A Note on the Inverse Relationship Between Hazard and Life Expectancy

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Abstract:

Intuitively, life expectancy and hazard rate should be inversely related to each other. Whereas life expectancy, or mean time to failure, is determinable as a simple descriptive statistic, the concept of hazard is defined as an instantaneous failure rate and involves taking limits. This note investigates "inverting" life expectancy as a method for estimating the hazard rate. The main result is that given any finite collection of (internally consistent) pairs of age and associated life expectancy values, there is a uniquely determined step function that determines a "gauntlet" survival model with the given life expectancies at their respective ages. The Appendix provides a simple computer algorithm for implementing this model in practice.

I. Introduction

In general, life expectancy is determinable as a simple descriptive statistic. It is both easier to interpret and to estimate than the hazard rate, which is defined as an instantaneous failure rate and involves taking limits. When working with insurance data, "claim life expectancy" is often available as a reserve (c.f. [3]). In practice, reserves may be related with claim survival data only to the extent that closed, i.e. "dead", cases are characterized by having no reserves. On the other hand, knowledge of the hazard rate function is useful for many insurance applications (c.f. [6]). It might be very useful, therefore, to be able to go directly from life expectancy to the hazard rate.

In the exponential decay survival model, for example, life expectancy and hazard are both constant and inverse to each other. If you were confronted with survival data, you might observe the expectation of life early on to get an intuitive feel. If the life expectation were fairly constant, you would naturally gravitate to the exponential decay model and you would already know to assign the reciprocal of the mean time to failure as the constant hazard. This note suggests a generalization of this simple approach, detailing how to approximate hazard with a step function directly from information on life expectancy.

While this approach is just an alternative organization to the usual way of empirically calculating hazard, it has some technical and conceptual advantages. In particular, the approach is simple to explain and amenable to implementation on a computer. Censored observations are handled in a transparent fashion. Moreover, the technique can be extended to higher dimensions (c.f. [4]). As noted, in the case of insurance applications, reserves can be regarded as life expectancies and so the method provides a direct way of incorporating reserves into hazard models.

II. Notation and Background

Let $f(t)$ denote a continuous function on the nonnegative real numbers $\Re_+ = [0,\infty)$ satisfying:

$$\int_0^\infty f(t)dt = 1$$

Regard $f(t)$ as a probability density of failure times and define the function:

$$S(t) = 1 - \int_0^t f(s)ds = \int_t^\infty f(s)ds$$

As is customary, we refer to $S(t)$ as the *survival function*, $f(t)$ as the *probability density function [PDF]* and t as "time". We also let T denote the random variable for the distribution of survival times and $\mu = E(T)$ the mean duration, which we assume throughout to be finite. Survival analysis refers to the following function:

$$h(t) = \frac{f(t)}{S(t)}$$

as the *hazard rate function* or sometimes as the *force of mortality*. The hazard rate function measures the instantaneous rate of failure at time t and can be expressed as a limit of conditional probabilities:

$$h(t) = \lim_{\Delta t \to 0} \frac{\Pr\{t \le T < t + \Delta t \mid T \ge t\}}{\Delta t}$$

There are many well-known relationships and interpretations of these functions—refer to Allison[1] for a particularly succinct discussion;. It is convenient to recall that setting

$$g(t) = \int_0^t h(s)ds \text{ then } S(t) = e^{-g(t)}.$$

Fix t and restrict attention to values of time $w > t$. The conditional probability of survival to w, given survival to t, is $S_t(w) = \dfrac{S(w)}{S(t)}$. In this context (see [3]), the *expectation of life at time t*, given survival to time t, is just:

$$\rho(t) = \frac{\int_t^\infty (w-t)f(w)dw}{\int_t^\infty f(w)dw} = \int_t^\infty S_t(w)dw = \int_t^\infty \frac{S(w)}{S(t)}dw$$

Observe that under our assumptions, $\rho(0) = \mu$ and the function $\rho(t)$ is well defined for all t>0. . Observe too that for any a<b with $S(a) > 0$ we have the relation:

$$\rho(a)S(a) = \int_a^\infty S(t)dt = \int_a^b S(t)dt + \int_b^\infty S(t)dt$$

$$\le \int_a^b S(a)dt + \int_b^\infty S(t)dt = S(a)(b-a) + \rho(b)S(b)$$

$$\Rightarrow \quad a + \rho(a) \le b + \frac{\rho(b)S(b)}{S(a)} \le b + \rho(b)$$

with strict inequality exactly when $S(b) < S(a)$.

This paper concerns itself with how the two functions $h(t), \rho(t)$ relate to each other. While we might expect an inverse relationship of some sort, note that the two are conceptually quite different: h is local while ρ is global. Still, it is reasonable to expect that the average values of h over an appropriate interval might relate with the values of ρ over that interval.

Example: Suppose the expectation of life (mean time to failure) is constant on the interval $[a,b)$, $\rho(t)=\alpha, a\le t<b$, including the case $b=\infty$. Then

$$\alpha S(t)=\rho(t)S(t)=\int_t^\infty S(w)dw$$

$$\Rightarrow \quad \alpha\frac{dS}{dt}=-S(t) \quad \Rightarrow S(t)=e^{-\frac{t}{\alpha}}$$

$$\Rightarrow g(t)=\frac{t}{\alpha}\Rightarrow h(t)=\frac{dg}{dt}=\frac{1}{\alpha} \quad a\le t<b$$

The following proposition generalizes this:

Proposition 1: For any real numbers $a<b$ with $S(a)>0$, there exists a $\varsigma\in[a,b]$ with:

$$h(\varsigma)=\frac{S(a)-S(b)}{S(a)\rho(a)-S(b)\rho(b)}$$

Proof: Consider the integral $\int_a^b S(t)h(t)dt$. Because $S(t)$ is nonnegative, the intermediate value theorem for integrals implies there is $\varsigma\in[a,b]$ with:

$$\int_a^b S(t)h(t)dt=h(\varsigma)\int_a^b S(t)dt=h(\varsigma)\left(\int_a^\infty S(t)dt-\int_b^\infty S(t)dt\right)=h(\varsigma)(S(a)\rho(a)-S(b)\rho(b))$$

On the other hand, taking $u(t)=-g(t)=-\int_0^t h(w)dw,\quad \frac{du}{dt}=-h(t)$ and we have:

$$\int_a^b S(t)h(t)dt=-\int_{-g(a)}^{-g(b)} e^u du=e^{-g(a)}-e^{-g(b)}=S(a)-S(b)$$

and the result follows.

Not surprisingly, there are formal relationships between hazard $h(t)$ and life expectancy $\rho(t)$, as in:

Proposition 2:

$i)\qquad 1+\dfrac{d\rho}{dt}=h(t)\rho(t)$

$ii)\qquad \rho(t)>0\Rightarrow h(t)=\dfrac{1}{\rho(t)}+\dfrac{d(\ln\rho)}{dt}$

$iii)\qquad \rho(t)>0\Rightarrow \dfrac{1}{\rho(t)}=-\dfrac{d(\ln\rho S)}{dt}$

$iv)\qquad \lim_{t\to\infty}\rho(t)=\lim_{t\to\infty}\dfrac{1}{h(t)}$

Proof: The verification is straightforward: from the definition of $\rho(t)$ and the formula for differentiating a ratio:

$$\frac{d\rho}{dt}=\frac{S(t)(-S(t))-\int_t^{\infty}S(w)dw\left(\frac{dS}{dt}\right)}{S(t)^2}=\frac{f(t)\int_t^{\infty}S(w)dw-S(t)^2}{S(t)^2}$$

$$=\frac{f(t)}{S(t)}\frac{\int_t^{\infty}S(w)dw}{S(t)}-1=h(t)\rho(t)-1$$

$$\Rightarrow 1+\frac{d\rho}{dt}=h(t)\rho(t)$$

establishing i); ii) is immediate from i):

$$\rho(t)>0\Rightarrow h(t)=\frac{h(t)\rho(t)}{\rho(t)}=\frac{1+\frac{d\rho}{dt}}{\rho(t)}=\frac{1}{\rho(t)}+\frac{\frac{d\rho}{dt}}{\rho(t)}=\frac{1}{\rho(t)}+\frac{d\ln(\rho)}{dt}$$

And iii) can be readily derived from ii):

$$\rho(t)>0\Rightarrow -\frac{d\ln(S)}{dt}=h(t)=\frac{1}{\rho(t)}+\frac{d\ln(\rho)}{dt}$$

$$\Rightarrow \frac{1}{\rho(t)}=-\frac{d\ln(S)}{dt}-\frac{d\ln(\rho)}{dt}=-\frac{d(\ln(S)+\ln(\rho))}{dt}=-\frac{d\ln(\rho S)}{dt}$$

Finally iv) is a straightforward application of L'Hôpital's rule (see [5] p.90): indeed, under our assumptions we have:

$$\mu = \int_0^\infty S(t)dt = \lim_{t\to\infty}\int_0^\infty S(w)dw = \lim_{t\to\infty}\int_0^t S(w)dw + \int_t^\infty S(w)dw$$

$$= \int_0^\infty S(t)dt + \lim_{t\to\infty}\int_t^\infty S(w)dw = \mu + \lim_{t\to\infty}\int_t^\infty S(w)dw$$

$$\Rightarrow 0 = \lim_{t\to\infty}\int_t^\infty S(w)dw = \lim_{t\to\infty} S(t)$$

So invoking L'Hôpital's rule:

$$\lim_{t\to\infty}\rho(t) = \lim_{t\to\infty}\frac{\int_t^\infty S(w)dw}{S(t)} = \lim_{t\to\infty}\frac{-S(t)}{-f(t)} = \lim_{t\to\infty}\frac{1}{h(t)}$$

completing the proof.

It is easy to see that the expectation of life function uniquely determines the survival model. Indeed, Proposition 2 shows that the function $\rho(t)$ determines the hazard function $h(t)$ and whence specifies the complete survivorship model. Proposition 2 also generalizes the inverse relationship between survival and hazard noted for the exponential decay model. Indeed, it shows that in general hazard and life expectancy do *not* follow a simple inverse relationship. Indeed, $h(t)$ is the sum of *two* components, one inversely related and the other directly related to $\rho(t)$. More precisely, hazard consists of a "first order" component in fact being the inverse of $\rho(t)$ and a "second order" component responding to the proportional change in $\rho(t)$ as captured by the latter's logarithmic derivative.

Our interest is in finding a more "elementary" relationship between $h(t)$ and $\rho(t)$ -- preferably one amenable to calculation from empirical discrete data and, in particular, one that avoids derivatives.

The following technical lemma is the key result needed to invert life expectancy to hazard and its proof blueprints an algorithm for the calculation.

Lemma: For any triplet of positive real numbers $\alpha, \beta, \gamma > 0$ with $\gamma > 1 - \frac{\beta}{\alpha}$, there exists a *unique* $\eta > 0$ such that:

$$\alpha\eta = \frac{e^{\beta\eta} - 1}{e^{\beta\eta} - \gamma}$$

Proof: Consider the function

$$\psi(x) = \psi(\alpha, \beta, \gamma; x) = \alpha x - \frac{e^{\beta x} - 1}{e^{\beta x} - \gamma}$$

the lemma asserts that $\psi(x)$ has exactly 1 positive real root. Define

$$\varphi(x) = \frac{d\psi}{dx} = \alpha - \frac{(e^{\beta x} - \gamma)(\beta e^{\beta x}) - (e^{\beta x} - 1)(\beta e^{\beta x})}{\left(e^{\beta x} - \gamma\right)^2}$$

$$= \alpha + \frac{\beta e^{\beta x}(\gamma - 1)}{\left(e^{\beta x} - \gamma\right)^2}$$

We consider three cases:

Case $\gamma = 1$: Here $\psi(x) = \alpha x - 1$ clearly has unique positive root $\frac{1}{\alpha}$.

Case $\gamma < 1$: In this case, we first verify that $\varphi(x)$ has a unique positive root. Indeed, noting that for $x>0$, $e^{\beta x} > 1 > \gamma \Rightarrow e^{\beta x} - \gamma > 0$, we find that:

$$\varphi(x) = 0$$

$$\Leftrightarrow \alpha(e^{\beta x} - \gamma)^2 = \beta e^{\beta x}(1 - \gamma)$$

$$\Leftrightarrow e^{\beta x} - \gamma = \sqrt{\frac{\beta e^{\beta x}(1-\gamma)}{\alpha}} = \sqrt{e^{\beta x}}\sqrt{\frac{\beta(1-\gamma)}{\alpha}}$$

Letting $y = \sqrt{e^{\beta x}}$ this equation becomes:

$$y^2 - \sqrt{\frac{\beta(1-\gamma)}{\alpha}}y - \gamma = 0$$

which has roots:

$$\sqrt{\frac{\beta(1-\gamma)}{4\alpha}} \pm \sqrt{\frac{\beta(1-\gamma)}{4\alpha} + \gamma}$$

only one of which is >0, and so

$$
\begin{aligned}
y &= \sqrt{\frac{\beta(1-\gamma)}{4\alpha}} + \sqrt{\frac{\beta(1-\gamma)}{4\alpha} + \gamma} \\
\Rightarrow e^{\beta x} &= y^2 = \gamma + y\sqrt{\frac{\beta(1-\gamma)}{4\alpha}} \\
&= \gamma + \left(\sqrt{\frac{\beta(1-\gamma)}{4\alpha}} + \sqrt{\frac{\beta(1-\gamma)}{4\alpha} + \gamma}\right)\sqrt{\frac{\beta(1-\gamma)}{4\alpha}} \\
&= \gamma + \frac{\beta(1-\gamma)}{2\alpha} + \sqrt{\frac{\beta(1-\gamma)}{2\alpha}\left(\frac{\beta(1-\gamma)}{2\alpha} + 2\gamma\right)}
\end{aligned}
$$

It follows that setting

$$
\tau = \frac{\ln\left(\gamma + \frac{\beta(1-\gamma)}{2\alpha} + \sqrt{\frac{\beta(1-\gamma)}{2\alpha}\left(\frac{\beta(1-\gamma)}{2\alpha} + 2\gamma\right)}\right)}{\beta}
$$

then τ is the unique positive root of $\varphi(x) = \frac{d\psi}{dx}$. Note that

$$
\begin{aligned}
&\gamma > 1 - \frac{\beta}{\alpha} \Rightarrow \frac{\beta}{\alpha} > 1 - \gamma > 0 \Rightarrow \frac{\beta}{1-\gamma} > \alpha \\
&\Rightarrow \varphi(0) = \alpha + \frac{\beta(\gamma - 1)}{(\gamma - 1)^2} = \alpha - \frac{\beta}{1-\gamma} < 0
\end{aligned}
$$

and it follows that $\psi(x)$ is decreasing on $(0, \tau)$. The next claim is that $\psi(x) < 0$ for x positive and near 0. To verify this, consider:

$$
\lambda(x) = \frac{\alpha x e^{\beta x} - \alpha\gamma x}{e^{\beta x} - 1}
$$

Combining the assumption that $\gamma > 1 - \frac{\beta}{\alpha}$ with L'Hospital's rule, we find that:

$$
\lim_{x \to 0} \lambda(x) = \lim_{x \to 0} \frac{\alpha\beta x e^{\beta x} + \alpha e^{\beta x} - \alpha\gamma}{\beta e^{\beta x}} = \frac{\alpha}{\beta}(1 - \gamma) < 1
$$

This means there exists $\varepsilon > 0$ such that $\lambda(x) < 1$ for $0 < x < \varepsilon$. Since $e^{\beta x} > 1 > \gamma$, we have that

$$1 > \lambda(x) = \frac{\alpha x e^{\beta x} - \alpha\gamma x}{e^{\beta x} - 1}$$

$$\Leftrightarrow e^{\beta x} - 1 > \alpha x e^{\beta x} - \alpha\gamma x = \alpha x(e^{\beta x} - \gamma) > 0$$

$$\Leftrightarrow \frac{e^{\beta x} - 1}{e^{\beta x} - \gamma} > \alpha x \Leftrightarrow \psi(x) = \alpha x - \frac{e^{\beta x} - 1}{e^{\beta x} - \gamma} < 0$$

proving the claim. It follows that $\psi(x)$, which is negative near 0, remains negative and can have no root in $(0, \tau)$ since $\psi(x)$ is decreasing over that interval. On the other hand, observe that

$$1 > \gamma \Rightarrow 0 < e^{\frac{\beta}{\alpha}} - 1 < e^{\frac{\beta}{\alpha}} - \gamma$$

$$\Rightarrow \psi(\frac{1}{\alpha}) = \frac{\alpha}{\alpha} - \frac{e^{\frac{\beta}{\alpha}} - 1}{e^{\frac{\beta}{\alpha}} - \gamma} = 1 - \frac{e^{\frac{\beta}{\alpha}} - 1}{e^{\frac{\beta}{\alpha}} - \gamma} > 0$$

Which means that $\psi(x)$ increases from negative to positive with a unique root on $[\tau, \frac{1}{\alpha}]$ and remains positive and increasing on $(\frac{1}{\alpha}, \infty)$. In particular, $\psi(x)$ has a unique positive root and the lemma is established for the case $\gamma < 1$. This leaves only the remaining:

Case $\gamma > 1$: In this case $\gamma - 1 > 0$ clearly implies that:

$$\varphi(x) = \frac{d\psi}{dx} = \alpha + \frac{\beta e^{\beta x}(\gamma - 1)}{\left(e^{\beta x} - \gamma\right)^2} > 0$$

and so $\psi(x)$ is monotonic increasing and can therefore have at most one root in any interval in its domain. We therefore need to investigate the behavior of $\psi(x)$ at 0 and $\delta = \frac{\ln(\gamma)}{\beta}$. We evidently have the following one-sided limits:

$$\lim_{x>0, x\to 0} \psi(x) = \lim_{x>0, x\to 0} \alpha x - \frac{e^{x} - 1}{e^{x} - \gamma} = 0 - \frac{0}{1 - \gamma} = 0$$

$$\lim_{x>\delta, x\to\delta} \psi(x) = \frac{\alpha \ln(\gamma)}{\beta} - (\gamma - 1)\left(\lim_{e^{t}>\gamma, t\to\ln(\gamma)} \frac{1}{e^{xt} - \gamma}\right) = \frac{\alpha \ln(\gamma)}{\beta} - (+\infty) = -\infty$$

$$\lim_{x<\delta, x\to\delta} \psi(x) = \frac{\alpha \ln(\gamma)}{\beta} - (\gamma-1)\left(\lim_{e^t<\gamma, t\to\ln(\gamma)} \frac{1}{e^{xt}-\gamma} \right) = \frac{\alpha \ln(\gamma)}{\beta} - (-\infty) = +\infty$$

Let $\omega = 3\gamma + e^{\frac{2\beta}{\alpha}}$ and $\varepsilon = \frac{\ln(\omega)}{\beta} > \frac{\ln(\gamma)}{\beta} = \delta$, the claim is that $\psi(\varepsilon) > 0$. To verify this, note that

$$\begin{aligned} &\omega > 3\gamma \Rightarrow \omega - \gamma > 2\gamma > 2\gamma - 2 \\ &\Rightarrow \frac{\omega - \gamma}{2} > \gamma - 1 > 0 \\ &\Rightarrow \frac{1}{2} > \frac{\gamma - 1}{\omega - \gamma} > 0 \end{aligned}$$

Similarly:

$$\begin{aligned} &\omega > e^{\frac{2\beta}{\alpha}} \Rightarrow \ln(\omega) > \frac{2\beta}{\alpha} \\ &\Rightarrow \frac{\alpha}{\beta}\ln(\omega) > 2 > \frac{3}{2} > 1 + \frac{\gamma-1}{\omega-\gamma} = \frac{\omega-1}{\omega-\gamma} \end{aligned}$$

From the definitions we find that:

$$\psi(\varepsilon) = \frac{\alpha}{\beta}\ln(\omega) - \frac{\omega-1}{\omega-\gamma} > 0,$$

which establishes the claim. We have shown that $\psi(x)$ is positive, in fact is monotonic increasing from 0 upward on $(0,\delta)$, that $\psi(x)$ increases monotonically from negative to positive with a unique root in $(\delta,\varepsilon]$, and $\psi(x)$ is positive and monotonic increasing on (ε,∞). This proves the assertion in the case $\gamma > 1$ and completes the proof of the lemma.

Now consider a positive interval $[a,b)$ on which the hazard is flat:

$$\begin{aligned} &h(t) = \eta, \quad a \le t < b \\ &\Rightarrow g(b) - g(a) = \int_a^b h(t)dt = \eta(b-a) \\ &\Rightarrow \frac{S(a)}{S(b)} = e^{\eta(b-a)} \end{aligned}$$

Clearly $\eta = 0 \Leftrightarrow S(a) = S(b)$ so consider the case $\eta > 0$. Proposition 1 implies that:

$$\eta = \frac{S(a)-S(b)}{S(a)\rho(a)-S(b)\rho(b)}$$

$$\Leftrightarrow \rho(a)\eta = \frac{S(a)-S(b)}{S(a)-S(b)\left(\frac{\rho(b)}{\rho(a)}\right)} = \frac{\frac{S(a)}{S(b)}-1}{\frac{S(a)}{S(b)}-\left(\frac{\rho(b)}{\rho(a)}\right)} = \frac{e^{\eta(b-a)}-1}{e^{\eta(b-a)}-\left(\frac{\rho(b)}{\rho(a)}\right)}$$

$$\Leftrightarrow \psi(\rho(a), b-a, \frac{\rho(b)}{\rho(a)}; \eta) = 0$$

Note too that since $\eta > 0$:

$$a+\rho(a) < b+\rho(b) \Leftrightarrow 1-\frac{b-a}{\rho(a)} < \frac{\rho(b)}{\rho(a)}$$

In [2], a survivorship model whose hazard is a step function is quite naturally described as a *gauntlet* survivorship model. The main result of this note is that any collection of life expectations that is finite and satisfies the above inequality can be approximated by a gauntlet survivorship model. In fact, the associated gauntlet is essentially a canonical form hazard approximation and the Appendix provides a computer algorithm for determining it.

Theorem: Given an ordered sequence of pairs of real numbers $\{(a_i, \alpha_i) \mid 1 \le i \le n\}$ such that:

i) $0 = a_1 < \ldots < a_i < a_{i+1} < \ldots < a_n$

ii) $\alpha_i > 0, 1 \le i \le n$

iii) $1-\frac{a_{i+1}-a_i}{\alpha_i} \le \frac{\alpha_{i+1}}{\alpha_i}, 1 \le i \le n$

And with the function ψ as in the lemma, define the step function $h:\Re_+ \to \Re_+$ as follows:

$$h(t) = \frac{1}{\alpha_n}, \quad t \ge a_n$$

$$h(t) = \begin{cases} 0 & 1-\frac{a_{i+1}-a_i}{\alpha_i} = \frac{\alpha_{i+1}}{\alpha_i} \\ \psi^{-1}(\alpha_i, a_{i+1}-a_i, \frac{\alpha_{i+1}}{\alpha_i}; \{0\}) & 1-\frac{a_{i+1}-a_i}{\alpha_i} < \frac{\alpha_{i+1}}{\alpha_i} \end{cases} \quad a_i \le t < a_{i+1}, \quad 1 \le i < n$$

Then the survivorship model determined by the hazard function $h(t)$ has expectation of life function $\rho(t)$ satisfying $\rho(a_i) = \alpha_i, 1 \le i \le n$.

Proof. The lemma guarantees that the function $h:\Re_+ \to \Re_+$ is well defined and the above example shows that:

$$\rho(t)=\frac{1}{\frac{1}{\alpha_n}}=\alpha_n,\quad t\geq a_n$$

The proof is by contradiction. Assuming the result false means that there is an $i<n$ such that:

$$\rho(a_j)=\alpha_j,\, i+1\leq j\leq n \qquad \rho(a_i)\neq\alpha_i$$

Set

$$\beta=a_{i+1}-a_i,\quad \gamma=\frac{\alpha_{i+1}}{\alpha_i}=\frac{\rho(a_{i+1})}{\alpha_i},\quad h(t)\equiv\eta \text{ on } [a_i,a_{i+1})$$

Suppose first that

$$1-\frac{a_{i+1}-a_i}{\alpha_i}<\frac{\alpha_{i+1}}{\alpha_i}$$

By definition of h, this implies that

$$\psi(\alpha_i,\beta,\gamma;\eta)=0$$
$$\Rightarrow \eta\alpha_i=\frac{e^{\beta\eta}-1}{e^{\beta\eta}-\gamma}$$

The comments just proceeding the statement of the theorem applied to the hazard function $h(t)\equiv\eta$ on $[a_i,a_{i+1})$ show that:

$$\psi(\rho(a_i),\beta,\frac{\rho(a_{i+1})}{\rho(a_i)};\eta)=0$$
$$\Rightarrow \eta\rho(a_i)=\frac{e^{\beta\eta}-1}{e^{\beta\eta}-\frac{\rho(a_{i+1})}{\rho(a_i)}}=\frac{e^{\beta\eta}-1}{e^{\beta\eta}-\gamma\left(\frac{\alpha_i}{\rho(a_i)}\right)}$$

It follows that:

$$\eta\rho(a_i)\left(e^{\beta\eta}-\gamma\left(\frac{\alpha_i}{\rho(a_i)}\right)\right)=e^{\beta\eta}-1=\eta\alpha_i\left(e^{\beta\eta}-\gamma\right)$$
$$\Rightarrow \rho(a_i)e^{\beta\eta}-\alpha_i\gamma=\alpha_i e^{\beta\eta}-\alpha_i\gamma$$
$$\Rightarrow \rho(a_i)e^{\beta\eta}=\alpha_i e^{\beta\eta}\Rightarrow \rho(a_i)=\alpha_i$$

which contradicts the choice of i. We must therefore have:

$$1-\frac{a_{i+1}-a_i}{\alpha_i}=\frac{\alpha_{i+1}}{\alpha_i}$$
$$\Leftrightarrow \alpha_i-(a_{i+1}-a_i)=\alpha_{i+1}$$
$$\Leftrightarrow \alpha_i=a_{i+1}-a_i+\alpha_{i+1}$$

However, from our earlier observations on the hazard function $h(t)\equiv 0$ on $[a_i,a_{i+1})$, in this event:

$$S(a_i)=S(a_{i+1})$$
$$\Rightarrow a_i+\rho(a_i)=a_{i+1}+\rho(a_{i+1})=a_{i+1}+\alpha_{i+1}$$
$$\Rightarrow \rho(a_i)=a_{i+1}+\alpha_{i+1}-a_i=\alpha_i$$

This contradiction completes the proof of the theorem.

Remark: Compare the definition

$$h(t)=\frac{1}{\alpha_n},\quad t\ge a_n$$

of the Theorem with Proposition 2 (iv).

Remark: The discussion in [5; pp148-156] points out some shortcomings in the state of the art as regards the application of bivariate loss distributions. In [4] the survival model structure is generalized to higher dimensions using the concept of a hazard vector field $\eta:\Im\to\Im$ and its associated survival vector field $\rho:\Im\to\Im$, using the notation of that paper. Among the observations in that paper is the relationship:

$$b\in\Im_a\Rightarrow b+\rho(b)\in\Im_{a+\rho(a)}$$

Given any assignment of survival vectors to a finite discrete rectangular lattice $L\subset\Im$ that satisfies this consistency condition, the methods derived here can be applied to determine a "gauntlet" hazard vector field whose associated survival vector field coincides with the original assignment of survival vectors on L. Indeed, the primary motivation for this note was to seek a way of determining hazard that was amenable to vector arithmetic.

References

[1] Allison, Paul D., *Survival Analysis Using the SAS® System: A Practical Guide,* The SAS Institute, Inc., 1995.

[2] Corro, Dan, *Determining the Change in Mean Duration Due to a Shift in the Hazard Rate Function*, CAS Forum, Winter 2001.

[3] Corro, Dan, *Modeling Loss Development with Micro Data*, CAS Forum, Fall 2000.

[4] Corro, Dan, *Modeling Multi-Dimensional Survival with Hazard Vector Fields*, CAS Forum, Winter 2001.

[5] Klugman, Stuart A., Panjer, Harry H., Gordon E. Willmot, *Loss Models,* Wiley Series in Probability and Statistics, John Wiley & Sons, Inc.,1998..

[6] Wang, Shaun, *"Implementation of Proportional Hazards Transforms in Ratemaking"*, PCAS LXXXV, pp. 940-979.

APPENDIX

The SAS LOG includes both source code and annotations of a sample run whose output is in the SAS LISTING that follows the log. The SAS syntax is readily adapted to any programming context that supports conditional loop processing.

```
SAS LOG:
 NOTE: The initialization phase used 0.07 CPU seconds and 6068K.
 1          ***************************************************************;
 2          ***       INVERSTING MEAN FAILURE TIME    ***********************;
 3          ***************************************************************;
 4          OPTIONS MPRINT LS=131 PS=59 NOCENTER;
 5          TITLE 'INVERTING MEAN FALURE TIME';
 6          DATA ONE;
 7          INPUT A ALPHA;
 8          CARDS;

 NOTE: The data set WORK.ONE has 6 observations and 2 variables.
 NOTE: The DATA statement used 0.01 CPU seconds and 6952K.

 15         ;
 16         PROC SORT  DATA=ONE;
 17         BY DESCENDING A;

  NOTE: HOST sort chosen, but SAS sort recommended.
 NOTE: There were 6 observations read from the dataset WORK.ONE.
 NOTE: The data set WORK.ONE has 6 observations and 2 variables.
 NOTE: The PROCEDURE SORT used 0.11 CPU seconds and 7044K.

 18         DATA ONE;
 19         SET ONE;
 20         KEEP A ALPHA BETA GAMMA ERROR;
 21         BETA = LAG(A) - A;
 22         GAMMA = LAG(ALPHA)/ALPHA;
 23         IF GAMMA < 1 - (ALPHA/BETA) THEN ERROR = 1;
 24                                        ELSE ERROR = 0;

 NOTE: Missing values were generated as a result of performing an operation on missing
values.
      Each place is given by: (Number of times) at (Line):(Column).
      1 at 21:15   1 at 22:19   1 at 23:14   1 at 23:22
 NOTE: There were 6 observations read from the dataset WORK.ONE.
 NOTE: The data set WORK.ONE has 6 observations and 5 variables.
 NOTE: The DATA statement used 0.01 CPU seconds and 7044K.

 25         PROC SORT  DATA=ONE;
 26         BY A;

 NOTE: HOST sort chosen, but SAS sort recommended.
 NOTE: There were 6 observations read from the dataset WORK.ONE.
 NOTE: The data set WORK.ONE has 6 observations and 5 variables.
 NOTE: The PROCEDURE SORT used 0.02 CPU seconds and 7044K.
```

```
DATA ONE;SET ONE;
KEEP A ALPHA ETA ERROR;
IF BETA = . THEN DO;
   ETA = 1/ALPHA;
   END;
ELSE IF (ABS(GAMMA - 1 + (ALPHA/BETA)) < 0.00005) THEN DO;*TOLERANCE;
   ETA = 0;
   END;
ELSE DO;
   IF (ABS(GAMMA - 1) < 0.00005) THEN DO;*TOLERANCE;
      ETA = 1/ALPHA;
      END;
   ELSE DO;
      IF GAMMA < 1 THEN DO;
         TEMP = (BETA*(1 - GAMMA))/(2*ALPHA);
         LHS  = LOG(GAMMA + TEMP + SQRT(TEMP*(TEMP + 2*GAMMA)))/BETA;
         RHS  = 1/ALPHA;
         END;
      ELSE DO;
         LHS  = LOG(GAMMA)/BETA;
         TEMP = 3*GAMMA + EXP((2*BETA)/ALPHA);
         RHS  = LOG(TEMP)/BETA;
         END;
     ETA  = (RHS + LHS)/2;
     DO WHILE (RHS - LHS > 0.00005);*ADJUST TO DESIRED TOLERANCE;
        TEMP = EXP(BETA*ETA);
        PSI_ETA = ALPHA*ETA - (TEMP - 1)/(TEMP - GAMMA);
        IF PSI_ETA > 0 THEN RHS = ETA;
                       ELSE LHS = ETA;
        ETA  = (RHS + LHS)/2;
        END;
     END;
  END;

NOTE: There were 6 observations read from the dataset WORK.ONE.
NOTE: The data set WORK.ONE has 6 observations and 4 variables.
NOTE: The DATA statement used 0.03 CPU seconds and 7054K.

PROC PRINT DATA=ONE;
NOTE: There were 6 observations read from the dataset WORK.ONE.
NOTE: The PROCEDURE PRINT printed page 1.
NOTE: The PROCEDURE PRINT used 0.02 CPU seconds and 8062K.

NOTE: The SAS session used 0.30 CPU seconds and 8062K.
NOTE: SAS Institute Inc., SAS Campus Drive, Cary, NC USA 27513-2414
```

SAS LISTING:

INVERTING MEAN FALURE TIME

Obs	A	ALPHA	ERROR	ETA
1	0	9.0	0	0.16227
2	1	9.5	0	0.05405
3	2	9.0	0	0.05712
4	3	8.5	0	0.06059
5	4	8.0	0	0.06451
6	5	7.5	0	0.13333