}$$

For the particular case of a Pareto distribution ($X \sim Pa(A, \alpha)$) we obtain

$$
\begin{aligned}
f_{X_{dis}}(A) &= 1 - \left(\frac{A + h/2}{A}\right)^{-\alpha}, \\
f_{X_{dis}}(A + xh) &= \left(\frac{A - xh/2}{A}\right)^{-\alpha} - \left(\frac{A + xh/2}{A}\right)^{-\alpha}, \quad x = 1, 2, \ldots
\end{aligned}
$$

We choose to work with a lattice step $h = 20$.
The first masses points of the lattice distributions for our numerical example are

x	400	425	450	475	500	525	...
$\mathbb{P}[X = x]$	0.0451	0.0807	0.0699	0.0611	0.0537	0.0475	...
y	700	725	750	775	800	825	...
$\mathbb{P}[Y = y]$	0.0433	0.0790	0.0702	0.0626	0.0560	0.0503	...

Table 1: Lattice version of the original distributions

Using the Panjer's algorithm we are able to obtain the aggregate claims distributions of S and T :

x	0	25	50	75	100	125
$\mathbb{P}[S = x]$	0.0919	0.0185	0.0179	0.0174	0.0169	0.0164
$\mathbb{P}[T = x]$	0.0084	0.0033	0.0036	0.0039	0.0041	0.0044

Table 2: Aggregate claims distributions

Note that these distributions concern the ceding company whereas we are interested in the pricing of reinsurance covers. This will be discussed in the next section.

4. Detailed model

4.1. Attachment points and covers

Let us now define the liability of an excess of loss reinsurer i.r.o. the claims. Let us denote

- $P_{Fire} = 500$ as the deductible of the Fire claims,
- $P_{MTPL} = 1000$ as the deductible of the MTPL claims,
- $L_{Fire} = 2500$ as the cover of the Fire claims,
- $L_{MTPL} = 4000$ as the cover of the MTPL claims.

We obtain the reinsurer's liability for the individual claims as follows :

$$
\begin{aligned}
X_i^{Re} &= \min(L_{Fire}, \max(0, X_i - P_{Fire})), \\
Y_i^{Re} &= \min(L_{MTPL}, \max(0, X_i - P_{MTPL})).
\end{aligned}
$$

The aggregate liability of the reinsurer is :

$$
\begin{aligned}
S^{Re} &= X_1^{Re} + \cdots + X_N^{Re}, \\
T^{Re} &= T_1^{Re} + \cdots + T_M^{Re}.
\end{aligned}
$$

The distribution of the reinsurer's liability for the individual claims and for the aggregate claims is

x	0	25	50	75	100	125	
$\mathbb{P}[X = x]$	0.3105	0.0475	0.0423	0.0379	0.0340	0.0307	...
$\mathbb{P}[Y = x]$	0.6026	0.0235	0.0216	0.0199	0.0183	0.0170	...
$\mathbb{P}[S = x]$	0.1784	0.0212	0.0201	0.0192	0.0183	0.0175	...
$\mathbb{P}[T = x]$	0.1371	0.0161	0.0158	0.0154	0.0151	0.0148	...

Table 3: Reinsurer's claims and aggregate claims distributions

4.2. Long-tailed business and inflation

We now have to introduce the fact that, in an insurance context, claims are not paid outright. Especially in excess of loss reinsurance where large claims are involved, it may be very long before a claim is finally settled. Thus, we have to introduce this notion and a companion thereof : the future inflation. We will follow the presentation of Walhin et al. (2001).

We will assume that the payments of the claims occur at times $t_0, t_2, \ldots, t_n$ according to a given claims payment pattern : $c_{Fire}(t_0), \ldots, c_{Fire}(t_n)$ or $c_{MTPL}(t_0), \ldots, c_{MTPL}(t_n)$ where t_n is the time of final settlement. We will furthermore assume that the payments arise, on average, in the middle of the year, i.e. $t_j = j + 0.5$, $j = 0, 1, \ldots, n$.

The claims payment pattern is supposed to be estimated by using past data and adjusted for potential changes in the future payment patterns, e.g. due to changes in legislation or in the claims management.

Let us assume that the MTPL claims are completely settled in $n = 7$ years whereas the fire claims are completely settled in two years. We use the following payment patterns :

t	0	1	2	3	4	5	6	7
c_{Fire}	50%	40%	10%	0%	0%	0%	0%	0%
c^{Σ}_{Fire}	50%	90%	100%	100%	100%	100%	100%	100%
c_{MTPL}	5%	10%	10%	10%	25%	25%	10%	5%
c^{Σ}_{MTPL}	5%	15%	25%	35%	60%	85%	95%	100%

Table 4: Payment patterns

where c^{Σ} denotes the cumulative claims pattern payment.

Moreover the future payments will undergo future inflation. Indeed the losses X_i are assumed not to include any future inflation. Let us define an inflation index : $inf_{Fire}(t_0), \ldots, inf_{Fire}(t_n)$ and $inf_{MTPL}(t_0), \ldots, inf_{MTPL}(t_n)$. The future payments for a loss X_i or Y_i then read :

$$\begin{aligned} X_i(j + 0.5) &= c_{Fire}(t_j) X_i \frac{inf_{Fire}(t_j)}{inf_{Fire}(t_0)} \quad , \quad j = 0, 1, \ldots, n, \\ Y_i(j + 0.5) &= c_{MTPL}(t_j) Y_i \frac{inf_{MTPL}(t_j)}{inf_{MTPL}(t_0)} \quad , \quad j = 0, 1, \ldots, n. \end{aligned}$$

The future inflation will be modelled by a geometric growth and we furthermore assume the future inflation index to be constant between two times $t_j = j$, $j = 0, 1, \ldots, n$:

$$\frac{inf_{Fire}(j)}{inf_{Fire}(j-1)} - 1 = 3\% , \; j = 1, 2, \ldots, n,$$

$$
\begin{aligned}
inf_{Fire}(j+0.5) &= inf_{Fire}(j)\,,\; j=0,1,\ldots,n,\\
\frac{inf_{MTPL}(j)}{inf_{MTPL}(j-1)}-1 &= 3.5\%\,,\; j=1,2,\ldots,n,\\
inf_{MTPL}(j+0.5) &= inf_{MTPL}(j)\,,\; j=0,1,\ldots,n.
\end{aligned}
$$

Future claims payments then read

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
$X_i(t)$	400	200.00	164.80	42.44	0	0	0	0	0
	425	212.50	175.10	45.09	0	0	0	0	0
	450	225.00	185.40	47.74	0	0	0	0	0
	475	237.50	195.70	50.39	0	0	0	0	0
	Y_i								
$Y_i(t)$	700	35.00	72.45	74.99	77.61	200.82	207.85	86.05	44.53
	725	36.25	75.04	77.66	80.38	207.99	215.27	89.12	46.12
	750	37.50	77.63	80.34	83.15	215.16	222.69	92.19	47.71
	775	38.75	80.21	83.02	85.93	222.33	230.11	95.27	49.30

Table 5: Future claims payments (inflation only)

As we are interested in large losses, it is commonly observed on the market that this category of losses undergoes a higher inflation than usual. One speaks of the superimposed inflation. For the future payments, it is then more adequate to use another index, including inflation and superimposed inflation : $supinf_{Fire}(t_0),\ldots,supinf_{Fire}(t_n)$ or $supinf_{MTPL}(t_0),\ldots,supinf_{MTPL}(t_n)$. The future payments for a loss X_i or Y_i then read :

$$
\begin{aligned}
X_i(t_j) &= c_{Fire}(t_j)X_i\frac{supinf_{Fire}(t_j)}{supinf_{Fire}(t_0)} \quad, \quad j=0,1,\ldots,n,\\
Y_i(t_j) &= c_{MTPL}(t_j)Y_i\frac{supinf_{MTPL}(t_j)}{supinf_{MTPL}(t_0)} \quad, \quad j=0,1,\ldots,n.
\end{aligned}
$$

Let us assume that the future inflation and superimposed inflation is modelled by a geometric growth :

$$
\begin{aligned}
\frac{supinf_{Fire}(j)}{supinf_{Fire}(j-1)}-1 &= 3\%\,,\; j=1,2,\ldots,n,\\
supinf_{Fire}(j+0.5) &= supinf_{Fire}(j)\,,\; j=0,1,2,\ldots,n,\\
\frac{supinf_{MTPL}(j)}{supinf_{MTPL}(j-1)}-1 &= 5\%\,,\; j=1,2,\ldots,n,\\
supinf_{MTPL}(j+0.5) &= supinf_{MTPL}(j)\,,\; j=0,1,2,\ldots,n,
\end{aligned}
$$

that is we assume no superimposed inflation for the fire claims and 1.50% of superimposed inflation for the MTPL claims.

Future claims payments then read

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
$X_i(t)$	400	200.00	164.80	42.44	0	0	0	0	0
	425	212.50	175.10	45.09	0	0	0	0	0
	450	225.00	185.40	47.74	0	0	0	0	0
	475	237.50	195.70	50.39	0	0	0	0	0
	Y_i								
$Y_i(t)$	700	35.00	75.50	77.18	81.03	212.71	223.35	93.81	49.25
	725	36.25	76.13	79.93	83.93	220.31	231.33	97.16	51.01
	750	37.50	78.75	82.69	86.82	227.91	239.30	100.51	52.77
	775	38.75	81.38	85.44	89.72	235.50	247.28	103.86	54.53

Table 6: Future claims payments (including superimposed inflation)

It is also interesting to define the cumulative payments for a loss X_i or Y_i as :

$$X_i^{\Sigma}(j+0.5) = \sum_{k=0}^{j} X_i(k+0.5) \quad , \quad j = 0, 1, \ldots, n,$$

$$Y_i^{\Sigma}(j+0.5) = \sum_{k=0}^{j} Y_i(k+0.5) \quad , \quad j = 0, 1, \ldots, n.$$

The evolution of the cumulative payments for the reinsurer for a loss X_i or Y_i then reads :

$$X_i^{\Sigma Re}(j+0.5) = \min(L_{Fire}, \max(0, X_i^{\Sigma}(j+0.5) - P_{Fire})) \quad , \quad j = 0, 1, \ldots, n,$$

$$Y_i^{\Sigma Re}(j+0.5) = \min(L_{MTPL}, \max(0, Y_i^{\Sigma}(j+0.5) - P_{MTPL})) \quad , \quad j = 0, 1, \ldots, n.$$

Within our numerical example we have

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
	$\vdots$								
$X_i^{\Sigma}(t)$	500	250.00	456.00	509.05	509.05	509.05	509.05	509.05	509.05
	525	262.50	478.80	534.50	534.50	534.50	534.50	534.50	534.50
	550	275.00	501.60	559.95	559.95	559.95	559.95	559.95	559.95
	575	287.50	524.40	585.40	585.40	585.40	585.40	585.40	585.40
	$\vdots$								
	Y_i								
	$\vdots$								
$Y_i^{\Sigma}(t)$	3000	150.00	465.00	795.75	1143.04	2054.67	3011.88	3413.91	3624.97
	3025	151.25	468.88	802.38	1152.56	2071.79	3036.98	3442.36	3655.18
	3050	152.50	472.75	809.01	1162.09	2088.91	3062.08	3470.81	3685.39
	3075	153.75	476.63	815.64	1171.61	2106.03	3087.18	3499.25	3715.60
	$\vdots$								
	X_i								
$X_i^{\Sigma Re}(t)$	500	0	0	9.04	9.04	9.04	9.04	9.04	9.04
	525	0	0	34.50	34.50	34.50	34.50	34.50	34.50
	550	0	1.60	59.95	59.95	59.95	59.95	59.95	59.95
	575	0	24.40	85.40	85.40	85.40	85.40	85.40	85.40
	$\vdots$								
	Y_i								
	$\vdots$								
$Y_i^{\Sigma Re}(t)$	3000	0	0	0	143.04	1054.67	2011.88	2413.91	2624.97
	3025	0	0	0	152.56	1071.79	2036.98	2442.26	2655.18
	3050	0	0	0	162.09	1088.91	2062.08	2470.81	2685.39
	3075	0	0	0	171.61	1106.03	2087.18	2499.26	2715.60
	$\vdots$								

Table 7: Cumulative insurer's and reinsurer's payments

We show the evolution of the figures from 500 for Fire claims and from 3000 for MTPL claims in order to see figures different from 0 for the reinsurer's payments.

4.3. TECHNICAL RESERVES

In an ideal situation the claims manager is able to calculate exact reserves for a loss X_i or Y_i:

$$RX_i(j+0.5) = X_i^\Sigma(n+0.5) - X_i^\Sigma(j+0.5) \quad , \quad j=0,1,\ldots,n,$$
$$RY_i(j+0.5) = Y_i^\Sigma(n+0.5) - Y_i^\Sigma(j+0.5) \quad , \quad j=0,1,\ldots,n.$$

Within our numerical example, we have

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
	⋮								
$RX_i(t)$	500	259.05	53.05	0	0	0	0	0	0
	525	272.00	55.70	0	0	0	0	0	0
	550	284.95	58.35	0	0	0	0	0	0
	575	297.90	61.00	0	0	0	0	0	0
	⋮								
	Y_i								
	⋮								
$RY_i(t)$	3000	3474.97	3159.97	2829.22	2481.93	1570.30	613.09	211.07	0
	3025	3503.93	3186.31	2852.80	2502.62	1583.39	618.20	212.82	0
	3050	3532.89	3212.64	2876.38	2523.30	1596.48	623.31	214.58	0
	3075	3561.85	3238.97	2899.95	2543.98	1609.56	628.42	216.34	0
	⋮								

Table 8: Ideal reserves

However there may be systematic deviations from these exact reserves. Let us assume that we have observed a pattern of deviation of the incurred loss (overstatement or understatement) : $d_{Fire}(t_0),\ldots,d_{Fire}(t_n)$ or $d_{MTPL}(t_0),\ldots,d_{MTPL}(t_n)$ where $d(t_j)=100\%$ if there is no deviation of reservation at time t_j. The incurred loss and the outstanding, for a loss X_i or Y_i, may now be defined as follows :

$$IX_i(j+0.5) = d_{Fire}(j+0.5)X_i^\Sigma(n+0.5) \quad , \quad j=0,1,\ldots,n,$$
$$RX_i(j+0.5) = IX_i(j+0.5) - X_i^\Sigma(j+0.5) \quad , \quad j=0,1,\ldots,n,$$
$$IY_i(j+0.5) = d_{MTPL}(j+0.5)Y_i^\Sigma(n+0.5) \quad , \quad j=0,1,\ldots,n,$$
$$RY_i(j+0.5) = IY_i(j+0.5) - Y_i^\Sigma(j+0.5) \quad , \quad j=0,1,\ldots,n.$$

Let us assume that the overstatement pattern is given by

t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
d_{Fire}	100%	100%	100%	100%	100%	100%	100%	100%
d_{MTPL}	125%	125%	125%	125%	105%	105%	100%	100%

Table 9: Overstatement pattern

We then have the evolution of the outstanding and incurred losses :

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
	⋮								
$IX_i(t)$	500	509.05	509.05	509.05	509.05	509.05	509.05	509.05	509.05
	525	534.50	534.50	534.50	534.50	534.50	534.50	534.50	534.50
	550	559.95	559.95	559.95	559.95	559.95	559.95	559.95	559.95
	575	585.40	585.40	585.40	585.40	585.40	585.40	585.40	585.40
	⋮								
	⋮								
$RX_i(t)$	500	259.05	53.05	0	0	0	0	0	0
	525	272.00	55.70	0	0	0	0	0	0
	550	284.95	58.35	0	0	0	0	0	0
	575	297.90	61.00	0	0	0	0	0	0
	⋮								
	Y_i								
	⋮								
$IY_i(t)$	3000	4531.22	4531.22	4531.22	4531.22	3806.22	3806.22	3624.97	3624.97
	3025	4568.98	4568.98	4568.98	4568.98	3837.94	3837.94	3655.18	3655.18
	3050	4606.74	4606.74	4606.74	4606.74	3869.66	3869.66	3685.39	3685.39
	3075	4644.50	4644.50	4644.50	4644.50	3901.38	3901.38	3715.60	3715.60
	⋮								
	⋮								
$RY_i(t)$	3000	4381.22	4066.22	3735.47	3388.18	1751.55	794.34	211.07	0
	3025	4417.73	4100.10	3766.59	3416.41	1766.15	800.96	212.82	0
	3050	4454.24	4133.99	3797.72	3444.65	1780.75	807.58	214.58	0
	3075	4490.75	4167.87	3828.85	3472.88	1795.34	814.20	216.34	0
	⋮								

Table 10: Insurer's reserves and incurred losses with overstatement

From the evolution of the incurred losses, it is now possible to derive the evolution of the incurred losses for the excess of loss reinsurer :

$$\begin{aligned} IX_i^{Re}(j+0.5) &= \min(L_{Fire}, \max(0, IX_i(j+0.5) - P_{Fire})) \quad , \quad j = 0,1,\ldots,n. \\ IY_i^{Re}(j+0.5) &= \min(L_{MTPL}, \max(0, IY_i(j+0.5) - P_{MTPL})) \quad , \quad j = 0,1,\ldots,n. \end{aligned}$$

Within our numerical example we have

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
$IX_i^{Re}(t)$	0	9.04	9.04	9.04	9.04	9.04	9.04	9.04	9.04
	25	34.50	34.50	34.50	34.50	34.50	34.50	34.50	34.50
	50	59.95	59.95	59.95	59.95	59.95	59.95	59.95	59.95
	75	85.40	85.40	85.40	85.40	85.40	85.40	85.40	85.40
	$\vdots$								
	Y_i								
	$\vdots$								
$IY_i^{Re}(t)$	3000	3531.21	3531.25	3531.25	3531.25	2806.22	2806.22	2624.97	2624.97
	3025	3568.97	3568.97	3568.97	3568.97	2837.94	2837.94	2655.18	2655.18
	3050	3606.73	3606.73	3606.73	3606.73	2869.66	2869.66	2685.39	2685.39
	3075	3644.50	3644.50	3644.50	3644.50	2901.38	2901.38	2715.60	2715.60
	$\vdots$								

Table 11: Reinsurer's incurred losses

Our aim is to obtain the distribution of the paid claims and the distribution of the loss reserves at times $j + 0.5$, $j = 0, 1, \ldots, n$. This will allow us to obtain average values and so a cash flow model will be built in order to find the net present value of the business. This will allow us to determine if the business is worth the value or not. However before obtaining these distributions, we first have to consider some clauses that may affect the claims individually or in the aggregate.

It should be clear that the extension to multiple insurance lines is immediate. However, for educational purposes, we will limit ourselves to the methodology for two lines only.

4.4. Stability clause

If the attachment point (P) of the treaty is fixed, the reinsurer will take all future inflation during the development of the claim for his own account. Indeed once the loss is exceeds the attachment point, all future increases (except the part of the loss exceeding the cover of the treaty) due to inflation are borne by the reinsurer only. In order to protect themselves against this kind of possible moral hasard, reinsurers have introduced the stability clause. With this clause the reinsurer is willing to optimally share the future inflation between the ceding company and himself. There are several variants of the stability clause (see e.g. Gerathewohl (1980) for details). In this paper, and in particular in our numerical application, we will work with the so-called "date of payment" stability clause. When this clause is applied, the attachment point and/or the cover of the treaty are indexed each year with the following ratio

$$ratio = \frac{\text{sum of actual payments}}{\text{sum of adjusted payments}},$$

where adjusted payments means that each payment is discounted to the inception of the treaty with use of a conventional index, let us say the inflation index. The interested reader

is referred to Walhin et al. (2001) for further details.
We thus arrive at future attachment points and covers :
$P_{Fire}(t_0), P_{Fire}(t_1), \ldots, P_{Fire}(t_n), P_{MTPL}(t_1), P_{MTPL}(t_2), \ldots, P_{MTPL}(t_n)$,
$L_{Fire}(t_0), L_{Fire}(t_1), \ldots, L_{Fire}(t_n)$ and $L_{MTPL}(t_1), L_{MTPL}(t_2), \ldots, L_{MTPL}(t_n)$ instead of singles P_{Fire}, P_{MTPL}, L_{Fire} and L_{MTPL}.
In accordance with the hypotheses on inflation, we will assume that $P_{Fire}(j+0.5) = P_{Fire}(j)$, $P_{MTPL}(j+0.5) = P_{MTPL}(j)$, $L_{Fire}(j+0.5) = L_{Fire}(j)$ and $L_{MTPL}(j+0.5) = L_{MTPL}(j)$, $j = 0, 1, \ldots, n$.
The evolution of the cumulative paid loss and incurred loss, for a loss X_i or Y_i, for the reinsurer now reads :

$$\begin{aligned}
X_i^{\Sigma Re}(j+0.5) &= \min(L_{Fire}(j+0.5), \max(0, X_i^{\Sigma}(j+0.5) - P_{Fire}(j+0.5))) \quad , \quad j = 0, 1, \ldots, n, \\
Y_i^{\Sigma Re}(j+0.5) &= \min(L_{MTPL}(j+0.5), \max(0, Y_i^{\Sigma}(j+0.5) - P_{MTPL}(j+0.5))) \quad , \quad j = 0, 1, \ldots, n, \\
IX_i^{Re}(j+0.5) &= \min(L_{Fire}(j+0.5), \max(0, IX_i(j+0.5) - P_{Fire}(j+0.5))) \quad , \quad j = 0, 1, \ldots, n, \\
IY_i^{Re}(j+0.5) &= \min(L_{MTPL}(j+0.5), \max(0, IY_i(j+0.5) - P_{MTPL}(j+0.5))) \quad , \quad j = 0, 1, \ldots, n.
\end{aligned}$$

When the claim is finally settled, both situations lead to the same repartition of the loss between the insurer and the reinsurer. The only difference is in the evolution of the cash flows.
Let us assume that the date of payment stability clause is applied to the attachment point and to the limit of the MTPL claims with a margin of 10%, i.e. the payments will be adjusted only if the claims index shows an evolution larger than the margin (see Walhin et al. (2001) for formulae details or Gerathewohl (1980) for further general details on the subject). The selected index is the claims index. It is also assumed that the application of the stability clause is based on incurred losses, that is, outstanding losses are used, and discounted as if they were payments. The attachment point and limit for the Fire claims are fixed, which is not illogical since Fire is not long-tail business. The evolution of the attachment point and limit for the MTPL claims is the following :

t_j	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
P_{MTPL}	1000	1000	1000	1086.58	1108.74	1124.33	1129.94	1131.99
L_{MTPL}	4000	4000	4000	4346.30	4434.96	4497.32	4519.75	4527.95

Table 12: Evolution of the MTPL layer with stability clause

The payments and incurred losses of the reinsurer now read

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i								
	⋮								
$Y_i^{\Sigma Re}(t)$	3000	0	0	0	56.46	945.93	1887.55	2283.97	2492.98
	3025	0	0	0	65.99	963.05	1912.65	2312.42	2523.19
	3050	0	0	0	75.51	980.17	1937.75	2340.87	2553.40
	3075	0	0	0	85.03	997.29	1962.84	2369.32	2583.61
	⋮								
	X_i								
	⋮								
$IY_i^{Re}(t)$	3000	3531.21	3531.25	3531.25	3444.64	2697.48	2681.89	2495.03	2492.95
	3025	3568.97	3568.97	3568.97	3482.40	2729.20	2713.61	2525.24	2523.19
	3050	3606.73	3606.73	3606.73	3520.16	2760.92	2745.33	2555.45	2553.40
	3075	3644.50	3644.50	3644.50	3557.92	2792.64	2777.05	2585.66	2583.61
	⋮								

Table 13: Reinsurer's payments and incurred losses with stability clause (MTPL only)

4.5. Interests sharing clause / Loss adjustment expenses clause

When the claims development is long, it is expected that legal interests will have to be paid. The longer the claims development is, the higher the legal interests are. Once again for moral hazard reasons it may be tempting from the reinsurer's point of view to share the legal interests proportionnally between the cedent and the reinsurer. This is the aim of the interests sharing clause which is common practice, e.g. in Belgium.
The *interests sharing clause states that the legal interests have to be shared between the* ceding company and the reinsurer according to the pro rata liability of the reinsurer in the total liability of the loss excluding the legal interests. This means that the legal interests have to be excluded from the incurred loss before the application of the treaty. Afterwards they are divided between the ceding company and the reinsurer in accordance with the pro rata liability of both parties in the loss. Let us assume that on average a proportion δ_{Fire} or δ_{MTPL} of the incurred loss represents the interests. Note that it is reasonable to assume that this proportion is a function of the loss. However, in practice, it is extremely difficult to estimate the average proportion of the legal interests in such a way that it does not seem necessary to assume a varying proportion. Nevertheless it is possible to work within an extended model. The interested reader is referred to Walhin et al. (2001) for further details.
A common practice on North American markets is that loss adjustment expenses undergo the same treatment as the legal interests in Belgium, i.e. they are also shared on a pro rata basis between the insurer and the reinsurer. These expenses may thus be treated exactly as are the legal interests, within the loss adjustment expenses clause.
We will assume an interests sharing clause only for the MTPL claims and we assume that the portion of interests in the losses is $\delta = 15\%$.

The payments and incurred losses of the reinsurer now read

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	X_i	0	1	2	3	4	5	6	7
	⋮								
$Y_i^{\Sigma Re}(t)$	3000	0	0	0	0	750.27	1689.14	2084.57	2293.22
	3025	0	0	0	0	767.39	1714.24	2113.02	2323.43
	3050	0	0	0	0	784.51	1739.33	2141.47	2353.64
	3075	0	0	0	0	801.63	1764.43	2169.92	2383.85
	⋮								
	X_i								
	⋮								
$IY_i^{Re}(t)$	3000	3354.74	3354.74	3354.74	3252.89	2501.82	2483.48	2295.63	2293.22
	3025	3392.50	3392.50	3392.50	3290.65	2533.54	2515.20	2325.84	2323.43
	3050	3430.26	3430.26	3430.26	3328.41	2565.26	2546.92	2356.05	2353.64
	3075	3468.02	3468.02	3468.02	3366.17	2596.98	2578.63	2386.26	2383.85
	⋮								

Table 14: Reinsurer's payments and incurred losses with interests sharing clause (MTPL only)

4.6. Lattice distributions

Most probably the random variables derived above are not of lattice type. So it is necessary to make a rearithmetization of them. This is done again with the rounding method.
With the lattice version of the payments and incurred losses, we will be able to apply Panjer's algorithm in order to obtain the aggregate claims / incurred losses for each development year.

As an example, here are some rearithmetized distributions :

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	x								
$\mathbb{P}[X_i^{\Sigma Re}(t) = x]$	0	0.752	0.400	0.310	0.310	0.310	0.310	0.310	0.310
	25	0.017	0.038	0.048	0.048	0.048	0.048	0.048	0.048
	50	0.015	0.034	0.042	0.042	0.042	0.042	0.042	0.042
	75	0.014	0.031	0.038	0.038	0.038	0.038	0.038	0.038
	$\vdots$								
$\mathbb{P}[IX_i^{Re}(t) = x]$	0	0.310	0.310	0.310	0.310	0.310	0.310	0.310	0.310
	25	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046
	50	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042
	75	0.038	0.038	0.038	0.038	0.038	0.038	0.038	0.038
	$\vdots$								
$\mathbb{P}[Y_i^{\Sigma Re}(t) = x]$	0	0.999	0.997	0.990	0.981	0.919	0.802	0.733	0.686
	25	0.000	0.000	0.001	0.001	0.005	0.009	0.014	0.017
	50	0.000	0.000	0.000	0.001	0.002	0.008	0.000	0.016
	75	0.000	0.000	0.000	0.001	0.004	0.008	0.013	0.000
	$\vdots$								
$\mathbb{P}[IY_i^{Re}(t) = x]$	0	0.255	0.255	0.255	0.407	0.626	0.648	0.686	0.686
	25	0.056	0.056	0.056	0.000	0.022	0.000	0.017	0.017
	50	0.000	0.000	0.000	0.040	0.020	0.020	0.016	0.016
	75	0.050	0.050	0.050	0.037	0.000	0.018	0.000	0.000
	$\vdots$								

Table 15: Rearithmetized reinsurer's payments and incurred losses distributions

4.7. Clauses limiting the liability of the reinsurer

There are two clauses which may limit the liability of the reinsurer in an excess of loss treaty. The annual aggregate limit (Aal_{Fire} or Aal_{MTPL}) on the one hand is the maximal aggregate loss the reinsurer will pay. The annual aggregate deductible (Aad_{Fire} or Aad_{MTPL}) on the other hand is a deductible on the aggregate loss of the reinsurer. Both annual clauses may coexist. In such a case the aggregate loss of the reinsurer reads :

$$\begin{aligned}
S_{X^{\Sigma Re}}(j+0.5) &= \min(Aal_{Fire}, \max(0, \sum_{i=1}^{N} X_i^{\Sigma Re}(t+0.5) - Aad_{Fire})) \quad , \quad j = 0,1,\ldots,n, \\
S_{Y^{\Sigma Re}}(t+0.5) &= \min(Aal_{MTPL}, \max(0, \sum_{i=1}^{M} Y_i^{\Sigma Re}(t+0.5) - Aad_{MTPL})) \quad , \quad j = 0,1,\ldots,n, \\
S_{IX^{Re}}(t+0.5) &= \min(Aal_{Fire}, \max(0, \sum_{i=1}^{N} IX_i^{Re}(t+0.5) - Aad_{Fire})) \quad , \quad j = 0,1,\ldots,n, \\
S_{IY^{Re}}(t+0.5) &= \min(Aal_{MTPL}, \max(0, \sum_{i=1}^{M} IY_i^{Re}(t+0.5) - Aad_{MTPL})) \quad , \quad j = 0,1,\ldots,n.
\end{aligned}$$

Let us assume that there is no annual aggregate deductible and no annual aggregate limit for the separate treaties :

$$
\begin{aligned}
Aad_{Fire} &= 0, \\
Aad_{MTPL} &= 0, \\
Aal_{Fire} &\rightarrow \infty, \\
Aal_{MTPL} &\rightarrow \infty.
\end{aligned}
$$

We have the following distributions

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	x								
$\mathbb{P}[S_{X^{Re}}(t) = x]$	0	0.538	0.223	0.178	0.178	0.178	0.178	0.178	0.178
	25	0.023	0.021	0.021	0.021	0.021	0.021	0.021	0.021
	50	0.021	0.020	0.020	0.020	0.020	0.020	0.020	0.020
	75	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.019
$\mathbb{P}[S_{IX^{Re}}(t) = x]$	0	0.178	0.178	0.178	0.178	0.178	0.178	0.178	0.178
	25	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021
	50	0.020	0.020	0.020	0.020	0.020	0.020	0.020	0.020
	75	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.019
$\mathbb{P}[S_{Y^{Re}}(t) = x]$	0	0.999	0.987	0.953	0.908	0.667	0.371	0.263	0.208
	25	0.000	0.001	0.002	0.005	0.017	0.017	0.018	0.018
	50	0.000	0.001	0.002	0.003	0.008	0.016	0.001	0.017
	75	0.000	0.001	0.002	0.004	0.015	0.015	0.017	0.001
$\mathbb{P}[S_{IY^{Re}}(t) = x]$	0	0.024	0.024	0.024	0.051	0.154	0.172	0.208	0.208
	25	0.007	0.007	0.007	0.000	0.017	0.000	0.018	0.018
	50	0.001	0.001	0.001	0.011	0.016	0.017	0.017	0.017
	75	0.006	0.006	0.006	0.010	0.002	0.016	0.001	0.001

Table 16: Reinsurer's aggregate payments and incurred losses

5. Global distributions and global expected values

As we are interested in a global treaty combining Fire and MTPL claims, we have to obtain the global distributions of :

$$
\begin{aligned}
S_{(X+Y)^{\Sigma Re}}(j+0.5) &= \min(Aal, \max(0, S_{X^{\Sigma Re}}(j+0.5) + S_{Y^{\Sigma Re}}(j+0.5) - Aad)) \quad , \quad j = 0, 1, \ldots, n, \\
S_{(IX+IY)^{Re}}(j+0.5) &= \min(Aal, \max(0, S_{IX^{Re}}(j+0.5) + S_{IY^{Re}}(j+0.5) - Aad)) \quad , \quad j = 0, 1, \ldots, n,
\end{aligned}
$$

where Aal is a multiline annual aggregate limit and Aad is a multiline annual aggregate deductible.
We will assume that there is an annual aggregate deductible on the global treaty (multiline aggregate deductible) :

$$
\begin{aligned}
Aad &= 1000, \\
Aal &\rightarrow \infty.
\end{aligned}
$$

Note that Ribeaud (2000) used the terminology "Multiline aggregate deductible" / "Multiline aggregate limit".
These distributions are easily obtained by convolutions because for our model we assumed mutual independencies .
Note that in case of dependencies between the claim amounts or between the claim frequencies, algorithms exist, giving the joint distributions of $(S_{X^{\Sigma R_e}}, S_{Y^{\Sigma R_e}})$ or $(S_{IX^{R_e}}, S_{IY^{R_e}})$. See e.g. Walhin and Paris (2000a) for the first case of dependency and Walhin and Paris (2000b) for the second case of dependency. Having the joint distributions, it then becomes immediate to obtain the distributions of $S_{X^{\Sigma R_e}} + S_{Y^{\Sigma R_e}}$ or $S_{IX^{R_e}} + S_{IY^{R_e}}$.
Within our numerical example we obtain

	t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
	x								
$\mathbb{P}[S_{X^{\Sigma R_e}}(t) + S_{Y^{\Sigma R_e}}(t) = x]$	0	0.537	0.221	0.170	0.162	0.119	0.066	0.047	0.037
	25	0.023	0.021	0.021	0.020	0.017	0.011	0.009	0.008
	50	0.021	0.020	0.019	0.019	0.015	0.011	0.006	0.008
	75	0.019	0.019	0.019	0.018	0.016	0.011	0.008	0.005
	100	0.018	0.018	0.018	0.018	0.014	0.010	0.008	0.007
	125	0.016	0.017	0.017	0.017	0.015	0.010	0.008	0.007
	⋮								
$\mathbb{P}[S_{IX^{\Sigma R_e}}(t) + S_{IY^{\Sigma R_e}}(t) = x]$	0	0.004	0.004	0.004	0.009	0.028	0.031	0.037	0.037
	25	0.002	0.002	0.002	0.001	0.006	0.004	0.008	0.008
	50	0.001	0.001	0.001	0.003	0.006	0.007	0.008	0.008
	75	0.002	0.002	0.002	0.003	0.004	0.006	0.005	0.005
	100	0.002	0.002	0.002	0.002	0.006	0.007	0.007	0.007
	125	0.001	0.001	0.001	0.003	0.006	0.007	0.007	0.007
	⋮								

Table 17: Global payment and incurred losses distributions

As we will use a cash flow model that is introduced in section 6 (investment decision process) we are interested in obtaining the expected values of the future payments and outstanding.
The incremental payments are

$$\begin{aligned} Paid(0.5) &= S_{(X+Y)^{\Sigma R_e}}(0.5), \\ Paid(j+0.5) &= S_{(X+Y)^{\Sigma R_e}}(j+0.5) - S_{(X+Y)^{\Sigma R_e}}(t_{j-0.5}) \quad , \quad j = 1, 2, \ldots, n, \end{aligned}$$

and the loss reserves are

$$Reserve(j+0.5) = S_{(IX+IY)^{R_e}}(j+0.5) - S_{(X+Y)^{\Sigma R_e}}(j+0.5) \quad , \quad j = 0, 1, \ldots, n.$$

This is the situation where the reinsurer follows the information given by the cedent. Another situation might be that the reinsurer books the ultimate loss in such a way that he avoids overstatement and / or understatement of the ceding company's reserves. In this case the loss reserves read :

$$Reserve(j+0.5) = S_{(X+Y)^{\Sigma R_e}}(n+0.5) - S_{(X+Y)^{\Sigma R_e}}(j+0.5) \quad , \quad j = 0, 1, \ldots, n.$$

We are now able to obtain the average aggregate payments and average aggregate reserves for the reinsurer :

- paid losses :

$$PL(j+0.5) = -\mathbb{E}Paid(j+0.5) \quad , \quad j = 0, 1, \dots, n.$$

- reserve :

$$RES(j+0.5) = \mathbb{E}Reserve(j+0.5) \quad , \quad j = 0, 1, \dots, n.$$

Let us assume that the share of the reinsurer in the treaty is 20%. It is indeed common practice that several reinsurers take a share in a given treaty. Unless the ceded risk is really small, a cedent would not accept to work with only one reinsurer for solvency reasons.
The following table gives the expected aggregate payments and loss reserves of the reinsurer (for a share of 20%). We assume that the reinsurer follows the reserves of the cedent. Furthermore we will assume that all cash flows related to losses happen in the middle of the year.

t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
$-PL(t)$	27.19	59.78	21.70	5.78	35.10	76.38	49.29	30.04
$RES(t)$	533.50	473.72	452.01	387.61	192.03	110.69	30.41	0

Table 18: Expected aggregate payments and loss reserves of the reinsurer

Let us assume that the estimated premium income is 50000. This information is important as the reinsurance premium is usually expressed as a percentage of the cedent's premium income. One traditionally speaks of a rate.
By adding up the payments we immediately arrive at the technical rate (TR) :

$$TR = \frac{305.25}{20\% \times 50000} = 3.05\%.$$

This rate is not satisfactory because it does not take into account the investment income the reinsurer can obtain on loss reserves. On the other hand neither does it take into account the cost of reserving (in particular when there is overstatement). Finally, it does not take into account the fact that the total payment is a sum of different cash flows. This is the reason why we introduce the following cash flow model.

6. THE CASH FLOW MODEL

This section is adapted from Walhin et al. (2001).
When a reinsurer wants to write business he has to provide a solvency margin, or some allocated capital : C. Let us assume that the return after tax which the shareholders demand from this capital is coc. We call coc the cost of capital. It can be derived e.g. via the CAPM (Capital Asset Pricing Model, see e.g. Brealey and Myers (2000)) where $coc = r_F + \beta P_R$. r_F is the risk-free rate and P_R is the risk premium of the market. β measures the systematic risk, i.e. market sensitivity, associated to the investment.
In the present paper we assume the same cost of capital whatever the type of business is. This is clearly a simplifying hypothesis. One may be tempted to work within a more general model where each line of business has its own cost of capital. For example it is clear that cat business hardly correlated with the market, implying that the cost of capital for cat business should be about the risk free rate. Independently the required capital for writing cat business is large due to the high volatility of this kind of business and the risk of large deviations.
In our case we have two types of business to analyse : Motor Third Party Liability and Fire. Even if we had two different costs of capital, it is really not clear how we could use them. As we mix both types of business, we have to use one cost of capital, possibly some (weighted) average of the above-mentioned costs of capital. The present multiline cover shows a limitation of working with different costs of capital. There is clearly room for further research at this point.
Traditionally we say that the business is worth the value if the net present value of all future cash flows, including capital allocation and release, is positive. A nil value implies that the requirements of the shareholders are just fulfilled. A positive value implies some creation of value for the shareholders. In the latter case we have the following inequality :

$$0 < \sum_{j=0}^{n} \frac{CF(t_j)}{(1+coc)^{t_j}}.$$

We will use the cash flow model in this way and say that a treaty is acceptable if the net present value of all future cash flows, including the variations in allocated capital, is positive. Let us note that if the firm is not financed exclusively through equity capital but also through some debt or hybrid capital, coc becomes a weighted average cost of capital (see e.g. Brealey and Myers (2000) for details). This however is obviously not very important for insurers and reinsurers who are essentially financed through equity capital. We will assume the cost of capital to be $coc = 11\%$.

We have three types of cash flows related to losses :

- paid losses

$$PL(j+0.5) = -\mathbb{E}Paid(j+0.5) \quad , \quad j = 0, 1, \ldots, n.$$

- variation of the loss reserve : $VR(j+0.5)$, $j = 0, 1, \ldots n$:

$$\begin{aligned} RES(j+0.5) &= \mathbb{E}Reserve(j+0.5) \quad , \quad j = 0, 1, \ldots, n, \\ VR(0.5) &= -RES(0.5), \\ VR(j+0.5) &= RES(j+1.5) - RES(j+0.5) \quad , \quad j = 1, 2, \ldots, n. \end{aligned}$$

- investment income on reserve : $IR(j+0.5)$, $j = 0, 1, \ldots, n$:

$$\begin{aligned} IR(0.5) &= 0, \\ IR(j+0.5) &= rRES(j-0.5) \quad , \quad j = 1, \ldots, n. \end{aligned}$$

We logically assume that investment income on the reserves are paid with a one year delay. We will assume that the interest rate obtained on the loss reserve is $r = 5\%$. We observe a limitation of our model. It is not possible to account for two different interest rates on the loss reserves (note that it would be possible if there were no clauses on the global distribution, which seldom is the case).
We can now define the aggregate cash flow at the middle of the year :

$$CF(j+0.5) = PL(j+0.5) + VR(j+0.5) + IR(j+0.5) \quad , \quad j = 0, 1, \ldots, n.$$

We will assume that all the other cash flows occur at the beginning of the year : $t_j = j$, $j = 0, 1, \ldots, n+1$. These cash flows are :

- commercial premium ($CP(j)$).
  The premium may be thought to be incepted at time 0. This is not always the case. Often there is a minimum deposit premium at time 0. The balance is paid at time 1. We do not take into account (but it is not difficult to do so) the fact that the minimum deposit premium is often paid in different instalments (one quarter every three months or one half every six months). Moreover we will see in section 7 that premium adjustments may be necessary. Thus premium cash flows at times other than 0 and 1 are not excluded. We will assume that there is a minimum and deposit premium of 80% of the expected commercial reinsurance premium. By deposit we mean that 80% of the premium is paid at time $t = 0$ whereas the balance is paid at time $t = 1$. By minimum we mean that at least the reinsurance rate times 80% of the premium income (estimated by the cedent) will be paid. In case the actual premium income is lower than 80% of the estimated premium income, the minimum and deposit premium is due. We assume that the estimated premium income will be the actual one.
- brokerage ($B(j)$).
  Brokerage, if any, is traditionally a percentage of the commercial premium. It will thus be deducted at times premiums are paid. We will assume that brokerage is 10%.
- retrocession ($R(j)$).
  Cost of retrocession, if any, is not the premium paid to the retrocessionnaire but rather the expected value of this premium minus the aggregate loss paid by the retrocessionnaire. A possible modelization is a percentage of the commercial premium minus a fraction of the paid losses. The first percentage is the traditional rate demanded by the retrocessionnaire on commercial premiums. The latter fraction represents the share of the average claims the retrocessionnaire is expected to pay. We will assume that retrocession costs (premiums) are 3% of the commercial premium. We assume that on average 2% of the losses are paid by the retrocession (this is assumed to be estimated with the developped model). In other words we cede 2% of the losses to the retrocession and the premium we are asked for that risk is 3% of the commercial premium.

- administrative expenses ($AE(j)$).
  Administrative expenses may be of two types : fixed expenses and proportional expenses. The fixed expenses represent the fixed costs of the reinsurer (including the fixed costs of the priced treaty) whereas the proportional costs represent the costs directly associated with the management of the treaty. We assume that these proportional expenses are based on the paid losses (note that this is just an assumption that can be easily modified). It is not illogical to admit that the expenses will be paid during the course of the treaty (think of the accounting and claims management of the treaty). So there may be a cash flow of expenses for all times j. We will assume that administrative expenses are 5 for the fixed part and 4% of the paid losses each year (the proportional administrative expenses are assumed to be paid at the end of the year).
- variation in the allocated capital ($VC(j)$).
  As announced in the previous section, some capital has to be allocated in order to run the business. However, at last at the end of the development, this allocated capital is released to the shareholders. In practice, the allocation rule may be such that the allocated capital is given back after x years or in function of the evolution of the loss reserves. So there will be variations in the allocated capital, exactly as there are in the loss reserves. Within our numerical example the allocated capital, $C(j)$, $j = 0, 1, \ldots, n+1$ is assumed to be 1.25 times the standard deviation of the ultimate aggregate claims, i.e. $\sqrt{Var(1-\gamma)S_{(X+Y)^{ERe}}(n+0.5)}$ where γ is the fraction of the claims paid by the retrocessionnaire. We assume e.g. that the capital allocation is based on the standard deviation premium principle (see Walhin et al. (2001) for further details). We make the hypothesis that capital has to be allocated during three years. See Walhin et al. (2001) for further details on capital allocation.
- investment income on the allocated capital ($IC(j)$).
  As allocated capital is mobilized, an auto-remuneration of this capital is possible. Indeed the mobilized capital will be invested and will produce an investment income. Moreover one might think that this auto-remuneration is higher than the remuneration on the loss reserves because the latter are probably invested in risk-free assets. So, while capital is allocated there is a cash flow of investment income on it at a return rate $l = 7\%$.

We are then able to define the cash flows at integer times :

$$CF(j) = CP(j) + B(j) + R(j) + AE(j) + VC(j) + IC(j) \quad , \quad j = 0, 1, \ldots, n+1.$$

The problem of taxes remains to be treated. In order to find the tax we first have to define the taxable profit at times j and $j + 0.5$:

$$\begin{aligned} TaxProfit(j) &= CP(j) + B(j) + R(j) + AE(j) + IC(j) \quad , \quad j = 0, 1, \ldots, n+1, \\ TaxProfit(j+0.5) &= PL(j-0.5) + VR(j-0.5) + IR(j-0.5) \quad , \quad j = 0, 1, \ldots, n. \end{aligned}$$

The tax cash flows are then

$$\begin{aligned} Tax(j) &= \tau TaxProfit(j) \quad , \quad j = 0, 1, \ldots, n+1, \\ Tax(j+0.5) &= \tau TaxProfit(j+0.5) \quad , \quad j = 0, 1, \ldots, n. \end{aligned}$$

where $\tau = 30\%$ is an average tax rate. It assumes all cash flows, including financial return, to be taxed at the same rate. This is obviously not always true and specific corrections are easy to include in the model according to the tax regime of the reinsurer's domicile.
The treaty will be acceptable if

$$\sum_{j=0}^{n+1} \frac{CF(j) - Tax(j)}{(1+coc)^j} + \sum_{j=0}^{n} \frac{CF(j+0.5) - Tax(j+0.5)}{(1+coc)^{j+0.5}} > 0.$$

The following table gives the cash flow model with the technico-financial premium. This table takes into account a reinsurer's share of 20%.

t	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
TFP	**294.69**	0	0	0	0	0	0	0	0
PL	0	−27.19	−59.78	−21.70	−5.78	−35.10	−76.38	−49.29	−30.04
VR	0	−533.50	59.78	21.70	64.40	195.59	81.33	80.28	30.41
IR	0	0	26.67	23.69	22.60	19.38	9.60	5.53	1.52
$CF(j)$	294.69								
$CF(j+0.5)$	0	−560.69	26.67	23.69	81.22	179.87	14.56	36.53	1.89
$\frac{CF(j)}{(1+coc)^j}$	294.69	0	0	0	0	0	0	0	0
$\frac{CF(j+0.5)}{(1+coc)^{j+0.5}}$	0	−532.18	22.81	18.25	56.37	112.46	8.20	18.54	0.86
NPV	**0**								

Table 19: Cash flow model for the technico-financial premium

The technico-financial premium (TFP) is 294.69.
The technico-financial rate is thus given by

$$TFR = \frac{294.69}{50000 \times 20\%} = 2.95\%.$$

It may seem surprising that the technico-financial premium is so close to the technical premium. This is due to the fact that there is a lot of overstatement by the ceding company and that overstatement is followed by the reinsurer. We will make some sensitivity analysis on this aspect.

We now obtain the commercial premium :

j	0	1	2	3	4	5	6	7	8
$j+0.5$	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
CP	384.22	96.05	0	0	0	0	0	0	0
AE	−5	−1.09	−2.39	−0.87	−0.23	−1.40	−3.06	−1.97	−1.20
B	−38.42	−9.61	0	0	0	0	0	0	0
R	−11.53	−2.34	1.20	0.43	0.12	0.70	1.53	0.99	0.60
PL	0	−27.19	−59.78	−21.70	−5.78	−35.10	−76.38	−49.29	−30.04
VR	0	−533.50	59.78	21.70	64.40	195.59	81.33	80.28	30.41
IR	0	0	26.67	23.69	22.60	19.38	9.60	5.53	1.52
VC	−497.94	0	0	497.94	0	0	0	0	0
IC	0	34.86	34.86	34.86	0	0	0	0	0
$CF(j)$	−168.67	117.88	33.66	532.36	−0.12	−0.70	−1.53	−0.99	−0.60
$CF(j+0.5)$	0	−560.69	26.67	23.68	81.22	179.87	14.56	36.53	1.89
$TaxPr(j)$	329.27	117.88	33.66	34.42	−0.12	−0.70	−1.53	−0.99	−0.60
$TaxPr(j+0.5)$	0	−560.69	26.67	23.69	81.22	179.87	14.56	36.53	1.89
$Tax(j)$	98.78	35.36	10.10	10.33	−0.03	−0.21	−0.46	−0.30	−0.18
$Tax(j+0.5)$	0	−168.21	8.00	7.11	24.36	53.96	4.37	10.96	0.57
$\frac{CF(j)-Tax(j)}{(1+coc)^j}$	−267.45	74.34	19.12	381.71	−0.05	−0.29	−0.57	−0.33	−0.18
$\frac{CF(j+0.5)-Tax(j+0.5)}{(1+coc)^{j+0.5}}$	0	−372.53	15.97	12.77	39.46	78.72	5.74	12.98	0.61
NPV	0								

Table 20: Cash flow model for the commercial premium

The total commercial premium is then

$$384.22 + 96.05 = 480.27,$$

which produces a rate of

$$\frac{480.27}{50000 \times 20\%} = 4.80\%.$$

Summarizing we have the following rates

TR	3.05%
TFR	2.95%
CR	4.80%

Table 21: Rates

It is now easy to provide some sensitivity analyses. Let us compare the rates for different multiline aggregate deductibles (MAD). We will also give the rate in the case where there is no overstatement for the MTPL claims :

	with overstatement			without overstatement		
MAD	TR	TFR	CR	TR	TFR	CR
1000	3.05%	2.95%	4.80%	3.05%	2.56%	4.35%
2000	1.90%	1.89%	3.55%	1.90%	1.58%	3.18%
3000	1.13%	1.13%	2.69%	1.13%	0.92%	2.40%

Table 22: Sensitivity analysis 1

We observe the effect of the overstatement on the technico-financial rate. The effect of the multiline aggregate deductible is equally important. Note that it would be difficult to obtain these rates without the comprehensive model we use.

Let us now assume that there is an annual aggregate deductible for the MTPL and Fire claims of $Aad_{Fire} = Aad_{MTPL} = 500$. To compensate, the multiline aggregate deductible becomes $Aad = 500$. We obtain :

TR	2.65%
TFR	2.60%
CR	4.29%

Table 23: Sensitivity analysis 2

7. Special clauses

It is often observed in excess of loss treaties that the reinsurance premium is a function of the excess of loss amounts. In these situations, governed by typical clauses, the reinsurance premium is a random variable :

$$P_{Re} = P^{Init} + P^{Rand}.$$

where P^{init} denotes the initial premium, which is not random whereas P^{Rand} denotes the random part of the premium.
The clauses are

- Paid reinstatements
- Sliding scale premium
- Profit commission.

The practical pricing proceeds in two steps. The first one is easy : we merely calculate the commercial premium necessary to cover the treaty if there is no "random" clause. We then obtain the evolution of paid losses, loss reserves, investment income on loss reserves, allocated capital, investment income on allocated capital and administrative expenses. There is no reason to believe that these elements will be different in the cash flow model with "random clause". We now move to the second step, i.e. the cash flow model with the "random" clause. The previous elements are fixed. Other elements may vary : premiums, brokerage, retrocession, and taxes. The process will be iterative. As a first guess we choose an initial premium (or one limit of the scale in the case of a sliding scale). According to the evolution of the incurred losses, this premium will be split in several premiums in the future, i.e.

- $CP(0) = P^{Init}$ (or, more exactly, the minimum and deposit premium, the balance of it which will be paid in $t = 1$) for a treaty with paid reinstatements. $CP(j)$ = future adjustments for reinstatements due to incurred losses hitting the layer for $j = 1, 2, \ldots, n+1$.
- $CP(0) = P^{Init} = P_{min}$ (or, more exactly, the minimum deposit premium, the balance of which will be paid in $t = 1$) for a treaty with sliding scale. $CP(j)$ = future adjustments for $j = m, m+1, \ldots, n+1$ where m is the first year for which a premium adjustment is contractually agreed.
- $CP(0) = P^{Init}$ (or, more exactly, the minimum deposit premium, the balance of which will be in $t = 1$) for a treaty with profit commission. $CP(j)$ = future adjustments for profit commission for $j = m, m+1, \ldots, n+1$ where m is the first year for which a premium adjustment is contractually agreed.

With this pattern of premium payments, we immediately obtain the pattern of brokerage, retrocession and as a result the pattern of tax. We are then able to calculate the net present value of the business. If it is positive we try a new premium lower than the previous one. If it is negative we try a new premium higher than the previous one. The trial and error scheme is continued until the net present value of the business is 0.
The interested reader will find more details in Walhin et al. (2001).
We now present the pricing for the case of a sliding scale. We always assume the same conditions. The sliding scale has a minimum rate $R_{min} = 3.75\%$, a loading $f = \frac{100}{70}$ and we look for the maximum rate R_{max}. We also assume that the first premium adjustement is foreseen after three years. The solution is given by $R_{max} = 5.91\%$. The following table gives the cash flows related to the commercial premium :

j	0	1	2	3	4	5	6	7	8
CP	300.00	75.00	0	180.28	−10.08	−29.61	−1.51	−7.42	−0.10

Table 24: Cash flow related to the commercial premium with a sliding scale

We observe the particular pattern of premium payment. At time $t = 0$, 80% of the minimum premium is paid. At time $t = 1$, 20% of the minimal premium is paid. There are no adjustments until time $t = 3$. At that time a huge positive adjustment is needed after which smaller negative adjustments follow. This shows an important fact for the sliding scale : a fraction of the premium may be paid late and this must have an influence on the pricing.

In the next table we give R_{max} in function of R_{min} and the first time for premium adjustments (m) :

R_{min}/m	1	2	3	4	5
4.50%	4.98%	5.04%	5.12%	5.21%	5.31%
4.25%	5.12%	5.24%	5.39%	5.55%	5.75%
4.00%	5.27%	5.44%	5.65%	5.89%	6.17%
3.75%	5.41%	5.64%	5.91%	6.24%	6.60%

Table 25: Sensitivity analysis 3

This table confirms what was said above. We observe a dramatic effect of the variable first year of premium adjustment. This aspect is however traditionally neglected by reinsurers when pricing sliding scale covers.
Many more sensitivity analyses are possible : see Walhin et al. (2001) for more analyses in the single branch pricing.

8. Conclusion

We have shown in this paper that a comprehensive methodology is of great help when pricing excess of loss treaties, even multiline treaties.
All the elements of a pricing are combined in a unique tool : actuarial elements (the severities $X, Y, \ldots$, the frequencies $N, M, \ldots$, the clauses, the retrocession), financial elements (the financial advantage when claims are paid long after the premium instalment, the remuneration of the shareholders at the cost of capital, the use of a cash flow model), economic elements (inflation, superinflation) and commercial elements (brokerage, administrative expenses).
The Panjer's algorithm is a powerful tool we often use (in fact as many times as there are periods between claims payments in our model) in order to find the aggregate situation of one line in the future. Obviously this has a computing cost which is really low nowadays. The aggregate claims distribution of the multiline is simply obtained by convolution.
The notion of cost of capital has been used in order to provide a fair price for the shareholders. A lot of parameters are necessary in order to run our model. Note that these parameters would also be necessary within a simplified model. In case some parameters are difficult to estimate, our methodology provides a solution in the sense that it easily allows for sensitivity analyses.

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