Determining the Change in Mean Duration Due to a Shift in the Hazard Rate Function

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Abstract:

From a major world event (such as a military action) to a seemingly minor detail (such as the use of a new plastic washer in a faucet design) change must be accounted for when collecting, interpreting and analyzing data. Indeed, the intervention itself may be the focus of the study. Theoretically, the best way to model some interventions, especially time-dependent ones, is via the hazard function. On the other hand, it may be necessary to translate into simpler concepts in order to answer practical questions. The average duration, for example, may have well-understood relationships with costs, making it the best choice for presenting the result.

For example, Shuan Wang [3] discusses deforming the hazard function by a constant multiplicative factor—proportional hazard transform—as a way to price risk load, with the mean playing the role of the pure loss premium.

This paper investigates how a shift in the hazard rate impacts the mean. The primary focus of the discussion is the case of bounded hazard rate functions of finite support. A formal framework is defined for that case and a practical calculation is described for measuring the impact on the mean duration of any deformation of the hazard function. The primary tool is the Cox Proportional Hazard model. Several formal results are derived and concrete illustrations of the calculation are provided in an Appendix, using the SAS implementation. The paper establishes that the method can be applied in a very general context and, in particular, to deformations which are not globally proportional shifts. Indeed, the method demands no assumed form for either the survival distribution or the deformation. The discussion begins with a case study that illustrates the application of these ideas to assess the cost impact of a TPA referral program.

Introduction

Recall that the survival function, $S(t)$, is just the probability of surviving to maturity time t and that the hazard function , $h(t)$, is the rate of failure at time t. We assume some general familiarity with these concepts in this discussion—they are introduced formally in Section II. While both functions equally well determine a model of survivorship, the survival function is the more common and the hazard function the more arcane. Often though, the best way to model a change in circumstances, especially a time-dependent intervention, is via the hazard function. On the other hand, it may be necessary to translate into simpler concepts in order to answer practical questions. The average duration, for example, may have a well-understood relationship with costs which makes it the best choice for presenting the result.

For example, Shuan Wang [3] discusses deforming the hazard function by a constant multiplicative factor—proportional hazard transform—as a way to price risk load, with the mean playing the role of the pure loss premium.

This paper investigates how a shift in the hazard rate impacts the mean. The primary focus of the discussion is the case of bounded hazard rate functions of finite support. A formal framework is defined for that case and a practical calculation is described for measuring the impact on the mean duration of any deformation of the hazard function. The primary tool is the Cox Proportional Hazard model (see [1]). Several formal results are derived and concrete illustrations of the calculation are provided in an Appendix, using the Statistical Analysis System [SAS] implementation (c.f. [1]) of the Cox model. The paper establishes that the method can be applied in a very general context and, in particular, to deformations which are not globally proportional shifts. Indeed, the method demands no assumed form for either the survival distribution or the deformation.

The paper begins with a case study that illustrates how these ideas were used to assess the cost impact of a Third Party Administrator (TPA) referral program. While this paper has a distinctly theoretical focus, the best way to explain the basic concepts is through a real world example. Indeed, most of the ideas are a direct consequence of attempts to achieve a better understanding of the case study outlined in Section I. The study illustrates that for most practical issues it is sufficient to determine the mean duration to failure via numerical integration. For many purposes, there is little need to invoke the more esoteric results developed in the subsequent sections. Still, the example illustrates the potential value of building a survivorship model whose hazard structure is designed to accommodate the issues under consideration. Among the technical results of the paper is a description of just such a survivorship model. While the discussion of the case study is largely self-contained for anyone generally familiar with the terminology of survivorship models, the discussion does make an occasional reference to the notation and observations developed in the subsequent sections.

Section II introduces the notation and formal set-up. The language shifts from rather discursive to decidedly technical. Section III discusses some well known examples. The remainder of the paper is devoted to several technical findings on how duration is impacted by a hazard shift. Specifically, Section IV discusses the case of finite support that is the case of primary interest. Section V considers how to combine hazards of finite support into more complex models suited to empirical data and the kind of investigation described in the case study.

Section I: A Case Study

Consider the following situation (while the data is based on a real world study, some liberties are taken in this discussion; in particular, the thought process, as described, follows hindsight more than foresight). The context is workers compensation (WC) insurance. We are required to assess whether a third party claims administrator (TPA) is saving money for two of its clients that have been selectively referring a portion of their WC claims over to be managed by the TPA. These clients are both large multi-state employers that are "self-insured" inasmuch as they do not purchase a WC insurance policy. The medical bills and loss of wages benefits are the direct responsibility of the employers and each has built internal systems to process their WC claims. The data captured by these systems is designed for administering claims, however, rather than for analytical use. As such, the data is comparatively crude relative to claim data of insurance companies or TPA's. They do, however, capture the date and jurisdiction of the injury, a summary of payments made to date, as well as if and when the claim is settled. There are, however, no "case reserves" available nor are there sufficient details, such as impairment rating or diagnosis, to adequately assess the severity of the claim.

Over the past few years, the employers have selectively farmed out the more complex claims to the TPA. The TPA has its own claim data on the cases referred to it and there is sufficient overlap to identify common claims within the TPA and the employer files. Moreover, the TPA files are more like insurance carrier data files and contain considerably more information, including the date of the referral, impairment ratings, claimant demographics and other claim characteristics.

A major problem is referral selection bias. The selection process itself is not well defined, even within an employer. Also, when the TPA first entered the picture, a greater percentage of referred claims were older, outstanding cases. Simply comparing the average cost per case of referred versus retained cases would not yield any meaningful information. Indeed, the selection process refers claims that are more expensive. Not only does this result in a higher severity for the referred cases, it renders the retained cases less severe over time. In such a circumstance, no matter how successful the TPA is in reducing costs, its mean cost per case will be comparatively high.

One fact that stood out for both employers is that the percentage of cases that closed within one year had more than doubled since the TPA became involved. Also, the referral rate shot up dramatically, suggesting that the TPA is, at some pragmatic level, viewed as being effective. Of course, that could also be the effect of imposed cost reductions on the staff the employer is now willing to maintain for WC claims handling, given the money spent on the TPA.

Another complicating issue is that the benefits that will be paid on some WC claims are paid out over many years. Without any consistent reserves it is very problematic to find comparable data. The challenge here is to make an assessment using the currently available payment data.

Without the presence of case reserves or enough claim characteristics to grade the severity of the claims, conventional actuarial approaches do not work well. As noted, the employer data, being collected largely for administrative purposes, did include the key dates of injury and settlement. This, combined with what was noted in regard to claim closure rates, suggested an approach based on survival analysis. In this context, a "life" corresponds with a claim, beginning at the date of injury and "failing" at claim settlement. Information on unsettled (open) claims is then "right censored". It was hoped that the survival analysis models would enable us to deal with censored data, since there were no case reserves available for that purpose.

Merging the TPA data together with the employer claim data, we built a data set that included an indicator of referral and, where so indicated, the date of referral. Other covariates captured are:

Explanatory Variables Used in the Proportional Hazard Duration Models	
Description	**Variable Name(s)**
Indicator of which of the two employers the claimant worked for	EMPL2
Indicators of the year of the injury (year 1992 as base)	AY93,AY94
Indicator whether a medical fee schedule applies in the state of jurisdiction	MF01
Indicator whether employer choice of physician applies in the state of jurisdiction	EC01
Indicator whether the nature of the injury is a sprain or strain (subjective)	NOI_SPR
Indicator whether the nature of the injury is a cut or laceration (objective)	NOI_CUT
Indicator whether the claim was referred to the TPA	TPA
Time dependent indicator whether the claim was referred to the TPA	TxTPA = 0 prior to TPA referral TxTPA =1 after TPA referral. The x refers to 3 time frames of referral from date of injury: x=1 within 1st 6 months x=2 within 2nd 6 months x=3 after 1 year

A claim survivorship model was constructed from this data. As defined in later sections in a formal way, the conceptual base of the model is a "hazard" function. The model assumes that the various explanatory variables impact the hazard function as a proportional shift, i.e., multiplication by a constant proportionality factor. Such survivorship models are referred to as proportional hazard models. Referral to the TPA is an exception in as much as it is captured as a so-called "time-dependent" intervention.

Instead of a constant value for the explanatory variable, the TPA referral indicator is allowed to take on two values so as to be able to capture into the model the time frame of referral (=0 prior to referral, =1 afterward). The proportional adjustment factor associated with TPA referral confirmed the expectation that referral was associated with a greater hazard, i.e., shortened claim duration. While the effect on the hazard was measured, the assignment demanded that it be translated into savings. In order to do that, it was necessary to convert the result back into factors related to claim costs. Whence the basic question of this paper: how to translate a change in hazard into a change in (mean) duration.

The task is to assess the cost impact of the TPA program, but that is not clearly defined. Due to the limited time frame of the data, the lack of case reserves or multiple loss valuations, it was clear that the "ultimate" cost impact could not be assessed using the available data, at least not directly. Also, "ultimate cost impact" is a more complicated notion than what the clients were after. We interpreted the task more simply: since we had the actual payments made on TPA referred cases, what we needed to measure is hypothetical: what would the payments on those claims have been without use of the TPA?

There is a catch, however. Consider a simplified case: the "original" payout pattern is $1 per day for 100 days on all claims. Assume that the referral to the TPA results in a single $100 payment on the first day. A little thought will convince the reader that at any point in time, ignoring discounting and the prospect that the business fails, the TPA will appear more costly. The comparison will not be fair unless it takes into account the unpaid balance: no matter how simply you frame the issue, reserves cannot be completely ignored.

The data included payment and duration, so there were ways available to translate a change in mean duration to dollars. Our choice was to use the non-referred claims to build a regression model in which the dependent variable is (log of) the benefits paid to date. The explanatory variables would include available claim characteristics together with the (log of) the payment duration. The characteristics (such as employer, accident year, jurisdiction or nature of injury, as above, together with perhaps additional covariates if available like age, wage, gender, part of body) are assumed independent of TPA referral and their mean values over the TPA-referred claims are readily determined. The only missing piece is the duration variable. Again, the question reduces to the topic of this paper: determining the impact on the mean duration.

The Cox proportional hazard model is well suited to this context. The model was run on pooled TPA-referred and non-referred data, with TPA-referral included among the explanatory variables in the model. This captures TPA-referral as a deformation of the hazard function and the methods of the paper can be applied to finish the job. Appendix 2 provides output that details the calculations.

The case study, however, illustrates an additional complexity. More precisely, the TPA-referral was incorporated into the Cox proportional hazard model as a time-dependent intervention (both the date of injury and date of referral being available). Also, as in the paper, the deformation of the hazard function was modeled as a combination of proportional shifts over three time intervals, as shown in the following table (refer to Appendix 2, page 5 of the listing):

Time Period I	Hazard Ratio φ_i
1st 6 months	1.424
2nd 6 months	1.203
After 1 year	1.122

The pattern of the hazard ratios supports the TPA's contention that its early intervention is more cost effective. Indeed, TPA intervention has its greatest, and most statistically significant, impact during the first six months. Although not critical to this context, that was an important finding of the study.

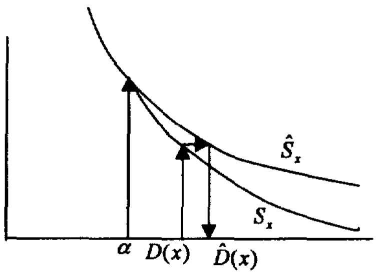
The difference in the values of the hazard ratios suggests that not only is it appropriate to model TPA referral as a time-dependent intervention, it is also appropriate to mitigate the global proportional hazard assumption by specializing to several time intervals. This is a very direct approach to that issue; the technical discussion of the subsequent sections follows that approach. An alternative way to mitigate that assumption—the one in fact used in the study report—is to group the TPA intervention by the lag time to referral. That formulation produces similar results and more directly supports the greater impact of early intervention. Conceptually, it is easy to regard TPA-referral within a few days of the injury as being an essentially different intervention than referral after several months.

The remainder of this discussion is somewhat more technical and makes reference to some of the notation and results presented in the subsequent sections of the paper.

The SAS PHREG procedure is used not only to estimate the three proportional hazard ratios φ_i. It optionally outputs paired values $(t, S(t))$ of a "baseline" survival function $S(t)$ at time t as well. We chose to determine a baseline survival function, $S(t)$, corresponding to the value of 0 for all covariates in the model. In particular, it applies to the case of non-referral as defined by the vanishing of the TPA-referral indicator variable. Observe that for the purpose of determining the baseline survival, only the non-time dependent TPA-referral indicator is used, since the baseline option is not available in the presence of time-dependent interventions.

This baseline survival function provides the expected duration distribution for the non-referred claims at the formal value 0 for the other explanatory variables in the model. Because referral is captured as a time dependent intervention, the deformation of the hazard function is itself dependent on the lag time to referral of the individual claims. Consequently, no single survival function of the form $S_\delta(t)$ (see Section II) can suffice to measure the impact on mean duration. This presents a somewhat more complicated situation than that considered in this paper.

To deal with this, let x represent a TPA-referred claim and $\beta = \beta(x)$ be the proportional hazard ratio associated to x by the model, which therefore includes the factor $\varphi = \varphi(x)$ for the TPA-referral as a time-dependent intervention. Let $D(x)$ represent the claim duration function; recall that we seek a hypothetical alternative $\hat{D}(x)$ which associates what the duration would have been had x not been referred. Letting $S_x, \hat{S}_x$ denote, respectively, the survival curves for x with and without referral, and $\alpha = \alpha(x)$ the lag time to TPA-referral, we have the following picture:



The idea is adjust duration so as to hold "maturity" constant. It follows from observations in Section II that:

$$\hat{S}_x(t) \approx S(t)^{\frac{\beta}{\varphi}} \quad \text{and} \quad S_x(t) = \begin{cases} \hat{S}_x(t) & t \le \alpha \\ \hat{S}_x(t)^{\varphi} \hat{S}_x(\alpha)^{1-\varphi} & t \ge \alpha \end{cases}$$

It follows, taking the $\left(\frac{\beta}{\varphi}\right)^{th}$ root, that:

$$S_x(D(x)) \approx \hat{S}_x(\hat{D}(x)) \quad \Leftrightarrow \quad S(\hat{D}(x)) \approx \left(S(D(x))\right)^{\varphi} S(\alpha)^{1-\varphi}$$

Since the baseline survival curve $S(t)$ is known, this provides a way to determine $\hat{D}(x)$ for any TPA-referred claim x. The methods described in the paper can now be invoked to estimate what the mean payment duration of those claims would have been had they not been referred. Again, the details of the calculation can be found in Appendix 2. The following table summarizes the findings in the case study (pages 10 and 15 of the listing):

Assumption	Mean Duration
As Referred to TPA (actual)	0.737 years
No Referral (hypothetical)	0.826 years

Note that the application of the logic used to define $D(x) \mapsto \hat{D}(x)$ becomes somewhat problematic when crossing a boundary of the time intervals used to define the φ_i. That is another reason that, in the study, we chose to partition the TPA-intervention by layer of referral lag $\alpha = \alpha(x)$.

Finally, these mean duration figures can be plugged into the cost models and translated into dollar savings attributable to TPA-referral. This case study is included to illustrate a non-traditional application of survival analysis to an insurance problem, emphasizing the power that manipulating the hazard function can bring to the analysis. The remainder of the paper develops a formal context in which this can be done. The focus is on formal relationships between the more "arcane" changes in hazard and the more "presentable" effect on mean duration.

Section II: Basic Terminology and Notation

Let $\Re^+$ denote the set of nonnegative real numbers. Let $h(t)$ denote a function from some subinterval $\Sigma \subseteq \Re^+$ to $\Re^+$. The set Σ is called the *support*. We assume throughout that $h(t)$ is (Lebesgue) integrable on Σ and that $0 \in \overline{\Sigma}$ is in the closure of the support. Any such $h : \underset{t \;\mapsto\; h(t)}{\Sigma \rightarrow \Re^+}$ can be viewed as a hazard rate function and survival analysis associates the following three functions:

$$g : \underset{t \;\mapsto\; g(t)}{\Sigma \rightarrow \Re^+} \text{ where } g(t) = \int_0^t h(s)ds$$

$$S : \underset{t \;\mapsto\; S(t)}{\Sigma \rightarrow [0,1]} \text{ where } S(t) = e^{-g(t)}$$

$$f : \underset{t \;\mapsto\; f(t)}{\Sigma \rightarrow \Re^+} \text{ where } f(t) = -\frac{dS}{dt} = h(t)S(t)$$

As is customary, we refer to $S(t)$ as the *survival function*, $f(t)$ as the *probability density function [PDF]* and t as time. We also let T denote the random variable for the distribution of survival times and $\mu = E_T(T)$ the mean duration. When we adorn $h(t)$ with a subscript, superscript, etc., we make the convention that these associated functions all follow suit. There are many well-known relationships and interpretations of these functions—refer to Allison[1] for a particularly succinct discussion which also discusses the SAS implementation of the Cox proportional hazard model.

Provided f is differentiable at t, it is readily determined whether the hazard rate is increasing or decreasing at t:

$$\frac{dh}{dt} = \frac{\frac{df}{dt}}{S} + h^2;\ h \text{ is decreasing at } t \Leftrightarrow \frac{df}{dt} < -\frac{f^2}{S}$$

In particular, it is a necessary—but by no means sufficient—condition that the density be decreasing in order for the hazard to be decreasing.

We are concerned with what happens when $h(t)$ is changed or "shifted" in some fashion. This paper deals particularly with proportional shifts as the Cox model provides a viable way to measure that type of shift (c.f.[2]). More precisely, we are interested in shifts of the form:

$$\delta = \delta(\alpha, \varphi) \text{ for } \alpha, \varphi \geq 0 \text{ where } \delta(h) = h_\delta \text{ is defined as } h_\delta(t) = \begin{cases} h(t) & t \leq \alpha \\ \varphi h(t) & t > \alpha \end{cases}$$

The following are immediate consequences of this definition and our notational conventions:

$$g_\delta(t) = \begin{cases} g(t) & t \le \alpha \\ g(\alpha) + \varphi(g(t) - g(\alpha)) \\ = (1-\varphi)g(\alpha) + \varphi g(t) & t \ge \alpha \end{cases}$$

$$S_\delta(t) = \begin{cases} S(t) & t \le \alpha \\ S(\alpha)^{1-\varphi} S(t)^\varphi & t \ge \alpha \end{cases}$$

$$f_\delta(t) = \begin{cases} f(t) & t \le \alpha \\ \varphi \left[\dfrac{S(t)}{S(\alpha)} \right]^{\varphi-1} f(t) & t \ge \alpha \end{cases}$$

We are particularly interested in the effect that such a shift has on mean duration, which is formally captured in the function:

$$\Delta(h;\alpha,\varphi) : \underset{\mu}{\Re^+} \underset{\mapsto}{\rightarrow} \underset{\mu_{\delta(\alpha,\varphi)}}{\Re^+}$$

While at first these shifts may seem restrictive, one of the main results of this paper is to show that the ability to measure these shifts is sufficient for handling very general problems. In fact, it will be shown that even when dealing with time-dependent interventions one can generally make do with the ability to handle the case $\alpha = 0$, in which case we make the common identification $\delta(0,\varphi) = \varphi$ of scalar multiplication with the scalar itself. Accordingly, we have

$$h_\varphi(t) = \varphi h(t),\ S_\varphi(t) = S(t)^\varphi \quad \textit{and} \quad f_\varphi(t) = \varphi S(t)^{\varphi-1} f(t) \quad \textit{for all } \varphi \ge 0, t \in \Sigma$$

Section III illustrates this notation in the case of two of the (infinite support) distributions commonly used in survival analysis. However, we choose to deal exclusively with the case of hazard functions with finite support in the remainder of the paper. Section IV discusses the additional assumptions, notation and conventions applicable specifically to finite support hazard functions and presents some examples. Section V discusses decomposing and combining finite support hazards and presents the main result: a formula for calculating the effect on mean duration of a shift in the hazard rate function. We also provide two appendices that detail the calculations referenced in the paper using SAS and, in particular, illustrate how the SAS proportional hazards model procedure (PHREG) can be used to do all the heavy lifting.

Section III: Familiar Examples

In this section we illustrate our notation with some distributions with infinite support $\Sigma = (0,\infty)$ which have found common application in survival analysis. The first three are selected to present straightforward illustrations of the notation and concepts and for those we only consider the case $\alpha = 0$ (recall the identification $\delta(0,\varphi) = \varphi$). We begin with the simplest example of a hazard function:

Example III.1. *Constant hazard function*: Let $h(t) \equiv 1$, then:

$$h_\varphi(t) = \varphi \quad g_\varphi(t) = \varphi t \quad S_\varphi(t) = e^{-\varphi t} \quad f_\varphi(t) = \varphi e^{-\varphi t}$$

and a straightforward integration by parts yields $\Delta(0,\varphi) = \mu_\varphi = \int_0^\infty \varphi t e^{-\varphi t}\, dt = \frac{1}{\varphi}$.

Example III.2. *Increasing hazard function*: Let $h(t) = t$, then:

$$h_\varphi(t) = \varphi t \quad g_\varphi(t) = \frac{\varphi t^2}{2} \quad S_\varphi(t) = e^{-\frac{\varphi t^2}{2}} \quad f_\varphi(t) = \varphi t e^{-\frac{\varphi t^2}{2}}$$

The motivated reader may readily verify, via another integration by parts and exploiting the symmetry of the normal PDF, that:

$$\Delta(0,\varphi) = \mu_\varphi = \int_0^\infty \varphi t^2 e^{-\frac{\varphi t^2}{2}}\, dt = \sqrt{\frac{\pi}{2\varphi}}$$

Example III.3. *Decreasing hazard function*: Let $h(t) = \frac{1}{1+t}$, then:

$$h_\varphi(t) = \frac{\varphi}{1+t} \quad g_\varphi(t) = \varphi \ln(1+t) \quad S_\varphi(t) = \frac{1}{(1+t)^\varphi} \quad f_\varphi(t) = \frac{\varphi}{(1+t)^{\varphi+1}}$$

In this case, integration by parts together with l'Hospital's rule gives:

$$\Delta(0,\varphi) = \mu_\varphi = \int_0^\infty \frac{\varphi t}{(1+t)^{\varphi+1}}\mathrm{dt} = \int_0^\infty \frac{\mathrm{dt}}{(1+t)^\varphi} - \lim_{t\to\infty} \frac{1}{\varphi(1+t)^{\varphi-1}}$$

in which the right hand side limits both exist for $\varphi > 1$. For $\varphi = 1$ the right hand side diverges to $+\infty$, whence $\mu_\varphi \geq \mu_1$ is infinite for $\varphi \leq 1$. This illustrates that a proportional increase in the hazard function can reduce an infinite mean duration to a finite number and, conversely, that a proportional decrease can make a finite mean duration become infinite.

The next example describes one of the most popular survival distributions, often defined via its PDF:

Example III.4. *Weibulll density with parameters* $a,b > 0$. In this example, define

$$f(a,b;t) = abt^{b-1}e^{-at^b}$$

then (see, e.g. [2] Hogg-Klugman, pp. 231-232)

$$S(t) = e^{-at^b}; \; h(t) = abt^{b-1}; \quad \text{and} \quad \mu = \frac{\Gamma\left(\frac{1}{b}\right)}{ba^{\frac{1}{b}}}$$

This distribution conforms to a proportional hazard model, indeed:

$$f_\varphi(a,b;t) = f(\varphi a,b;t),$$

$$S_\varphi(t) = e^{-\varphi a t^b}, \; h_\varphi(t) = \varphi abt^{b-1} \quad \textit{and} \quad \mu_\varphi = \frac{\Gamma\left(\frac{1}{b}\right)}{b(\varphi a)^{\frac{1}{b}}} = \frac{\mu}{\varphi^{\frac{1}{b}}}$$

Letting $\Gamma(\alpha)\Gamma(\alpha;t) = \int_0^t s^{\alpha-1}e^{-s}ds$ define the incomplete gamma function (as in [2], p. 217), we leave to the reader the verification that for the Weibull density:

$$\Delta(a,b;\alpha,\varphi) = \mu_{\delta(\alpha,\varphi)}$$

$$= a^{-\frac{1}{b}}\Gamma\left(\frac{b+1}{b}\right)\left[\Gamma\left(\frac{b+1}{b};a\alpha^b\right) + e^{a\alpha^b(\varphi-1)}\varphi^{-\frac{1}{b}}\left[1-\Gamma\left(\frac{b+1}{b};\varphi a\alpha^b\right)\right]\right]$$

When $\alpha = 0, a = b = 1$ this reduces to Example III.1; when $\alpha = 0, a = \frac{1}{2}, b = 2$ this reduces to Example III.2.

Example III.5. *Pareto density with parameters* $a,b > 0$. In this example, define

$$f(a,b;t) = ab^a(b+t)^{-a-1}$$

then (see, e.g. [2], pp. 222-223)

$$S(t) = \left(\frac{b}{b+t}\right)^a, \; h(t) = \frac{a}{b+t} \quad \text{and for } a > 1 \quad \mu = \frac{b}{a-1}$$

This distribution conforms to a proportional hazard model, indeed:

$$f_\varphi(a,b;t) = f(\varphi a,b;t),$$

$$S_\varphi(t) = \left(\frac{b}{b+t}\right)^{\varphi a}, \; h_\varphi(t) = \frac{\varphi a}{b+t} \quad \text{and for } \varphi a > 1, \; \mu_\varphi = \frac{b}{\varphi a - 1}$$

We again leave to the reader the verification that for the Pareto denisty:

$$\Delta(a,b;\alpha,\varphi)=\mu_{\delta(\alpha,\varphi)}$$

$$=\frac{b}{a-1}-\left(\frac{b}{b+\alpha}\right)^{a}\left(\frac{b+\alpha}{a-1}\right)+\left(\frac{b}{b+\alpha}\right)^{\varphi a}\left(\frac{b+\alpha}{\varphi a-1}\right)$$

$$+\alpha\left[\left(\frac{b}{b+\alpha}\right)^{\varphi a}-\left(\frac{b}{b+\alpha}\right)^{a}\right]$$

When $\alpha=0$, $a=b=1$ this reduces to Example III.3.

The last two examples are suggestive of the common approach to performing calculations in survival analysis: first, we select a form for the distribution, then we fit parameters to the data. Finally, we calculate whatever statistics are needed using formulas specific to that distribution (e.g. as found in [2]). This paper suggests the expediency of a simpler more empirical approach to calculating μ_δ that avoids making any assumptions as to the form of the distribution as well as any parameter estimation. Also, we can use the method with time-dependent interventions and it is especially easy to do in practice.

Section IV: Hazard with Finite Support

Most survival analysis discussions use distributions whose natural support is the set of positive real numbers, as in the previous section. The impetus for this work came from insurance, particularly claims analysis. Although actuaries customarily employ the usual collection of survival distributions--with their infinite supports--in practical applications claim duration is subject to limits. Moreover, the specific structure of the very far "tail" is either intrinsically unknowable, irrelevant, or both. Accordingly, this study focuses on the situation in which the data is limited to a finite time interval.

As described in the case study section, the insurance problem prompting this investigation arose in the line of workers compensation insurance. A very small percentage of those claims involve pension benefits that can continue for decades. Even the best insurance data bases, however, rarely track a coherent set of losses for more than 10 annual evaluations. That study concerned the implementation of a new program and the available data consisted of a one snapshot evaluation of claims captured into various automated systems. The data typically went back only four years and even the most matured cohort included a high percentage of open ("right censored") cases.

In this section we introduce the assumptions and notation for our case of interest: support $\Sigma = (0,1]$. We make the assumption that $h(t)$ is piecewise continuous. Observe that $g(t)$ and $S(t)$ are both continuous on $[0,1]$, the former nondecreasing and the latter nonincreasing. Let $p = S(1), 0 \le p \le 1$. The distribution T has a point mass of probability p at $\{1\}$. We will make extensive use of the following:

Proposition IV.1: *For any positive integer n:*

$$E(T^n) = n\int_0^1 t^{n-1}S(t)dt.$$

Proof: The proof is really just the integration by parts the diligent reader would have done a few times already in the previous section:

$$u = -S(t) \quad du = f(t)dt; \quad v = t^n \quad dv = nt^{n-1}dt$$

$$E(T^n) = \int_0^1 t^n f(t)dt + p = \int_0^1 vdu + p = uv\Big]_0^1 - \int_0^1 udv + p$$

$$= -t^n S(t)\Big]_0^1 + n\int_0^1 t^{n-1}S(t)dt + p = -p + n\int_0^1 t^{n-1}S(t)dt + p = n\int_0^1 t^{n-1}S(t)dt$$

completing the proof.

Letting σ^2 denote the variance of T, the following two corollaries are apparent:

Corollary IV.1:

$$i) \quad \mu = \int_0^1 S(t)dt$$

$$ii) \quad \mu^2 + \sigma^2 = 2\int_0^1 tS(t)dt$$

Corollary IV.2:

$$\Delta(\alpha,\varphi) = \mu_\delta = \int_0^\alpha S(t)dt + S(\alpha)^{1-\varphi}\int_\alpha^1 S(t)^\varphi dt$$

In particular, observe that $\mu_\varphi = \int_0^1 S(t)^\varphi dt$. It is intuitively clear that increasing the hazard decreases the mean duration, i.e., that $\Delta = \mu_\delta$ is a decreasing function of φ. A bit more thought should convince the reader that $\Delta = \mu_\delta$ is an increasing function of α for $\varphi > 1$ and decreasing for $\varphi < 1$. Since $g(t)$ is increasing, the following result formalizes this:

Proposition IV.2:

$$i) \quad \frac{\partial\Delta}{\partial\alpha} = (\varphi-1)f(\alpha)S(\alpha)^{-\varphi}\int_\alpha^1 S(t)^\varphi dt$$

$$ii) \quad \frac{\partial\Delta}{\partial\varphi} = S(\alpha)^{1-\varphi}\left[g(\alpha)\int_\alpha^1 S(t)^\varphi dt - \int_\alpha^1 g(t)S(t)^\varphi dt\right]$$

Proof: i) From Corollary IV.2, the fundamental theorem of calculus and the product rule for differentiation:

$$\frac{\partial\Delta}{\partial\alpha} = \frac{\partial}{\alpha}\left[\int_0^\alpha S(t)dt - S(\alpha)^{1-\varphi}\int_1^\alpha S(t)^\varphi dt\right]$$

$$= S(\alpha) - \left[S(\alpha)^{1-\varphi}S(\alpha)^\varphi + \int_1^\alpha S(t)^\varphi dt\big((1-\varphi)S(\alpha)^{-\varphi}(-f(\alpha))\big)\right]$$

$$= S(\alpha) - S(\alpha) + (1-\varphi)S(\alpha)^{-\varphi}f(\alpha)\int_1^\alpha S(t)^\varphi dt$$

$$= (\varphi-1)S(\alpha)^{-\varphi}f(\alpha)\int_\alpha^1 S(t)^\varphi dt$$

ii) Observe that:

$$S_\delta(t)=\begin{cases} S(t) & t\le\alpha \\ S(\alpha)^{1-\varphi}S(t)^{\varphi} & t\ge\alpha \end{cases}$$

$$\Rightarrow\quad \frac{\partial S}{\partial\varphi}=\begin{cases} 0 & t\le\alpha \\ S(\alpha)^{1-\varphi}\left(\ln(S(t))S(t)^{\varphi}\right) & \\ -S(t)^{\varphi}\left(\ln(S(\alpha))S(\alpha)^{1-\varphi}\right) & t\ge\alpha \end{cases}$$

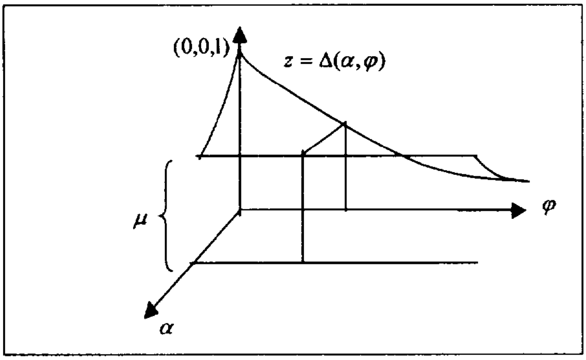
Noting that our assumptions enable us to differentiate under the integral, and recalling that $g(t)=-\ln(S(t))$, we find that:

$$\frac{\partial\Delta}{\partial\varphi}=\frac{\partial}{\partial\varphi}\left[\int_0^1 S_\delta(t)dt\right]=\int_0^1\frac{\partial S_\delta(t)}{\partial\varphi}dt=-S(\alpha)^{1-\varphi}\left[\int_\alpha^1 S(t)^{\varphi}g(t)dt-g(\alpha)\int_\alpha^1 S(t)^{\varphi}dt\right]$$

which completes the proof.

The graph of $\Delta=\mu_\delta$ is a tent with a single "pole" of unit height at the origin, a front wall of infinite length and constant height μ and a back wall of decreasing height:

$\Delta(0,0)=1\quad and\quad \forall\alpha,\varphi\ge 0,\quad \Delta(1,\varphi)=\Delta(\alpha,1)=\mu$



Since $g(t)$ is nondecreasing, we clearly have:

$$-\ln(p)\mu=g(1)\mu\ge\int_0^1 g(t)S(t)dt\quad and\quad \mu=\int_0^1 tf(t)dt+p\ge p$$

The following refines this:

Lemma IV.1: $-\ln(p)\mu + p - \mu \geq \int_0^1 g(t)S(t)dt$

Proof: Set

$$u = g(t) \quad du = h(t)dt; \quad v = \int_0^t S(w)dw \quad dv = S(t)dt$$

$$\int_0^1 g(t)S(t)dt = g(t)\int_0^t S(w)dw\Bigg]_0^1 - \int_0^1 h(t)\int_0^t S(w)dwdt$$

$$= g(1)\mu - \int_0^1 f(t)\int_0^t \frac{S(w)}{S(t)}dwdt$$

$$\leq -\ln(p)\mu - \int_0^1 f(t)\int_0^t 1dwdt \quad as \quad \frac{S(w)}{S(t)} \geq 1 \ for \ w \leq t$$

$$= -\ln(p)\mu - \int_0^1 f(t)tdt = -\ln(p)\mu - (\mu - p).$$

Applying the lemma to the hazard function $h_\varphi(t)$:

$$0 \geq \frac{\partial \Delta}{\partial \varphi}\bigg|_{\alpha = 0} = -\int_0^1 g(t)S(t)^\varphi dt \geq \frac{\mu_\varphi(1 + \varphi \ln(p)) - p^\varphi}{\varphi}.$$

which formally confirms how $\Delta(\alpha, \varphi)$ flattens as $\varphi \to \infty$. On the other hand, observe that if $h(a) > 0$ for some $a \geq 0$, then,

$$g(t) > 0 \ for \ t \geq a \Rightarrow S(t) < 1 \ \ for \ t \geq a$$

$$\Rightarrow \lim_{\varphi \to \infty} \mu_\varphi = \lim_{\varphi \to \infty} \Delta(0, \varphi) = \lim_{\varphi \to \infty} \int_0^1 S(t)^\varphi dt \leq \int_0^a dt = a$$

While the effect of an increase (decrease) of the hazard function clearly has the opposite affect on the mean duration, the effect on the variance is unclear. Indeed, the reader can use Corollary IV.1 to verify that:

$$\lim_{\varphi \to \infty} \sigma_\varphi = \lim_{\varphi \to 0} \sigma_\varphi = 0$$

Before we discuss some examples, we note the following integration formula, in which

$\mathrm{B}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \forall\, a,b>0$, the usual beta and gamma functions.

Lemma IV.2: For $a,c>-1, b>0$

$$\int_0^1 t^a(1-t^b)^c dt=\frac{\mathrm{B}\left(\frac{a+1}{b},c+1\right)}{b}$$

Proof: Letting $x=t^b \Rightarrow dx=bt^{b-1}dt$, then

$$\int_0^1 t^a(1-t^b)^c dt=\frac{1}{b}\int_0^1 t^{a-b+1}(1-t^b)^c bt^{(b-1)}dt$$

$$=\frac{1}{b}\int_0^1\left(x^{\frac{1}{b}}\right)^{a-b+1}(1-x)^c dx$$

$$=\frac{1}{b}\int_0^1 x^{\frac{a+1}{b}-1}(1-x)^c dx=\frac{\mathrm{B}\left(\frac{a+1}{b},c+1\right)}{b},$$

as claimed.

We next present some examples. The first, while especially simple, will play a major role in later findings.

Example IV.1. *Constant hazard function*, let $h(t)\equiv 1, 0\le t\le 1$. Then, as in Example III.1, we have:

$$h_\varphi(t)=\varphi;\quad g_\varphi(t)=\varphi t;\quad S_\varphi(t)=e^{-\varphi t};\quad f_\varphi(t)=\varphi e^{-\varphi t}$$

and we observe that

$$p_\varphi=S_\varphi(1)=e^{-\varphi}=p^\varphi,\quad \text{where } p=p_1=\frac{1}{e}$$

More generally, for $0\le\alpha\le 1$, we find:

$$h_\delta(t)=\begin{cases}1 & t\in[0,\alpha]\\ \varphi & t\in[\alpha,1]\end{cases}$$

$$g_\delta(t)=\begin{cases}t & t\in[0,\alpha]\\ \varphi(t-\alpha)+\alpha & t\in[\alpha,1]\end{cases}$$

$$S_\delta(t)=\begin{cases}e^{-t} & t\in[0,\alpha]\\ p^\alpha e^{-\varphi(t-\alpha)} & t\in[\alpha,1]\end{cases}$$

$$\Delta(\alpha,\varphi)=\mu_\delta=\int_0^1 S_\delta(t)dt=\int_0^\alpha e^{-t}dt+p^\alpha\int_\alpha^1 e^{-\varphi(t-\alpha)}dt=1-p^\alpha+\frac{p^\alpha}{\varphi}\left(1-p^{\varphi(1-\alpha)}\right)$$

In particular, we find that $\Delta(0,\varphi)=\mu_\varphi=\dfrac{1-p^\varphi}{\varphi}$. We will make considerable use of this example in later sections where we deal with combining hazards and show how to use the Cox Proportional Hazard model to approximate any hazard of finite support by a step function.

Example IV.2. *Increasing hazard function*, select $p\in[0,1]$, and define $f(t)=1-p,\ t\in[0,1]$; then:

$$S(t)=1-(1-p)t \quad and \quad h(t)=\frac{1-p}{1-(1-p)t}$$

This is an example of an increasing hazard that is not a proportional hazard model. We note that $h(t)$ is defined and continuous on $[0,1]$ for $p>0$, while the case $p=0$ is reminiscent of the infinite support case via the transformation $t\leftrightarrow\dfrac{t}{t+1}$. Finally, we note that the case $p=1\Rightarrow S(t)\equiv 1$ is of little interest, so we require $p<1$.

We leave to the reader the straightforward verification that in this case;

$$\Delta(\alpha,\varphi)=\mu_\delta=\alpha-\frac{(1-p)\alpha^2}{2}+\frac{p^{\varphi+1}(1-(1-p)\alpha)^{1-\varphi}-(1-(1-p)\alpha)^2}{(\varphi+1)(p-1)}$$

In particular,

$$\mu_\varphi=\frac{1-p^{\varphi+1}}{(\varphi+1)(1-p)}$$

For the special case $p=0$, the formulas simplify considerably and we have:

$$\mu_\varphi = \frac{1}{\varphi+1} \quad and \quad S_\varphi(t) = (1-t)^\varphi$$

$$\Rightarrow \mu_\varphi^{\,2} + \sigma_\varphi^{\,2} = 2\int_0^1 t(1-t)^\varphi dt = \frac{2\Gamma(2)\Gamma(\varphi+1)}{\Gamma(\varphi+3)} = \frac{2}{(\varphi+2)(\varphi+1)}$$

$$\Rightarrow \sigma_\varphi^{\,2} = \frac{\varphi}{(\varphi+1)^2(\varphi+2)}$$

In this case,

$$\sigma^{\,2} = \frac{1}{12}; \qquad \sigma_\varphi = \sigma \Leftrightarrow \varphi \in \left\{1, \frac{\sqrt{33}-5}{2}\right\}$$

and the variance is maximized exactly when $\varphi = \frac{\sqrt{5}-1}{2}$, giving a specific illustration of the relationship between a proportional shift in the hazard and the variance.

The next example is a simple way to define a new hazard function from an old one.

Example IV.3. *Reversed hazard function,* let $h(t)$ be any hazard function of finite support and define $\bar{h}(t) = h(1-t)$, then clearly

$$\left(\bar{h}\right)_\varphi(t) = \varphi h(1-t) = \overline{\left(h_\varphi\right)}(t) \quad \text{for every } \varphi > 0$$

which shows that the reverse of a proportional hazard model is also one. Clearly, the reverse of an increasing (decreasing) hazard function is decreasing (increasing) and $\bar{\bar{h}} = h$. Letting $u = 1-t$, we find that

$$\bar{g}(t) = \int_0^t \bar{h}(s)ds = -\int_1^{1-t} h(u)du = g(1) - g(1-t)$$

$$\bar{p} = p; \quad \bar{S}(t) = \frac{p}{S(1-t)}; \quad \bar{\mu} = p\int_0^1 \frac{dt}{S(1-t)}$$

The reverse of Example IV.1 is, of course, again Example IV.1. The reverse of Example IV.2 is a decreasing hazard function with survival function $\frac{p}{1-(1-p)(1-t)}$ and mean $\frac{p\ln(p)}{p-1}$.

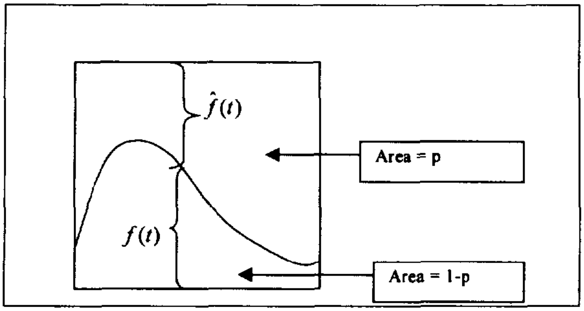
The next example is another simple way to define a new hazard function from an old one.

Example IV.4. *Complement hazard function,* let $h(t)$ be any hazard function of finite support such that $f(t) \le 1, 0 \le t \le 1$, and define $\hat{f}(t) = 1 - f(t)$, then

$$\hat{S}(t) = 1 - \int_0^t \hat{f}(s)ds = 1 - \int_0^t (1 - f(s))ds = 1 - t + (1 - S(t)) = 2 - t - S(t)$$

$$\hat{p} = 1 - p; \quad \hat{h}(t) = \frac{1 - f(t)}{2 - t - S(t)}$$

We again clearly have $\hat{\hat{f}}(t) = f(t)$; the picture is:



Example IV.5: *Modified Beta density with parameters* a, b, c, p. Assume $a, c > -1, b > 0, 0 \le p < 1$ and define

$$f(a,b,c,p;t) = \frac{b(1-p)t^a(1-t^b)^c}{\mathrm{B}\left(\frac{a+1}{b}, c+1\right)}$$

Then, clearly, $f(t) \ge 0$, when $0 \le t \le 1$ and the above lemma implies that

$$\int_0^1 f(a,b,c,p;t)dt = 1 - p$$

The binomial theorem enables us to write:

$$f(a,b,c,p;t) = \frac{b(1-p)}{\mathrm{B}\left(\frac{a+1}{b}, c+1\right)} \sum_{k=0}^{\infty} (-1)^k \binom{c}{k} t^{a+bk}$$

$$S(a,b,c,p;t) = 1 - \int_0^t f(s)ds = 1 + \frac{b(1-p)}{\mathrm{B}\left(\frac{a+1}{b}, c+1\right)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\binom{c}{k} t^{a+bk+1}}{a+bk+1}$$

When c is an **integer**, the reduction formula for the Gamma function gives:

$$\mathrm{B}\left(\frac{a+1}{b}, c+1\right) = \frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma(c+1)}{\Gamma\left(\frac{a+1}{b}+c+1\right)} = \frac{\Gamma\left(\frac{a+1}{b}\right)c!}{\Gamma\left(\frac{a+1}{b}\right)\prod_{j=0}^{c}\left(\frac{a+1}{b}+j\right)}$$

$$= \frac{c!}{\prod_{j=0}^{c}\left(\frac{a+bj+1}{b}\right)} = \frac{b^{c+1}c!}{\prod_{j=0}^{c} a+bj+1}$$

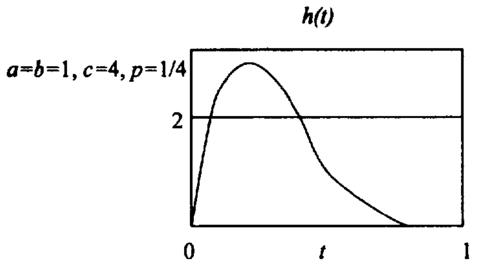
from which we find that for c an **integer:**

$$f(a,b,c,p;t) = \frac{b(1-p)\prod_{j=0}^{c}(a+bj+1)}{b^{c+1}c!} \sum_{k=0}^{c} (-1)^k \binom{c}{k} t^{a+bk}$$

$$= \frac{(1-p)\prod_{j=0}^{c}(a+bj+1)}{b^c} \sum_{k=0}^{c} \frac{(-1)^k t^{a+bk}}{k!(c-k)!}$$

$$S(a,b,c,p;t) = 1 + \frac{(1-p)}{b^c} \sum_{k=0}^{c} \prod_{\substack{j=0 \\ j \neq k}}^{c} (a+bj+1) \frac{(-1)^{k+1} t^{a+bk+1}}{k!(c-k)!}$$

which expresses $f(t)$ and $S(t)$ as polynomials. When $ac \neq 0$, $f(t) = 0 \Leftrightarrow t \in \{0,1\}$. In fact, it is readily verified that $f(t)$ is positive on $(0,1)$ with a unique maximum at $t = \left(\frac{a}{a+cb}\right)^{\frac{1}{b}}$. It follows that the hazard function in this example is generally $\bigcap$-shaped. The following picture illustrates this:

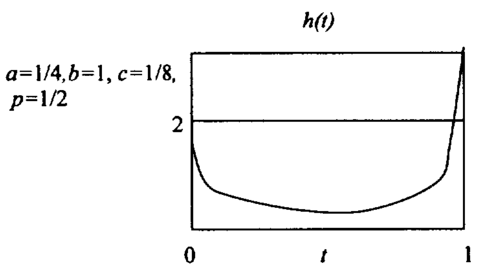


The final example is a slight variation of the previous one.

Example IV.6: Assume $a,c > -1$, $b > 0$, $0 \le p < 1$ and define

$$f(a,b,c,p;t) = \frac{b(1-p)(1-t^a(1-t^b)^c)}{b - \mathrm{B}\left(\frac{a+1}{b}, c+1\right)}$$

Again $f(t) \ge 0$, when $0 \le t \le 1$ and the above lemma implies that $\int_0^1 f(a,b,c,p;t)dt = 1-p$. In this case, the hazard function is generally $\bigcup$-shaped. The following picture illustrates this:



In the event that a particular shape of the hazard function is required, the last two examples provide candidates for parameter estimation. The following section argues that, for most purposes, a simple step function is preferable, from both the conceptual and computational perspectives.

This section concludes with two results. The first is one more observation on the difference $\mu - \mu_\delta$. The second revisits how for a finite hazard the survival function is a convenient device for computing moments, in this case relating it with the moment generating function.

In Example IV.2, note that

$$\mu-\mu_{\varphi}=\frac{p+1}{2}-\frac{p^{\varphi+1}-1}{(\varphi+1)(p-1)}=\frac{1}{1-p}\left[\frac{1-p^{2}}{2}+\frac{p^{\varphi+1}-1}{\varphi+1}\right]$$

The following generalizes this:

Proposition IV.3: Assume $f(t)$ is continuous on (0,1), then
$\forall\varphi>0,\quad 0\le\alpha<1,\quad \exists\zeta\in(\alpha,1)$ such that

$$f(\zeta)(\mu-\mu_{\delta})=p_{\alpha}{}^{2}\left[\frac{1-\left(\frac{p}{p_{\alpha}}\right)^{2}}{2}+\frac{\left(\frac{p}{p_{\alpha}}\right)^{\varphi+1}-1}{\varphi+1}\right] \quad \text{where } p_{\alpha}=S(\alpha).$$

Proof:

$$\mu-\mu_{\delta}=\int_{0}^{1}S(t)-S_{\delta}(t)dt=\int_{0}^{\alpha}S(t)-S(t)dt+\int_{\alpha}^{1}S(t)-S(\alpha)^{1-\varphi}S(t)^{\varphi}dt$$

$$=\int_{\alpha}^{1}S(t)-p_{\alpha}{}^{1-\varphi}S(t)^{\varphi}dt$$

Consider first the case $\mu=\mu_{\delta}$. Observe that

$$S(t)-p_{\alpha}{}^{1-\varphi}S(t)^{\varphi}\begin{cases}\le 0 & \varphi\le 1\\ \ge 0 & \varphi\ge 1\end{cases}$$

It follows, therefore, by continuity and the preceding equation, that $\mu=\mu_{\delta}$ would force

$$S(t)-p_{\alpha}{}^{1-\varphi}S(t)^{\varphi}=0\quad \forall t\in(\alpha,1)$$

Now if $p=p_{\alpha}$, then the right hand side is clearly 0 and the result holds. So consider the case $\mu=\mu_{\delta},\ p_{\alpha}<p$. We then have both:

$$p-p_{\alpha}{}^{1-\varphi}p^{\varphi}=\lim_{t\to 1}\left\{S(t)-p_{\alpha}{}^{1-\varphi}S(t)^{\varphi}\right\}=\lim_{t\to 1}\{0\}=0$$

$$\text{and}\quad p-p_{\alpha}{}^{1-\varphi}p^{\varphi}\begin{cases}<0 & \varphi<1\\ >0 & \varphi>1\end{cases}$$

which clearly forces $\varphi = 1$. The result again follows since $\varphi = 1$ makes the right hand side 0.

The upshot is that we may now assume that $\mu \neq \mu_\delta$. Because f is continuous and does not change sign, the generalized intermediate value theorem for integrals $\Rightarrow \exists \zeta \in (\alpha,1)$ such that

$$f(\zeta)\int_\alpha^1 S(t) - p_\alpha^{1-\varphi} S(t)^\varphi \, dt = \int_\alpha^1 \left(S(t) - p_\alpha^{1-\varphi} S(t)^\varphi\right) f(t)dt, \quad f(\zeta) > 0$$

Noting that $dS = -f(t)dt; \quad t = \alpha \Leftrightarrow S(t) = p_\alpha; \quad t = 1 \Leftrightarrow S(t) = p$. With the change of variable we have:

$$f(\zeta)(\mu - \mu_\delta) = \int_\alpha^1 \left(S(t) - p_\alpha^{1-\varphi} S(t)^\varphi\right) f(t)dt = \int_p^{p_\alpha} \left(S - p_\alpha^{1-\varphi} S^\varphi\right) dS$$

$$= \left.\frac{S^2}{2} - p_\alpha^{1-\varphi} \frac{S^{\varphi+1}}{\varphi+1}\right]_p^{p_\alpha} = \frac{p_\alpha^2}{2} - p_\alpha^{1-\varphi} \frac{p_\alpha^{\varphi+1}}{\varphi+1} - \frac{p^2}{2} + p_\alpha^{1-\varphi} \frac{p^{\varphi+1}}{\varphi+1}$$

$$= p_\alpha^2 \left[\frac{1 - \left(\frac{p}{p_\alpha}\right)^2}{2} + \frac{\left(\frac{p}{p_\alpha}\right)^{\varphi+1} - 1}{\varphi+1} \right]$$

which completes the proof.

Proposition IV.4. $M_T(x) = 1 + x\int_0^1 e^{xt} S(t)dt$

Proof: By definition:

$$M_T(x) = E(e^{xT}) = \int_0^1 e^{xt} f(t)dt + pe^x$$

Under our assumptions, we can interchange summation with integration, whence:

$$
\begin{aligned}
M_T(x) - pe^x &= \int_0^1 e^{xt} f(t)dt \\
&= \int_0^1 \sum_{k=0}^{\infty} \frac{(xt)^k}{k!} f(t)dt \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \int_0^1 t^k f(t)dt \\
&= \int_0^1 f(t)dt + \sum_{k=1}^{\infty} \frac{x^k}{k!}\left[k\int_0^1 t^{k-1} S(t)dt - p\right] \\
&= 1 - p + x\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}\left[\int_0^1 t^{k-1} S(t)dt - \frac{p}{k}\right] \\
&= 1 - p + x\sum_{k=1}^{\infty}\left[\int_0^1 \frac{x^{k-1}t^{k-1}}{(k-1)!} S(t)dt\right] - \sum_{k=1}^{\infty} \frac{px^k}{k!} \\
&= 1 + x\sum_{k=0}^{\infty}\left[\int_0^1 \frac{x^k t^k}{k!} S(t)dt\right] - p\sum_{k=0}^{\infty} \frac{x^k}{k!} \\
&= 1 + x\left[\int_0^1 \sum_{k=0}^{\infty} \frac{(xt)^k}{k!} S(t)dt\right] - pe^x \\
&= 1 + x\int_0^1 e^{xt} S(t)dt - pe^x.
\end{aligned}
$$

Section V: Combining Finite Support Hazard Functions

We continue with the notation and assumptions of the previous section. Consider first the case of two hazard functions, $h_1(t)$ and $h_1(t)$. If these represent independent causes of failure, then their sum $h_1 + h_2$ provides the corresponding hazard function. In this case, we clearly have:

$$g = g_1 + g_2 ; \quad S = S_1 S_2 ; \quad f = S_1 f_2 + f_1 S_2 ; \quad \mu = \int_0^1 S_1(t) S_2(t) dt ,$$

and we can readily generalize this to the case of compounding together any finite number of hazards.

Consider the case of adding a constant hazard, i.e., the case $h_2(t) \equiv a > 0$. While this will clearly decrease the mean duration to failure, the issue is by how much. From Example IV.1, we have $S_2(t) = e^{-at}$, and from Proposition IV.4 we find:

$$M_{T_1}(-a) = 1 - a \int_0^1 e^{-at} S_1(t) dt = 1 - a \int_0^1 S_1(t) S_2(t) dt = 1 - a\mu \quad \Rightarrow \mu = \frac{1 - M_{T_1}(-a)}{a}$$

While adding hazards is formally very simple, this suggests that the effect of the mean duration can become complicated in even the simplest contexts. Moreover, the more useful and challenging task would be to reverse this process: to decompose a compound hazard into mutually independent hazards. Fortunately, our needs are much less demanding.

In this section we detail a very simple and straightforward way to combine hazard functions. This provides the framework needed to exploit the Cox Proportional hazard model to approximate hazard functions with step functions. The approach also fits in well within the context of time-dependent interventions.

Begin with a finite support hazard function $h(t)$ and let $\{0 = \alpha_0 < \alpha_1 < ... < \alpha_n = 1\}$ be a partition of $[0,1]$ into n subintervals. We can readily decompose $h(t)$ into n finite support hazard functions:

$$h_i(t) = h(\alpha_{i-1} + t(\alpha_i - \alpha_{i-1})) \quad 0 \le t \le 1, i = 1,2,...,n$$

Fortunately, this process is readily reversed, i.e. given an ordered set of n finite support hazard functions $\{h_i(t), i = 1,2,...,n\}$ together with a partition $\{0 = \alpha_0 < \alpha_1 < ... < \alpha_n = 1\}$ of $[0,1]$ into n subintervals, we define their *gauntlet hazard function* on $[0,1]$ by

$$h(t) = \{h_1, h_2, ..., h_n; 0 = \alpha_0 < \alpha_1 < ... < \alpha_n = 1\}(t) = h_i\left(\frac{t - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}\right) \quad \text{where } \alpha_{i-1} \le t < \alpha_i$$

We observe that when the $h_i, i = 1,2,...,n$ are all constant hazard functions ($h_i(t) \equiv \varphi_i = h_{\varphi_i}, i = 1,2,...,n$ from Example IV.1) their gauntlet hazard function is a step function. Conversely, any hazard step function is the gauntlet of constant hazard functions in an essentially unique way.

The interpretation is straightforward. As suggested by the name, we can think of the hazards being lined up in sequence, much like a gauntlet. Survival becomes a matter of passing successively through the hazards, in sequence. A concern arises when any $p_i = 0, i < n$, since failure is assured during the corresponding interval, rendering the rest of the gauntlet essentially moot and introduces a singularity in the hazard function. As was noted before, the case $p=0$ is akin to infinite support hazards. In general, the probability of surviving the *i-th* interval of the gauntlet hazard is $\prod_{k=1}^{i} p_k$.

From these definitions, combined with our notational conventions, we have:

$$g(t) = \sum_{k=1}^{i-1} (\alpha_k - \alpha_{k-1}) g_k(1) + (\alpha_i - \alpha_{i-1}) g_i\left(\frac{t - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}\right) \quad \text{where } \alpha_{i-1} \le t < \alpha$$

$$S(t) = \prod_{k=1}^{i-1} p_k^{\alpha_k - \alpha_{k-1}} S_i\left(\frac{t - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}\right)^{\alpha_i - \alpha_{i-1}} \quad \text{where } \alpha_{i-1} \le t < \alpha$$

$$\mu = \int_0^1 S(t)dt = \sum_{i=1}^{n} \int_{\alpha_{i-1}}^{\alpha_i} S(t)dt = \sum_{i=1}^{n} \prod_{k=1}^{i-1} p_k^{\alpha_k - \alpha_{k-1}} \int_{\alpha_{i-1}}^{\alpha_i} S_i\left(\frac{t - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}\right)^{\alpha_i - \alpha_{i-1}} dt$$

$$= \sum_{i=1}^{n} \prod_{k=1}^{i-1} p_k^{\alpha_k - \alpha_{k-1}} (\alpha_i - \alpha_{i-1}) \int_0^1 S_i(u)^{\alpha_i - \alpha_{i-1}} du$$

$$= \sum_{i=1}^{n} \prod_{k=1}^{i-1} p_k^{\alpha_k - \alpha_{k-1}} (\alpha_i - \alpha_{i-1})(\mu_i)_{\alpha_i - \alpha_{i-1}}$$

It is instructive to note two special cases of this formula for μ:

Case 1: Assume the partition is uniform, that is, $\alpha_i = \frac{i}{n}$ then the formula becomes:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} p_k \right)^{\frac{1}{n}} (\mu_i)_{\frac{1}{n}}$$

Case 2: Assume the hazard is constant on all the intervals (step function). Then by Example IV.1,

$$h_i \equiv \varphi_i \Rightarrow (\mu_i)_{\alpha_i - \alpha_{i-1}} = \frac{1 - e^{\varphi_i(\alpha_{i-1} - \alpha_i)}}{\varphi_i(\alpha_i - \alpha_{i-1})},$$

and the formula becomes

$$\mu = \sum_{i=1}^{n} \prod_{k=1}^{i-1} p_k^{\alpha_k - \alpha_{k-1}} (\alpha_i - \alpha_{i-1})(\mu_i)_{\alpha_i - \alpha_{i-1}}$$

$$= \sum_{i=1}^{n} \prod_{k=1}^{i-1} e^{\varphi_k(\alpha_{k-1} - \alpha_k)} (\alpha_i - \alpha_{i-1}) \left(\frac{1 - e^{\varphi_i(\alpha_{i-1} - \alpha_i)}}{\varphi_i(\alpha_i - \alpha_{i-1})} \right)$$

$$= \sum_{i=1}^{n} e^{\sum_{k=1}^{i-1} \varphi_k(\alpha_{k-1} - \alpha_k)} \left(\frac{1 - e^{\varphi_i(\alpha_{i-1} - \alpha_i)}}{\varphi_i} \right)$$

Finally, when both apply, in the case of a step function with uniform partition, the formula simplifies to:

$$\mu = \sum_{i=1}^{n} e^{-\frac{1}{n}\sum_{k=1}^{i-1} \varphi_k} \left(\frac{1 - e^{-\frac{\varphi_i}{n}}}{\varphi_i} \right)$$

In the example below, we consider how to make use of this, given a set of empirical observations. The formulas suggest that it may prove useful to approximate the hazard function by a step function. In that regard, notice that the natural choice for $\varphi_i \approx h_i(t)$ is the average value of the hazard function over the ith interval. This, in turn, is readily determined from the survival function:

$$\frac{1}{\alpha_i - \alpha_{i-1}} \int_{\alpha_{i-1}}^{\alpha_i} h(t)dt = \frac{g(\alpha_i) - g(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}} = \frac{\ln(S(\alpha_{i-1})) - \ln(S(\alpha_i))}{\alpha_i - \alpha_{i-1}}$$

We conclude with a simple example that illustrates how, despite the awkwardness of the formulas, the calculations can be quite simple in practice.

Example V.1 Let $h_0(t) \equiv 1$, for $0 \le t \le 1$ be the constant unit hazard function and $\delta = \delta(\frac{1}{3}, 2) \circ \delta(\frac{2}{3}, \frac{1}{2})$ be the composite of the two shifts. Consider the hazard step function defined from:

$$h(t) = (h_0)_s(t) = \begin{cases} 1 & 0 \le t < \frac{1}{3} \\ 2 & \frac{1}{3} \le t < \frac{2}{3} \\ 1 & \frac{2}{3} \le t \le 1 \end{cases}$$

SAS was used to simulate two survival data sets One and Two, conforming to the hazard functions h_0 and h, respectively. The PDFs are readily determined from earlier examples and were used to perform the simulations (refer to Appendix 1 for details). A survival function was produced from Two. An excerpt of the output is provided below (page 10 of the listing),

t	$S(t)$	$g(t)$
0	1	0
$\frac{1}{3}$	0.71665	$0.33317 \approx \frac{1}{3}$
$\frac{2}{3}$	0.36770	$1.00048 \approx 1$
1	0.26359	$1.33335 \approx \frac{1}{3}$

The estimation of the hazard function h(t) from the survival function is:

$$t \in [0, \tfrac{1}{3}], \quad h(t) \approx \frac{g(\frac{1}{3}) - g(0)}{\frac{1}{3} - 0} \approx \frac{\frac{1}{3} - 0}{\frac{1}{3}} = 1$$

$$t \in [\tfrac{1}{3}, \tfrac{2}{3}], \quad h(t) \approx \frac{1 - \frac{1}{3}}{\frac{1}{3}} = 2$$

$$t \in [\tfrac{2}{3}, 1], \quad h(t) \approx \frac{\frac{4}{3} - 1}{\frac{1}{3}} = 1$$

The simple average of an upper and a lower Riemann sum of the survival function over [0,1] (equivalent to the trapezoidal rules since the survival function is monotonically decreasing) was used to estimate the mean duration to failure to be 0.56193 (page 16 of the listing):

$$\mu = \int_0^1 S(t)dt \approx 0.56193$$

Compare this with the value determined using the above formula:

$\varphi_1 = \varphi_3 = 1, \varphi_2 = 2:$

$$\mu = \left(1 - e^{-\frac{1}{3}}\right) + e^{-\frac{1}{3}}\left(\frac{1 - e^{-\frac{2}{3}}}{2}\right) + e^{-\frac{1}{3}-\frac{2}{3}}\left(1 - e^{-\frac{1}{3}}\right)$$

$$= 0.562077$$

Finally, data set Two observations were flagged and pooled with set One survival data. The SAS PHREG procedure was then run on the combined data set with the flagged data modeled as a time-dependent intervention applicable to the middle interval. The PHREG procedure produced a hazard ratio of 2.000 (page 4 of the listing) for that intervention, illustrating how the Cox proportional hazard model can be used to approximate a hazard function by a step function. By the same token, it illustrates how that procedure may provide the means to unpack this process. More precisely, the procedure results may reveal a change in hazard as (approximated by) a combination of shifts like the ones considered here: $\delta \; = \; \delta\,(\frac{1}{3}, 2) \circ \delta\,(\frac{2}{3}, \frac{1}{2})$. From that, the results of this paper can be used to translate this into the effect on the mean time to failure.

References:

[1] Allison, Paul D., *Survival Analysis Using the SAS® System: A Practical Guide*, The SAS Institute, Inc., 1995.

[2] Hogg, Robert V. and Klugman, Stuart A., *Loss Distributions*, John Wiley & Sons, 1984.

[3] Wang, Shuan, "Implementation of Proportional Hazards Transforms in Ratemaking," PCAS LXXXV, pp. 940-979.

APPENDIX 1

SASLOG:

```
********************************;
OPTIONS MPRINT LS=131 PS=59 NOCENTER;
*OPTIONS OBS = 100;
DATA ZERO;
INPUT Z;
CARDS;

NOTE: The data set WORK.ZERO has 1 observations and 1 variables.

;
DATA ONE;SET ZERO;
KEEP T CLOSED SHOCK;
RETAIN COUNT;
IF _N_ = 1;
CLOSED = 1;
SHOCK  = 0;
COUNT = 0;
DO I = 1 TO 1000;
   T = I/1000;
   DO J = 1 TO ROUND(50*EXP(-T),1);
      COUNT + 1;OUTPUT;END;END;
T = 1;P = EXP(-1);
CLOSED = 0;
DO J = 1 TO (P/(1-P))*COUNT;
   OUTPUT;END;

NOTE: The data set WORK.ONE has 49980 observations and 3 variables.

DATA TWO;SET ZERO;
KEEP T CLOSED SHOCK;
RETAIN COUNT;
IF _N_ = 1;
CLOSED = 1;
SHOCK  = 1;
COUNT = 0;
DO I = 1 TO 333;
   T = I/1000;
   DO J = 1 TO ROUND(50*EXP(-T),1);
      COUNT + 1;OUTPUT;END;END;
DO I = 334 TO 666;
   T = I/1000;
   DO J = 1 TO ROUND(100*EXP(-2*T + 1/3),1);
      COUNT + 1;OUTPUT;END;END;
DO I = 667 TO 1000;
   T = I/1000;
   DO J = 1 TO ROUND(50*EXP(-T - 1/3),1);
      COUNT + 1;OUTPUT;END;END;
T = 1;P = EXP(-4/3);
CLOSED = 0;
DO J = 1 TO (P/(1-P))*COUNT;
   OUTPUT;END;

NOTE: The data set WORK.TWO has 49956 observations and 3 variables.
```

```
          DATA THREE; SET ONE TWO;
          TITLE 'PHREG PAPER:TEST';

NOTE: The data set WORK.THREE has 99936 observations and 3 variables.
          PROC PHREG SIMPLE DATA=THREE;
               MODEL T*CLOSED(0)= SHOCK /CORRB COVB;

NOTE: The PROCEDURE PHREG printed pages 1-2.

          PROC PHREG SIMPLE DATA=THREE;
               MODEL T*CLOSED(0)= TSHOCK /CORRB COVB;
               IF 2/3 >= T > 1/3 THEN TSHOCK = SHOCK;ELSE TSHOCK = 0;
          %MACRO MEANDUR;
          PROC PHREG SIMPLE DATA=ZDATA;
               MODEL T*CLOSED(0)= /CORRB COVB;
               BASELINE OUT=BASE SURVIVAL=S;
          DATA BASE;SET BASE END = EOF;
          IF _N_ = 1 THEN DO;T = 0;S = 1;OUTPUT;END;
          IF T < 1 THEN OUTPUT;
          IF EOF OR T >= 1 THEN DO;T = 1;OUTPUT;END;
          PROC SORT NODUP DATA = BASE; BY T;
          DATA SUBBASE;SET BASE;
          IF ABS(T -   0) < .01 OR
             ABS(T - 1/3) < .01 OR
             ABS(T - 2/3) < .01 OR
             ABS(T -   1) < .01;
          G = -LOG(S);
          PROC PRINT DATA=SUBBASE;
          DATA MEAN;SET BASE END=EOF;KEEP  UPPER LOWER MEAN;
          KEEP  UPPER LOWER MEAN;
          RETAIN UPPER LOWER OLD_S OLD_T;
          IF _N_ = 1 THEN DO;
             OLD_T  = 0;
             UPPER = 0;
             LOWER = 0;
             OLD_S = 1;
             END;
          D = T - OLD_T;
          IF D > 0 THEN DO;
             UPPER + D*OLD_S;
             LOWER + D*S;
             END;
             OLD_T = T;
             OLD_S = S;
          IF EOF THEN DO;
             MEAN = (UPPER + LOWER)/2;OUTPUT;
             END;
          PROC PRINT DATA = MEAN;
          %MEND MEANDUR;
          DATA ZDATA;SET ONE;
NOTE: The PROCEDURE PHREG printed pages 3-4.
NOTE: The PROCEDURE PHREG used 3001K.

          TITLE 'PHREG PAPER:TEST BASE ONE';
          %MEANDUR;

NOTE: The data set WORK.ZDATA has 49980 observations and 3 variables.
```

```
MPRINT(MEANDUR):    PROC PHREG SIMPLE DATA=ZDATA;
MPRINT(MEANDUR):    MODEL T*CLOSED(0)= /CORRB COVB;
MPRINT(MEANDUR):    BASELINE OUT=BASE SURVIVAL=S;
NOTE: There are no explanatory variables in the MODEL statement.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.
NOTE: The PROCEDURE PHREG printed page 5.
MPRINT(MEANDUR):    DATA BASE;
MPRINT(MEANDUR):   SET BASE END = EOF;
MPRINT(MEANDUR):    IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):   T = 0;
MPRINT(MEANDUR):   S = 1;
MPRINT(MEANDUR):   OUTPUT;
MPRINT(MEANDUR):   END;
MPRINT(MEANDUR):    IF T < 1 THEN OUTPUT;
MPRINT(MEANDUR):    IF EOF OR T >= 1 THEN DO;
MPRINT(MEANDUR):   T = 1;
MPRINT(MEANDUR):   OUTPUT;
MPRINT(MEANDUR):   END;

NOTE: The data set WORK.BASE has 1002 observations and 2 variables.

MPRINT(MEANDUR):    PROC SORT NODUP DATA = BASE:
MPRINT(MEANDUR):    BY T;

NOTE: 1 duplicate observations were deleted.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.
MPRINT(MEANDUR):    DATA SUBBASE;
MPRINT(MEANDUR):   SET BASE;
MPRINT(MEANDUR):    IF ABS(T - 0) < .01 OR ABS(T - 1/3) < .01 OR ABS(T - 2/3) <
01 OR ABS(T - 1) < .01;
MPRINT(MEANDUR):    G = -LOG(S);

NOTE: The data set WORK.SUBBASE has 60 observations and 3 variables.
MPRINT(MEANDUR):    PROC PRINT DATA=SUBBASE;

NOTE: The PROCEDURE PRINT printed pages 6-7.
MPRINT(MEANDUR):    DATA MEAN;
MPRINT(MEANDUR):    SET BASE END=EOF;
MPRINT(MEANDUR):    KEEP UPPER LOWER MEAN;
MPRINT(MEANDUR):    RETAIN UPPER LOWER OLD_S OLD_T;
MPRINT(MEANDUR):    IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):    OLD_T = 0;
MPRINT(MEANDUR):    UPPER = 0;
MPRINT(MEANDUR):    LOWER = 0;
MPRINT(MEANDUR):    OLD_S = 1;
MPRINT(MEANDUR):    END;
MPRINT(MEANDUR):    D = T - OLD_T;
MPRINT(MEANDUR):    IF D > 0 THEN DO;
MPRINT(MEANDUR):    UPPER + D*OLD_S;
MPRINT(MEANDUR):    LOWER + D*S;
MPRINT(MEANDUR):    END;
MPRINT(MEANDUR):    OLD_T = T;
MPRINT(MEANDUR):    OLD_S = S;
MPRINT(MEANDUR):    IF EOF THEN DO;
MPRINT(MEANDUR):    MEAN = (UPPER + LOWER)/2;
MPRINT(MEANDUR):    OUTPUT;
MPRINT(MEANDUR):    END;
```

```
NOTE: The data set WORK.MEAN has 1 observations and 3 variables.
MPRINT(MEANDUR):    PROC PRINT DATA = MEAN;

NOTE: The PROCEDURE PRINT printed page 8.
96          DATA ZDATA;SET TWO;
97          TITLE 'PHREG PAPER:TEST BASE TWO';
98          %MEANDUR;

NOTE: The data set WORK.ZDATA has 49956 observations and 3 variables.
MPRINT(MEANDUR):    PROC PHREG SIMPLE DATA=ZDATA;
MPRINT(MEANDUR):    MODEL T*CLOSED(0)= /CORRB COVB;
MPRINT(MEANDUR):    BASELINE OUT=BASE SURVIVAL=S;
NOTE: There are no explanatory variables in the MODEL statement.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.
NOTE: The PROCEDURE PHREG printed page 9.
MPRINT(MEANDUR):    DATA BASE;
MPRINT(MEANDUR):   SET BASE END = EOF;
MPRINT(MEANDUR):    IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):   T = 0;
MPRINT(MEANDUR):   S = 1;
MPRINT(MEANDUR):   OUTPUT;
MPRINT(MEANDUR):   END;
MPRINT(MEANDUR):    IF T < 1 THEN OUTPUT;
MPRINT(MEANDUR):    IF EOF OR T >= 1 THEN DO;
MPRINT(MEANDUR):   T = 1;
MPRINT(MEANDUR):   OUTPUT;
MPRINT(MEANDUR):   END;

NOTE: The data set WORK.BASE has 1002 observations and 2 variables.
MPRINT(MEANDUR):    PROC SORT NODUP DATA = BASE;
MPRINT(MEANDUR):    BY T;

NOTE: HOST sort chosen, but SAS sort recommended.
NOTE: 1 duplicate observations were deleted.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.
MPRINT(MEANDUR):    DATA SUBBASE;
MPRINT(MEANDUR):   SET BASE;
MPRINT(MEANDUR):    IF ABS(T - 0) < .01 OR ABS(T - 1/3) < .01 OR ABS(T - 2/3) <
01 OR ABS(T - 1) < .01;
MPRINT(MEANDUR):    G = -LOG(S);

NOTE: The data set WORK.SUBBASE has 60 observations and 3 variables.
MPRINT(MEANDUR):    PROC PRINT DATA=SUBBASE;

NOTE: The PROCEDURE PRINT printed pages 10-11.
MPRINT(MEANDUR):    DATA MEAN;
MPRINT(MEANDUR):   SET BASE END=EOF;
MPRINT(MEANDUR):    KEEP UPPER LOWER MEAN;
MPRINT(MEANDUR):    RETAIN UPPER LOWER OLD_S OLD_T;
MPRINT(MEANDUR):    IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):    OLD_T = 0;
MPRINT(MEANDUR):    UPPER = 0;
MPRINT(MEANDUR):    LOWER = 0;
MPRINT(MEANDUR):    OLD_S = 1;
MPRINT(MEANDUR):    END;
MPRINT(MEANDUR):    D = T - OLD_T;
```

```
MPRINT(MEANDUR):   IF D > 0 THEN DO;
MPRINT(MEANDUR):   UPPER + D*OLD_S;
MPRINT(MEANDUR):   LOWER + D*S;
MPRINT(MEANDUR):   END;
MPRINT(MEANDUR):   OLD_T = T;
MPRINT(MEANDUR):   OLD_S = S;
MPRINT(MEANDUR):   IF EOF THEN DO;
MPRINT(MEANDUR):   MEAN = (UPPER + LOWER)/2;
MPRINT(MEANDUR):   OUTPUT;
MPRINT(MEANDUR):   END;

NOTE: The data set WORK.MEAN has 1 observations and 3 variables.

MPRINT(MEANDUR):   PROC PRINT DATA = MEAN;

NOTE: The PROCEDURE PRINT printed page 12.

99          DATA ZDATA;SET THREE;
100         TITLE 'PHREG PAPER:TEST BASE THREE';
101         %MEANDUR;

NOTE: The data set WORK.ZDATA has 99936 observations and 3 variables.

MPRINT(MEANDUR):   PROC PHREG SIMPLE DATA=ZDATA;
MPRINT(MEANDUR):   MODEL T*CLOSED(0)= /CORRB COVB;
MPRINT(MEANDUR):   BASELINE OUT=BASE SURVIVAL=S;
NOTE: There are no explanatory variables in the MODEL statement.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.
NOTE: The PROCEDURE PHREG printed page 13.

MPRINT(MEANDUR):   DATA BASE;
MPRINT(MEANDUR):  SET BASE END = EOF;
MPRINT(MEANDUR):   IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):  T = 0;
MPRINT(MEANDUR):  S = 1;
MPRINT(MEANDUR):  OUTPUT;
MPRINT(MEANDUR):  END;
MPRINT(MEANDUR):   IF T < 1 THEN OUTPUT;
MPRINT(MEANDUR):   IF EOF OR T >= 1 THEN DO;
MPRINT(MEANDUR):  T = 1;
MPRINT(MEANDUR):  OUTPUT;
MPRINT(MEANDUR):  END;

NOTE: The data set WORK.BASE has 1002 observations and 2 variables.

MPRINT(MEANDUR):   PROC SORT NODUP DATA = BASE;
MPRINT(MEANDUR):   BY T;

NOTE: 1 duplicate observations were deleted.
NOTE: The data set WORK.BASE has 1001 observations and 2 variables.

MPRINT(MEANDUR):   DATA SUBBASE;
MPRINT(MEANDUR):  SET BASE;
MPRINT(MEANDUR):   IF ABS(T - 0) < .01 OR ABS(T - 1/3) < .01 OR ABS(T - 2/3) <
01 OR ABS(T - 1) < .01;
MPRINT(MEANDUR):   G = -LOG(S);
```

```
NOTE: The data set WORK.SUBBASE has 60 observations and 3 variables.

MPRINT(MEANDUR):   PROC PRINT DATA=SUBBASE;

NOTE: The PROCEDURE PRINT printed pages 14-15.
MPRINT(MEANDUR):   DATA MEAN;
MPRINT(MEANDUR):  SET BASE END=EOF;
MPRINT(MEANDUR):  KEEP UPPER LOWER MEAN;
MPRINT(MEANDUR):   KEEP UPPER LOWER MEAN;
MPRINT(MEANDUR):   RETAIN UPPER LOWER OLD_S OLD_T;
MPRINT(MEANDUR):   IF _N_ = 1 THEN DO;
MPRINT(MEANDUR):   OLD_T = 0;
MPRINT(MEANDUR):   UPPER = 0;
MPRINT(MEANDUR):   LOWER = 0;
MPRINT(MEANDUR):   OLD_S = 1;
MPRINT(MEANDUR):   END;
MPRINT(MEANDUR):   D = T - OLD_T;
MPRINT(MEANDUR):   IF D > 0 THEN DO;
MPRINT(MEANDUR):   UPPER + D*OLD_S;
MPRINT(MEANDUR):   LOWER + D*S;
MPRINT(MEANDUR):   END;
MPRINT(MEANDUR):   OLD_T = T;
MPRINT(MEANDUR):   OLD_S = S;
MPRINT(MEANDUR):   IF EOF THEN DO;
MPRINT(MEANDUR):   MEAN = (UPPER + LOWER)/2;
MPRINT(MEANDUR):  OUTPUT;
MPRINT(MEANDUR):   END;

NOTE: The data set WORK.MEAN has 1 observations and 3 variables.

MPRINT(MEANDUR):   PROC PRINT DATA = MEAN;
NOTE: The PROCEDURE PRINT printed page 16.
```

SAS LISTING

PHREG PAPER:TEST page 1

The PHREG Procedure

Data Set: WORK.THREE
Dependent Variable: T
Censoring Variable: CLOSED
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
99936	68382	31554	31.57

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
SHOCK	99936	0.49988	0.50000	0	1.00000

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	1510536.08	1509248.84	1287.233 with 1 DF (p=0.0001)
Score	.	.	1290.337 with 1 DF (p=0.0001)
Wald	.	.	1282.328 with 1 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
SHOCK	1	0.275302	0.00769	1282	0.0001	1.317

Estimated Covariance Matrix

	SHOCK
SHOCK	0.0000591043

Estimated Correlation Matrix

	SHOCK
SHOCK	1.000000000

The PHREG Procedure

Data Set: WORK.THREE
Dependent Variable: T
Censoring Variable: CLOSED
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
99936	68382	31554	31.57

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
TSHOCK	99936	0.17425	0.37933	0	1.00000

WARNING: Simple statistics listed for the time-dependent explanatory variables have limited value.

PHREG PAPER:TEST **page 4**

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	1510536.08	1507331.64	3204.433 with 1 DF (p=0.0001)
Score	.	.	3197.243 with 1 DF (p=0.0001)
Wald	.	.	3073.086 with 1 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
TSHOCK	1	0.693074	0.01250	3073	0.0001	2.000

Estimated Covariance Matrix

	TSHOCK
TSHOCK	0.0001563092

Estimated Correlation Matrix

	TSHOCK
TSHOCK	1.000000000

PHREG PAPER:TEST page 5

The PHREG Procedure

Data Set: WORK.ZDATA
Dependent Variable: T
Censoring Variable: CLOSED
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
49980	31594	18386	36.79

NOTE: There are no explanatory variables in this model.

-2 LOG L = 657271.7

PHREG PAPER: TEST BASE ONE pages 6-7

OBS	T	S	G
1	0.000	1.00000	0.00000
2	0.001	0.99900	0.00100
3	0.002	0.99800	0.00200
4	0.003	0.99700	0.00301
5	0.004	0.99600	0.00401
6	0.005	0.99500	0.00501
7	0.006	0.99400	0.00602
8	0.007	0.99300	0.00703
9	0.008	0.99200	0.00804
10	0.009	0.99100	0.00904
11	0.324	0.72327	0.32397
12	0.325	0.72255	0.32497
13	0.326	0.72183	0.32597
14	0.327	0.72111	0.32697
15	0.328	0.72039	0.32797
16	0.329	0.71967	0.32897
17	0.330	0.71895	0.32997
18	0.331	0.71823	0.33097
19	0.332	0.71751	0.33197
20	0.333	0.71679	0.33298
21	0.334	0.71607	0.33398
22	0.335	0.71535	0.33499
23	0.336	0.71463	0.33600
24	0.337	0.71391	0.33700
25	0.338	0.71319	0.33801
26	0.339	0.71247	0.33902
27	0.340	0.71174	0.34004
28	0.341	0.71102	0.34105
29	0.342	0.71030	0.34206
30	0.343	0.70960	0.34305
31	0.657	0.51845	0.65692
32	0.658	0.51793	0.65792
33	0.659	0.51741	0.65893
34	0.660	0.51689	0.65993
35	0.661	0.51637	0.66094
36	0.662	0.51585	0.66195
37	0.663	0.51533	0.66296
38	0.664	0.51481	0.66397
39	0.665	0.51429	0.66498
40	0.666	0.51377	0.66599
41	0.667	0.51325	0.66700
42	0.668	0.51273	0.66802
43	0.669	0.51220	0.66903
44	0.670	0.51168	0.67005
45	0.671	0.51116	0.67106
46	0.672	0.51064	0.67208
47	0.673	0.51012	0.67310
48	0.674	0.50962	0.67408
49	0.675	0.50912	0.67506
50	0.676	0.50862	0.67605
51	0.991	0.37117	0.99110
52	0.992	0.37079	0.99212
53	0.993	0.37041	0.99315
54	0.994	0.37003	0.99418
55	0.995	0.36967	0.99515
56	0.996	0.36931	0.99612
57	0.997	0.36895	0.99710
58	0.998	0.36859	0.99808
59	0.999	0.36823	0.99905
60	1.000	0.36787	1.00003

```
PHREG PAPER: TEST BASE ONE                      page 8

 OBS      UPPER       LOWER        MEAN

  1      0.63244     0.63180     0.63212



PHREG PAPER: TEST BASE TWO                      page 9

 The PHREG Procedure

 Data Set: WORK.ZDATA
 Dependent Variable: T
 Censoring Variable: CLOSED
 Censoring Value(s): 0
 Ties Handling: BRESLOW

           Summary of the Number of
           Event and Censored Values

                                             Percent
    Total        Event     Censored        Censored

    49956        36788        13168           26.36
 NOTE: There are no explanatory variables in this model.

 -2 LOG L = 757604.6
```

OBS	T	S	G
1	0.000	1.00000	0.00000
2	0.001	0.99900	0.00100
3	0.002	0.99800	0.00200
4	0.003	0.99700	0.00301
5	0.004	0.99600	0.00401
6	0.005	0.99500	0.00502
7	0.006	0.99399	0.00602
8	0.007	0.99299	0.00703
9	0.008	0.99199	0.00804
10	0.009	0.99099	0.00905
11	0.324	0.72314	0.32416
12	0.325	0.72242	0.32515
13	0.326	0.72170	0.32615
14	0.327	0.72097	0.32715
15	0.328	0.72025	0.32815
16	0.329	0.71953	0.32915
17	0.330	0.71881	0.33015
18	0.331	0.71809	0.33116
19	0.332	0.71737	0.33216
20	0.333	0.71665	0.33317
21	0.334	0.71521	0.33518
22	0.335	0.71379	0.33717
23	0.336	0.71237	0.33916
24	0.337	0.71095	0.34116
25	0.338	0.70952	0.34316
26	0.339	0.70810	0.34517
27	0.340	0.70668	0.34717
28	0.341	0.70526	0.34919
29	0.342	0.70386	0.35118
30	0.343	0.70246	0.35317
31	0.657	0.37473	0.98155
32	0.658	0.37399	0.98353
33	0.659	0.37325	0.98551
34	0.660	0.37251	0.98750
35	0.661	0.37177	0.98949
36	0.662	0.37103	0.99148
37	0.663	0.37029	0.99348
38	0.664	0.36955	0.99548
39	0.665	0.36880	0.99749
40	0.666	0.36806	0.99950
41	0.667	0.36770	1.00048
42	0.668	0.36734	1.00146
43	0.669	0.36698	1.00244
44	0.670	0.36662	1.00342
45	0.671	0.36626	1.00441
46	0.672	0.36590	1.00539
47	0.673	0.36554	1.00637
48	0.674	0.36518	1.00736
49	0.675	0.36482	1.00835
50	0.676	0.36446	1.00934
51	0.991	0.26593	1.32451
52	0.992	0.26567	1.32549
53	0.993	0.26541	1.32647
54	0.994	0.26515	1.32745
55	0.995	0.26489	1.32843
56	0.996	0.26463	1.32941
57	0.997	0.26437	1.33040
58	0.998	0.26411	1.33138
59	0.999	0.26385	1.33237
60	1.000	0.26359	1.33335

PHREG PAPER: TEST BASE TWO page 12

OBS	UPPER	LOWER	MEAN
1	0.56229	0.56156	0.56193

PHREG PAPER: TEST BASE THREE page 13

The PHREG Procedure

Data Set: WORK.ZDATA
Dependent Variable: T
Censoring Variable: CLOSED
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
99936	68382	31554	31.57

NOTE: There are no explanatory variables in this model.

-2 LOG L = 1510536

PHREG PAPER: TEST BASE THREE pages 14-15

OBS	T	S	G
1	0.000	1.00000	0.00000
2	0.001	0.99900	0.00100
3	0.002	0.99800	0.00200
4	0.003	0.99700	0.00301
5	0.004	0.99600	0.00401
6	0.005	0.99500	0.00502
7	0.006	0.99400	0.00602
8	0.007	0.99300	0.00703
9	0.008	0.99199	0.00804
10	0.009	0.99099	0.00905
11	0.324	0.72320	0.32407
12	0.325	0.72248	0.32506
13	0.326	0.72176	0.32606
14	0.327	0.72104	0.32706
15	0.328	0.72032	0.32806
16	0.329	0.71960	0.32906
17	0.330	0.71888	0.33006
18	0.331	0.71816	0.33106
19	0.332	0.71744	0.33207
20	0.333	0.71672	0.33307
21	0.334	0.71564	0.33458
22	0.335	0.71457	0.33608
23	0.336	0.71350	0.33758
24	0.337	0.71243	0.33908
25	0.338	0.71136	0.34058
26	0.339	0.71028	0.34209
27	0.340	0.70921	0.34360
28	0.341	0.70814	0.34511
29	0.342	0.70708	0.34661
30	0.343	0.70603	0.34809
31	0.657	0.44661	0.80608
32	0.658	0.44598	0.80749
33	0.659	0.44535	0.80891
34	0.660	0.44471	0.81032
35	0.661	0.44408	0.81174
36	0.662	0.44345	0.81316
37	0.663	0.44282	0.81458
38	0.664	0.44219	0.81601
39	0.665	0.44156	0.81744
40	0.666	0.44093	0.81886
41	0.667	0.44049	0.81986
42	0.668	0.44005	0.82086
43	0.669	0.43961	0.82186
44	0.670	0.43917	0.82287
45	0.671	0.43873	0.82387
46	0.672	0.43829	0.82487
47	0.673	0.43785	0.82588
48	0.674	0.43742	0.82686
49	0.675	0.43699	0.82785
50	0.676	0.43656	0.82883
51	0.991	0.31856	1.14393
52	0.992	0.31824	1.14494
53	0.993	0.31792	1.14594
54	0.994	0.31760	1.14695
55	0.995	0.31729	1.14793
56	0.996	0.31698	1.14891
57	0.997	0.31667	1.14989
58	0.998	0.31636	1.15087
59	0.999	0.31605	1.15185
60	1.000	0.31574	1.15283

PHREG PAPER: TEST BASE THREE page 16

OBS	UPPER	LOWER	MEAN
1	0.59737	0.59669	0.59703

APPENDIX 2

SASLOG:

```
350         *****************************************************************;
351         ***BEGIN CODE FOR  CASE STUDY SECTION****************************;
352         %MACRO VLIST;
353                EMPL2
354                AY93-AY94
355                MF01 EC01
356                NOI_SPR NOI_CUT
357         %MEND VLIST;

NOTE: The data set WORK.ONE has 12512 observations and 100 variables.
358         DATA ONE;SET ONE;
359         KEEP T CLOSED01 TPA LAG2TPA %VLIST ;
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
360         *CREATE BASELINE SURVIVAL FUNCTION FOR VANISHING COVARIATES';
361         *TPA IS NON TIME-DEPENDENT REFERRAL VARIABLE';
362         TITLE 'PROPORTIONAL HAZARD MODEL FOR BASELINE';

NOTE: The data set WORK.ONE has 12512 observations and 11 variables.
363         PROC SORT DATA=ONE;BY TPA;

NOTE: The data set WORK.ONE has 12512 observations and 11 variables.

364         DATA INRISK;
365         INPUT %VLIST TPA;
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
366         CARDS;

NOTE: The data set WORK.INRISK has 1 observations and 8 variables.
368         ;
369         PROC PHREG SIMPLE DATA=ONE;
370              MODEL T*CLOSED01(0)= %VLIST TPA;
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
371              BASELINE COVARIATES=INRISK OUT=BASE SURVIVAL=S / NOMEAN;
NOTE: The data set WORK.BASE has 958 observations and 10 variables.
NOTE: The PROCEDURE PHREG printed pages 1-2.

372         DATA BASE;SET BASE;KEEP  T  S;IF T > 0;
NOTE: The data set WORK.BASE has 957 observations and 2 variables.
 373         PROC SORT DATA = BASE; BY T;

NOTE: The data set WORK.BASE has 957 observations and 2 variables.
374         DATA BASE;SET BASE END = EOF;
375         IF _N_ = 1 THEN DO;T = 0;S = 1;OUTPUT;END;
376         IF T < 1 THEN OUTPUT;
377         IF EOF OR T >= 1 THEN DO;T = 1;OUTPUT;END;

NOTE: The data set WORK.BASE has 959 observations and 2 variables.
378         PROC SORT NODUP DATA = BASE; BY T;
379         *CAPTURE BASELINE SURVIVAL FUNCTION ON [0,1] TO ARRAY TABLE';


NOTE: The data set WORK.BASE has 958 observations and 2 variables.
```

```
380         DATA BASE;SET BASE END=EOF;
381         ARRAY MATT(I) T1-T1000;
382         ARRAY MATS(I) S1-S1000;
383         KEEP   T1-T1000 S1-S1000;
384         RETAIN T1-T1000 S1-S1000;
385         I = MIN(_N_,1000);
386         MATT = T;
387         MATS = S;
388         IF EOF THEN DO;
389            DO I = _N_+ 1 TO 1000;
390               MATT = 1;
391               MATS = 0;
392               END;
393            OUTPUT;
394            END;
395         *RUN PROPOTIONAL HAZARD MODEL';
396         *TXPA X=1,2,3 ARE TIME-DEPENDENT REFERRAL VARIABLES';
397         TITLE 'PROPORTIONAL HAZARD MODEL WITH TIME DEPENDENT REFERRAL';

NOTE: The data set WORK.BASE has 1 observations and 2000 variables.
398         PROC PHREG SIMPLE DATA=ONE OUTEST=PARMS;
399              MODEL T*CLOSED01(0)= %VLIST
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
400                                   T1TPA T2TPA T3TPA;
401              IF TPA=1 & T >= LAG2TPA THEN TTPA=1;ELSE TTPA = 0;
402              IF 1/6  > T        THEN T1TPA=TTPA;ELSE T1TPA = 0;
403              IF 1/3  > T >= 1/6 THEN T2TPA=TTPA;ELSE T2TPA = 0;
404              IF        T >= 1/3 THEN T3TPA=TTPA;ELSE T3TPA = 0;
405         *DETERMINE PHI=REFERRAL RISK RATIO BY TLAYER;
406         TITLE 'HAZARD RATIO PHI BY TIME LAYER';
407         DATA PARMS;SET PARMS;
NOTE: The data set WORK.PARMS has 1 observations and 14 variables.
NOTE: The PROCEDURE PHREG printed pages 3-4.

408         KEEP TLAYER PHI;
409         TLAYER = 1;PHI = EXP(T1TPA);OUTPUT;
410         TLAYER = 2;PHI = EXP(T2TPA);OUTPUT;
411         TLAYER = 3;PHI = EXP(T3TPA);OUTPUT;

NOTE: The data set WORK.PARMS has 3 observations and 2 variables.
412         PROC PRINT DATA = PARMS;

NOTE: The PROCEDURE PRINT printed page 5.
413         DATA ONE;SET ONE;
414              IF 1/6  > T        THEN TLAYER = 1;
415                                 ELSE
416              IF 1/3  > T >= 1/6 THEN TLAYER = 2;
417                                 ELSE
418                                      TLAYER = 3;

NOTE: The data set WORK.ONE has 12512 observations and 12 variables.
419         PROC SORT DATA=ONE  ;BY TLAYER;


NOTE: The data set WORK.ONE has 12512 observations and 12 variables.
420         PROC SORT DATA=PARMS;BY TLAYER;
```

```
NOTE: The data set WORK.PARMS has 3 observations and 2 variables.
        DATA ONE;MERGE ONE(IN=INO) PARMS(IN=INP);BY TLAYER;
        IF INO & INP;

NOTE: The data set WORK.ONE has 12512 observations and 13 variables.
        PROC SORT DATA=ONE; BY TPA;
        *USE PHI AND BASELINE SURVIVAL ARRAY TO ADJUST T;

NOTE: The data set WORK.ONE has 12512 observations and 13 variables.
        DATA ONE;SET ONE;
        RETAIN T1-T1000 S1-S1000;
        ARRAY MATT(I) T1 - T1000;
        ARRAY MATS(I) S1 - S1000;
        DROP T1-T1000 S1-S1000;
        IF _N_ =1 THEN SET BASE;
        IF (TPA = 1) & (T < 1) THEN DO;
           ALPHA = LAG2TPA;
           LOOKUP = 0;I = 1;
           DO WHILE(LOOKUP = 0);
              LHT = MATT;LHS = MATS;
              I + 1;RHT = MATT;RHS = MATS;
              IF LHT <= ALPHA <= RHT THEN DO;
                 S_ALPHA = LHS + ((ALPHA - LHT)/(RHT - LHT))*(RHS - LHS);
                 LOOKUP = 1;
              END;
           END;
           LOOKUP = 0;I = 1;
           DO WHILE(LOOKUP = 0);
              LHT = MATT;LHS = MATS;
              I + 1;RHT = MATT;RHS = MATS;
              IF LHT <= T <= RHT THEN DO;
                 S_T = LHS + ((T - LHT)/(RHT - LHT))*(RHS - LHS);
                 LOOKUP = 1;
              END;
           END;
           S_ADJT = (S_T**PHI)*(S_ALPHA**(1-PHI));
           LOOKUP = 0;I = 1;
           DO WHILE(LOOKUP = 0);
              LHT = MATT;LHS = MATS;
              I + 1;RHT = MATT;RHS = MATS;
              IF LHS >= S_ADJT >= RHS THEN DO;
                 ADJT = RHT + ((S_ADJT - RHS)/(LHS - RHS))*(LHT - RHT);
                 LOOKUP = 1;
              END;
           END;
        END;
        ELSE DO;
           ADJT = T;
        END;
        *USE PHREG TO MAKE SURVIVAL FUNCTION AT MEANS FOR ACTUAL DURATION T;
        TITLE 'ACTUAL MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS';

NOTE: The data set WORK.ONE has 12512 observations and 24 variables.
        PROC PHREG SIMPLE DATA=ONE;BY TPA;
             MODEL T*CLOSED01(0)= %VLIST;
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
             BASELINE OUT=BASE SURVIVAL=S;
```

```
NOTE: The data set WORK.BASE has 1325 observations and 10 variables.
NOTE: The PROCEDURE PHREG printed pages 6-9.

          DATA MEAN;SET BASE;BY TPA;KEEP  TPA UPPER LOWER MEAN;
          RETAIN UPPER LOWER OLD_T OLD_S;
          IF FIRST.TPA THEN DO;
             UPPER = 0;
             LOWER = 0;
             OLD_T = 0;
             OLD_S = 1;
             END;
          IF OLD_T < T THEN DO;
             UPPER + (T - OLD_T)*OLD_S;
             LOWER + (T - OLD_T)*S;
             END;
             OLD_T = T;
             OLD_S = S;
          IF LAST.TPA THEN DO;
             MEAN = (UPPER + LOWER)/2;OUTPUT;
             END;

NOTE: The data set WORK.MEAN has 2 observations and 4 variables.

          DATA MEAN;SET MEAN;*CONVERT TO YEARS;
          UPPER = 3*UPPER;
          LOWER = 3*LOWER;
          MEAN  = 3*MEAN;

NOTE: The data set WORK.MEAN has 2 observations and 4 variables.
          PROC PRINT DATA = MEAN;
          *USE PHREG TO MAKE SURVIVAL FUNCTION AT MEANS FOR ADJUSTED DURATION;

NOTE: The PROCEDURE PRINT printed page 10.
          PROC PHREG SIMPLE DATA=ONE;BY TPA;
               MODEL ADJT*CLOSED01(0)= %VLIST;
MPRINT(VLIST):   EMPL2 AY93-AY94 MF01 EC01 NOI_SPR NOI_CUT
          BASELINE OUT=BASE SURVIVAL=S;
          TITLE 'ADJUSTED MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS';

NOTE: The data set WORK.BASE has 2311 observations and 10 variables.
NOTE: The PROCEDURE PHREG printed pages 11-14.

          DATA BASE;
          SET BASE;*RENAME ADJT TO BE SIMPLY T AS ABOVE;
          T = ADJT;

NOTE: The data set WORK.BASE has 2311 observations and 11 variables.
          DATA MEAN;SET BASE;BY TPA;KEEP  TPA UPPER LOWER MEAN;
          RETAIN UPPER LOWER OLD_T OLD_S;
          IF FIRST.TPA THEN DO;
             UPPER = 0;
             LOWER = 0;
             OLD_T = 0;
             OLD_S = 1;
             END;
          IF OLD_T < T THEN DO;
             UPPER + (T - OLD_T)*OLD_S;
```

```
510            LOWER + (T - OLD_T)*S;
511            END;
512            OLD_T = T;
513            OLD_S = S;
514         IF LAST.TPA THEN DO;
515            MEAN = (UPPER + LOWER)/2;OUTPUT;
516            END;

NOTE: The data set WORK.MEAN has 2 observations and 4 variables.
517         DATA MEAN;SET MEAN;*CONVERT TO YEARS;
518         UPPER = 3*UPPER;
519         LOWER = 3*LOWER;
520         MEAN  = 3*MEAN;

NOTE: The data set WORK.MEAN has 2 observations and 4 variables.
521         PROC PRINT DATA = MEAN;
NOTE: The PROCEDURE PRINT printed page 15.
```

SAS LISTING:

```
PROPORTIONAL HAZARD MODEL FOR BASELINE      page 1
 The PHREG Procedure

 Data Set: WORK.ONE
 Dependent Variable: T
 Censoring Variable: CLOSED01
 Censoring Value(s): 0
 Ties Handling: BRESLOW
```

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
12512	9961	2551	20.39

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	12512	0.10526	0.30690	0	1.00000
AY93	12512	0.33840	0.47318	0	1.00000
AY94	12512	0.33048	0.47041	0	1.00000
MF01	12512	0.57297	0.49467	0	1.00000
EC01	12512	0.56138	0.49624	0	1.00000
NOI_SPR	12512	0.68782	0.46340	0	1.00000
NOI_CUT	12512	0.24680	0.43117	0	1.00000
TPA	12512	0.17927	0.38359	0	1.00000

PROPORTIONAL HAZARD MODEL FOR BASELINE page 2
The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	171122.136	170068.900	1053.235 with 8 DF (p=0.0001)
Score	.	.	1052.856 with 8 DF (p=0.0001)
Wald	.	.	1034.622 with 8 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.275576	0.03598	58.67093	0.0001	0.759
AY93	1	0.347298	0.02421	205.72217	0.0001	1.415
AY94	1	0.591461	0.03204	340.80216	0.0001	1.807
MF01	1	-0.208174	0.02216	88.26442	0.0001	0.812
EC01	1	0.245834	0.02227	121.89395	0.0001	1.279
NOI_SPR	1	0.140682	0.04307	10.66704	0.0011	1.151
NOI_CUT	1	0.186939	0.04580	16.65742	0.0001	1.206
TPA	1	0.134552	0.03398	15.68328	0.0001	1.144

The PHREG Procedure

Data Set: WORK.ONE
Dependent Variable: T
Censoring Variable: CLOSED01
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
12512	9961	2551	20.39

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	12512	0.10526	0.30690	0	1.00000
AY93	12512	0.33840	0.47318	0	1.00000
AY94	12512	0.33048	0.47041	0	1.00000
MF01	12512	0.57297	0.49467	0	1.00000
EC01	12512	0.56138	0.49624	0	1.00000
NOI_SPR	12512	0.68782	0.46340	0	1.00000
NOI_CUT	12512	0.24680	0.43117	0	1.00000
T1TPA	12512	0.09407	0.29194	0	1.00000
T2TPA	12512	0.06554	0.24748	0	1.00000
T3TPA	12512	0.01966	0.13884	0	1.00000

WARNING: Simple statistics listed for the time-dependent explanatory variables have limited value.

HAZARD RATIO PHI BY TIME LAYER page 4

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	171122.136	170012.885	1109.251 with 10 DF (p=0.0001)
Score	.	.	1125.216 with 10 DF (p=0.0001)
Wald	.	.	1101.663 with 10 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.276409	0.03598	59.00264	0.0001	0.759
AY93	1	0.343170	0.02424	200.35078	0.0001	1.409
AY94	1	0.535551	0.03216	277.34071	0.0001	1.708
MF01	1	-0.207791	0.02216	87.95253	0.0001	0.812
EC01	1	0.245800	0.02227	121.79944	0.0001	1.279
NOI_SPR	1	0.143863	0.04308	11.15294	0.0008	1.155
NOI_CUT	1	0.189546	0.04581	17.11765	0.0001	1.209
T1TPA	1	0.353487	0.04301	67.54561	0.0001	1.424
T2TPA	1	0.184415	0.05265	12.26748	0.0005	1.203
T3TPA	1	0.114674	0.10755	1.13683	0.2863	1.122

HAZARD RATIO PHI BY TIME LAYER page 5

OBS	TLAYER	PHI
1	1	1.42402
2	2	1.20251
3	3	1.12151

TPA=0

The PHREG Procedure

Data Set: WORK.ONE
Dependent Variable: T
Censoring Variable: CLOSED01
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of Event and Censored Values

Total	Event	Censored	Percent Censored
10269	8544	1725	16.80

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	10269	0.11014	0.31308	0	1.00000
AY93	10269	0.38631	0.48693	0	1.00000
AY94	10269	0.21180	0.40861	0	1.00000
MF01	10269	0.57162	0.49487	0	1.00000
EC01	10269	0.56062	0.49634	0	1.00000
NOI_SPR	10269	0.68410	0.46490	0	1.00000
NOI_CUT	10269	0.25280	0.43464	0	1.00000

ACTUAL MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 7

TPA=0

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	143854.615	143106.464	748.151 with 7 DF (p=0.0001)
Score	.	.	743.549 with 7 DF (p=0.0001)
Wald	.	.	733.247 with 7 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.289695	0.03776	58.86290	0.0001	0.748
AY93	1	0.341809	0.02451	194.49158	0.0001	1.407
AY94	1	0.573640	0.03373	289.16531	0.0001	1.775
MF01	1	-0.195073	0.02391	66.55687	0.0001	0.823
EC01	1	0.230103	0.02395	92.26850	0.0001	1.259
NOI_SPR	1	0.149435	0.04673	10.22562	0.0014	1.161
NOI_CUT	1	0.211587	0.04956	18.22332	0.0001	1.236

ACTUAL MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 8

TPA=1

The PHREG Procedure

Data Set: WORK.ONE
Dependent Variable: T
Censoring Variable: CLOSED01
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
2243	1417	826	36.83

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	2243	0.08292	0.27583	0	1.00000
AY93	2243	0.11904	0.32390	0	1.00000
AY94	2243	0.87383	0.33212	0	1.00000
MF01	2243	0.57914	0.49381	0	1.00000
EC01	2243	0.56487	0.49588	0	1.00000
NOI_SPR	2243	0.70486	0.45621	0	1.00000
NOI_CUT	2243	0.21935	0.41390	0	1.00000

ACTUAL MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 9

TPA=1

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	19580.571	19429.633	150.938 with 7 DF (p=0.0001)
Score	.	.	138.667 with 7 DF (p=0.0001)
Wald	.	.	131.971 with 7 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.033574	0.11954	0.07888	0.7788	0.967
AY93	1	1.451228	0.38774	14.00880	0.0002	4.268
AY94	1	1.734945	0.38948	19.84265	0.0001	5.669
MF01	1	-0.281972	0.05898	22.85722	0.0001	0.754
EC01	1	0.342128	0.06101	31.44825	0.0001	1.408
NOI_SPR	1	0.091231	0.11133	0.67149	0.4125	1.096
NOI_CUT	1	0.033698	0.12062	0.07805	0.7800	1.034

ACTUAL MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 10

OBS	TPA	UPPER	LOWER	MEAN
1	0	1.02820	1.02542	1.02681
2	1	0.74414	0.73050	0.73732

ADJUSTED MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 11
TPA=0

The PHREG Procedure

Data Set: WORK.ONE
Dependent Variable: ADJT
Censoring Variable: CLOSED01
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
10269	8544	1725	16.80

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	10269	0.11014	0.31308	0	1.00000
AY93	10269	0.38631	0.48693	0	1.00000
AY94	10269	0.21180	0.40861	0	1.00000
MF01	10269	0.57162	0.49487	0	1.00000
EC01	10269	0.56062	0.49634	0	1.00000
NOI_SPR	10269	0.68410	0.46490	0	1.00000
NOI_CUT	10269	0.25280	0.43464	0	1.00000

TPA=0

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	143854.615	143106.464	748.151 with 7 DF (p=0.0001)
Score	.	.	743.549 with 7 DF (p=0.0001)
Wald	.	.	733.247 with 7 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.289695	0.03776	58.86290	0.0001	0.748
AY93	1	0.341809	0.02451	194.49158	0.0001	1.407
AY94	1	0.573640	0.03373	289.16531	0.0001	1.775
MF01	1	-0.195073	0.02391	66.55687	0.0001	0.823
EC01	1	0.230103	0.02395	92.26850	0.0001	1.259
NOI_SPR	1	0.149435	0.04673	10.22562	0.0014	1.161
NOI_CUT	1	0.211587	0.04956	18.22332	0.0001	1.236

ADJUSTED MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 13

TPA=1

The PHREG Procedure

Data Set: WORK.ONE
Dependent Variable: ADJT
Censoring Variable: CLOSED01
Censoring Value(s): 0
Ties Handling: BRESLOW

Summary of the Number of
Event and Censored Values

Total	Event	Censored	Percent Censored
2243	1417	826	36.83

Simple Statistics for Explanatory Variables

Total Sample

Variable	N	Mean	Standard Deviation	Minimum	Maximum
EMPL2	2243	0.08292	0.27583	0	1.00000
AY93	2243	0.11904	0.32390	0	1.00000
AY94	2243	0.87383	0.33212	0	1.00000
MF01	2243	0.57914	0.49381	0	1.00000
EC01	2243	0.56487	0.49588	0	1.00000
NOI_SPR	2243	0.70486	0.45621	0	1.00000
NOI_CUT	2243	0.21935	0.41390	0	1.00000

ADJUSTED MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 14

TPA=1

The PHREG Procedure

Testing Global Null Hypothesis: BETA=0

Criterion	Without Covariates	With Covariates	Model Chi-Square
-2 LOG L	19564.220	19414.420	149.800 with 7 DF (p=0.0001)
Score	.	.	137.480 with 7 DF (p=0.0001)
Wald	.	.	130.785 with 7 DF (p=0.0001)

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Wald Chi-Square	Pr > Chi-Square	Risk Ratio
EMPL2	1	-0.027966	0.11955	0.05472	0.8150	0.972
AY93	1	1.456546	0.38758	14.12275	0.0002	4.291
AY94	1	1.738075	0.38932	19.93071	0.0001	5.686
MF01	1	-0.281598	0.05895	22.81847	0.0001	0.755
EC01	1	0.339608	0.06097	31.02579	0.0001	1.404
NOI_SPR	1	0.089939	0.11134	0.65252	0.4192	1.094
NOI_CUT	1	0.033703	0.12061	0.07808	0.7799	1.034

ADJUSTED MEAN DURATION FROM SURVIVAL FUNCTION AT MEANS page 15

OBS	TPA	UPPER	LOWER	MEAN
1	0	1.02820	1.02542	1.02681
2	1	0.83222	0.81883	0.82552