A Random Walk Model for Paid Loss Development

Daniel D. Heyer

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Abstract

Traditional loss development techniques focus on estimating the expected ultimate loss but do not generally indicate the magnitude of possible deviation from this estimate. In a variety of circumstances, however, point reserve estimates are not sufficient. In particular, loss portfolio transfers, commutations, novations, and reserve margin securitization all typically require an estimate of the range of possible loss outcomes.

By adjusting a paid loss model described in Foundations of Casualty Actuarial Science to incorporate a random fluctuation component, a stochastic differential equation model is obtained. This model is analogous to the stock price model used to develop the Black-Scholes option pricing formula. Furthermore, this differential equation has an explicit solution that yields Lognormal distributed development factors similar to the Lognormal link-ratio model published by Roger Hayne.

A slight modification to the model for undiscounted reserves provides a differential equation that accounts for variation in both the amount and timing of loss payments. This equation does not have an explicit solution but can be solved numerically to yield the distribution of the present value reserve.

Introduction

Traditional loss development techniques focus on estimating expected ultimate losses but do not generally indicate the magnitude of possible deviation from this estimate. Typically, a reasonable point-estimate reserve is selected after evaluating the range of estimates produced by several projection techniques. Barring significant calendar year effects, this approach is quite effective when reserves from many accident periods are combined into a single aggregate reserve. In this case, the development on any single reserve may be offset by development on the remaining reserves.

In a variety of circumstances, however, reserve point-estimates are insufficient. In particular, loss portfolio transfers, commutations, novations, and reserve margin securitization often involve a single reserve. Furthermore, these contracts are typically priced on an economic basis. Economic pricing requires valuation of the uncertainty arising from both payment amount and timing.

By adjusting a paid loss model described in *Foundations of Casualty Actuarial Science* to incorporate a random fluctuation component, a stochastic differential equation (SDE) for paid loss development is obtained. This model is analogous to the random walk stock price model used to develop the Black-Scholes option pricing formula. This differential equation has an explicit solution that yields Lognormal distributed development factors similar to a loss development model published by Roger Hayne. This distribution may be used to compute prediction intervals for the indicated reserve, and expected adverse deviation from the carried reserve.

A slight modification to the model for undiscounted reserves provides a differential equation for discounted reserves. This equation does not have an explicit solution but may be solved numerically to yield the distribution of the present value reserve.

Historical Motivation for Model Approach

The model developed here is a generalization of two models already familiar to the actuarial profession. The most straightforward model is the Lognormal Age-to-Age Factor model developed by Roger Hayne[1]. This model assumes that age-to-age factors are Lognormal distributed and uses the properties of compounded Lognormal variates to project ultimate losses. As we shall see later, this is an entirely appropriate model for loss development. Implementation of Hayne's model, however, is complicated by several limitations...

- Parameters are estimated for each development age using losses observed at each age. This data becomes sparse at later development ages.
- Tail factors must be estimated.
- Two parameters must be estimated for each development age. This creates a significant potential for over-fitting. (*i.e.* the model has so much flexibility that it is fitting parameters to the noise in the data as well as to the underlying relationship of interest.)

These issues, however, can be addressed by uniting the Hayne model with the Loss Function Model detailed by Ronald Wiser[2]. In this model, Wiser discusses loss rate functions that can be integrated to yield the expected incremental paid losses during any specified period. In general differential equation form...

$$dP = m(t)dt \qquad (1)$$

...where dP is the incremental paid loss over each time dt, P is paid losses and $m(t)$ is the loss rate function. The choice of loss rate function is governed by incurred and reporting patterns, timing of salvage and subrogation recoveries, etc. In general, however, the loss rate function should tend to zero over time. Under this model, age-to-age factors are no longer a practical necessity. Once the parameters have been estimated for the loss rate function, however, age-to-age factors may be computed directly by...

$$\textit{Age-to-Age Factor}(t_1,t_2) = \frac{\int_0^{t_2} m(s)ds}{\int_0^{t_1} m(s)ds} \qquad (2)$$

Typically, $m(t)$ will have far fewer parameters than Hayne's model so there is less opportunity for overfitting. Furthermore, the model already incorporates an implicit tail factor so there is no need to estimate this separately. Note, however, that this tail factor is based solely upon the characteristics of the selected loss rate function. This model does not address the development variability that was the crux of Hayne's model.

The technical question becomes, then, how can we modify Equation (1) to incorporate random variation. The statistical tool for accomplishing this is called stochastic differential equations (SDEs). SDEs allow us to write differential equations with random coefficients or constants. These equations have found application in a variety of engineering, biological and financial systems subject to "noisy growth". In an insurance reserving setting, paid loss development is an example of noisy growth.[3] By assumption, losses follow a "development pattern" and it is the actuary's charge to assess whether deviations from the development pattern are random or systematic. SDEs are one approach for quantifying the paid loss development pattern and statistically testing deviations from that pattern.

Unfortunately, standard Riemann integration techniques cannot be used to solve SDEs. The next section details the basic technical apparatus required to specify and evaluate the equations used in this model. This explanation, however, should not be taken as either a general or complete presentation of the topic.

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[1] Roger Hayne, An Estimate of Statistical Variation in Development Factor Methods, 1985 Proceedings of the Casualty Actuarial Society, Volume LXXII

[2] Ronald Wiser, *Loss Reserving*, Foundations of Casualty Actuarial Science, Third Edition

[3] By contrast, incurred loss development is subject to systematic manipulation by the actuary and does not constitute noisy growth.

Stochastic Differential Equations

The differential equation that forms the basis of this projection method is an extension of Equation (1)...

$$dP = \mu(t)Pdt + \sigma(t)PdB_t \quad \text{...or...} \quad \frac{dP}{P} = \mu(t)dt + \sigma(t)dB_t \tag{3}$$

Here μ(t) is the loss log-growth rate, dB_t is a Brownian motion noise function (Brownian motion will be discussed in further detail below) and $\sigma(t)$ is a noise scale factor. Solving this equation for $P(t)$ is somewhat problematic as P is a stochastic process rather than a normal function. Was this a Riemann integral we would make the substitution...

$$G(P) = \ln(P) \Rightarrow dG(P) = \frac{dP}{P} \tag{4}$$

This substitution would make the solution of Equation (3) relatively straightforward. When dealing with a stochastic process, however, we cannot so easily use the derivative "chain-rule" to go from $G(P)$ to $dG(P)$. The chain-rule for stochastic processes is given by Îto's lemma.[4] Without proof, a form of this lemma states...

Let X_t be an Îto process given by $dX_t = u(t,x)\cdot dt + v(t,x)\cdot dB_t$. Let $Y_t = g(t, X_t)$ be a twice continuously differentiable transformation of X_t. Then Y_t is also an Ito process and

$$dY_t = \left(\frac{dg(t,x)}{dx}u(t,x) + \frac{dg(t,x)}{dt} + \frac{1}{2}\frac{d^2g(t,x)}{dx^2}v^2(t,x) \right)\cdot dt + v(t,x)\cdot dB_t$$

After applying this lemma, the log-transformation $G(P)$ yields the following solution to Equation (3)...

$$\ln\left(\frac{P_T}{P_0}\right) = \int_0^T \left(\mu(t) - \frac{1}{2}\sigma^2(t) \right)dt + \int_0^T \sigma(t)dB_t \tag{5}$$

This model is called geometric Brownian motion and is frequently used in financial models: a famous example being the Black-Scholes option pricing formula. How do we interpret this result in a loss development context? The left-hand side of the equation may be interpreted as the log link-ratio between two development ages. The log link-ratio is equal to a fixed component given by the integral of $\mu(t) - \frac{1}{2}\sigma^2(t)$ over time, and a random component given by the integral of $\sigma(t)$ over the random noise process. Although not required in theory, the fixed integral $\int_0^\infty \left(\mu(t) - \frac{1}{2}\sigma^2(t) \right)dt$ should generally be finite to ensure a finite ultimate loss.

To understand the random component, we must first understand the basic behaviors of Brownian motion. Brownian motion is a continuous-time random walk process. Conceptually, this is a process that generates Normal random increments for each time increment dt and sums these increments over time. When a function such as $\sigma(t)$ is integrated over a Brownian motion path, we have what is called an Îto integral. Îto integrals have two basic, statistical properties that we will use to understand Equation (5)[5]...

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[4] For a complete discussion of Ito's lemma see Øksendal, Stochastic Differential Equations, Chapter 4.

[5] These properties only hold for "nice" functions $\sigma(t)$. For a complete discussion of Brownian Motion and its relationship to Îto Integrals see Øksendal, Stochastic Differential Equations, Chapter 3.

$$E\left[\left(\int \sigma(t)dB_t\right)\right] = 0 \tag{6}$$
$$E\left[\left(\int \sigma(t)dB_t\right)^2\right] = \int \sigma(t)^2 dt$$

From these properties we can show that the random noise process is Normal distributed, has an expected value of zero, and a variance of $\int \sigma^2(t)dt$.[6] This yields the following distribution model for Equation (5)...

$$\left(\frac{P_{t_2}}{P_{t_1}}\right) \sim Lognormal\left(\int_{t_1}^{t_2}\left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt, \sqrt{\int_{t_1}^{t_2}\sigma^2(t)dt}\right) \tag{7}$$

In other words, the link-ratios between any two ages are Lognormal distributed with the distribution parameters indicated in Equation (7). Using the results of Hayne, this also implies that the paid loss development between any two ages is also Lognormal distributed. A benefit to this approach is that once the model has been fit, development factors for any time interval may be computed regardless of the increment in the underlying data.

Applying the Model

The primary steps in applying the random walk model are verifying that observed age-to-age factors are independently, identically, Lognormal distributed; identifying appropriate functions for $\mu(t)$ and $\sigma^2(t)$; and estimating the parameters for those functions. Paid loss development data representative of non-standard, personal auto, bodily injury liability coverage is used to demonstrate the application of this model.

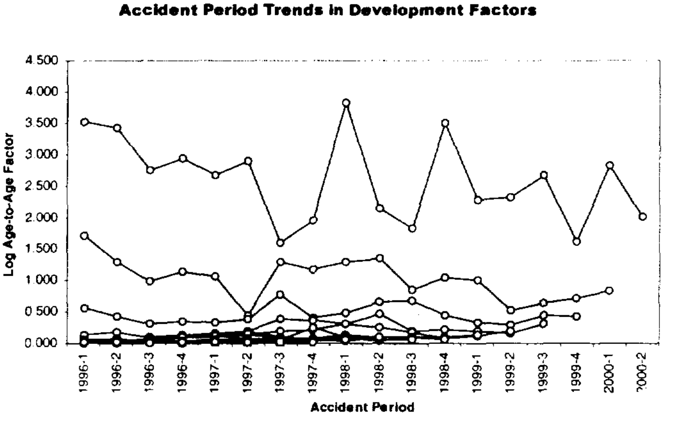
Data Diagnostics - Testing Model Assumptions

This section tests whether the data satisfies the assumptions underlying the random walk model. This is done using the raw data and prior to any model selection or fitting. Note that a violation of the model assumptions does not necessarily imply that the subsequent model fit will be poor. Rather, a violation of the model assumptions means that any statistical tests based upon the model results are biased. The magnitude of that bias depends upon the seriousness of the violation.

The data are shown in Exhibit 1. This data has not been adjusted for any changes in reporting, claim handling, inflation, etc. so the first step is to verify that the age-to-age factors do not show any significant accident year trends. (*i.e.* that within each development age, the age-to-age factors are independently, identically distributed.) This is shown in Figure 1 below...

[6] For the interested reader, this entire derivation is presented in detail in Pliska, Mathematics of Derivative Securities, Chapter 1.

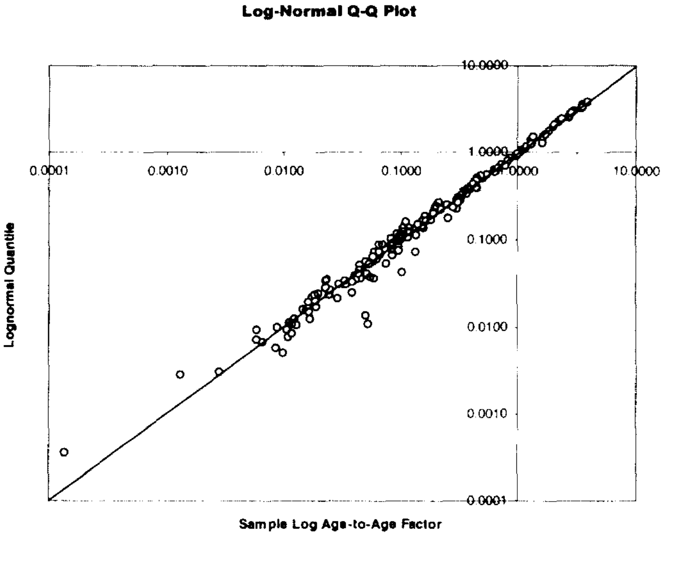
Figure 1



Each line on this plot is the observed log age-to-age factor for a common development age. Although the early development periods (largest development factors) exhibit a slight downward trend in the first few accident periods, this is insignificant given the large, random fluctuations observed in later periods. Accordingly, we can reasonably assume that the development factors at each age are independent. Note, however, that these uncorrected trends will increase the volatility of projections made at early development ages. If these trends could be removed through "data-leveling", the precision of the ultimate loss projections could be greatly improved.

A Q-Q plot was used to verify that the age-to-age factors at each age are Lognormal distributed. This is shown in Figure 2 below...

Figure 2



This plot shows the sample log age-to-age factor and the theoretical sample quantile under the Lognormal distribution; a perfect distribution fit yields a straight line. Although this plot obscures the fit for individual development ages, we can readily see that the Lognormal assumption is quite reasonable. At later development ages (lower, left corner), however, the Lognormal assumption is generally poorer. There are several reasons for this...

- At later ages, the small number of observations makes the data less stable.
- For small samples, the sample quantile is a poor measure of the underlying distribution quantile.
- At later ages, the actual likelihood of favorable development arising from salvage and subrogation recoveries is smaller than predicted by a Lognormal model.

The last point will be particularly important when computing reserve estimates; at later development ages, the lower prediction limit for the required reserve may be negative. In other words, the model recognizes that favorable development could reduce the ultimate loss below the current paid loss. This behavior is probably inconsistent with most lines of business. Fortunately, however, the lower limit is not typically of concern when evaluating reserve estimates.

Curve Family Selection

The next step in the modeling process is to select appropriate families of curves for $\mu(t)$ and $\sigma^2(t)$. This is a non-trivial task: polynomial functions will generally not be appropriate and, consequently, standard sequential model selection techniques cannot be used. The following procedure is presented as a practical approach for streamlining the model selection process. Of course other more theoretically accurate, and computationally more difficult, approaches are possible.

For this data, both $\mu(t)$ and $\sigma^2(t)$ have the same restrictions imposed upon them: they must be positive, decreasing functions that tend to zero over time. This is shown graphically in Figure 3 below. These types of functions are generically referred to as "tail-functions". In this example, three classes of tail function were considered. These functions were...

$$\alpha \cdot e^{-\left(\frac{t}{\beta}\right)^{\gamma}} \tag{8.1}$$

$$\alpha\left[1+\gamma\left(\frac{t}{\beta}\right)\right]^{-\frac{1}{\gamma}} \tag{8.2}$$

$$\alpha \cdot t^{-\beta} + \gamma \tag{8.3}$$

In this example, these specific functions were selected because they encompass a wide range of tail decay rates. In practice, a varied catalogue of tail functions may be obtained by scaling the survival function of various statistical distributions.[7] The tail-functions given above correspond to the scaled tail functions for the Weibull, Generalized Extreme Value, and Power distributions respectively. Also in order, these functions vary from lightest to heaviest tailed. Selecting the most appropriate curve form is complicated by the fact that we cannot directly observe the rate functions $\mu(t)$ and $\sigma^2(t)$. Rather, we can only observe the integrated values of these functions (*i.e.* the log age-to-age factors) as shown by the integrals on the right side of Equation (5). Furthermore, both the rate functions and the resulting log age-to-age factors vary by orders of magnitude. These complications, however, were exploited to develop a model selection procedure.

First, least-squares estimation was used to estimate the parameters of each curve form by fitting each curve's integral to the mean and variance of the observed log age-to-age factors. Typically, the least-squares approach would be inappropriate for this data because the fitted values vary by several orders of magnitude; the least-squares approach fits parameters to the largest values and ignores the smallest values. This characteristic, however, was used to justify the curve family selection. A curve that is fitted to the largest values and coincidentally fits the smallest values, too, is probably capturing the true underlying relationship in the data. By placing the empirical and fitted log age-to-age factors on a log-plot, the curves may be evaluated at both the

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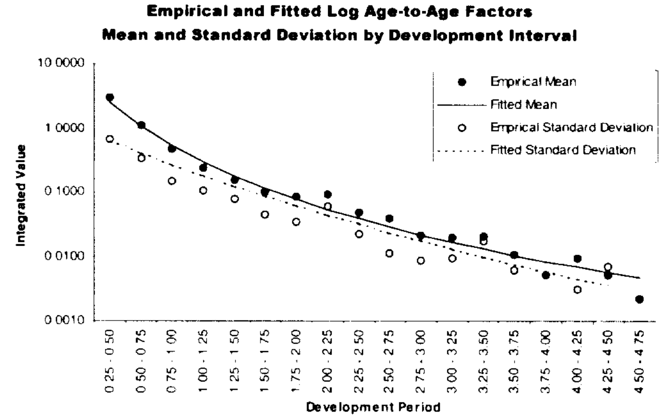
[7] A concise reference for statistical distributions, distribution functions, transformations, etc. is Evans *et al*, Statistical Distributions.

largest and smallest values. This is shown in Exhibit 2. Here the Generalized Extreme Value tail function generally provides the best overall fit for both $\mu(t)$ and $\sigma^2(t)$. In general, however, the same tail function need not be selected for both components. The final parameterization of these curves is shown in Figure 3 below...

Figure 3

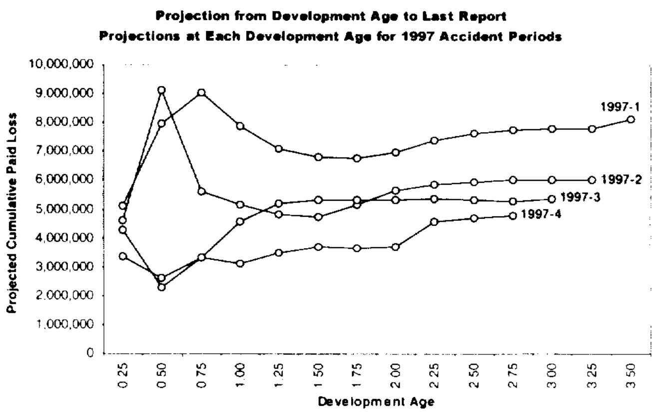


Parameter Estimation

The least-squares parameters used to select the tail functions are not the parameters for paid loss projection; rather maximum likelihood estimation was used to select the parameters for the $\mu(t)$ and $\sigma^2(t)$ tail functions. The maximum likelihood estimation procedure allows the model to be tuned for long-term projections.

With the case study data in triangular form, we can use the model to project the paid losses from each development age to the last reported value (*i.e.* the last diagonal in the development triangle). We can then use the observed value, the projected value, and the projection distribution given by Equation (7) to compute a likelihood statistic for every such projection. The final model parameters, then, are selected to maximize the overall likelihood that the observed losses could be generated by the modeled distribution. The maximum likelihood estimation procedure and the resulting projections are summarized in Exhibit 3 and in Figure 4 below...

Figure 4



In this plot, the losses at each development age are projected to the last diagonal of the development triangle. Each line on the plot shows these projected values for a single accident period. If the model made perfect projections at each development age, this plot would consist of horizontal lines. In reality, however, early projections are relatively inaccurate but quickly converge within a few periods.

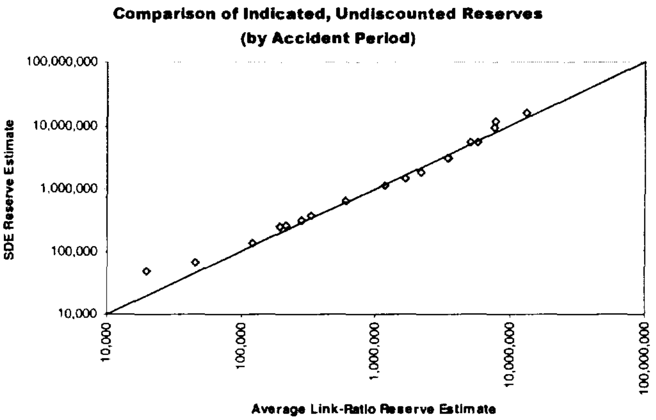
In this application, maximum-likelihood and least-squares estimation differ in one key respect. Least-squares estimation seeks to minimize the volatility of the left-hand side of Figure 4 where the development factors are largest. This creates a large potential for overfitting if there is significant noise in this immature data. Maximum-likelihood estimation does not seek to minimize this volatility *per se*. Rather, maximum-likelihood seeks to ensure that the volatility conforms to an assumed distribution. To the extent that the assumed distribution model is correct, maximum-likelihood will also minimize volatility in the same fashion as least-squares estimation. If the assumed model is incorrect, however, the volatility will be increased due to the bias arising from the model mis-specification.

The parameter estimation technique presented here was chosen for its tractability rather than its statistical properties. In fact, the parameters produced by this procedure will be neither unbiased nor minimum variance. More sophisticated estimation techniques incorporating censored data analysis would rectify these issues.

Model Results

By subtracting the paid-to-date losses from the projected ultimate losses, we have the indicated reserve. A first test for the model is that the expected reserves should be consistent with the reserves indicated by traditional actuarial analysis. These results are shown in Exhibit 4 and in Figure 5 below...

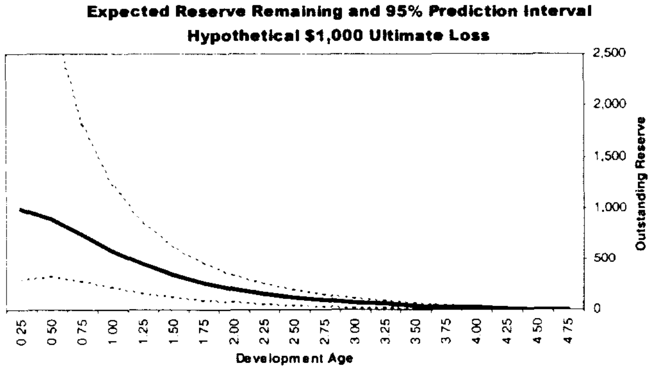
Figure 5



As expected the reserves indicated by traditional and SDE projection methods are similar. Although not readily apparent on the log-log plot above, the largest dollar deviation between the two methods occurs in the largest, least mature reserves. These deviations are consistent with the volatility component of the SDE model. Under the SDE model, large fluctuations are likely during immature development periods. Furthermore, due to the skewness of the Lognormal distribution, these are likely to be large upward fluctuations. This also results in large prediction intervals for the least mature reserves. This is the same effect that C.K. Khury modeled using an arbitrary reserve radius G-function.[8] This is depicted in Figure 6 below...

[8] C.K. Khury, Loss Reserves: Performance Standards, 1980 Proceedings of the Casualty Actuarial Society, Volume LXVII

Figure 6



Here the ultimate loss is $1000 but at the time the reserves are set, this amount is unknown. We can, however, use the model to estimate the probable range of required reserves at each development age. In the plot, this is shown as an expected reserve that declines as losses are paid out, and a prediction interval that contracts as the ultimate loss becomes more certain.

Finally, having a distribution for the required reserve allows calculation of the expected value of future adverse or favorable deviation from the selected reserve amount. The values are computed as tail expected values in the same manner as an excess pure premium or deductible savings is computed. In statistical terms...[9]

$$\text{Favorable Development} = E\left[R_{carried} - R_{req'd} \mid R_{carried} \geq R_{req'd}\right] \cdot P\left[R_{carried} \geq R_{req'd}\right] \quad (9.1)$$

$$\text{Adverse Development} = E\left[R_{req'd} - R_{carried} \mid R_{carried} \leq R_{req'd}\right] \cdot P\left[R_{carried} \leq R_{req'd}\right] \quad (9.2)$$

These results are shown in Exhibit 4 on an undiscounted basis assuming that the carried reserve is set at the average link-ratio reserve.

Discounted Reserves

A small modification to Equation (3) allows similar treatment of discounted (present value) reserves. To motivate this treatment, consider a continuous annuity that pays benefits at a varying rate b_t and force of interest δ...[10]

$$\bar{a}_x = \int e^{-\delta t} \cdot b_t \cdot dt$$
$$d\bar{a}_x = e^{-\delta t} \cdot b_t \cdot dt$$

Discounted loss reserves may be treated analogously if we treat the incremental loss development $\frac{dP}{dt}$ as the "benefit". This is given by...

$$dV = e^{-\delta(t-t_0)} \cdot \frac{dP}{dt} \cdot dt$$
$$dV = e^{-\delta(t-t_0)} \cdot dP \qquad (10)$$
$$dV = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right) e^{-\delta(t-t_0)} P dt + \sigma(t) e^{-\delta(t-t_0)} P dB_t$$

...where *V* is the present value loss reserve and δ is the force of interest used for discounting. Unfortunately, however, this expression does not lend itself to explicit solution in the same manner as Equation (3). Instead, numerical methods must be employed to compute the distribution of present value reserves. These methods can be somewhat difficult to implement.[11] To continue the example from above, the expected present value reserve and reserve volatility computed from Equation (10) are shown in Exhibit 4 and in Figure 7 below...

Figure 7

Implicit Margin in Average Link-Ratio Reserves
(Losses Discounted at 7.0% *per annum* Continuous Compounding)

Accident Period	Average Link-Ratio Reserve (Undiscounted)	Expected Discounted SDE Reserve	Expected Standard Deviation of Discounted SDE Reserve	Expected Margin in Average Link-Ratio Reserve
1996-1	0			
1996-2	19,948	48,252	32,262	-28,304
1996-3	45,365	62,719	30,563	-17,354
1996-4	122,715	128,511	52,979	-5,796
1997-1	194,942	229,769	85,414	-34,828
1997-2	217,319	237,904	82,712	-20,585
1997-3	286,525	281,997	93,894	4,527
1997-4	335,073	338,204	109,595	-3,131
1998-1	611,160	601,721	191,808	9,439
1998-2	1,183,357	1,024,669	323,292	158,688
1998-3	1,666,092	1,362,136	426,181	303,956
1998-4	2,210,746	1,700,183	526,117	510,563
1999-1	3,511,724	2,802,555	851,006	709,169
1999-2	3,426,796	2,805,902	824,190	620,894
1999-3	5,729,009	4,984,688	1,385,324	744,321
1999-4	5,078,453	5,146,741	1,311,692	-68,288
2000-1	7,739,817	8,782,127	1,978,193	-1,042,310
2000-2	7,914,469	11,252,088	2,196,198	-3,337,618
2000-3	13,337,789	14,916,718	2,649,990	-1,578,929
	53,631,298	56,706,885		-3,075,587

[10] Bowers *et al*, Actuarial Mathematics, Chapter 5

[11] For more information on numerical solutions to stochastic integrals see Tavella and Randall, Pricing Financial Instruments.

As this figure makes clear the overall margin is negative, and the positive reserve margins are quite small compared to the volatility of the underlying reserve estimates. Accordingly, there is little practical margin in the average link ratio reserves. This is due largely to the inherent characteristics of the business presented in this example...

- The extreme growth at early development ages makes early reserve estimates highly volatile.
- There is little development at later ages. This decreases the duration of immature reserves and consequently, the magnitude of the implicit margin in the undiscounted reserves.
- Similarly, the magnitude of the discount margin tends to be small at later ages because the indicated reserves are themselves small.

Lines of business characterized by protracted development with significant payments throughout the life of the reserve should contain larger implicit margins.

Conclusions

The model presented here unites common actuarial practice with a basic financial model, and provides concrete justification for the utility of link-ratio techniques. As presented however, this model is relatively crude and there are several areas for enhancement and further research.

- Parameter estimation techniques with more statistically desirable properties (*e.g* unbiased, minimum variance, etc.) should be employed.
- The model treats each accident period separately. Ito's lemma, however, is easily extended to multiple dimensions. This would allow joint modeling of each accident period in the reserve, etc. Significant research, however, would be required to understand the correlation structure between accident periods.
- The model can only be applied to positive, non-zero paid losses. This issue cannot easily be addressed within the geometric Brownian motion framework. For lines with a significant payment lag, additive Brownian motion or Poisson jump (frequency-severity) process may be a more appropriate model.
- Adjusting the model for report lag, calendar-year effects, and other sources of volatility could significantly enhance the precision of reserve estimates made at early development ages.
- Under the geometric Brownian motion model, all random deviations persist. In other words, an increase in the loss payment rate is always due to adverse deviation, never to accelerated claim payment. There are other stochastic differential equations that can accommodate claim payment volatility.
- Having a distribution for the ultimate loss allows common derivative security pricing techniques to be applied to loss portfolio transfers, commutations, and reserve margin securitization. This is an important area for further research if traditional insurance is to remain competitive with the capital markets.

Exhibit 1

Non-Standard Auto Bodily Injury Liability

Cumulative Paid Losses

Development Age

Accident Period	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	Average Link-Ratio Ultimate Loss
1996-1	10,996	369,493	2,044,411	3,545,999	4,066,094	4,346,452	4,449,499	4,660,135	4,946,500	5,031,364	5,207,964	5,272,084	5,479,258	5,559,845	5,575,454	5,581,591	5,642,002	5,642,767	5,655,215	5,655,215
1996-2	30,379	923,577	3,358,105	5,149,806	6,065,390	6,434,770	6,940,143	7,217,139	7,710,346	7,937,009	8,272,704	8,558,125	8,718,245	8,742,750	8,800,511	8,852,858	8,953,325	9,042,539		9,062,488
1996-3	63,239	988,450	2,615,749	3,528,112	3,909,767	4,109,248	4,187,078	4,551,604	4,682,945	4,876,102	5,097,885	5,196,030	5,264,130	5,309,388	5,372,076	5,379,097	5,410,513			5,455,878
1996-4	39,093	744,327	2,315,143	3,263,545	3,733,518	4,206,960	4,674,586	5,260,543	5,779,476	6,062,782	6,214,310	6,269,460	6,414,850	6,564,550	6,673,776	6,753,290				6,876,005
1997-1	65,664	942,312	2,713,647	3,734,512	4,354,517	4,891,834	5,384,858	5,910,456	6,592,840	7,039,673	7,358,421	7,541,425	7,667,061	8,072,516	8,204,816					8,399,758
1997-2	59,937	1,091,178	1,699,740	2,473,150	2,992,361	3,446,278	4,138,586	4,846,277	5,268,678	5,539,482	5,773,741	5,885,018	5,978,090	6,098,299						6,315,618
1997-3	56,224	279,527	1,018,077	2,212,843	3,256,599	3,912,537	4,316,166	4,615,991	4,880,118	5,044,104	5,133,113	5,287,851	5,345,446							5,631,970
1997-4	45,392	321,463	1,038,532	1,547,997	2,229,086	2,779,582	3,028,933	3,290,122	4,239,955	4,492,564	4,692,742	4,813,607								5,148,680
1998-1	8,363	387,219	1,398,819	2,267,289	3,059,885	4,156,629	4,772,668	5,265,137	5,850,179	6,122,812	6,465,182									7,076,342
1998-2	47,153	409,959	1,580,925	3,031,628	4,764,035	6,120,361	6,790,407	7,108,747	7,535,256	8,290,525										9,473,882
1998-3	158,143	985,728	2,306,657	4,532,235	5,545,023	6,547,284	7,144,524	7,791,978	8,217,594											9,883,887
1998-4	32,825	1,067,637	3,062,441	4,810,670	5,935,970	6,373,358	7,060,202	7,512,141												9,722,887
1999-1	151,425	1,492,994	4,046,201	5,822,294	6,830,335	7,669,164	8,818,850													12,330,574
1999-2	169,819	1,937,953	3,227,429	4,260,763	4,987,630	6,027,712														9,454,508
1999-3	118,032	1,719,811	3,245,182	5,032,184	6,850,862															12,579,871
1999-4	256,006	1,297,036	2,665,528	4,070,187																9,148,650
2000-1	85,211	1,442,047	3,297,787																	11,037,604
2000-2	175,762	1,330,312																		9,244,782
2000-3	180,400																			13,516,166

Log Cumulative Age-to-Age Factors

Development Interval

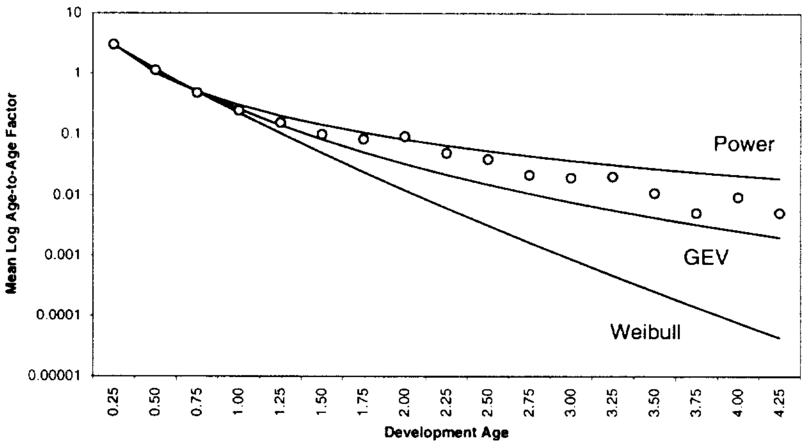
Accident Period	0.25 - 0.50	0.50 - 0.75	0.75 - 1.00	1.00 - 1.25	1.25 - 1.50	1.50 - 1.75	1.75 - 2.00	2.00 - 2.25	2.25 - 2.50	2.50 - 2.75	2.75 - 3.00	3.00 - 3.25	3.25 - 3.50	3.50 - 3.75	3.75 - 4.00	4.00 - 4.25	4.25 - 4.50	4.50 - 4.75
1996-1	3.515	1.711	0.551	0.137	0.087	0.023	0.046	0.060	0.017	0.034	0.012	0.039	0.015	0.003	0.001	0.011	0.000	0.002
1996-2	3.414	1.291	0.428	0.164	0.059	0.076	0.039	0.066	0.029	0.041	0.034	0.019	0.003	0.007	0.006	0.011	0.010	
1996-3	2.749	0.973	0.299	0.103	0.050	0.019	0.083	0.028	0.040	0.044	0.019	0.013	0.009	0.012	0.001	0.006		
1996-4	2.947	1.135	0.343	0.135	0.119	0.105	0.118	0.094	0.048	0.025	0.009	0.023	0.023	0.017	0.012			
1997-1	2.664	1.058	0.319	0.154	0.118	0.096	0.093	0.109	0.066	0.044	0.025	0.017	0.052	0.016				
1997-2	2.902	0.443	0.375	0.191	0.141	0.183	0.158	0.084	0.050	0.041	0.019	0.016	0.020					
1997-3	1.604	1.293	0.778	0.387	0.183	0.099	0.067	0.056	0.033	0.017	0.030	0.011						
1997-4	1.958	1.173	0.399	0.365	0.221	0.086	0.083	0.254	0.058	0.044	0.025							
1998-1	3.835	1.284	0.483	0.300	0.308	0.138	0.098	0.105	0.046	0.054								
1998-2	2.163	1.350	0.651	0.452	0.251	0.104	0.046	0.058	0.096									
1998-3	1.830	0.850	0.675	0.202	0.166	0.087	0.087	0.053										
1998-4	3.507	1.035	0.452	0.210	0.071	0.102	0.062											
1999-1	2.288	0.997	0.329	0.195	0.116	0.140												
1999-2	2.323	0.510	0.282	0.153	0.189													
1999-3	2.679	0.635	0.439	0.309														
1999-4	1.623	0.720	0.423															
2000-1	2.829	0.827																
2000-2	2.024																	
2000-3																		

Exhibit 2

Potential Curve Families for Rate Functions

Least-Squares Fit to Observed Log Age-to-Age Factors

Observed and Fitted Mean Log Age-to-Age Factors



Observed and Fitted Log Age-to-Age Factor Standard Deviations

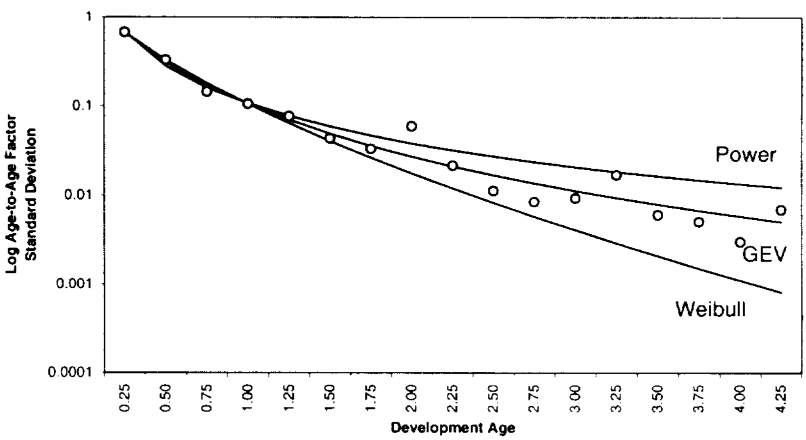


Exhibit 3

Results of Maximum-Liklihood Parameter Estimation

Generalized Extreme Value Tail Function Parameters (for Equation 8.2)
Maximum Likelihood Estimates

	$\mu(t)$	$\sigma^2(t)$
α	58.2410	4.0810
β	0.1550	0.2730
γ	0.2848	0.0678

Fitted Log Age-to-Age Factors (from Integration of Equation 8.2 Over Indicated Time Interval)

Development Interval	0.25 - 0.50	0.50 - 0.75	0.75 - 1.00	1.00 - 1.25	1.25 - 1.50	1.50 - 1.75	1.75 - 2.00	2.00 - 2.25	2.25 - 2.50	2.50 - 2.75	2.75 - 3.00	3.00 - 3.25	3.25 - 3.50	3.50 - 3.75	3.75 - 4.00	4.00 - 4.25	4.25 - 4.50	4.50 - 4.75
Fitted Mean	2.3489	1.0087	0.5104	0.2891	0.1767	0.1148	0.0782	0.0553	0.0403	0.0302	0.0231	0.0180	0.0143	0.0115	0.0094	0.0077	0.0064	0.0054
Fitted Variance	0.2830	0.1249	0.0575	0.0276	0.0137	0.0070	0.0037	0.0020	0.0011	0.0006	0.0004	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000

Negative Log-Likelihood Statistics (from Equation 7) for Projection from Development Age to Last Report

Accident Period	Development Age 0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75
1996-1	2.7328	0.4295	0.4443	0.9926	0.3253	-0.4234	-1.2578	-1.6810	-1.8484	-2.2923	-2.4840	-2.8366	-2.2755	-1.9434	-2.6751	-3.5892	-2.8833	-4.3170	
1996-2	1.4637	0.2301	0.5166	0.3426	-0.2511	-1.0140	-1.4209	-1.7582	-2.0495	-2.2928	-2.5900	-2.7116	-2.8560	-3.3861	-3.6166	-3.5307	-4.1843		
1996-3	0.6731	1.1926	1.6963	1.1996	0.2659	-0.6382	-1.4151	-1.6214	-2.0340	-2.2955	-2.2578	-2.4605	-2.7409	-3.1549	-3.1799	-4.2117			
1996-4	0.7653	0.2672	0.2301	-0.3587	-0.8435	-0.9709	-1.0148	-1.5257	-2.0447	-2.3404	-2.5931	-2.4383	-2.7027	-3.3427	-3.9755				
1997-1	0.6052	0.2925	0.1197	-0.4802	-0.7647	-0.6063	-0.1119	0.2347	-0.7680	-1.2579	-1.5438	-1.0425	0.8777	-3.6581					
1997-2	0.6008	1.0559	-0.1433	-0.4831	-0.4046	0.0968	-0.4779	-1.5232	-2.0021	-2.3463	-2.7037	-3.0492	-3.4335						
1997-3	0.6122	0.9721	0.2885	-0.4907	-0.8252	-1.1031	-1.4306	-1.7547	-2.0214	-2.3646	-2.8047	-3.1802							
1997-4	0.5936	0.5042	0.0362	0.4299	0.2466	0.2617	2.0606	4.6896	-1.8996	-2.4125	-3.0300								
1998-1	4.4221	0.7509	0.0756	0.0971	0.2662	-0.6767	-0.8655	-1.0308	-2.0085	-2.2995									
1998-2	0.8801	1.3322	0.5104	0.0442	-0.7341	-1.2131	-1.5261	-1.4234	-1.1127										
1998-3	0.9753	0.2247	-0.1680	-0.4089	-0.8787	-1.2458	-1.6604	-2.1861											
1998-4	1.3941	0.3221	0.1506	-0.1227	-0.5684	-1.3108	-1.8455												
1999-1	0.7295	0.4126	0.2556	-0.4196	-0.9879	-1.5164													
1999-2	1.3357	1.7855	0.3869	-0.4893	-1.2201														
1999-3	0.5747	0.5707	-0.2966	-0.8686															
1999-4	1.8405	0.4546	-0.4428																
2000-1	0.5796	0.0106																	
2000-2	0.4742																		
2000-3																			

Overall Negative Log-Likelihood -142.7721

NOTES

(1) Given a set of parameters and using Equation 7, we can compute the negative log-likelihood of the observed development factors. Above, the likelihoods for the development from each age to the last reported value (i.e. the last diagonal in the development triangle) are tabulated. The values of μ and σ entering Equation 7 are generated by integrating Equation 8.2 over the appropriate time period.

(2) The parameters for Equation 8.2 were generated by maximizing the likelihood (minimizing the negative log-likelihood) of the observed development factors.

Exhibit 4

Indicated Reserves

Discounted and Undiscounted Basis

Undiscounted Reserves (from Equation 7)

Accident Period	Last Recorded Loss	Average Link-Ratio Reserve	95% Lower Prediction Interval	SDE Expected Reserve	95% Upper Prediction Interval	(1) Expected Favorable Deviation	(2) Expected Adverse Deviation
1996-1	5,655,215	0					
1996-2	9,042,539	19,948	-15,082	48,783	113,218	7,578	-36,413
1996-3	5,410,513	45,365	2,972	64,126	126,153	7,344	-26,105
1996-4	6,753,290	122,715	25,719	132,768	241,957	17,640	-27,693
1997-1	8,204,816	194,942	65,892	239,938	418,837	21,902	-66,898
1997-2	6,098,299	217,319	80,566	250,750	426,885	23,272	-56,703
1997-3	5,345,446	286,525	104,864	299,617	503,224	34,374	-47,467
1997-4	4,813,607	335,073	132,621	361,776	604,475	36,570	-63,273
1998-1	6,465,182	611,160	242,665	647,186	1,082,941	68,850	-104,877
1998-2	8,290,525	1,183,357	418,009	1,106,584	1,865,212	189,147	-112,374
1998-3	8,217,594	1,666,092	556,069	1,474,769	2,518,086	317,999	-126,676
1998-4	7,512,141	2,210,746	690,446	1,842,303	3,205,692	508,489	-140,046
1999-1	8,818,850	3,511,724	1,130,708	3,033,491	5,418,658	731,261	-253,028
1999-2	6,027,712	3,426,796	1,128,044	3,026,927	5,607,256	675,099	-275,230
1999-3	6,850,862	5,729,009	2,008,387	5,346,004	10,436,596	985,589	-602,584
1999-4	4,070,197	5,078,453	2,085,288	5,472,294	11,586,239	650,237	-1,044,077
2000-1	3,297,787	7,739,817	3,533,966	9,217,365	22,403,695	901,234	-2,378,782
2000-2	1,330,312	7,914,469	4,290,492	11,674,815	36,888,928	783,212	-4,543,557
2000-3	180,400	13,337,789	4,870,474	15,853,918	85,620,182	2,020,744	-4,536,874
		53,631,298		60,093,412		7,980,541	-14,442,656

Discounted Reserves and Implicit Margin in Average Link-Ratio Reserve (from Numerical Solution of Equation 10)

Accident Period	Last Recorded Loss	Average Link-Ratio Reserve (Undiscounted)	(3) SDE Expected Discounted Reserve	Expected Margin in Average Link-Ratio Reserve
1996-1	5,655,215	0		
1996-2	9,042,539	19,948	48,252	-28,304
1996-3	5,410,513	45,365	62,719	-17,354
1996-4	6,753,290	122,715	128,511	-5,796
1997-1	8,204,816	194,942	229,769	-34,828
1997-2	6,098,299	217,319	237,904	-20,585
1997-3	5,345,446	286,525	281,997	4,527
1997-4	4,813,607	335,073	338,204	-3,131
1998-1	6,465,182	611,160	601,721	9,439
1998-2	8,290,525	1,183,357	1,024,669	158,688
1998-3	8,217,594	1,666,092	1,362,136	303,956
1998-4	7,512,141	2,210,746	1,700,183	510,563
1999-1	8,818,850	3,511,724	2,802,555	709,169
1999-2	6,027,712	3,426,796	2,805,902	620,894
1999-3	6,850,862	5,729,009	4,984,688	744,321
1999-4	4,070,197	5,078,453	5,146,741	-68,288
2000-1	3,297,787	7,739,817	8,782,127	-1,042,310
2000-2	1,330,312	7,914,469	11,252,088	-3,337,618
2000-3	180,400	13,337,789	14,916,718	-1,578,929
		53,631,298	56,706,885	-3,075,587

NOTES

(1) Measures expected favorable development on Average Link-Ratio reserve amount
Similar to a loss elimination value P[Carried Reserve>Required Reserve] * E[(Carried Reserve-Required Reserve)| Carried Reserve>Required Reserve]

(2) Measures expected adverse development on Average Link-Ratio reserve amount
Similar to an excess pure premium. P[Carried Reserve<Required Reserve] * E[(Carried Reserve-Required Reserve)| Carried Reserve<Required Reserve)

(3) Paid losses discounted at 7.0% continuous compounding