A Dynamic Method for the Valuation of Fair Value Insurance Liabilities

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Abstract

This paper presents a dynamic method to estimate fair value insurance liabilities for the whole book (with separate but correlated lines) of business. The model studies the aggregate liability without assuming independence of individual losses. A non-traditional approach is proposed which estimates the fair value liability based on a stochastic model of individual losses. Using the contingent claim analysis, the fair value liability are approximated by solving a partial differential equation. Parameters estimation, correlations measurement and applications of the model are also discussed in the study. Comparisons of the proposed method to the existing methods are given for application purpose.

1. Introduction

This study addresses the evaluation of insurance liabilities on a fair value basis. The fair value of liabilities is, as stated in the white paper by the Casualty Actuarial Society's Task Force on Fair Value Liabilities: "the fair value of the market value, if a sufficiently active market exists, OR an estimated market value, otherwise"(CAS 2000).

Fair value estimates of insurance liability reflect expected cash flows, the time value of money and an adjustment for risk. Over last fifteen years, many methods for estimating the fair value of property/casualty insurance liabilities has been introduced. All of these methods have their own advantages and disadvantages as summarized in the Casualty Actuarial Society's Task Force white paper (CAS 2000). Among various methods, there are two major approaches used to compute risk loads for the fair value liability that are represented in the literatures: the finance approach and the actuarial approach. The classical finance approach, is used in such methods as CAPM (D'Arcy and Doherty (1988), Fairley (1987), Feldblum (1990), Mahler (1998), and Myers and Cohn (1987)), the internal rate of return (Cummins (1990)), the single-period risk-adjusted discount method (Butsic (1988), and D'Arcy (1988)), the method based on underwriting data (Myers and Cohn (1987)), and the direct estimation of market values method (Allen, Cummins and Philips (1998), Ronn and Verma (1986)). The finance approach evaluates systematic risk by measuring the correlation between insurance companies returns from underwriting and market returns on its shareholder's equity.

The traditional actuarial approach is to use the aggregate probability distribution-based risk loads for the market risk adjustment of the liabilities. The actuarial based methods often explicitly incorporate process (diversifiable) and parameter (nondiversifiable) risk components into the risk load formulas. For a multiple line insurance company, liability (includes aggregate claim and expenses, taxes, et.c.) analysis estimates the total random losses for a book of insurance product line by studying possible aggregate claim distributions. Such distributions are probability distributions of the total dollar amount of loss under one or more insurance policies. They combine the separate effects of the underlying frequency and severity distributions. Assuming families of distributions (e.g. lognormals or shifted gammas) such that if each separate distribution is a member of these families, a closed form and elegant solution is possible. These methods can also be used to value unearned premium reserve and incurred but not reported reserves. (See Beard, Pesonen and Pentikâinen (1984), Bhlmann (1970), Embrechts (1995), Hayne (1989), Heckman and Meyers (1983), Heckman (1999), Kreps (1990 and 1998), Meyers and Nathaniel (1983), Meyers (1991, 1994 and 1998), Panjer (1992), Philbrick (1994), Wang (1997)).

Among all the existing methods, this approach is most widely used in actuarial practice and it continues to develop. The method can be used with company-specific data and can be used by line to reflect unique line of business risks. As indicated in the Casualty Actuarial Society's Task Force white paper, there are some unsolved problems associated with this approach such as measuring correlations of lines or segments of the business with other segments, estimating/ calibrating model parameters, and establishing a guideline for the applications of available methods. This paper presents a dynamic method to estimate the fair value of insurance liabilities for the whole book (with separate but correlated multiple lines) of business. The model studies the aggregate liability without assuming independent individual losses based on a non-traditional version of the collective risk theory. A new approach is proposed which estimates the fair value of insurer's liability based on a stochastic model of individual losses. To reflect the changing of the aggregate liability over time, a continuous model is presented using contingency claim analysis. By using the contingent claim analysis, the fair value liablity are approximated by solving a partial differential equation. Parameters estimation, correlations measurement and applications of the model are also discussed in the study.

The paper is organized as follows: The mathematical model for fair value of liability is presented in the next section. Several applications of the model and case studies are presented in Section 3. In the following section, the comparison of the new method to the existing methods will be addressed. Section 5 summarizes and concludes the paper.

2. Theory

This section presents the mathematical model for the valuation of fair value liability. To reflect the changing of the aggregate liability over time, a continuous model is presented using contingency claim analysis. We begin with the simplest case, where it is assumed that correlation among the classes of business are all a result of one underlying force (risk source) that affects different classes.

2.1 Mono-line of Business

For a specific line of business and a specifific accident year t, we define $\{X(t), t \geq 0\}$, as the instantaneous ulltimate loss (includes claim, expenses and taxes) process, and $\{L(t), t \geq 0\}$, as the aggregate of fair value liability process over the period of $[0, t]$.

Assume the instantaneous loss amount $X(t)dt$ between time t and time $t+dt$ is described by a general stochastic process of the form:

$$dX = \mu(t, X)dt + \sigma(t, X)dW \tag{2.1}$$

where μ is the drift of X , W is a standard Brownian motion (Wiener process),
and the local volatility σ is a deterministic function that may depend on both the loss X and the time t.

Over the time period [0,T], the aggregate of fair value liability $L(T)$ is defined by the equation

$$L(T) = \int_0^T X(\xi)e^{-r\xi}d\xi + F(X(T))e^{-rT},$$

where r is the discount rate (see Section 3.1 for the detail discussion), and F is assumed to be a continuous terminal function.

Remark: In many cases, there may be some delay in claims: information might not be available until the end of the evaluation period (time T). Therefore, in our definition, F is introduced, as a function of $X(T)$, to reflect situations like this. Notice that if F is the zero function, the definition above is the same as the conventional definition for the present value of the aggregate loss. Notice also, that it is possible for $X(t)$ to be negative, reflecting the release of reserves upon deaths of annuitants. Similarly, the aggregate of fair value liability over $[0,t]$, $L(t)$, is defined as

$$L(t) = \int_0^t X(\xi)e^{-r\xi}d\xi + F(X(t))\,e^{-rt}.$$

Remark: The claim reserve process is $R(t) = L(t) - C$ where C is either the claims paid to date or the case incurred claims to date. Since C is a known value, so we focus our analysis on L in this paper.

Next, we define the function $u(t,x)$ as the expected present value of the fair value liability over [0,t],

$$u(t,x) = E[L(t) \,|\, X(0) = x] \tag{2.2}$$

where $x = X(0)$.

Remark: The function $u(t,x)$ is the conditional expectational of the aggragate of fair value liability, conditioned by $X(0) = x$. When $t = T$, $u(T,x)$ is the expected present value of the fair value liability over $[0,T]$.

THEOREM 1 *Suppose that σ and μ satisfy the linear growth condition*

$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 \le K^2(1+|x|^2) \tag{2.3}$$

for every $0 \le t < \infty$, $x \in R$,

$$|F(x)| \le K^2(1+|x|^2)$$

for every $x \in R$, where K is a positive constant; and

suppose that $u(t,x)$ is continuous and is of class $C^{1,2}([0,T] \times R)$. Then the expected present value of the fair value liability $u(t,x)$ can be calculated by solving the following Cauchy problem

$$u_t = \frac{1}{2}\sigma^2 u_{xx} + \mu u_x - ru + x; \quad in\,[0,T) \times R \tag{2.4}$$

and

$$u(0,x) = F(x); \quad x \in R \tag{2.5}$$

as well as the polynomial growth condition:

$$max_{0 \leq t} \quad |u(t, x)| \leq M(1 + |x|^{2\eta}); \quad x \in R \tag{2.6}$$

for some $M > 0, \eta \geq 1$.

Proof This is a special case of Theorem 2, when $d = 1$. See the proof of Theorem 2.

In the following examples, we consider several simple applications of Theorem 1.

EXAMPLE 1

We first consider a mono-line liability reserve with the amount of cash flows being certain: the instantaneous loss amount $X(t)dt$ satisfy $dX = \mu_0 X dt$, where μ_0 is a constant.

Therefore $\mu(t, X) = \mu_0 X$, and $\sigma = 0$ in equation (2.1). We also ignore the investment income, i.e. $r = 0$. Furthemore, we assume $F(x) = 0$.

According to equation (2.2), given that $x = X(0)$, the expected present value of the fair value liability is

$$u(t, x) = E[\int_0^t X(\xi) d\xi \,|X(0) = x] = \int_0^t (x e^{\mu_0 \xi})\, d\xi = \frac{x}{\mu_0}(e^{\mu_0 t} - 1).$$

The following figure (Figure 1) provides a graphic view of $X(t)$ and $u(t, x)$ in this example.

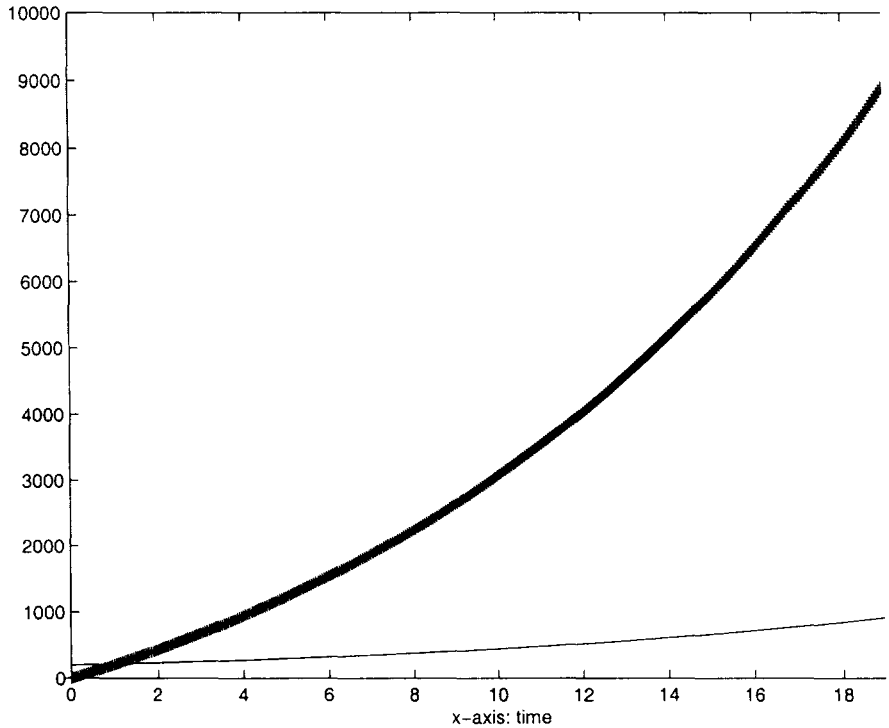

Figure 1. The expected fair value liability u(t,x) ('+++') v.s. the individual claims X(t) ('—,).

$u(t,x)$ satisfies

$$u_t = xe^{\mu_0 t},$$

$$u_x = \frac{1}{\mu_0} e^{\mu_0 t},$$

$$u_{xx} = 0.$$

It follows that

$$\frac{1}{2}\sigma^2\, u_{xx} + \mu u_x - ru + x = 0 + \mu_0 x u_x - 0 + x = \mu_0 x u_x + x = u_t,$$

and

$$u(0,x) = 0 = F(x).$$

Therefore, Equation (2.4) and (2.5) hold.

Accoording to Theorem 1, the fair value liability can be estimated by solving the partial differential equations:

$$u_t = \mu_0 x u_x + x,$$

and

$$u(0,x) = 0.$$

EXAMPLE 2

Consider a mono line liability reserve with uncertain cash flows:

$\mu = 0, \sigma = 1$ in Equation (2.1).

In this case, we have $dX = dW$.

Furthemore, ignore investment income ($r = 0$) and assume $F(x) = F_0$, a constant function.

According to Equation (2.2),

$$u(t,x) = E[\int_0^t W(\xi)d\xi \,+ F_0\,|X(0) = x] \,=\, xt + F_0.$$

It is easy to see that
$u_t = x$, $u_{xx} = 0$, and $u(0,x) = F_0$.

Therefore, $u(t,x)$ satisfies $u_t = \frac{1}{2}u_{xx} + x$ and $u(0,x) = F_0$ which are Equations (2.4) and (2.5) when $r = \mu = 0, \sigma = 1$.

Accoording to Theorem 1, the fair value liability can be estimated by solving the partial differential equations:

$$u_t = \frac{1}{2}u_{xx} + x,$$

and

$$u(0,x) = F_0.$$

EXAMPLE 3

Consider a monoline liability reserve with uncertain cash flows, when $\mu = 0, \sigma = 1$ and $r = 0$.

Let $F(x)$ be a bounded and continuous function, and consider a special case of Equation (2.2):

$$u(t,x) = E[\int_0^t W(\xi)\, d\xi + F(W(t))\,|X(0) = x] = xt + E[F(x + W(t))]$$

First,

$$u(t,x) = xt + \int_{-\infty}^{\infty} F(y)\, p(t;x,y)dy,$$

where

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}}\, e^{\frac{-(y-x)^2}{2t}}$$

is the transition density of the one-dimensional Brownian family.

Then $u(t,x)$ satisfies Equations (2.5):

$$u(0,x) = \lim_{t\to 0, y\to x} u(t,y) = F(x).$$

Next, one can varifies that

$$u_t = x + \int_{-\infty}^{\infty} F(y)\, p_t(t;x,y)dy \;=\; x + \int_{-\infty}^{\infty} F(y)\, p_{xx}(t;x,y)\, dy.$$

Therefore

$$u_t \;=\; x \;+\; \frac{1}{2}\, u_{xx},$$

which is Equation (2.4) when $u(t,x) \;=\; xt \;+\; \int_{-\infty}^{\infty} F(y)\, p(t;x,y)dy$ (see the proof of Theorem 2 as to why (2.4) reduces to $u_t = \frac{1}{2}u_{xx} + x$ in this case).

Accoording to Theorem 1, the fair value liability can be estimated by solving the partial differential equations:

$$u_t \;=\; \frac{1}{2}u_{xx} \;+\; x,$$

and

$$u(0,x) \;=\; \lim_{t\to 0,\, y\to x} u(t,y) \;=\; F(x).$$

2.2 Multi-line of Business

In general, the correlation among the lines of business might be a result of several underlying forces that affect different classes in different ways. For example, risk sources might include economic inflation, judicial climate, tort reform, property catastrophes, health of the economy, and rate levels.

We now discuss multiple line business with correlated risk by generalizing the results in Sectin 2.1.

For a class of business consisting of n lines, we define

$$X(t) = (x^{(1)}(t), x^{(2)}(t), \cdots, x^{(n)}(t))^T \;=\; \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \\ \cdots \\ x^{(n)}(t) \end{pmatrix},$$

as the instantaneous loss process at time $t, t \geq 0$.

Assume the loss amount $X(t)$ at time t is described by a n-dimensional stochastic process of the form:

$$dx^{(i)} \;=\; \mu_i(t,X)dt \;+\; \sum_{j=1}^{d} \sigma_{i,j}(t,X)dW_j \tag{2.7}$$

for $i = 1, 2, \cdots, n$, where

$\mu = (\mu_1(t, X), \mu_2(t, X), \cdots, \mu_n(t, X))$
is the drift of X,

W is a d-dimensional Wiener process,

and the local volatility $\sigma = (\sigma_{i,j}(t, X))$ is a n-by-d matrix that may depend on both the claim X and the time t.

Next, let $L(t), t \geq 0$, be the present value of aggregate fair value liability over the period of $[0, t]$, defined as

$$L(t) = \int_0^t (\sum_{i=1}^{n} x^{(i)}(\xi))e^{-r\xi}d\xi + F(\sum_i^{n} x^{(i)}(t))e^{-rt},$$

and let $u(t, X) = E^X[L]$ be the expected value of the fair value liability given that $X = X(0)$.

As a general case of one risk source (equation (2.2)), $u(t, X)$ is defined as

$$u(t, X) = E[\int_0^t (\sum_i^{n} x^{(i)}(\xi))e^{-r\xi}\, d\xi + F(\sum_i^{n} x^{(i)}(t))e^{-rt} \,|\, X = X(0)] \quad (2.8)$$

where $X = (x^{(1)}(0), x^{(2)}(0), \cdots, x^{(n)}(0))$ is the vector of losses at time 0 from the n risk sources.
Let $a(t, X) = (a_{i,j}(t, X))$ be a $n \times n$ matrix defined as $a(t, X) = \sigma\sigma^T$:

$$a_{i,j}(t, X) = \sum_{k=1}^{d} \sigma_{i,k}(t, X)\, \sigma_{k,j}(t, X),$$

$$g(X) \equiv \sum_{i=1}^{n} x^{(i)}(0),$$

and

$$\mathcal{A}u \equiv \frac{1}{2} \sum_{i,k=1}^{n} a_{i,k}(t, X)\, u_{x_i x_k} + \sum_{i=1}^{n} \mu_i(t, X)\, u_{x_i} \quad (2.9)$$

THEOREM 2 *Suppose that σ and μ satisfy the linear growth condition*

$$||\mu(t,X)||^2 + ||\sigma(t,X)||^2 \leq K^2(1 + ||X||^2) \tag{2.10}$$

for every $0 \leq t < \infty$, $x \in R^n$,

$$|F(X)| \leq K^2(1 + ||X||^2)$$

for every $x \in R$,
where K is a positive constant; and assume that $u(t,X)$ is continuous, and is of class $C^{1,2}([0,T] \times R^n)$.
Then
$u(t,X)$ satisfies the Cauchy problem

$$u_t = \mathcal{A}u - ru + g(X); \quad in\,[0,T) \times R^n \tag{2.11}$$

and

$$u(0,X) = F(g(X)); \quad X \in R^n \tag{2.12}$$

as well as the polynomial growth condition:

$$max_{0 \leq t}\, |u(t,X)| \leq M(1 + ||X||^{2\eta}); X \in R^n \tag{2.13}$$

for some $M > 0, \eta \geq 1$.

The proof of the Theorem 2 is given in Appendix 1.

Theorem 2 indicates that an estimate for the fair value insurance liability could be obtained by solving a partial differential equation (2.11)-(2.12).

The model presented here is a dynamic model: the fair value liability can be evaluated in a multi-period setting. Consider a sequence of time periods: $[0,T_1], [T_1,T_2], \ldots, [T_{k-1},T_k]$ and apply our model in every one of the k periods, a system of partial differential equations like (2.11) – (2.12) can be solved sequentially for the valuation of the fair value liability over the k periods.

Finally to conclude the section, we present a mathematical formula for the solution of partial differential equation (2.11)-(2.12).

2.3 Theoretical solution

To derive a closed-form solution, several conditions are introduced.

First, let us define

(i) *Uniform ellipticity*: There exists a positive constant δ such that

$$\sum_{i,k=1}^{n} a_{i,k}(t,x)\,\eta_i\eta_k \geq \delta||\eta||^2 \tag{2.14}$$

holds for every $\eta \in R^d$ and $(t,x) \in [0,\infty) \times R^d$.

(ii) *Boundedness*:
The functions $a_{i,k}(t,x)$ and $\mu_i(t,x)$ are bounded in $[0,T] \times R^d$.

(iii) *Hölder continuity*:
The functions $a_{i,k}(t,x)$ and $\mu_i(t,x)$ are Hölder-continuous in $[0,T] \times R^d$.

THEOREM 3 *Under the conditions (i)-(iii) and (2.10), $u_t = \mathcal{A}u - ru$ has a unique fundamental solution $G(t,x;\tau,\xi)$; the solution of equations (2.11)-(2.12) is*

$$\begin{aligned} u(t,X) &= \int_{R^d} G(t,X;0,\xi)F(\,g(X)\,)\,d\xi \\ &+ \int_0^t \int_{R^d} G(t,X;\tau,\xi)\,g(X)\,d\xi d\tau \end{aligned} \tag{2.15}$$

The proof of the Theorem 3 is given in Appendix 2. Theorem 3 provides a theoretical basis for the solution of equations (2.11)-(2.12). In practice, however, numerical solution of equations (2.11)-(2.12) should be seeked for any fair value liability valuation.

3. Applications

In this section, we consider the implementation issues of the model presented in previous section and its applications.

3.1 Discount Rate

We start with discussion on the discount rate, r, used in defining fair value liability process

$$L(t) = \int_0^t (\sum_{i=1}^{n} x^{(i)}(\xi))e^{-r\xi}d\xi + F(\sum_{i}^{n} x^{(i)}(t))e^{-rt}. \tag{3.1}$$

The discount rate is the interest rate at which the investment funds earn interest. The simplest way to implement the model is to use the risk-free interest rate as the discount rate r. Although the risk-adjusted rate is not used directly, the estimated fair value liability $u(t, X)$ is risk adjusted. The equation (2.11) is risk adjusted since its coefficients includes the covariance matrix $a(t, X)$ (see the definition of $\mathcal{A}$ in equation (2.9)).

The discount rate r can also be risk-adjusted as

$$r = r_f + \pi$$

by assuming that the short rate $R(t)$ follows process

$$dR(t) = rR(t)dt + \sigma_R(t, R)dW$$

where π is the market risk premium and σ_R is the local volatility of $R(t)$. There are many literatures in finance and economics on valuation and hedge of interest rate risk. Examples inlcude Duffie (1992), Hull (2000), Heath, Jarrow and Morton (1992).

3.2 Parameter Estimation

In order to solve equations (2.11) – (2.12), the parameters $\{\mu_i, i = 1, \ldots, n;\}$ and $\{a_{i,k}, i, k = 1, \ldots, n\}$ in Equation (2.11) need to be selected first. Simulation techenique are the methods most widely used today by actuaries to solve this problem. Recent advance in computing technology has significantly increased the accuracy and reduced the cost of the simulation. Patel and Raws (1999) presented a simulation approach in reserve valuation. As far as the data used for the simulation, we recommend a weighted average of simulation base on public data and company-specific data.

3.3 Case Studies

We now show some numerical examples of estimating fair value liability by solving equation (2.11) – (2.12) in case studies.

Case Study of Mono-line Business

We first consider a mono-line liability reserve with uncertain cash flows: assuming the instantaneous loss amount $X(t)dt$ satisfy

$$dX = 0.08dt + 2dW.$$

Assume that the investment return is 4% ($r = 4\%$) and $F(X) = X^{1.5}$. Using Theorem 1, we calculated the fair value liability by solving Equation (2.4) and (2.5). We used finite differences method to solve $(2.4) and (2.5)$ numerically. The estimated fair value liability with different initial individual loss levels are given in Figure 2.

Next, we consider a mono-line liability paid out over a longer period of time has higher uncertainty:

instead of constant volatility, we consider varying volatility:

assuming the instantaneous loss amount $X(t)dt$ satisfy

$$dX = 0.08dt + \sigma(t) = 2\sqrt{1+t}dW,$$

with all the other parameters remaining unchanged.

Figure 3 presents the computed values of fair value liability in this case.

Our estimates show that the fiar value liability with nonconstant volatility is more sensitive to the initial claim levels. Figure 4 makes a comparison of the two situations.

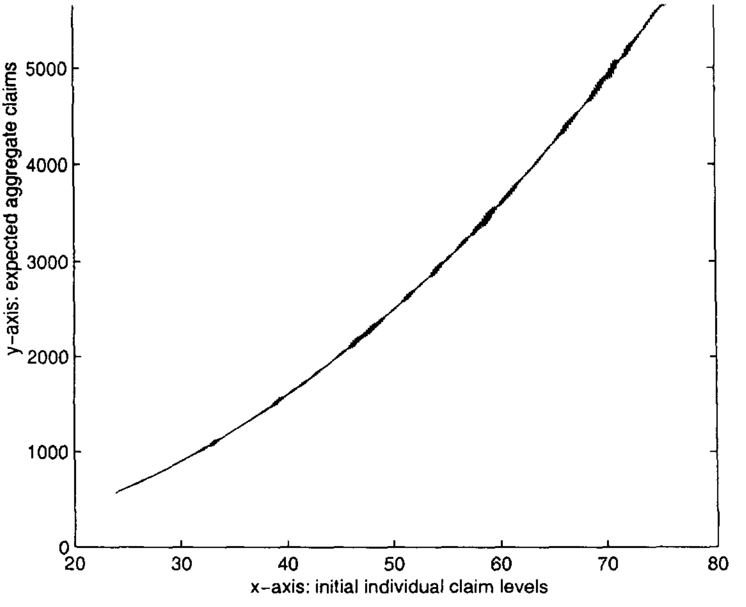

Figure 2. The expected fair value liability with variance $\sigma = 2.0$.

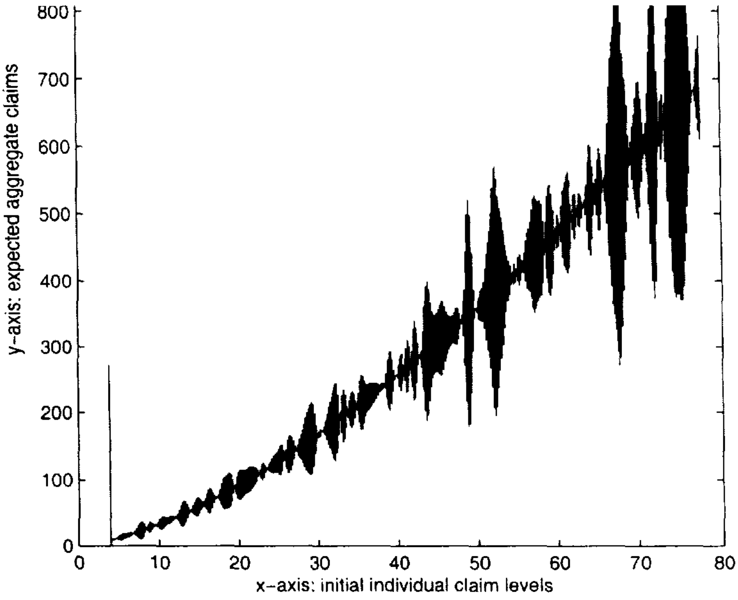

Figure 3. The expected fair value liability with variance $\sigma(t) = 2\sqrt{1+t}$.

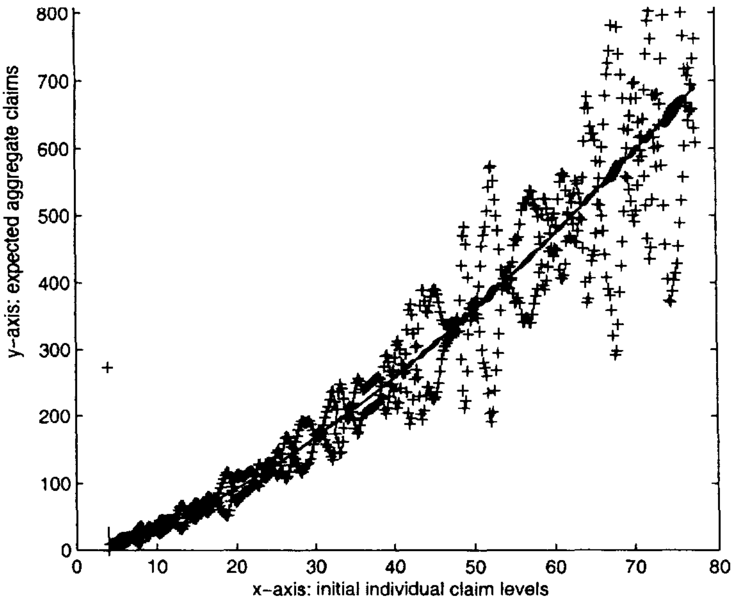

Figure 4. The differences in the expected fair value liability with covariance 2.0

Case Study of Multi-line Business

Assume an insurer writes two lines of business with uncertain cashflows. Let the loss process be:

$$X = (X^{(1)}(t), X^{(2)}(t)),$$

Assume $X^{(1)}(t)$ represent a property reserve with drift $\mu = 0.08$ and local volatility of $\sigma = 2$. Assume $X^{(2)}(t)$ represent a liability reserve with drift $\mu = 0.1$ and local volatility of $\sigma = 5$. Assume the correlation between the property reserve and the liability reserve be 1.5.

Therefore the drift μ and the covarance matrix $\sigma(t, X)$ are

$$\mu(t, X) = \begin{pmatrix} .08 \\ .1 \end{pmatrix}.$$

$$\sigma(t, X) = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 5 \end{pmatrix}.$$

Let the discount rate remain at 4% and the function F be defined as

$$F(X) = ((x^{(1)})^3 + (x^{(2)})^{1.5})^2.$$

Using Theorem 2 in Section 2.2, we calculated the fair value liability by solving Equations (2.11), (2.12).

Again, we used a finite difference method to calculate the estimated fair value liability. Figure 5 shows the computed values of the fair value liability.

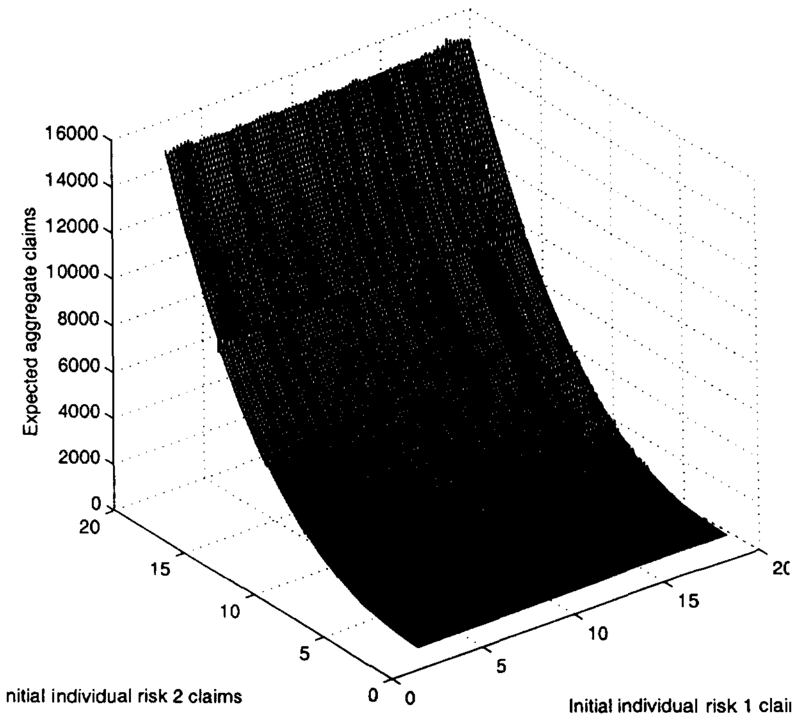

Figure 5. The fair value liability with constant variance.

Next, we checked how different levels of the correlation affect the estimated liabilities. As indicated in Table 1 , our estimates show that, in majority of cases, the fair value liability are lower when the loss claims between the lines of business are less correlated.

Table 1. Expected Fair Value Liability

(x_1, x_2)	$\sigma_{12} = 0$	$\sigma_{12} = 0.5$	$\sigma_{12} = 1.5$
(5, 5)	290.8	291.4	303.0
(5, 10)	330.4	332.7	372.9
(5, 15)	404.5	409.1	463.5
(5, 18)	434.6	434.4	427.6
(10, 5)	1956.9	1927.4	1657.5
(10, 10)	1908.7	1993.1	1565.11
(10, 15)	2171.7	2205.5	2438.1
(10, 18)	2283.8	2346.1	2902.2
(15, 5)	6687.2	6675.2	6583.5
(15, 10)	7134.8	7179.2	7540.8
(15, 15)	6903.7	6904.1	6947.6
(15, 18)	6845.7	6835.2	6759.6

Table 1 also shows that, for a fixed level of covariance, the calculated fair value liability increase as the initial loss amounts increase.

Finally, we considered the case when volatility varied with time. Assume all the other parameters remain the same and let

$$\sigma(t, X) = \begin{pmatrix} .08\sqrt{1+t} & .5 \\ .5 & 2(.5+t)^{\frac{1}{3}} \end{pmatrix}.$$

The estimated liabilities are shown in Figure 6.

The comparision of the estimated fair value liability (when the initial risk 1 claim level is x=9) between the constant volatility and non-constant volatility is shown in Figure 7.

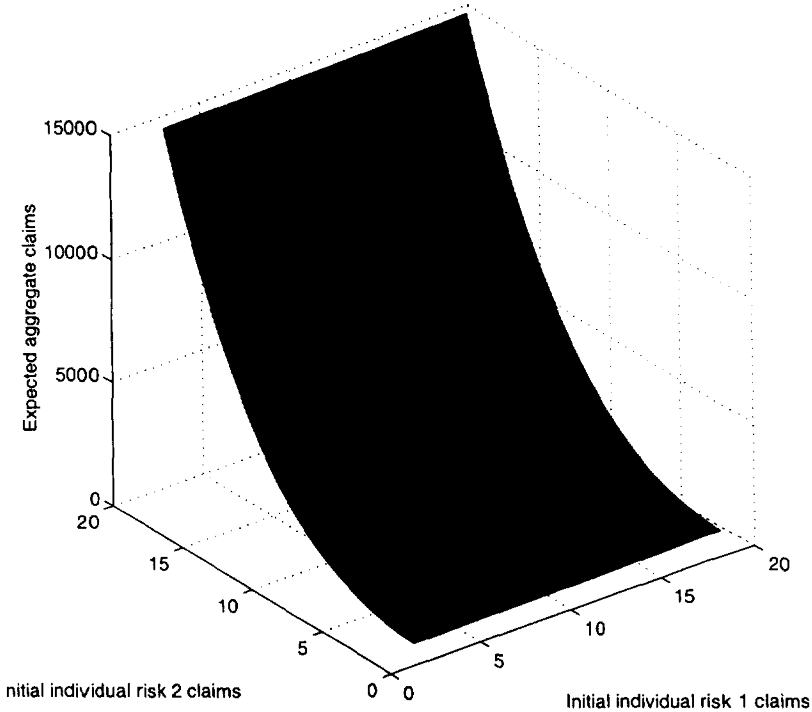

Figure 6. The fair value liability with NON constant variance.

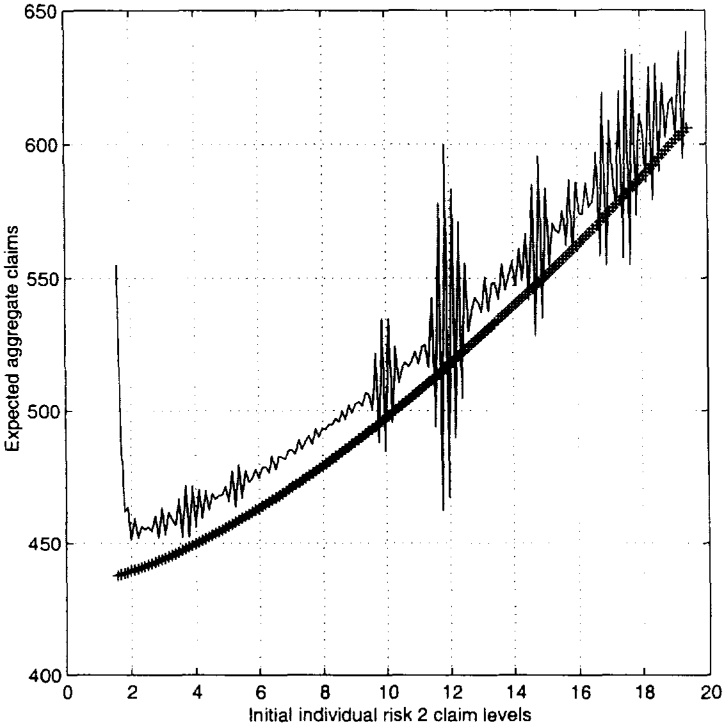

Figure 7. The fair value liability with constant variance (x=9): —
The aggregate claims with NON constant variance (x=9): +++

3.4 Applications in Reinsurance

In Section 2, a new method is provided for the estimation of the expected fair value liability without assuming independence of the individual losses. There are a number of applications of the method other than estimating fair value insurance liability. In the following, we discuss the applications of our method in reinsurance.

First we consider the problem of calculating stop-loss premiums.

Let p be the stop-loss premium, K be the cap. and L the fair value liability as defined in section 2.1:

$$L(t) = \int_0^t X(\xi)\, e^{-r\xi} d\xi + F(X(t))e^{-rt}$$

Assume L follows

$$dL(t) = \rho(t, L)dt + \nu(t, L)dW \tag{3.2}$$

At time T, the benefit is $max\{0, L(T) - K\} \equiv (L - K)^+$.

Define $v(t, L) = E[e^{-r(T-t)}\,(L - K)^+ \,|L(0) = L]$,
where r is the risk-free interest rate.

Then the fair value of the stop-loss premium should be $p = v(0, L)$. Using the analogue of Theorem 2 in Section 2, $v(t, L)$ is solved from the following:

$$v_t = \frac{1}{2}\nu^2 v_{LL} + \rho v_L - rv, \tag{3.3}$$

$$v(0, L) = (L - K)^+. \tag{3.4}$$

Remark: Note that the above partial differential equation is different from the Black-Scholes' partial differential equation or its type. Since L is not tradable, there is no risk neutral measure. Therefore ρ can't be replaced by a riskfree rate in equation (3.3).

Remark: In theory, p can be calculated from equations (3.3) – (3.4). However, there is no explicit formula to estimate ρ and ν without assuming the independence or some specific form of the dependence of the individual claims. One can, however, use the solution of (2.11)-(2.12) as an estimate of ρ.

In the following, we show a numerical example of calculating the stop-loss premiums, $p = U(0, S)$, and assume there is one risk source.

Recall that in the Case Study of Mono-line Business, where we consider a mono-line liability reserve with uncertain cash flows: assuming the instantaneous loss amount $X(t)dt$ satisfy

$$dX = 0.08dt + 2dW.$$

Assume that the investment return is 4% ($r = 4\%$) and $F(X) = X^{1.5}$.

Assume the initial individual claim is $x_0 = 30.8$. Using the estimates calculated in Section 3.3 as an approximation for ρ: $\rho = 178.4952$. We solved Equations (3.3) and (3.4) numerically. For the stop-loss cap $K = 160$, the stop-loss premiums calculated based on different aggregate claim levels are given in Figure 8.

We again looked at the case that the liability cash flows are more uncertian. Figure 9 compares the stop-loss premiums with constant volatility and varying volatility.

Finally, we tested how much change in stop-loss premium is due to the change of the value of ρ which is presented in Figure 10.

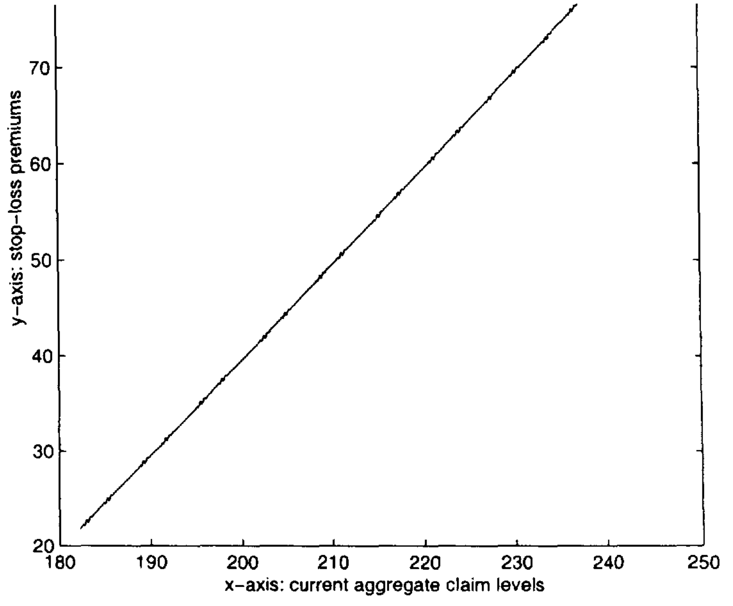

Figure 8. The stop-loss premiums with constant variance

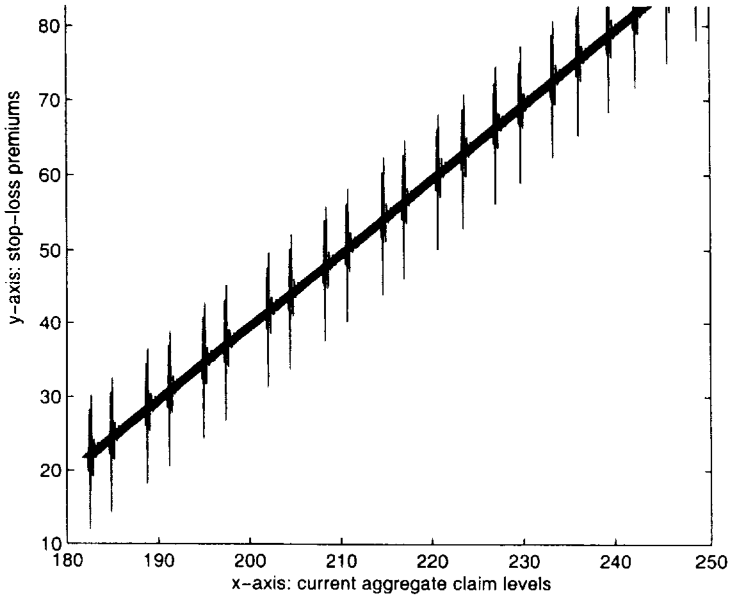

Figure 9. The stop-loss premiums with NON constant variance.

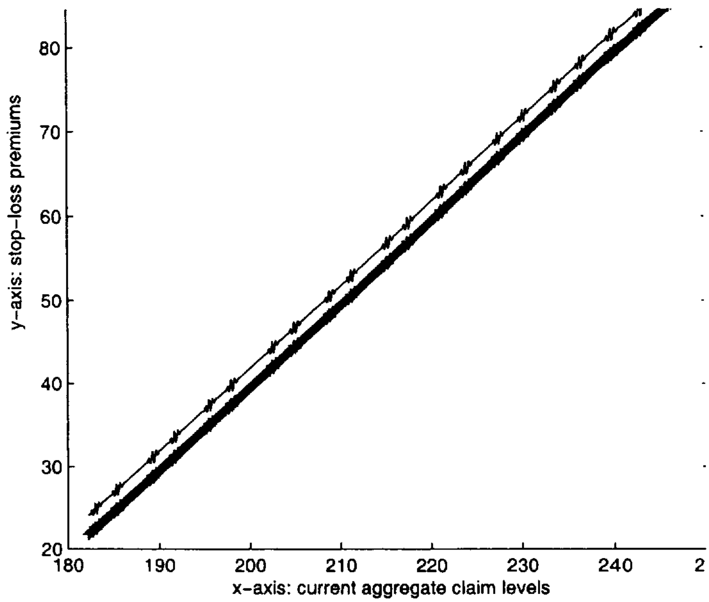

Figure 10. The differences in stop-loss premiums with changes in drifts.

Another application in reinsurance is the valuation of CATS index options. The price of a Catastrophe Insurance Futures and options (CATS) could be estimated using this approach. For a detailed discussion, see Guo (2000).

4. Discussion of the Method

In this section, we provide our view on the comparision between our method and the existing methods.

Our method provides a direct estimation of fair value liability. It used a combination of the financial approach and the actuarial approach. Unlike the method of Allen, Cummins and Phillips (1998), our method considers the impact of a particular company at issue or even specific lines of business of the company. It doesn't rely on the CAPM model, which may not accurately predict returns for insurance firms and no need to estimate the underwriting betas. There is a component of risk-adjusted discount method in our approach when the discount rate r in Equation (2.11) is risk-adjusted. The derivation of our method start with study individual loss risk process like actuarial distribution-based risk loads methods. Instead of calculating the risk loadhowever, our method estimate the risk-loaded fair value liability directly using the contigent-claim analysis in modern financial theory. Finally, the application of our method in valuation of stop-loss premium and CATS premium might provide some connection to the method of using the reinsurance market to estimate the fair value of liabilities.

5. Summary

This study provided a new dynamic method to estimate $E[L(T)]$, the expected fair value liability for a multiple line business.

The paper adopted the contingent claim analysis in modern finance theory to model the aggregate fair value liability for multiple lines of business. An important feature of the method is to concentrate on calculating the risk-loaded expectation of the aggregate liability instead of attempting to find the actual liability distribution in a complicated economic environment. The fair value liability was derived by solving a partial differential equation. Finite difference method was used to obtain the numerical solution as shown in the examples. The dynamic feature of the method make it possible to evaluate the fair value liability over the multiple periods by solving a system of partial differential equations sequentially. The effects of non-constant variance matrix on the liability estimate were discussed in the numerical examples. The paper also addressed some applications of the method including the evaluation of stop-loss premiums among others. The paper presents only the preliminary result of our study. A case study for the implementation of the new method and the comparison of other existing methods is under the way. Future research areas include creating a highly efficient and flexible simulation algorithm for the parameter estimation; deriving more accurate and stable numerical method for the partial differential equation; estimating the fair value liability with a stochastic interest rate process $\{r(t), 0 \leq t \leq T\}$; and extending the loss process to a more general risk process including a jump process, etc.

References

[1] Allen, F., J. D. Cummins and R. D. Phillips, 1998, "Financial Pricing of Insurance in a Multiple Line Insurance Company", Journal of Risk and Insurance, 1998, volume 65, pp. 597-636.

[2] Baxter, M. and A. Rennie, 1996, *Financial Calculus - An introduction to derivative pricing*, Springer-Verlag, Cambridge.

[3] Beard, R., T. Pentikäinen, and E. Pesonen, (1984). *Risk Theory*, 3rd edition, Chapman and Hall, London.

[4] Butsic, R., Determining the Proper Discount Rate for Loss Reserve Discounting: An Economic Approach, 1988 Casualty Actuarial Society Discussion Paper Program - Evaluating Insurance Company Liabilities, pp. 147-188.

[5] Butsic, R. P., 1994, "Solvency Measurement for Property-Liability Risk-Based Capital Applications", Journal of Risk and Insurance, 61: 656-690.

[6] Butsic, R. P., 1999, "Capital Allocation for Property Liability Insurers: A Catastrophe Reinsurance Application". Casualty Actuarial Society Forum, Fall 1999.

[7] CAS Task Force on Fair Value Liabilities, 2000. "White Paper on Fair Valuing Property/casualty Insurance Liabilities", Casualty Actuarial Society.

[8] Cornell, B., 1999. "Risk, Duration and Capital Budgeting: New Evidence on Some Old Questions", Journal of Business, vol. 72, pp 183-200.

[9] Cummins, J. D., "Multi-Period Discounted Cash Flow Ratemaking Models in Property-Liability Insurance", Journal of Risk and Insurance, March 1990, Volume 57:1, pp. 79-109.

[10] D'Arcy, S. P., 1988, "Use of the CAPM to Discount Property-Liability Loss Reserves", Journal of Risk and Insurance, Vol. 55:3, pp. 481-490.

[11] D'Arcy, S. P., and N. A. Doherty, 1988. "The Financial Theory of Pricing Property-Liability Insurance Contracts." Huebner Foundation.

[12] Derrig, R. A., 1994. "Theoretical Considerations of the Effect of Federal Income Taxes on Investment Income in Property-Liability Ratemaking", Journal of Risk and Insurance, 61: 691-709.

[13] Dhaene, J. and M. J. Goovaerts, 1997. "On the dependency of risks in the individual life model", *Insurance, Mathematics and Economics* 19, 243-253.

[14] Duffie, D., 1992. *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, New Jersey.

[15] Embrechts, P., 1995. "Risk theory of the second and third kind", *Scandinavian Actuarail Journal*, 1, 35-43.

[16] Fairley, W., 1987. Investment Income and Profit Margins in Property Liability Insurance: theory and Empirical Evidence, Fair Rate of Return in Property-Liability Insurance, Cummins, J.D., Harrington S.A., Eds, Kluwer-Nijhoff Publishing, pp. 1-26.

[17] Fama, E. and French, K., 1992. "The Cross Section of Expected Stock Returns", Journal of Finance, Vol 47, pp. 427-465

[18] Fama, E. and K. French, 1997. "Industry Costs of Equity", Journal of Financial Economics, Vol 43, pp. 153-193.

[19] Feldblum, S., 1990. "Risk Load for Insurers", PCAS LXXVII, pp. 160-195

[20] Friedman, A., 1964. *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Englewood Cliffs, N.J.

[21] Friedman, A., 1975. *Stochastic Differential Equations and Applications*, Volumn 1. Academic Press, New York.

[22] Hayne, R. M., 1989. "Application of Collective Risk Theory to Estimate Variability in Loss Reserves," Proceedings of the Casualty Actuarial Society (PCAS), LXXVI, p. 77-110

[23] Heath, D., R. Jarrow, and A. Morton, 1992. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation," *Econometrica*, 60(1), 77-105.

[24] Heckman, P. and G. Meyers, 1983. "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distribution", *Proceedings of the Casualty Actuarial Society*, 70, 22-61.

[25] Heckman, P., 1999. "Seriatim. Claim Valuation from Detailed Process Models," paper presented at Casualty Loss Reserve Seminar.

[26] Hull, J.C., 2000. *Options, Futures, and Other Derivative Securities*, (Fourth Edition). Prentice-Hall.

[27] Karatzas, I. and S. E. Shreve, 1991. *Brownian Motion and Stochastic Calculus*, second edition, Springer-Verlag.

[28] Kreps, R., 1998. "Investment Equivalent Reinsurance Pricing," PCAS.

[29] Kreps, R. E., 1990. "Reinsurer Risk Loads from Marginal Surplus Requirements," Proceedings of the Casualty Actuarial Society (PCAS), LXXVII, p. 196

[30] Mack, T., 1995. "Which Stochastic Model is Underlying the Chain Ladder Method," CAS Forum, Fall 1995, pp.229-240

[31] Mahler, H., 1998. "The Meyers-Cohn Profit Model, A Practical Application," PCAS LXXXV, pp. 689-774.

[32] Meyers, S and R. Cohn, 1987. "A Discounted Cash Flow Approach to Property-Liability Rate Regulation," Fair Rate of Return in Property-Liability Insurance, Cummins, J.D., Harrington S.A., Eds, Kluwer-Nijhoff Publishing, 1987, pp. 55-78.

[33] Meyers, G., 2000. "The Cost of Financing Insurance", paper presented to the NAIC's Insurance Securitization Working Group at the March 2000 NAIC quarterly meeting.

[34] Meyers, G., 1991. "The Competitive Market Equilibrium Risk Load Formula for Increased Limits Ratemaking," Proceedings of the Casualty Actuarial Society (PCAS), LXXVIII, pp 163-200.

[35] Meyers, G., 1989. "Risk Theoretic Issues in Loss Reserving: The Case of Workers Compensation Pension Reserves," Proceedings of the Casualty Actuarial Society (PCAS), LXXVI, p. 171

[36] Meyers, G. and N. Schenker, 1983. "Parameter Uncertainty in the Collective Risk Model," PCAS, 1983, pp. 111-143

[37] Patel, C. C. and A. Raws III, 1999. *Statistical Modeling Techniques for Reserve Ranges: A Simulation Approach*, Casualty Actuarial Society FORUM, Fall 1999, pp 367 -401.

[38] Panjer, H. and G. Willmot, 1992. *Insurance Risk Models*, The Society of Actuaries.

[39] Philbrick, S. W., 1994. "Accounting for Risk Margins," Casualty Actuarial Society Forum, Spring 1994, Volume 1, pp. 1-87

[40] Ronn, E. I., and A. K. Verma, 1986. "Pricing Risk-Adjusted Deposit Insurance: An Option-Based Model," Journal of Finance, 41(4): 871-895.

[41] Roth, R., 1992. "Analysis of Surplus and Rates of Return Using Leverage Ratios", 1992 Casualty Actuarial Society Discussion Paper Program - Insurer Financial Solvency, Volume 1, pp 439-464

[42] Venter, G., 1991. "Premium Implications of Reinsurance Without Arbitrage," ASTIN Bulletin, 21 No. 2: 223-232.

[43] Venter, G., 1998. "(Discusssion of) Implementation of the PH-Transform in Ratemaking, [by Shaun Wang]", presented at the Fall, 1998 meeting of the Casualty Actuarial Society.

[44] Wang, S., 1998. "Aggregation of Correlated Risk Portfolios: Models Algorithms", *Proceedings of the Casualty Actuarial Society* , pp 848-939, 1998.

[45] Wang, Shaun, 1998. "Implementation of the PH-Transform in Ratemaking", [Presented at the Fall, 1998 meeting of the Casualty Actuarial Society].

[46] Zehnwirth, B., 1994. "Probabilistic Development Factor Models with Application to Loss Reserve Variability, Prediction Intervals and Risk Based Capital," CAS Forum, Spring 1994, pp. 447-606.

6. Appendix 1

This appendix presents the proof for the Theorem 2 in Sectin 2.

Theorem 2

Suppose that σ and μ satisfy the linear growth condition

$$||\mu(t,X)||^2 + ||\sigma(t,X)||^2 \leq K^2(1 + ||X||^2) \tag{6.1}$$

for every $0 \leq t < \infty$, $x \in R^n$,

$$|F(X)| \leq K^2(1 + ||X||^2)$$

for every $x \in R$,
where K is a positive constant; and assume that $u(t,X)$ is continuous, and is of class $C^{1,2}([0,T] \times R^n)$.
Then
$u(t,X)$ satisfies the Cauchy problem

$$u_t = Au - ru + g(X\cdot); \quad in\,[0,T) \times R^n \tag{6.2}$$

and

$$u(0,X) = F(g(X)); \quad X \in R^n \tag{6.3}$$

as well as the polynomial growth condition:

$$max_{0\leq t}\,|u(t,X)| \leq M(1 + ||X||^{2\eta}); X \in R^n \tag{6.4}$$

for some $M > 0, \eta \geq 1$.

PROOF

Suppose v is a solution of $(6.2) - (6.3)$. We apply the Itô lemma and integration by parts to the process
$v(t-\xi, X_\xi)e^{-r\xi}; \xi \in [0,t]$, in conjunction with (2.11):

$$d[v(t-\xi, X_\xi)e^{-r\xi}] = e^{-r\xi}[-g(X_\xi)d\xi + \sum_{i=1}^{d} v_{x_i}(t-\xi, X_\xi)\sigma_i dW_{(i)}].$$

Let $\tau_n \equiv inf\{\xi \geq 0; ||X_\xi|| \geq n\}$;
we obtain

$$v(t,X) = E[F(g(X))e^{-rt}1_{\{\tau_n>t\}}|X(0)=X] + E[\int_0^{t\wedge\tau_n} g(X(\xi))e^{-r\xi}d\xi|X(0)=X]$$
$$+E[v(\tau_n, X_{\tau_n})e^{-r\tau_n}1_{\{\tau_n\leq t\}}|X(0)=X] \quad (6.5)$$

7. Appendix 2

This appendix presents the proof for the Theorem 3 in Sectin 2.

Theorem 3 Under the conditions (i)-(iii) and (2.10), $u_t = \mathcal{A}u - ru$ has a unique fundamental solution $G(t, x; \tau, \xi)$;
the solution of equation (2.11)-(2.12) is

$$u(t, X) = \int_{R^d} G(t, X; 0, \xi) F(\, g(X)\,)\, d\xi \\ + \int_0^t \int_{R^d} G(t, X; \tau, \xi)\, g(X)\, d\xi d\tau \tag{7.1}$$

PROOF

Under the conditions (i)-(iii), there is a fundamental solution $G(t, x : \tau, \xi)$ of

$$u_t = \mathcal{A}u - ru; \quad in\, [0, T) \times R^n \tag{7.2}$$

and

$$u(0, X) = F(X); \quad X \in R^n \tag{7.3}$$

(see Friedman (1975, pp141, 148 and Friedman (1964) Chapter I). For fixed $(\tau, \xi) \in (0, T] \times R^d$, the function $G(t, x : \tau, \xi)$ is of class $C^{1,2}((0, T] \times R^d)$ and

$$u(t, X) = \int_{R^d} G(t, X; 0, \xi) F(X)\, d\xi$$

satisfies (7.2) – (7.3). We recall from Theorem 2 that the solution of (7.2) – (7.3), with $r = 0$, is given by

$$u(t, X) = E[F(X(t))|X(0) = x]$$

This leads to the conclusion that any fundamental solution $G(t, x : \tau, \xi)$ is also the transition probability density for the process X; i.e.,

$$P[X(\tau)|X(t) = x \in A] = \int_A G(t, x : \tau, \xi) d\xi;\ 0 \leq t < \tau \leq T.$$

In particular, under the condition (2.10), this fundamental solution is unique, and

$$u(t,X) \;=\; E[\int_0^t (\sum_i^n x^{(i)}(\xi))e^{-r\xi}\,d\xi \;+\; F(\sum_i^n x^{(i)}(t))e^{-rt}\,|\,X = X(0)],$$

the solution to equation (2.11) and (2.12) now takes the form

$$u(t,X) = \int_{R^d} G(t,X;0,\xi)F(\,g(X)\,)\,d\xi + \int_0^t \int_{R^d} G(t,X;\tau,\xi)\,g(X)\,d\xi d\tau.$$